Q.1 Autocorrelation of y(n)

Given, $y(n) = u(n+a) - u(n-a)$

$$r_u(k) = E\left[u(n)\,u^*(n-k)\right]$$

$$r_y(k) = E\left(y(n)\,y^*(n-k)\right)$$

substituting y(n) in above,

$$r_y(k) = E\left[\left(u(n+a) - u(n-a)\right)\left(u^*(n+a-k) - u^*(n-a-k)\right)\right]$$

$$= E\left[u(n+a)\,u^*(n+a-k) - u(n+a)\,u^*(n-a-k) - u(n-a)\,u^*(n+a-k) + u(n-a).u^*(n-a-k)\right]$$

$$= E\left[u(n+a)\,u^*(n+a-k)\right] - E\left[u(n+a)\,u^*(n-a-k)\right] - E\left[u(n-a)\,u^*(n+a-k)\right] + E\left[u(n-a)\,u^*(n-a-k)\right]$$

$$= r_u(k) - r_u(2a+k) - r_u(-2a+k) + r_u(k)$$

$$= 2r_u(k) - r_u(2a+k) - r_u(-2a+k)$$

Q.2 1.7

$$u(n) = u(n-1) - 0.5\,u(n-2) + v(n)$$

$$a_1 = -1, \quad a_2 = 0.5$$

$$w_1 = 1, \quad w_2 = -0.5$$

Yule walker eqn^s →

$$\boxed{RW = r}$$

$$\begin{bmatrix} r(0) & r(1) \\ r(1) & r(0) \end{bmatrix} \begin{bmatrix} 1 \\ -0.5 \end{bmatrix} = \begin{bmatrix} r(1) \\ r(2) \end{bmatrix}$$

(b) expanding the above eq^n ⇒

$$\begin{cases} r(0) - 0.5 \cdot r(1) = r(1) \\ r(1) - 0.5 \cdot r(0) = r(2) \end{cases}$$

solving

$$r(1) = \frac{2}{3} r(0) \quad \& \quad r(2) = \frac{1}{6} r(0)$$

(c) $\mathrm{var}[u(n)] = r(0)$ [since $v(n)$ has zeromean, so will $u(n)$]

$$\& \quad \sigma_v^2 = \sum_{k=0}^{2} a_k r(k)$$

$$0.5 = r(0) + a_1 r(1) + a_2 r(2)$$

$$0.5 = r(0) + (-1) \times \frac{2}{3} r(0) + (0.5) \times \frac{1}{6} r(0)$$

$$\frac{1}{2} = \left(1 - \frac{2}{3} + \frac{1}{12}\right) r_0$$

$$r(0) \times \frac{5}{\cancel{12}\,6} = \frac{1}{2} \Rightarrow r(0) = \frac{6}{5} = \underline{\underline{1.2}}.$$

Q 3 $\boxed{2.2}$

$$R = \begin{bmatrix} 1 & 0.5 \\ 0.5 & 1 \end{bmatrix} \quad \left(E[u(n) u^T(n)] \right)$$

$$P = [0.5 \,,\, 0.25]^T \quad \left(E[d(n) . u(n)] \right)$$

(a) Tap weights of weiner filter?

let, $w = [w_1, w_2]^T$

from Weiner-Hopf equation $\Rightarrow$

$$w_0 = R^{-1} p$$

$$= \begin{bmatrix} 1 & 0.5 \\ 0.5 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 0.5 \\ 0.25 \end{bmatrix}$$

$$= \frac{1}{3} \begin{bmatrix} 4 & -2 \\ -2 & 4 \end{bmatrix} \begin{bmatrix} 0.5 \\ 0.25 \end{bmatrix}$$

$$= \frac{1}{3} \begin{bmatrix} 1.5 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0.5 \\ 0 \end{bmatrix} = \underline{\underline{[0.5, 0]^T}}$$

(b) $J_{min} = \sigma_d^2 - p^T w_0$

$$= \sigma_d^2 - \begin{bmatrix} 0.5 & 0.25 \end{bmatrix} \begin{bmatrix} 0.5 \\ 0 \end{bmatrix}$$

$$= \sigma_d^2 - 0.25$$

(c) Let eigenvalues of matrix R be λ

then,

$$|R - \lambda I| = 0$$

$$\left| \begin{bmatrix} 1 & 0.5 \\ 0.5 & 1 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} \right| = 0$$

$$(1-\lambda)^2 - (0.5)^2 = 0$$

$$(1-\lambda+0.5)(1-\lambda-0.5) = 0$$

$$(1.5-\lambda)(0.5-\lambda) = 0$$

$$\boxed{\lambda = 1.5,\ 0.5}$$

Let eigenvectors be $v = \begin{bmatrix} v_{11} \\ v_{12} \end{bmatrix}$

then

$$Rv = \lambda v$$

$$\begin{bmatrix} 1 & 0.5 \\ 0.5 & 1 \end{bmatrix} \begin{bmatrix} v_{11} \\ v_{12} \end{bmatrix} = \lambda \begin{pmatrix} v_{11} \\ v_{12} \end{pmatrix}$$

$$\left(\begin{bmatrix} 1 & 0.5 \\ 0.5 & 1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_{11} \\ v_{12} \end{bmatrix} = 0$$

$$\begin{bmatrix} 1-\lambda & 0.5 \\ 0.5 & 1-\lambda \end{bmatrix} \begin{bmatrix} v_{11} \\ v_{12} \end{bmatrix} = 0 \quad \text{--- } (1)$$

For $\lambda = 0.5$

$$\begin{bmatrix} 0.5 & 0.5 \\ 0.5 & 0.5 \end{bmatrix} \begin{bmatrix} v_{11} \\ v_{12} \end{bmatrix} = 0$$

$$\Downarrow$$

$$v_{11} + v_{12} = 0$$

$$\boxed{v_{11} = -v_{12}}$$

Normalizing v to 1,

$$v = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

For $\lambda = 1.5$, (1) gives,

$$\begin{bmatrix} -0.5 & 0.5 \\ 0.5 & -0.5 \end{bmatrix} \begin{bmatrix} v_{11} \\ v_{12} \end{bmatrix} = 0$$

$$v_{11} = v_{12},$$

Normalizing to 1,

$$v = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Matrix R can written in terms of its eigenvalues & eigenvectors as:

$$R = \phi \Lambda \phi^{-1},$$

Here, $\phi = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}$

$$\Lambda = \begin{bmatrix} 1.5 & 0 \\ 0 & 0.5 \end{bmatrix}, \Lambda^{-1} = \begin{bmatrix} 2/3 & 0 \\ 0 & 2 \end{bmatrix}$$

$$\phi^{-1} = \begin{bmatrix} 1/2\sqrt{2} & 1/2\sqrt{2} \\ 1/2\sqrt{2} & -1/2\sqrt{2} \end{bmatrix}$$

$$R^{-1} = \phi \Lambda^{-1} \phi^{-1} \quad \text{— (2)}$$

Following $w_0 = R^{-1} p$ & (2),

we get, $\boxed{w_0 = \phi \Lambda^{-1} \phi^{-1} p.}$

Q4. 2.6.

WH eq^n $\rightarrow$ $w_o = R^{-1} p \Rightarrow$ $p - R w_o = 0$

& $J_{min} = \sigma_d^2 - p^H R^{-1} p \Rightarrow \sigma_d^2 - p^H w_o = 0$

the above equations can be combined to give $\rightarrow$

$$\begin{bmatrix} \sigma_d^2 & p^H \\ p & R \end{bmatrix} \begin{bmatrix} 1 \\ -w_o \end{bmatrix} = \begin{bmatrix} J_{min} \\ 0 \end{bmatrix}$$

defining $A \triangleq \begin{bmatrix} \sigma_d^2 & p^H \\ p & R \end{bmatrix}$

expanding all matrices as expectations

$$\Rightarrow \quad A = \begin{bmatrix} E[d(n) d^*(n)] & E[d(n) u^H(n)] \\ E[u(n) d^*(n)] & E[u(n) u^H(n)] \end{bmatrix}$$

$$\Rightarrow \quad A = E\left\{ \begin{bmatrix} d(n) d^*(n) & d(n) u^H(n) \\ u(n) d^*(n) & u(n) u^H(n) \end{bmatrix} \right\}$$

$$A = E\left\{\begin{bmatrix} d(n) \\ u(n) \end{bmatrix}\begin{bmatrix} d^*(n) & u^H(n) \end{bmatrix}\right\}$$

Q5 $\boxed{2.7}$

$$J_{min} = \sigma_d^2 - p^H R^{-1} p.$$

eigenvalue decomposition of $R \rightarrow$

$$R = \phi \Lambda \phi^H$$

where $\phi = \begin{bmatrix} | & | & & | \\ q_1 & q_2 & \cdots & q_n \\ | & | & & | \end{bmatrix}$; e_i is eigenvector.

$\Lambda = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & & \vdots \\ \vdots & & \ddots & \\ 0 & \cdots & & \lambda_n \end{bmatrix}$; λ_i is the eigenvalue

$$\phi^{-1} = \begin{bmatrix} — & a_1^H & — \\ — & a_2^H & — \\ & \vdots & \\ — & a_n^H & — \end{bmatrix}$$

$$\therefore \; R = \sum_{k=1}^{n} \lambda_k q_k q_k^H$$

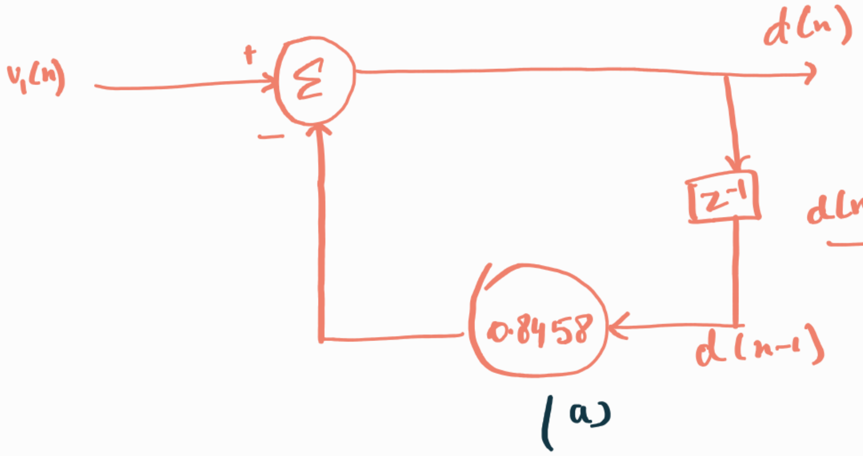
R^{-1} would similarly be

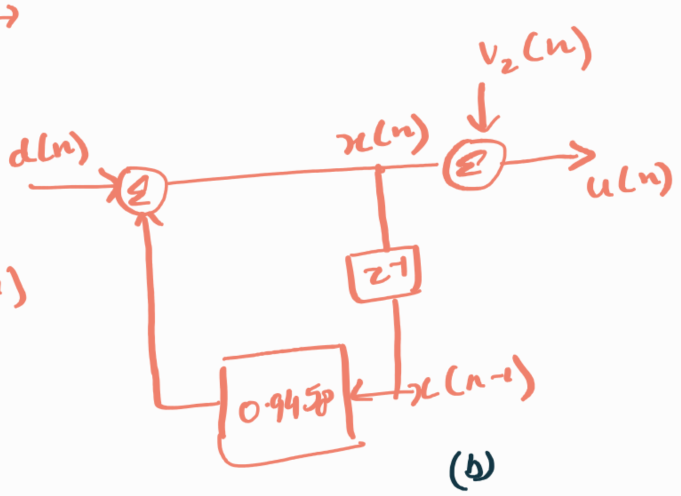
$$\boxed{R^{-1} = \sum_{k=1}^{n} \frac{1}{\lambda_k} q_k q_k^H}$$

$$J_{min} = \sigma_d^2 - \sum_{k=1}^{n} \frac{p^H q_k q_k^H p}{\lambda_k}$$

$$\boxed{J_{min} = \sigma_d^2 - \sum_{k=1}^{n} \frac{|q_k^H \cdot p|^2}{\lambda_k}}$$

Q6 $\boxed{2.11}$





(a) $u(n) = x(n) + v_2(n)$

(from image (b))

$$d(n) = v_1(n) - 0.8458\, d(n-1) \quad — (1)$$

$$x(n) = d(n) + 0.9458 \times x(n-1)$$

$$d(n) = x(n) - 0.9458\,x(n-1) \quad \text{--- (2)}$$

finding $d(n-1) \rightarrow$

$$d(n-1) = x(n-1) - 0.9458 \cdot x(n-2) \quad \text{--- (3)}$$

Putting (2) & (3) in (1)

$$x(n) - 0.9458\,x(n-1) = v_1(n) - 0.8458\left[x(n-1) - 0.9458\,x(n-2)\right]$$

$$x(n) = v_1(n) + 0.1\,x(n-1) + 0.8\,x(n-2)$$

$$\therefore \boxed{x(n) = 0.1\,x(n-1) + 0.8\,x(n-2) + v_1(n)}$$

(b) $u(n) = x(n) + V_2(n)$

Given, $x(n)$ & $V_2(n)$ are unrelated

$$\therefore R = R_x + R_{v_2}$$

$$R_x = \begin{bmatrix} r_x(0) & r_x(1) \\ r_x(1) & r_x(0) \end{bmatrix}; \quad a_1 = -0.1, \quad a_2 = -0.8$$
$$w_1 = 0.1, \quad w_2 = 0.8$$

Yule walker eqn's $\Rightarrow$

$$\begin{bmatrix} r_x(0) & r_x(1) \\ r_x(1) & r_x(0) \end{bmatrix} \begin{bmatrix} 0.1 \\ 0.8 \end{bmatrix} = \begin{bmatrix} r_x(1) \\ r_x(2) \end{bmatrix}$$

$$0.1\,r_x(0) + 0.8\,r_x(1) = r_x(1)$$
$$0.1\,r_x(1) + 0.8\,r_x(0) = r_x(2)$$

$$0.2\,r_x(1) = 0.1\,r_x(0)$$
$$r_x(1) = \frac{r_x(0)}{2}$$

$$0.1\,\frac{r_x(0)}{2} + 0.8\,r_x(0) = r_x(2)$$
$$r_x(2) = 0.85\,r_x(0)$$

$$\sigma_1^2 = r_x(0) + a_1 r_x(1) + a_2 r_x(2)$$

$$0.27 = r_x(0) - 0.1 \times 0.5\, r_x(0) - 0.8 \times 0.85 \times r_x(0)$$

$$= r_x(0)\left[1 - 0.05 - 0.68\right]$$

$$0.27 = 0.27\, r_x(0)$$

$$r_x(0) = 1$$

$$\therefore\ r_x(1) = 0.5$$

$$\therefore\ R_x = \begin{bmatrix} 1 & 0.5 \\ 0.5 & 1 \end{bmatrix} \qquad R_{v_2} = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.1 \end{bmatrix}$$

$$\therefore\ R = \begin{bmatrix} 1.1 & 0.5 \\ 0.5 & 1.1 \end{bmatrix}$$

let $p = \begin{bmatrix} P(0) \\ P(1) \end{bmatrix}$; $\therefore\ P(k) = E\left[u(n-k) \cdot d(n)\right]$; $k = 0, 1$

$$= E\left[u(n-k) \cdot \left(x(n) - 0.9458 \cdot x(n-1)\right)\right]$$

$$p(0) = E\left[u(n) \cdot \left(x(n) - 0.9458\, x(n-1)\right)\right]$$

$$p(0) = 1 - 0.9458 \times 0.5 = 0.5272$$

$$p(1) = E\left[u(n-1)\left(x(n) - 0.9458 \cdot x(n-1)\right)\right]$$

$$= 0.5 - 0.9458 \times 1 = -0.4458$$

Or, $p = \begin{bmatrix} 0.5272 \\ -0.4458 \end{bmatrix}$

(c) $w_o = R^{-1} p$

$$= \begin{bmatrix} 1.1 & 0.5 \\ 0.5 & 1.1 \end{bmatrix}^{-1} \begin{bmatrix} 0.5272 \\ -0.4458 \end{bmatrix}$$

$$= \begin{bmatrix} 0.8363 \\ -0.7853 \end{bmatrix}$$