CS 580 HW 7

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Q1

It's obvious that Enormous-Cut $\in$ NP, since given a certificate, we can easily check the certificate in polynomial time (just sum up the edge weights for edges with one endpoint in U and the other endpoint not in U and check whether the sum is $\geq$ K).

We'll reduce Set-Partition to Enormous-Cut thereby proving Enormous-Cut to be NP-Hard. We already showed Enormous-Cut *in* NP, so together we'll have that Enormous-Cut is NPC.

Let R = $\{r_1, r_2, ...r_n\}$ be an instance of Set-Partition (i.e a set of positive integers). We'll construct a graph G such that R contains a valid partition if and only if G contains a subset of vertices, U, s.t c(U) = $(\sum\limits_{u \in U} r_u)^2$ where c(U) denotes the sum of the weights of the edges with one endpoint in U and the other not in U and r_u denotes the uth integer in R.

We construct G as follows:

Let G be the complete graph on n vertices, one vertex for each integer in R (i.e the ith integer in R corresponds to the ith vertex in G). Then, let w_{ij}, the weight between vertex i and vertex j (i $\neq$ j), be $r_i * r_j$.

Claim: R contains a valid partition if and only if G contains a subset of vertices, U, s.t c(U) = $(\sum\limits_{u \in U} r_u)^2$

Proof:

($\Rightarrow$) Suppose R has a valid partition. That is, there exists some subset of R, S, s.t $\sum\limits_{r \in S} r = \sum\limits_{r \in R-S} r$. Consider the corresponding subset of vertices, U, in G. We then have that c(U) = $\sum\limits_{u \in U} \sum\limits_{v \in V-U} r_u r_v = \sum\limits_{u \in U} r_u \sum\limits_{v \in V-U} r_v = (\sum\limits_{u \in U} r_u)^2$ since $\sum\limits_{u \in U} r_u = \sum\limits_{v \in V-U} r_v$.

($\Leftarrow$) Suppose G contains a subset of vertices, U, s.t c(U) = $(\sum\limits_{u \in U} r_u)^2$. By definition, c(U) = $\sum\limits_{u \in U} \sum\limits_{v \in V-U} r_u r_v$ which by hypothesis equals $(\sum\limits_{u \in U} r_u)^2$. Thus, we have $\sum\limits_{u \in U} \sum\limits_{v \in V-U} r_u r_v - (\sum\limits_{u \in U} r_u)^2 = 0 \Rightarrow \sum\limits_{u \in U} r_u (\sum\limits_{v \in V-U} r_v - \sum\limits_{u \in U} r_u) =$

$0 \Rightarrow \sum_{v \in V-U} r_v - \sum_{u \in U} r_u = 0$ since $\sum_{u \in U} r_u > 0 \Rightarrow \sum_{u \in U} r_u = \sum_{v \in V-U} r_v$. Let S be the subset of integers in R corresponding to U, and we have the desired conclusion.

(Note: Technically, Enormous-Cut answers the question "Is c(U) $\geq$ K?" instead of "Is c(U) = K?", but if we just ask "Is c(U) $\geq$ K?" and "Is c(U) $\geq$ K + 1?", and the answer to the former is yes, but the answer to the latter is no, then that means c(U) = K)

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