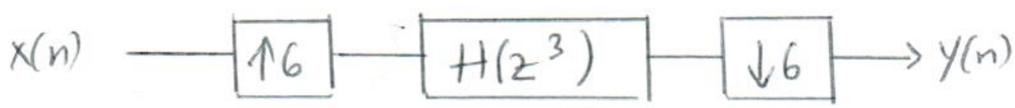
Midterm Exam — ECE 251C Fall 2019, Nguyen

Problem 1. (20pt) Consider the following LTI system $H(z)$:

$$H(z) = \frac{2z^{-1} + 5z^{-3}}{1 - \frac{3}{2}z^{-1} - z^{-2}}$$

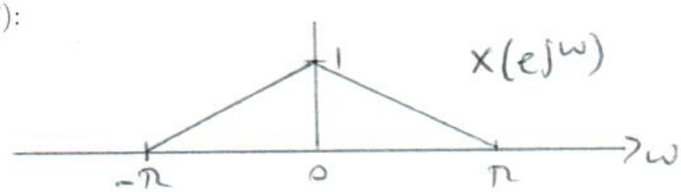
Find the two polyphases $H_{even}(z)$ and $H_{odd}(z)$, i.e., $H(z) = H_{even}(z^2) + z^{-1}H_{odd}(z^2)$

Problem 2. (40pt) Consider the multirate system below:



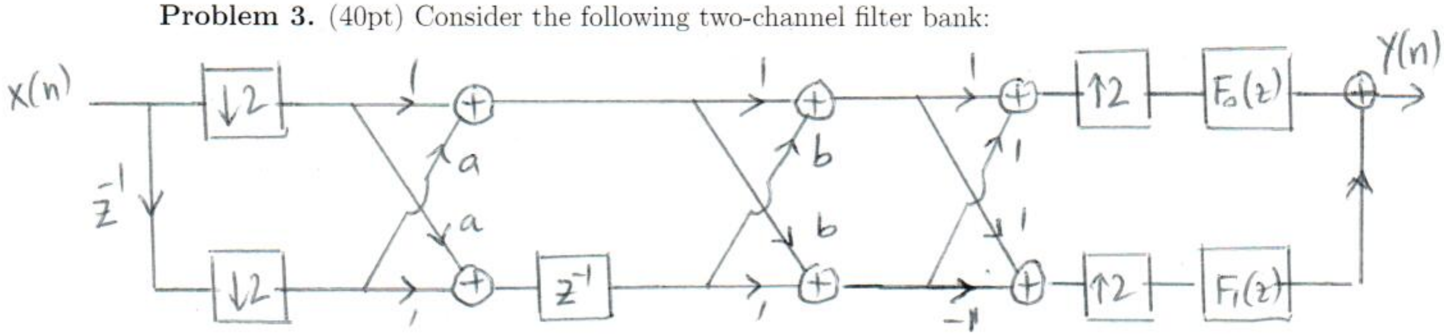
a. (10pt) Find $Y(z)$ in terms of $X(z)$ and $H(z)$.

b. (15pt) Sketch $|Y(e^{j\omega})|$ for $H(e^{j\omega})$ being an ideal lowpass filter with cutoff frequency at $\frac{\pi}{2}$ and $X(e^{j\omega})$:



c. (15pt) Sketch $|Y(e^{j\omega})|$ for $H(z) = -1 + 9z^{-2} + 16z^{-3} + 9z^{-4} - z^{-6}$ and $x(n) = (-1)^n$.

Problem 3. (40pt) Consider the following two-channel filter bank:



$a = 2, \ b = \frac{1}{2}$

a.(7pt) Find the polyphase matrix $\mathbf{H}_p(z)$

b.(7pt) Find the analysis filters $H_0(z)$ and $H_1(z)$.

c.(7pt) Find all zeros and poles of $H_0(z)$ and $H_1(z)$ and sketch their pole-zero plots.

d.(7pt) Find the PR synthesis filters $F_0(z)$ and $F_1(z)$ by inverting $\mathbf{H}_p(z)$.

e.(7pt) Verify that the system is PR by the aliasing condition and halfband condition.

f.(5pt) Find the delay L, i.e., $y(n) = x(n - L)$.

Problem 1

$$H(z) = \frac{2z^{-1} + 5z^{-3}}{1 - \frac{3}{2}z^{-1} - z^{-2}} = \frac{z^{-1}(2+5z^{-2})}{(1-z^{-2}) - \frac{3}{2}z^{-1}}$$

$$= \frac{z^{-1}(2+5z^{-2})\left[(1-z^{-2}) + \frac{3}{2}z^{-1}\right]}{\left[(1-z^{-2}) - \frac{3}{2}z^{-1}\right]\left[(1-z^{-2}) + \frac{3}{2}z^{-1}\right]}$$

$$= \frac{z^{-1}\left[(2+5z^{-2})(1-z^{-2}) + \frac{3}{2}z^{-1}(2+5z^{-2})\right]}{(1-z^{-2})^2 - \frac{9}{4}z^{-2}}$$

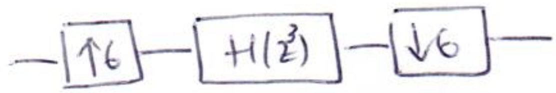
$$= \frac{\frac{3}{2}z^{-2}(2+5z^{-2}) + z^{-1}(2+5z^{-2})(1-z^{-2})}{(1-z^{-2})^2 - \frac{9}{4}z^{-2}}$$

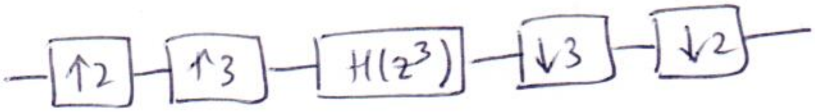
$$H_{even}(z) = \frac{\frac{3}{2}z^{-1}(2+5z^{-1})}{(1-z^{-1})^2 - \frac{9}{4}z^{-1}}$$

$$H_{odd}(z) = \frac{(2+5z^{-1})(1-z^{-1})}{(1-z^{-1})^2 - \frac{9}{4}z^{-1}}$$

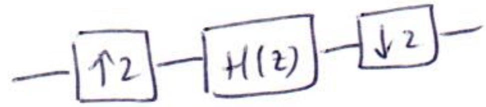
Problem 2

a-

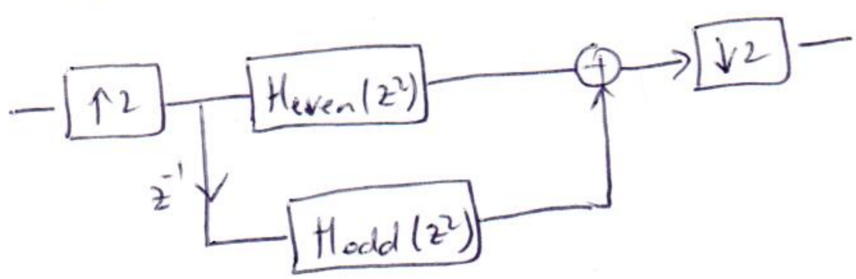




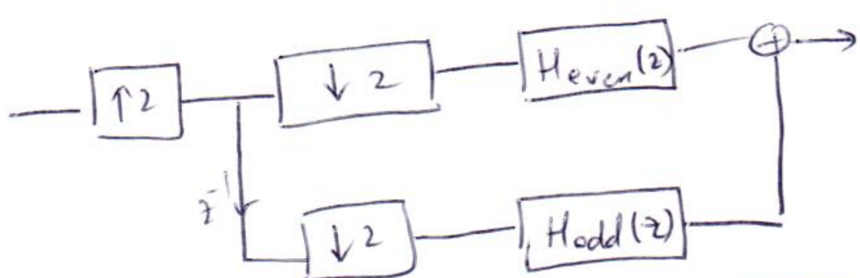
Using #1 Noble identity ⇒



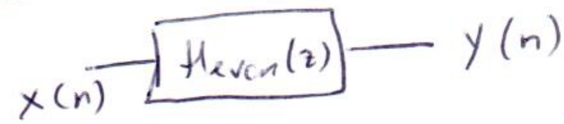
Using polyphase representation $H(z) = H_{even}(z^2) + z^{-1} H_{odd}(z^2)$.



Use #1 Noble identity yields



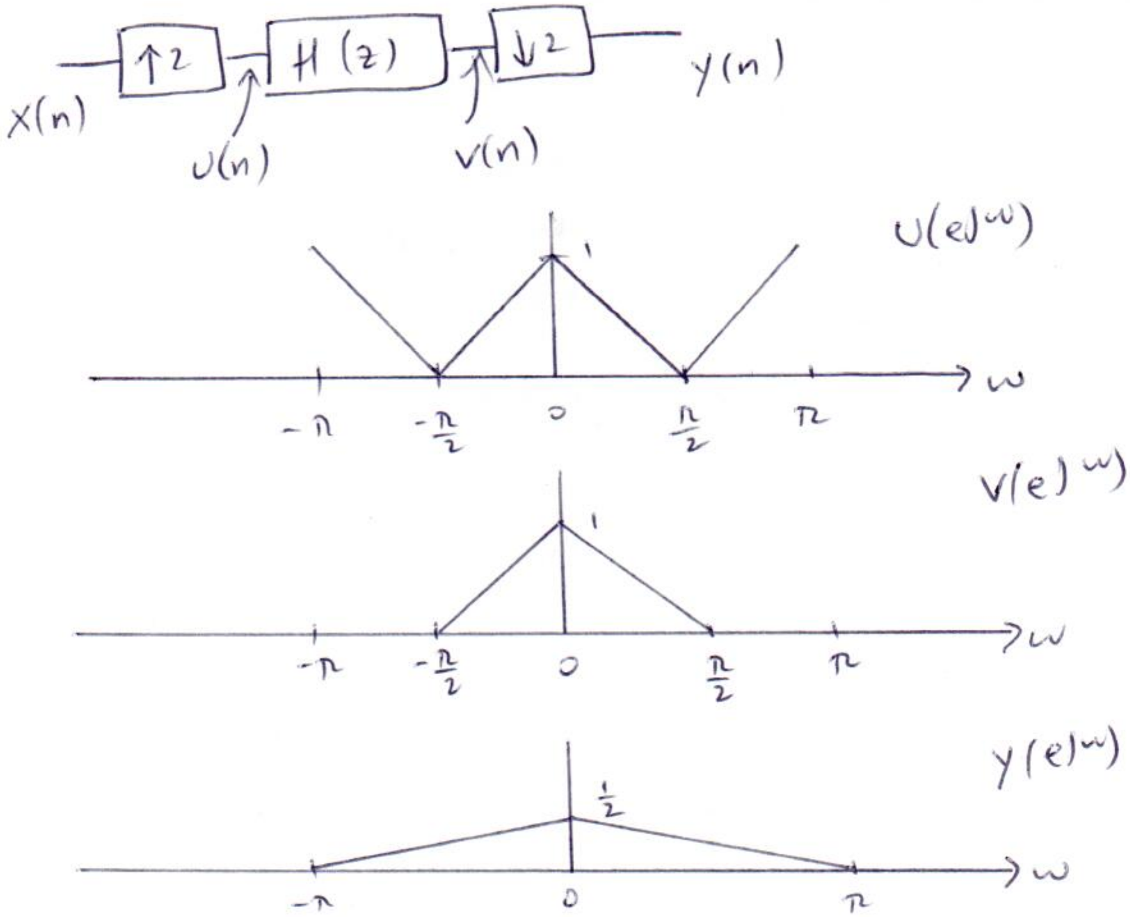
which is equivalent to



$$Y(z) = H_{even}(z)\, X(z)$$

where $h_{even}(n) = h(2n)$; the even polyphase comp. of $h(n)$

b.



c. $H(z) = -1 + 9z^{-2} + 16z^{-3} + 9z^{-4} - z^{-6}$

$H_{even}(z) = -1 + 9z^{-1} + 9z^{-2} - z^{-3}$ is a Type II linear phase

$H_{even}(z)$ has a zero @ π which means that

$H_{even}(e^{j\pi}) = 0$

$\Rightarrow y(n) = 0$.

Problem 3

a. $$H_p(z) = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & z^{-1} \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$$

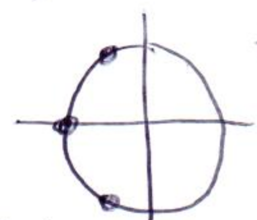
$$= \begin{pmatrix} \frac{3}{2} & \frac{3}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 2z^{-1} & z^{-1} \end{pmatrix} = \begin{pmatrix} \frac{3}{2} + 3z^{-1} & 3 + \frac{3}{2}z^{-1} \\ \frac{1}{2} - z^{-1} & 1 - \frac{1}{2}z^{-1} \end{pmatrix}$$

b. $$\begin{pmatrix} H_0(z) \\ H_1(z) \end{pmatrix} = H_p(z^2) \begin{pmatrix} 1 \\ z^{-1} \end{pmatrix} = \begin{pmatrix} \frac{3}{2} + 3z^{-2} & 3 + \frac{3}{2}z^{-2} \\ \frac{1}{2} - z^{-2} & 1 - \frac{1}{2}z^{-2} \end{pmatrix} \begin{pmatrix} 1 \\ z^{-1} \end{pmatrix}$$

$$\begin{pmatrix} H_0(z) \\ H_1(z) \end{pmatrix} = \begin{pmatrix} \frac{3}{2} + 3z^{-1} + 3z^{-2} + \frac{3}{2}z^{-3} \\ \frac{1}{2} + z^{-1} - z^{-2} - \frac{1}{2}z^{-3} \end{pmatrix}$$ ← Type II linear phase ← Type IV linear phase

c. * $H_0(z)$ is Type II linear phase + has a zero at π.

$$H_0(z) = (1 + z^{-1})(1 + z^{-1} + z^{-2})$$
$$= (1 + z^{-1})(1 - \rho e^{j\theta} z^{-1})(1 - \rho e^{-j\theta} z^{-1})$$
$$= (1 + z^{-1})(1 - 2\rho\cos\theta\, z^{-1} + \rho^2 z^{-2})$$
$$\Rightarrow \rho = 1, \quad \theta = \frac{2\pi}{3}$$



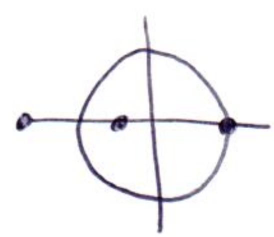
$H_0(z)$

* $H_1(z)$ is Type IV linear phase + has zero @ $z=1$

$$H_1(z) = (1 - z^{-1})(1 + 3z^{-1} + z^{-2})$$

zeros of $H_1(z)$ are at $z = 1$

$$z = \frac{-3 \pm \sqrt{5}}{2}$$



d. $F_p(z) = H_p^{-1}(z)$

$$= \frac{1}{\Delta(z)} \begin{pmatrix} 1 - \frac{1}{2}z^{-1} & -(3 + \frac{3}{2}z^{-1}) \\ -(\frac{1}{2} - z^{-1}) & \frac{3}{2} + 3z^{-1} \end{pmatrix}$$

$$\Delta(z) = \left(\frac{3}{2} + 3z^{-1}\right)\left(1 - \frac{1}{2}z^{-1}\right) - \left(3 + \frac{3}{2}z^{-1}\right)\left(\frac{1}{2} - z^{-1}\right)$$

$$= \frac{9}{2}z^{-1}$$

For causal solution,

$$F_p(z) = \frac{2}{9} \begin{pmatrix} 1 - \frac{1}{2}z^{-1} & -3 - \frac{3}{2}z^{-1} \\ -\frac{1}{2} + z^{-1} & \frac{3}{2} + 3z^{-1} \end{pmatrix}$$

$$\begin{pmatrix} F_0(z) & F_1(z) \end{pmatrix} = \begin{pmatrix} z^{-1} & 1 \end{pmatrix} F_p(z^2)$$

$$= \frac{2}{9} \begin{pmatrix} z^{-1} & 1 \end{pmatrix} \begin{pmatrix} 1 - \frac{1}{2}z^{-2} & -3 - \frac{3}{2}z^{-2} \\ -\frac{1}{2} + z^{-2} & \frac{3}{2} + 3z^{-2} \end{pmatrix}$$

$$= \frac{2}{9}\left[\left(-\frac{1}{2} + z^{-1} + z^{-2} - \frac{1}{2}z^{-3}\right) \quad \left(\frac{3}{2} - 3z^{-1} + 3z^{-2} - \frac{3}{2}z^{-3}\right)\right]$$

$$\begin{pmatrix} F_0(z) \\ F_1(z) \end{pmatrix} = \begin{pmatrix} \frac{2}{9}\left(-\frac{1}{2} + z^{-1} + z^{-2} - \frac{1}{2}z^{-3}\right) \\ \frac{2}{9}\left(\frac{3}{2} - 3z^{-1} + 3z^{-2} - \frac{3}{2}z^{-3}\right) \end{pmatrix}$$

e- Halfband $P_0(z) = F_0(z) H_0(z)$?

$$P_0(z) = \frac{2}{9}\left(-\frac{1}{2} + z^{-1} + z^{-2} - \frac{1}{2}z^{-3}\right)\left(\frac{3}{2} + 3z^{-1} + 3z^{-2} + \frac{3}{2}z^{-3}\right)$$

$$= \underbrace{\frac{2}{9}\left(\frac{1}{2}\right)\left(\frac{3}{2}\right)}_{\frac{1}{6}}\left(-1 + 2z^{-1} + 2z^{-2} - z^{-3}\right)\left(1 + 2z^{-1} + 2z^{-2} + z^{-3}\right)$$

$$\begin{array}{ccccccc}
 & & & -1 & 2 & 2 & -1 \\
 & & & 1 & 2 & 2 & 1 \\
\hline
 & & & -1 & 2 & 2 & -1 \\
 & & -2 & 4 & 4 & -2 & \\
 & -2 & 4 & 4 & -2 & & \\
-1 & 2 & 2 & -1 & & & \\
\hline
-1 & 0 & 4 & 6 & 4 & 0 & -1 \\
 & \uparrow & & \uparrow & & \uparrow & \\
 & z^{-1} & & z^{-3} & & z^{-5} &
\end{array}$$

$\underline{L = 3}$

Halfband

* <u>Checking aliasing cond</u>

Aliasing $= F_0(z) H_0(-z) - F_1(+z) H_1(-z)$ should be 0

$$= \frac{2}{9}\left(-\frac{1}{2} + z^{-1} + z^{-2} - \frac{1}{2}z^{-3}\right)\left(\frac{3}{2} - 3z^{-1} + 3z^{-2} - \frac{3}{2}z^{-3}\right)$$

$$- \frac{2}{9}\left(\frac{3}{2} - 3z^{-1} + 3z^{-2} - \frac{3}{2}z^{-3}\right)\left(\frac{1}{2} - z^{-1} - z^{-2} + \frac{1}{2}z^{-3}\right)$$

$$= 0$$

f- $L = 3$.