Homework 1 Solution – ECE 251C

2.2.5

5. (a) The frequency response of the ideal highpass filter is a "brick wall" with DC unity gain:

$$H_{LP}(e^{j\omega}) = \begin{cases} 1 & \text{for } 0 \le |\omega| < \omega_c \\ 0 & \text{for } \omega_c \le |\omega| < \pi \end{cases}$$

The coefficients of this filter can be found by taking the inverse discrete-time Fourier transform:

$$\begin{aligned} h_{LP}(n) &= \frac{1}{2\pi}\int_{2\pi} H_{LP} e^{j\omega n} d\omega = \frac{1}{2\pi}\int_{-\omega_c}^{\omega_c} e^{j\omega n} d\omega \\ &= \frac{1}{2\pi}\frac{e^{j\omega_c n} - e^{-j\omega_c n}}{jn} = \frac{1}{\pi n}\sin\omega_c n \end{aligned}$$

(b) From the given impulse response of the filter $h_{HP}(n)$:

$$\begin{aligned} H_{HP}(e^{j\omega}) &= \sum_{n=-\infty}^{\infty} h_{HP}(n)e^{-j\omega n} = \cdots + h_{HP}(-1)e^{j\omega} + h_{HP}(0) + h_{HP}(1)e^{j\omega} + \cdots \\ &= \cdots - h_{LP}(-1)e^{j\omega} + (1 - h_{LP}(0)) - h_{HP}(1)e^{j\omega} - \cdots \\ &= 1 - \sum_{n=-\infty}^{\infty} h_{LP}(n)e^{-j\omega n} = 1 - H_{LP}(e^{j\omega}) \\ &= \begin{cases} 0 & \text{for } 0 \le |\omega| < \omega_c \\ 1 & \text{for } \omega_c \le |\omega| < \pi \end{cases} \end{aligned}$$

(c) The desired frequency response is that of the ideal lowpass filter shifted by π:

$$\begin{aligned} h_{HP}(n) &= \frac{1}{2\pi}\int_{2\pi} H_{HP}(e^{j\omega})e^{j\omega n} d\omega = \frac{1}{2\pi}\int_{2\pi} H_{LP}\left(e^{j(\omega-\pi)}\right)e^{j\omega n} d\omega \\ &= \frac{1}{2\pi}\int_{2\pi} H_{LP}(e^{j\omega})e^{j(\omega+\pi)n} d\omega = e^{j\pi n}\frac{1}{2\pi}\int_{2\pi} H_{LP}(e^{j\omega})e^{j\omega n} d\omega \\ &= (-1)^n h_{LP}(n) \end{aligned}$$

3.1.3.

$(\downarrow 3)$ keeps every 3rd component and removes all the other components.
$(\uparrow 3)$ fills in two zeros between the input sample.
$(\uparrow 3)(\downarrow 3)$ keeps every 3rd component and replaces all the other components with zeros.

2.2.6.

If we let $z = e^{i\omega}$, we obtain the frequency response:

$$\begin{aligned} H(e^{i\omega}) &= 1 - 2e^{-i\omega} + 3e^{-2i\omega} - 3e^{-3i\omega} + 2e^{-4i\omega} - e^{-5i\omega} \\ &= e^{-\frac{5}{2}i\omega}(e^{\frac{5}{2}i\omega} - 2e^{\frac{3}{2}i\omega} + 3e^{\frac{1}{2}i\omega} - 3e^{-\frac{1}{2}i\omega} + 2e^{-\frac{3}{2}i\omega} - e^{-\frac{5}{2}i\omega}) \\ &= 2ie^{-\frac{5}{2}i\omega}\Big(\frac{e^{\frac{5}{2}i\omega} - e^{-\frac{5}{2}i\omega}}{2i} - 2\frac{e^{\frac{3}{2}i\omega} - e^{-\frac{3}{2}i\omega}}{2i} + 3\frac{e^{\frac{1}{2}i\omega} - e^{-\frac{1}{2}i\omega}}{2i}\Big) \\ &= e^{-\frac{5}{2}i\omega + \frac{\pi}{2}}\Big(2\sin\frac{5\omega}{2} - 4\sin\frac{3\omega}{2} + 6\sin\frac{\omega}{2}\Big) \end{aligned}$$

Hence:

$$|H(\omega)| = \Big|2\sin\frac{5\omega}{2} - 4\sin\frac{3\omega}{2} + 6\sin\frac{\omega}{2}\Big|$$

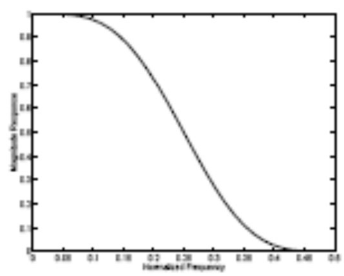
$$\phi(\omega) = -\frac{5}{2}\omega + \frac{\pi}{2} \quad \Rightarrow \quad \mathrm{GD} = \frac{5}{2}$$

The group delay is usually defined as $\mathrm{GD} = -\frac{d}{d\omega}\phi(\omega)$, with a minus sign.

2.3.5.

For $p = 2$

$$\begin{aligned} H_2(\omega) &= \Big(\frac{1+\cos\omega}{2}\Big)^2 \sum_{k=0}^{1} \binom{1+k}{k} \Big(\frac{1-\cos\omega}{2}\Big)^k \\ &= \Big(\frac{1+\cos\omega}{2}\Big)^2 + 2\Big(\frac{1-\cos\omega}{2}\Big)\Big(\frac{1+\cos\omega}{2}\Big)^2 = \frac{2 + 3\cos\omega - \cos^3\omega}{4} \end{aligned}$$



Frequency response of Daubechies filter ($p = 2$)

3.2.3.

$$\begin{aligned} &(3.11) \;\rightarrow\; V(\omega) = \sum \mathbf{v}(k)e^{-ik\omega} = \sum x(3k)e^{-ik\omega} \\ &(3.12) \;\rightarrow\; \mathbf{u}(n) = \begin{cases} \mathbf{x}(n), & \text{n devides 3} \\ 0, & \text{otherwise} \end{cases} = (\ldots, \mathbf{x}(0), 0, 0, \mathbf{x}(3), 0, 0, \ldots) \\ &\qquad\qquad (\downarrow 3)\mathbf{u} = (\downarrow 3)\mathbf{x} \\ &(3.13) \;\rightarrow\; \mathbf{u}(n) = \sum_{n=3k} \mathbf{x}(n)e^{-in\omega} \\ &\qquad\qquad = \frac{1}{3}\sum_{all\ n} \mathbf{x}(n)e^{-in\omega} + \frac{1}{3}\sum_{all\ n} \mathbf{x}(n)e^{-in(\omega+\frac{2\pi}{3})} + \frac{1}{3}\sum_{all\ n} \mathbf{x}(n)e^{-in(\omega+\frac{4\pi}{3})} \\ &(3.14) \;\rightarrow\; U(\omega) = \frac{1}{3}[X(\omega) + X(\omega + \frac{2\pi}{3}) + X(\omega + \frac{4\pi}{3})] \\ &(3.15) \;\rightarrow\; V(\omega) = U(\frac{\omega}{3}) \\ &(3.16) \;\rightarrow\; V(\omega) = \frac{1}{3}[X(\frac{\omega}{3}) + X(\frac{\omega+2\pi}{3}) + X(\frac{\omega+4\pi}{3})] \end{aligned}$$

3.3.2.

Verifying (3.27): $\mathbf{v} = (\downarrow M)\mathbf{x}$ and $\mathbf{u} = (\uparrow L)\mathbf{v}$ have components:

$$\mathbf{v}(k) = \mathbf{x}(Mk)$$

$$\mathbf{u}(n) = \begin{cases} \mathbf{v}(k) = \mathbf{x}(Mk) = \mathbf{x}(\frac{Mn}{L}), & \text{if } \frac{n}{L} = k \text{ is integer} \\ 0, & \text{otherwise} \end{cases}$$

Change the order of $(\downarrow M)$ and $(\uparrow L)$: upsampling first puts in $L-1$ zeros between each $\mathbf{x}(n)$ and $\mathbf{x}(n+1)$. It has components:

$$\mathbf{v}'(n) = \begin{cases} \mathbf{x}(k) = \mathbf{x}(\frac{n}{L}), & \text{if } \frac{n}{L} = k \text{ is an integer} \\ 0, & \text{otherwise} \end{cases}$$

Downsampling $\mathbf{u}' = (\downarrow M)\mathbf{v}'$ keeps every Mth components of $\mathbf{v}'$ and removes all the other components:

$$\mathbf{u}'(n) = \mathbf{v}'(Mn) = \begin{cases} \mathbf{x}(k) = \mathbf{x}(\frac{Mn}{L}), & \text{if } \frac{Mn}{L} = k \text{ is an integer} \\ 0, & \text{otherwise} \end{cases}$$

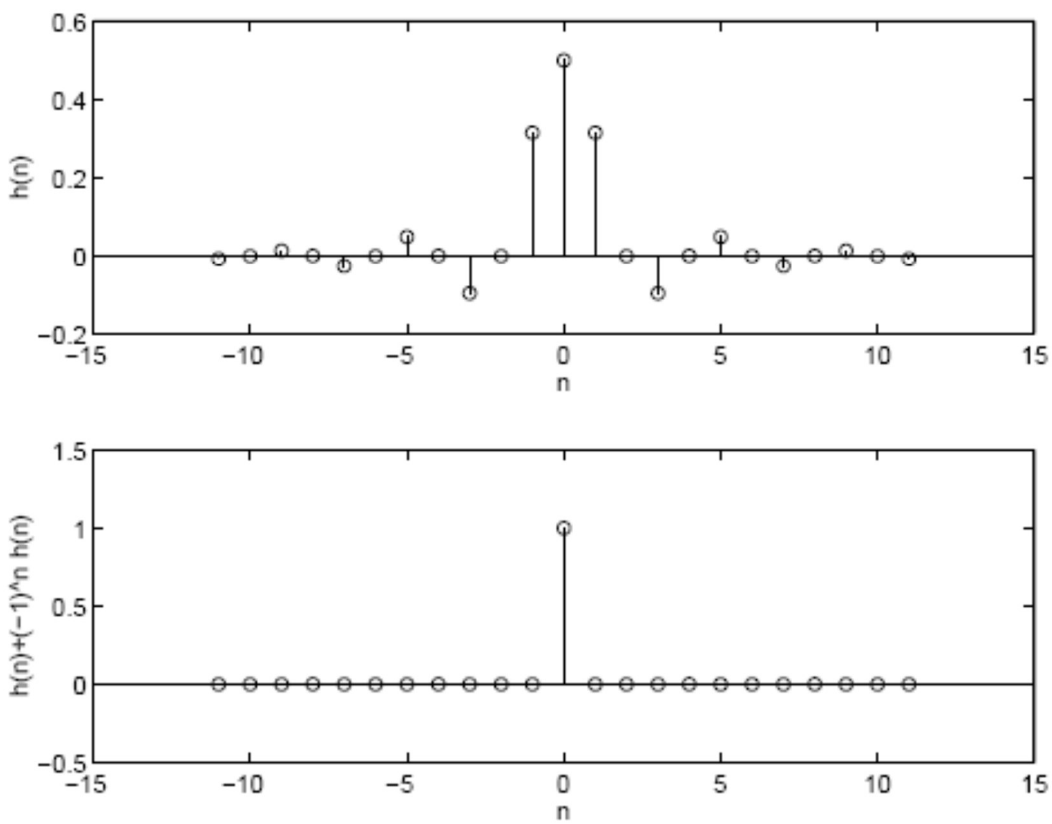
$$\begin{aligned} (M, L) = 1 \quad &\Longleftrightarrow \quad \frac{Mn}{L} \text{ is integer } if\ and\ only\ if\ \frac{n}{L} \text{ is integer} \\ &\Longleftrightarrow \quad \mathbf{u}'(n) = \mathbf{u}(n) \end{aligned}$$

Therefore, $(\uparrow L)(\downarrow M)\mathbf{x} = (\downarrow M)(\uparrow L)\mathbf{x}$ *if and only if* L and M are relatively prime.
The odd-numbered components become zeros after $(\uparrow 2)(\downarrow 2)$.
In $X(z) = \sum \mathbf{x}(n) z^{-n}$, the odd-numbered coefficients are zero, therefore $X(z) = \sum \mathbf{x}(2n) z^{-2n}$.

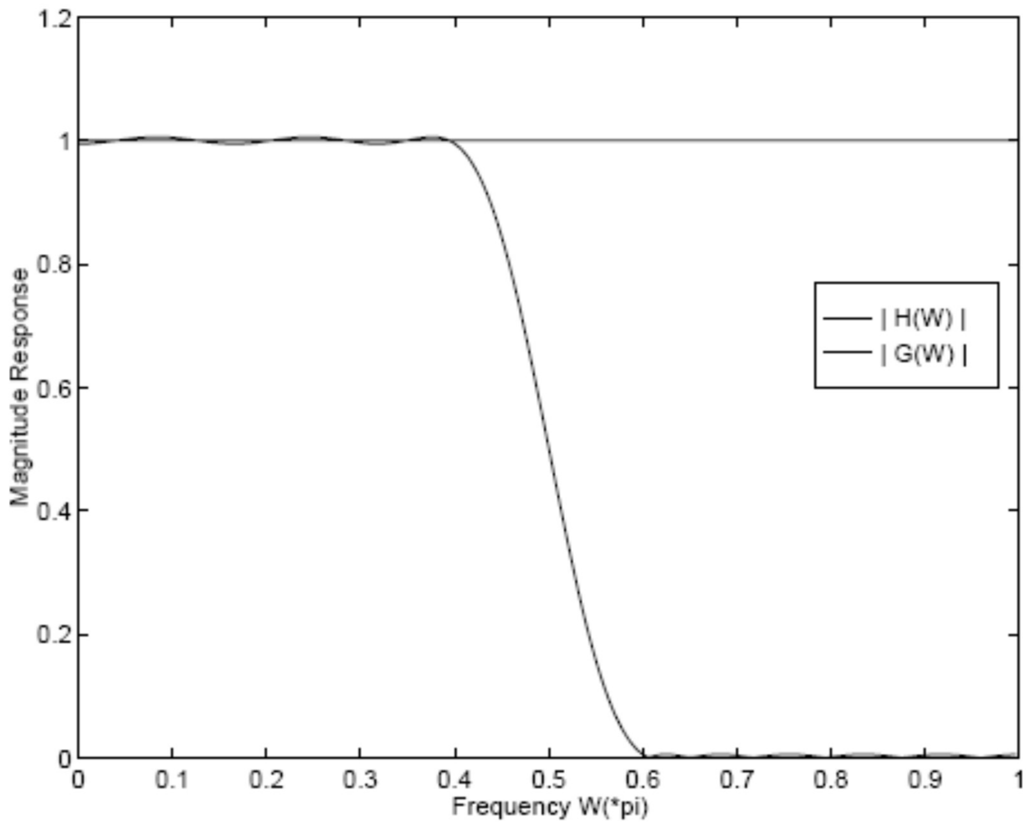
Matlab Solution:

Page 457. Problem 2.6.

```
% Problem 2.6

n = [-11:11];
h = remez(22,[0 0.4 0.6 1],[1 1 0 0],[1 1]);

% ws should be 0.6pi and deltas is the same as deltap

%%%%%   Part b   %%%%%

figure(1); subplot(211);
stem(n,h); xlabel('n'); ylabel('h(n)');

%%%%%   Part c   %%%%%

figure(1); subplot(212);
stem(n, (h + (-1).^n .* h)); xlabel('n'); ylabel('h(n)+(-1)^n h(n)');
```

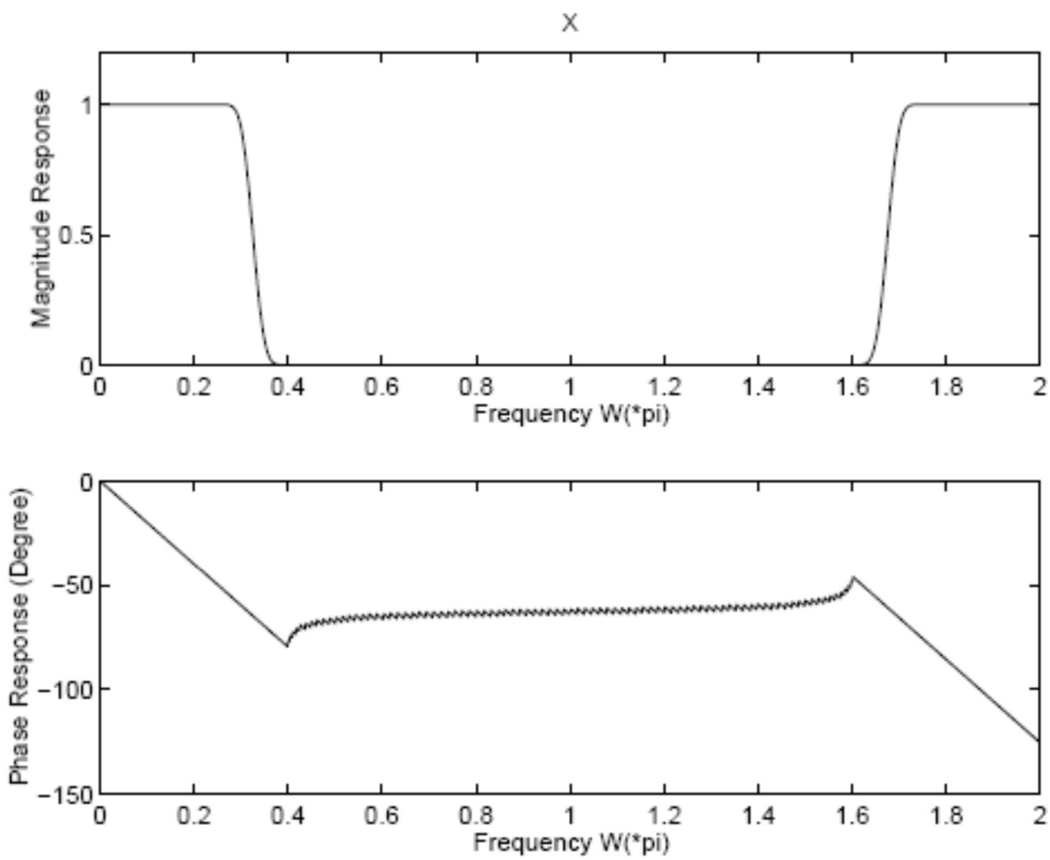


```
%%%%%   Part d   %%%%%

figure(2);
[H,W] = freqz(h,[zeros(1,11),1],512,'whole');
plot(W(1:256)/pi, abs(H(1:256))); hold on;
plot(W(1:256)/pi, abs(H(1:256)+H(257:512)),'r'); hold off;
xlabel('Frequency W(*pi)'); ylabel('Magnitude Response');
legend('y','| H(W) |','r','| G(W) |')
%axis([0,1,0,1]);
```



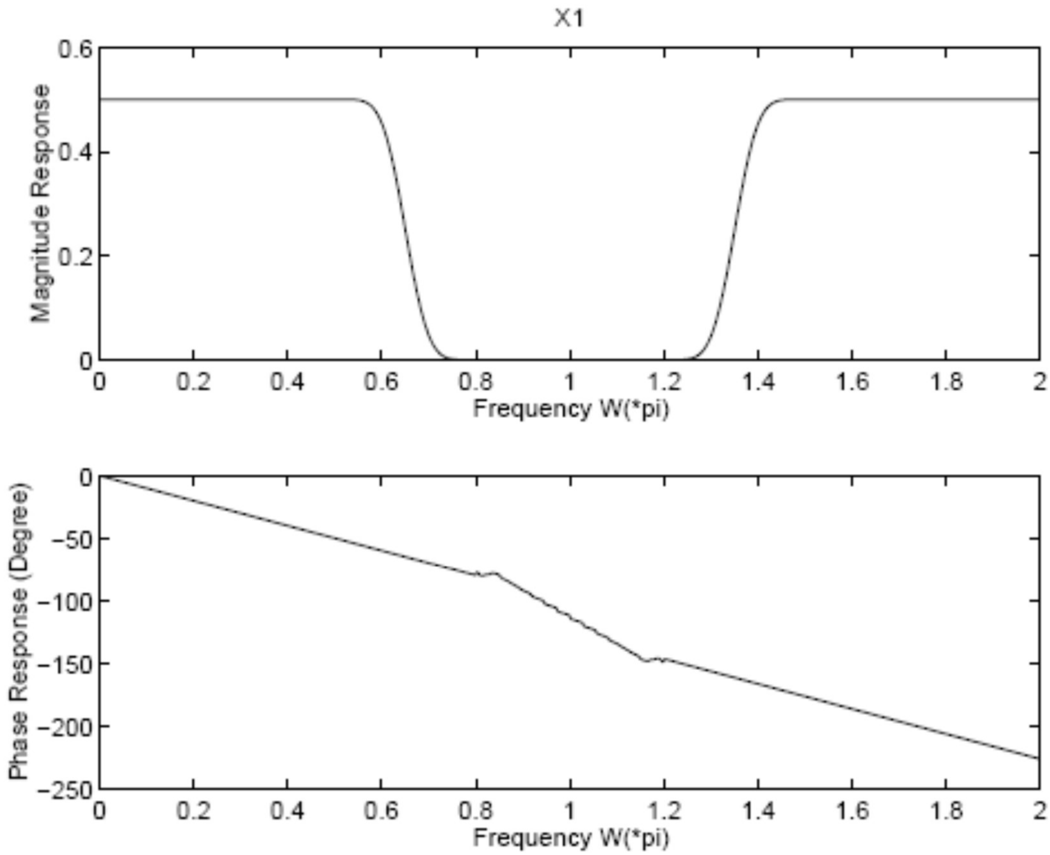
Page 458. Problem 3.1

```
% Problem 3.1

%%%%%   Part a   %%%%%
N = 1024;

figure(1); subplot(211)
x = remez(127, [0, 0.25, 0.4, 1], [1, 1, 0, 0],[1, 1]);
W = 2*pi/N*[0:(N-1)]';
X = fft(x, N);
plot(W/pi,abs(X));
xlabel('Frequency W(*pi)'); ylabel('Magnitude Response');
title('X');
```



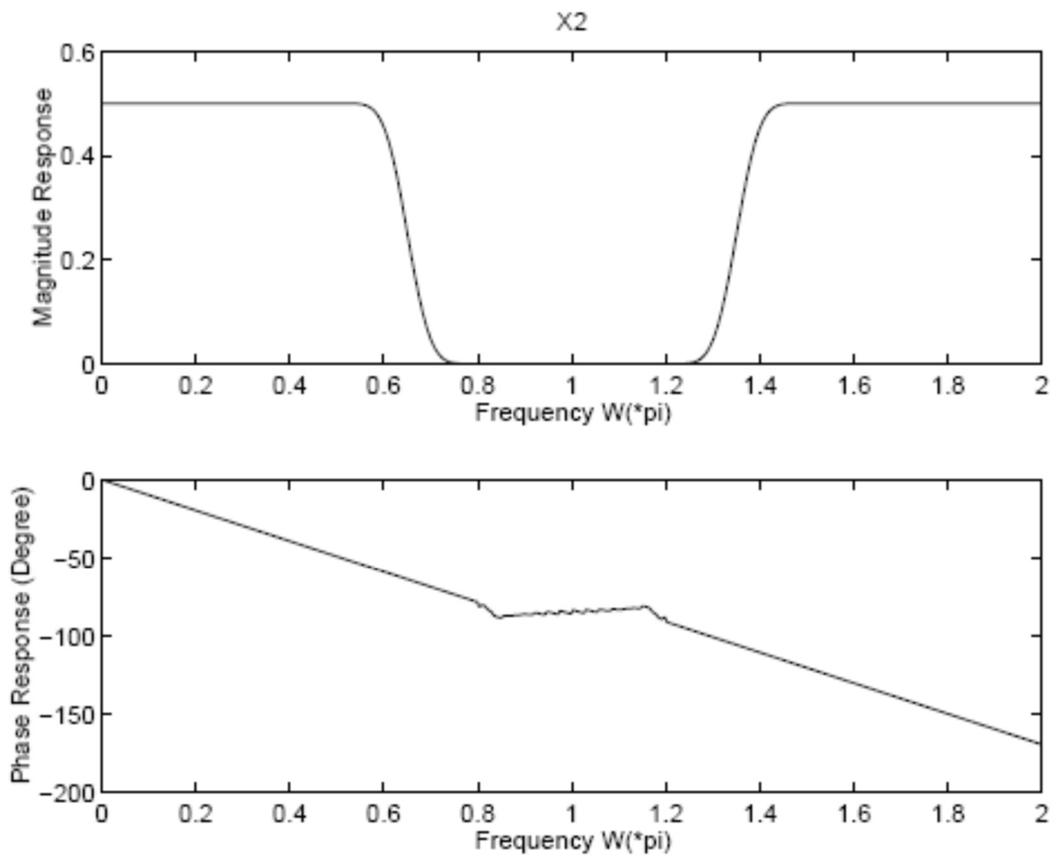

```
i_p = find(W >= 0.25*pi); i_p = i_p(1);
d_p = max(abs(abs(X(1:i_p))-1))
i_s = find(W >= 0.4*pi); i_s = i_s(1);
d_s = max(abs(X(i_s:N/2)))
%sprintf('delta_p = %6.5f, delta_s = %6.5f',d_p,d_s)
```

```
%%%%%   Part b   %%%%%

x1 = x(1:2:(length(x)));
X1 = fft(x1, N);
figure(2); subplot(211); plot(W/pi,abs(X1),'r'); hold on;
xlabel('Frequency W(*pi)'); ylabel('Magnitude Response');
title('X1');
```



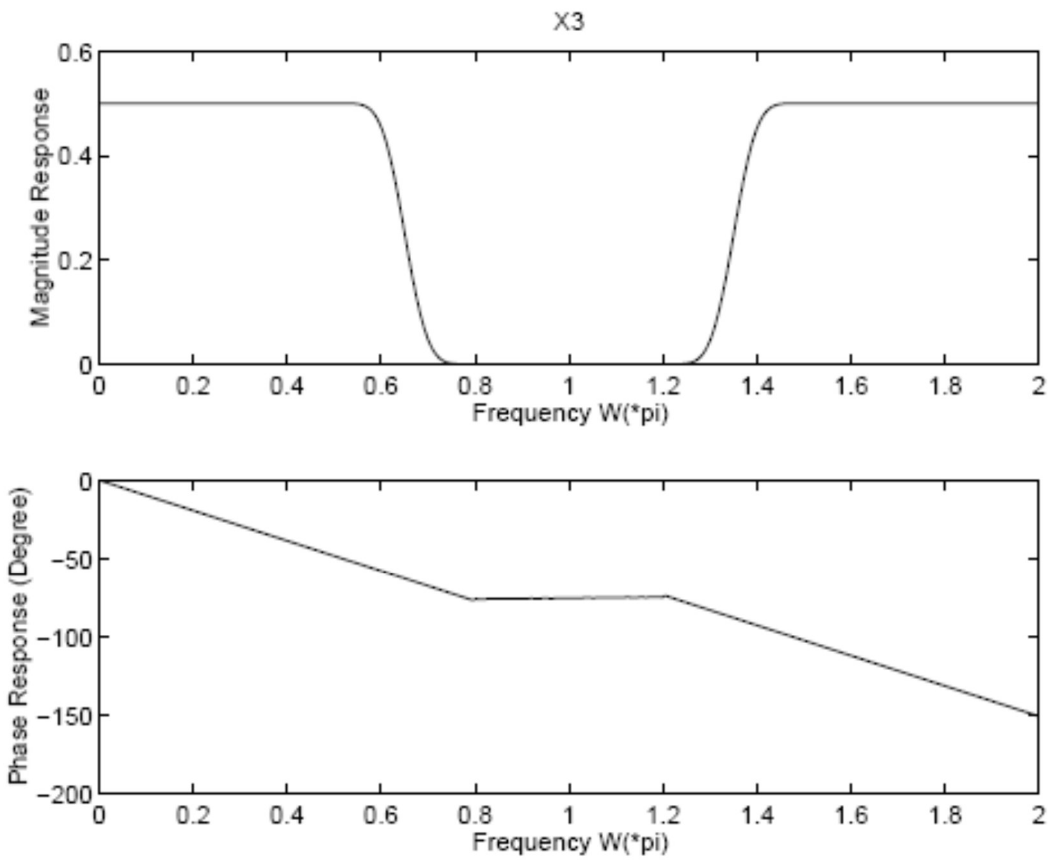

```
i_p = find(W >= 0.5*pi); i_p = i_p(1);
d_p1 = max(abs(abs(X1(1:i_p))-0.5))
i_s = find(W >= 0.8*pi); i_s = i_s(1);
d_s1 = max(abs(X1(i_s:N/2)))
```

```
%%%%%   Part c   %%%%%

x2 = x(2:2:length(x));
X2 = fft(x2, N);
figure(3); subplot(211); plot(W/pi,abs(X2),'g'); hold on
xlabel('Frequency W(*pi)'); ylabel('Magnitude Response');
title('X2');
```



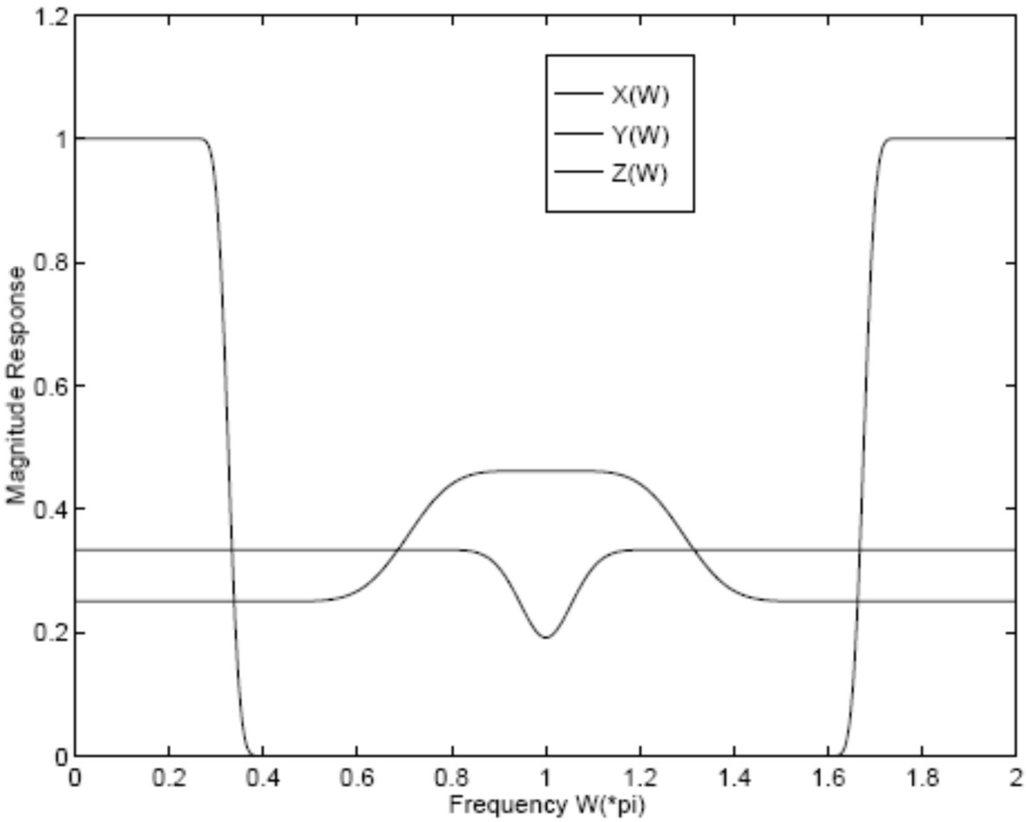

```
i_p = find(W >= 0.5*pi); i_p = i_p(1);
d_p2 = max(abs(abs(X1(1:i_p))-0.5))
i_s = find(W >= 0.8*pi); i_s = i_s(1);
d_s2 = max(abs(X1(i_s:N/2)))

%%%%%   Part e   %%%%%

x3 = x(3:2:length(x));
X3 = fft(x3,N);
figure(4); subplot(211); plot(W/pi,abs(X3),'m'); hold on;
xlabel('Frequency W(*pi)'); ylabel('Magnitude Response');
title('X3');

i_p = find(W >= 0.5*pi); i_p = i_p(1);
d_p3 = max(abs(abs(X1(1:i_p))-0.5))
i_s = find(W >= 0.8*pi); i_s = i_s(1);
d_s3 = max(abs(X1(i_s:N/2)))
```

```
%legend('y','X(W)','r','X1(W)','g','X2(W)','m','X3(W)');

figure(1); subplot(212);
plot(W/pi, unwrap(angle(X))); hold off;
xlabel('Frequency W(*pi)'); ylabel('Phase Response (Degree)');

figure(2); subplot(212);
plot(W/pi, unwrap(angle(X1)), 'r');hold off;
xlabel('Frequency W(*pi)'); ylabel('Phase Response (Degree)');

figure(3); subplot(212);
plot(W/pi, unwrap(angle(X2)), 'g'); hold off;
xlabel('Frequency W(*pi)'); ylabel('Phase Response (Degree)');

figure(4); subplot(212);
plot(W/pi, unwrap(angle(X3)), 'm'); hold off;
xlabel('Frequency W(*pi)'); ylabel('Phase Response (Degree)');
```



Page 458. Problem 3.2

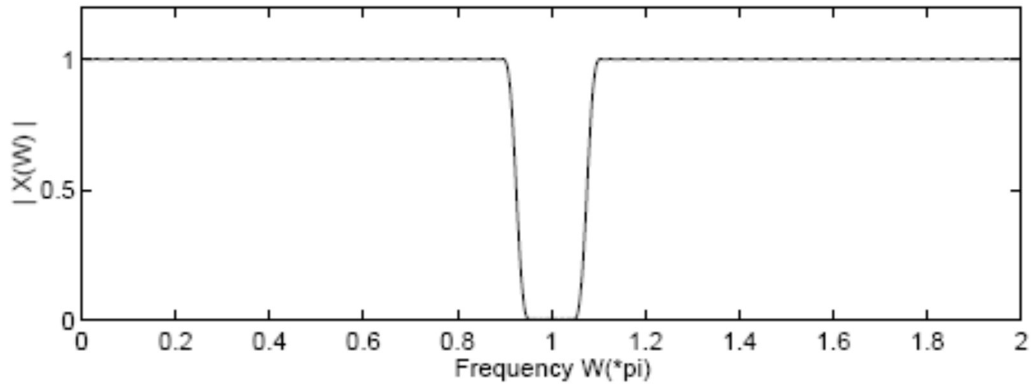
```
% Problem 3.2

%%%%%   Part a   %%%%%
N = 1024;

figure(1);
x = remez(127, [0, 0.25, 0.4, 1], [1, 1, 0, 0],[1, 1]);
W = 2*pi/N*[0:(N-1)]';
X = fft(x, N);
plot(W/pi,abs(X)); hold on;

i_p = find(W >= 0.25*pi); i_p = i_p(1);
d_p = max(abs(abs(X(1:i_p))-1))
i_s = find(W >= 0.4*pi); i_s = i_s(1);
d_s = max(abs(X(i_s:N/2)))
%sprintf('delta_p = %6.5f, delta_s = %6.5f',d_p,d_s)

%%%%%   Part b   %%%%%

y = x(1:3:(length(x)));
Y = fft(y, N);
plot(W/pi,abs(Y),'r');

i_p = find(W >= 0.25*3*pi); i_p = i_p(1);
d_p3 = max(abs(abs(Y(1:i_p))-1/3))
i_s = find(W >= 0.4*3*pi); i_s = i_s(1);
d_s3 = max(abs(Y(i_s:N/2)))

%No aliasing since the max freq is now 0.75 pi.
%Reconstruct by upsampling and filtering.

%%%%%   Part c   %%%%%

z = x(1:4:length(x));
Z = fft(z, N);
plot(W/pi,abs(Z),'g'); hold off;

xlabel('Frequency W(*pi)'); ylabel('Magnitude Response');
legend('y','X(W)','r','Y(W)','g','Z(W)');

%Aliasing occurs.  Can not reconstruct near edges.
```

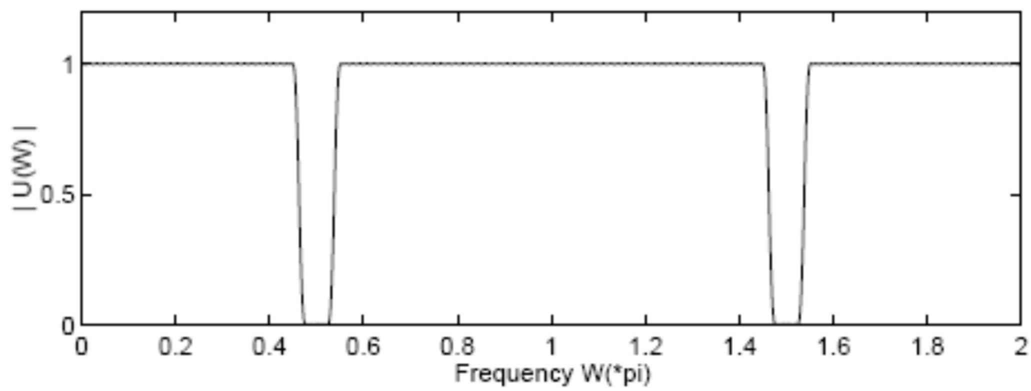


X(W)
Y(W)
Z(W)
Magnitude Response
Frequency W(*pi)
1.2
1
0.8
0.6
0.4
0.2
0
0
0.2
0.4
0.6
0.8
1
1.2
1.4
1.6
1.8
2

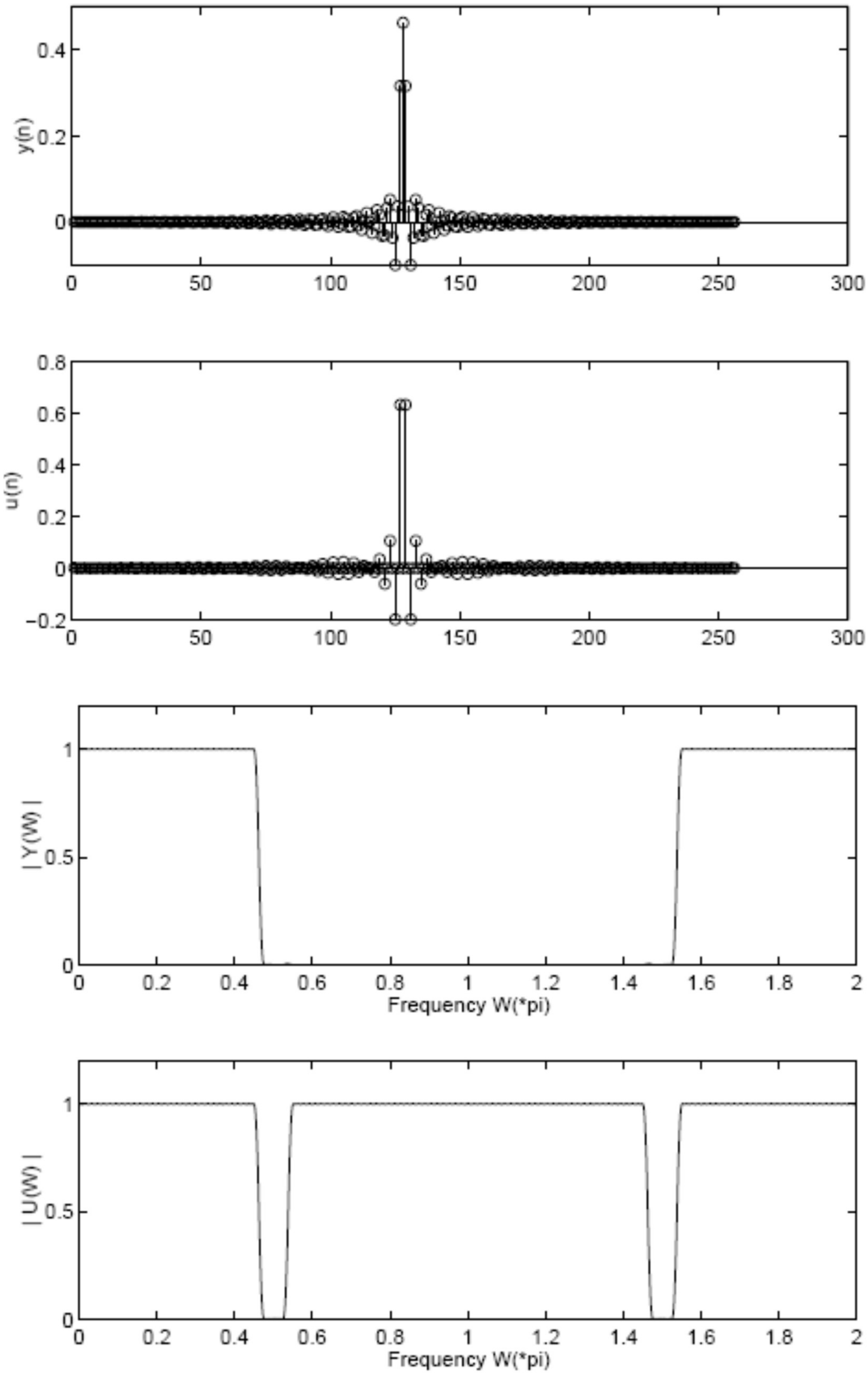
Page 458. Problem 3.3.

```
% Problem 3.3

N = 1024;

%%%%%   Part a   %%%%%

figure(1);
x = remez(127, [0, 0.9, 0.95, 1], [1, 1, 0, 0], [1, 1]);
W = 2*pi/N*[0:(N-1)]';
X = fft(x,N);
subplot(211); plot(W/pi,abs(X));
xlabel('Frequency W(*pi)'); ylabel('| X(W) |');

i_p = find(W >= 0.9*pi); i_p = i_p(1);
d_p = max(abs(abs(X(1:i_p))-1))
i_s = find(W >= 0.95*pi); i_s = i_s(1);
d_s = max(abs(X(i_s:N/2)))

%%%%%   Part b   %%%%%

u = zeros(1,2*length(x));
u(1:2:length(u)) = x;
U = fft(u,N);
subplot(212); plot(W/pi, abs(U), 'r');
xlabel('Frequency W(*pi)'); ylabel('| U(W) |');
```





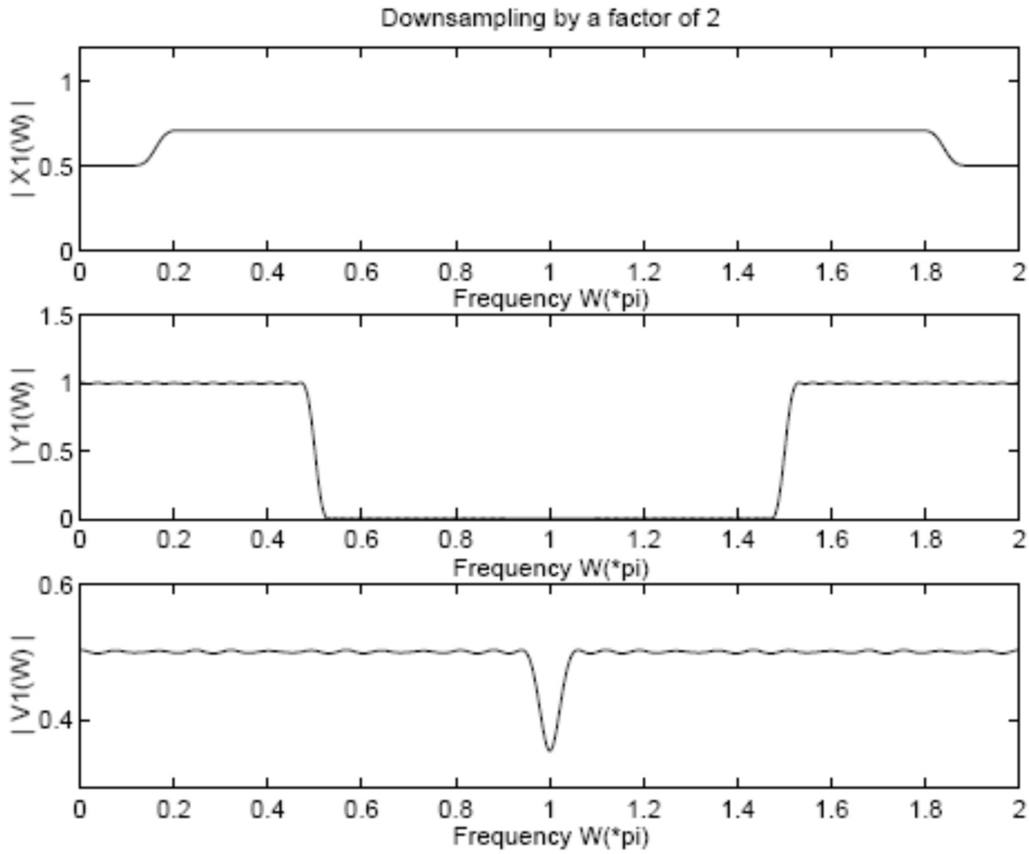

```
%%%%%   Part c   %%%%%

f = remez(100, [0,0.45,0.55,1], [1 1 0 0], [1 1]);
%The filter should be a halfband filter;
y = conv(u,f); y = y(51:length(y)-50);

figure(2);
subplot(211); stem(y); ylabel('y(n)');
subplot(212); stem(u); ylabel('u(n)');

Y = fft(y,N);
U = fft(u,N);
figure(3);
subplot(211); plot(W/pi,abs(Y)); xlabel('Frequency W(*pi)'); ylabel('|
Y(W) |');
subplot(212); plot(W/pi,abs(U)); xlabel('Frequency W(*pi)'); ylabel('|
U(W) |');
```



y(n)
0.4
0.2
0
0
50
100
150
200
250
300
u(n)
0.8
0.6
0.4
0.2
0
−0.2
0
50
100
150
200
250
300
| Y(W) |
1
0.5
0
0
0.2
0.4
0.6
0.8
1
1.2
1.4
1.6
1.8
2
Frequency W(*pi)
| U(W) |
1
0.5
0
0
0.2
0.4
0.6
0.8
1
1.2
1.4
1.6
1.8
2
Frequency W(*pi)

Page 459. Problem 3.4

```
% Problem 3.4


N = 1024;

%%%%%   Part a   %%%%%
figure(1);
x = remez(127, [0, 0.9, 0.95, 1], [1, 1, 0, 0], [1, 1]);
x1 = x(1:2:length(x));
W = 2*pi/N*[0:(N-1)];
X1 = fft(x1,N);
subplot(311); plot(W/pi,abs(X1)); title('Downsampling by a factor of
2');
xlabel('Frequency W(*pi)'); ylabel('| X1(W) |');
axis([0 2 0 1.2]);
% Severe aliasing since X is not bandlimited to pi/2

%%%%%   Part b   %%%%%

h1 = remez(100, [0 0.475,0.525,1], [1 1 0 0]);
y1 = conv(h1,x);
Y1 = fft(y1,N);
subplot(312); plot(W/pi,abs(Y1));
xlabel('Frequency W(*pi)'); ylabel('| Y1(W) |');

%Lowpass (bandlimited) to pi/2

%%%%%   Part c   %%%%%

v1 = y1(1:2:length(y1));
V1 = fft(v1,N);
subplot(313); plot(W/pi,abs(V1));
xlabel('Frequency W(*pi)'); ylabel('| V1(W) |');
% Some aliasing near pi
```



Downsampling by a factor of 2
| X1(W) |
| Y1(W) |
| V1(W) |
Frequency W(*pi)
Frequency W(*pi)
Frequency W(*pi)