Assignment 2

COL 352
Introduction to Automata & Theory of Computation

Problem 1

Design DFA for the following languages over $\{0, 1\}$

(a) The set of all strings such that every block of of five consecutive symbols have at least two 0's

(b) The set of strings with an equal number of 0's and 1's such that each prefix has at most one more 0 than 1's and at most one more 1 than 0's

Solution :

(a) The DFA can be specified as a 5-tuple $< Q, \Sigma, s, F, \delta >$. The set of all states Q is

$$Q = \{ s,\ q_0,\ q_1,\ q_{00},\ q_{01},\ q_{10},\ q_{11},\ q_{000},\ q_{001},\ q_{010},\ q_{011},\ q_{100},\ q_{101},\ q_{110},\ q_{111},\ q_{0000},\ q_{0001},\ q_{0010},\ q_{0011},\ q_{0100},\ q_{0101},\ q_{0110},\ q_{0111},\ q_{1000},\ q_{1001},\ q_{1010},\ q_{1011},\ q_{1100},\ q_{1101},\ q_{1110},\ q_{1111},\ q_{00000},\ q_{00001},\ q_{00010},\ q_{00011},\ q_{00100},\ q_{00101},\ q_{00110},\ q_{00111},\ q_{01000},\ q_{01001},\ q_{01010},\ q_{01011},\ q_{01100},\ q_{01101},\ q_{01110},\ q_{01111},\ q_{10000},\ q_{10001},\ q_{10010},\ q_{10011},\ q_{10100},\ q_{10101},\ q_{10110},\ q_{10111},\ q_{11000},\ q_{11001},\ q_{11010},\ q_{11011},\ q_{11100},\ q_{11101},\ q_{11110},\ q_{11111} \}$$

The start state is s. Assuming that all the strings of length less than 5 are accepted, the set of final states is

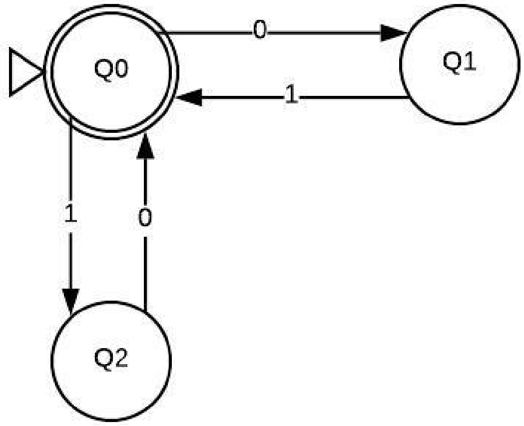
$$F = \{ s,\ q_0,\ q_1,\ q_{00},\ q_{01},\ q_{10},\ q_{11},\ q_{000},\ q_{001},\ q_{010},\ q_{011},\ q_{100},\ q_{101},\ q_{110},\ q_{111},\ q_{0000},\ q_{0001},\ q_{0010},\ q_{0011},\ q_{0100},\ q_{0101},\ q_{0110},\ q_{0111},\ q_{1000},\ q_{1001},\ q_{1010},\ q_{1011},\ q_{1100},\ q_{1101},\ q_{1110},\ q_{1111},\ q_{00000},\ q_{00001},\ q_{00010},\ q_{00011},\ q_{00100},\ q_{00101},\ q_{00110},\ q_{00111},\ q_{01000},\ q_{01001},\ q_{01010},\ q_{01011},\ q_{01100},\ q_{01101},\ q_{01110},\ q_{10000},\ q_{10001},\ q_{10010},\ q_{10011},\ q_{10100},\ q_{10101},\ q_{10110},\ q_{11000},\ q_{11001},\ q_{11010},\ q_{11100} \}$$

The transition function δ is given in following table

Current state	**Next state at input 0**	**Next state at input 1**
s	q_0	q_1
q_0	q_{00}	q_{01}
q_1	q_{10}	q_{11}
q_{00}	q_{000}	q_{001}
q_{01}	q_{010}	q_{011}
q_{10}	q_{100}	q_{101}
q_{11}	q_{110}	q_{111}
q_{000}	q_{0000}	q_{0001}
q_{001}	q_{0010}	q_{0011}
q_{010}	q_{0100}	q_{0101}
q_{011}	q_{0110}	q_{0111}
q_{100}	q_{1000}	q_{1001}
q_{101}	q_{1010}	q_{1011}

q_{110}	q_{1100}	q_{1101}
q_{111}	q_{1110}	q_{1111}
q_{0000}	q_{00000}	q_{00001}
q_{0001}	q_{00010}	q_{00011}
q_{0010}	q_{00100}	q_{00101}
q_{0011}	q_{00110}	q_{00111}
q_{0100}	q_{01000}	q_{01001}
q_{0101}	q_{01010}	q_{01011}
q_{0110}	q_{01100}	q_{01101}
q_{0111}	q_{01110}	q_{01111}
q_{1000}	q_{10000}	q_{10001}
q_{1001}	q_{10010}	q_{10011}
q_{1010}	q_{10100}	q_{10101}
q_{1011}	q_{10110}	q_{10111}
q_{1100}	q_{11000}	q_{11001}
q_{1101}	q_{11010}	q_{11011}
q_{1110}	q_{11100}	q_{11101}
q_{1111}	q_{11110}	q_{11111}
q_{00000}	q_{00000}	q_{00001}
q_{00001}	q_{00010}	q_{00011}
q_{00010}	q_{00100}	q_{00101}
q_{00011}	q_{00110}	q_{00111}
q_{00100}	q_{01000}	q_{01001}
q_{00101}	q_{01010}	q_{01011}
q_{00110}	q_{01100}	q_{01101}
q_{00111}	q_{01110}	q_{01111}
q_{01000}	q_{10000}	q_{10001}
q_{01001}	q_{10010}	q_{10011}
q_{01010}	q_{10100}	q_{10101}
q_{01011}	q_{10110}	q_{10111}
q_{01100}	q_{11000}	q_{11001}
q_{01101}	q_{11010}	q_{11011}
q_{01110}	q_{11100}	q_{11101}
q_{01111}	q_{11110}	q_{11111}
q_{10000}	q_{00000}	q_{00001}
q_{10001}	q_{00010}	q_{00011}
q_{10010}	q_{00100}	q_{00101}
q_{10011}	q_{00110}	q_{00111}
q_{10100}	q_{01000}	q_{01001}
q_{10101}	q_{01010}	q_{01011}
q_{10110}	q_{01100}	q_{01101}
q_{10111}	q_{01110}	q_{01111}
q_{11000}	q_{10000}	q_{10001}
q_{11001}	q_{10010}	q_{10011}
q_{11010}	q_{10100}	q_{10101}
q_{11011}	q_{10110}	q_{10111}
q_{11100}	q_{11000}	q_{11001}
q_{11101}	q_{11010}	q_{11011}
q_{11110}	q_{11100}	q_{11101}
q_{11111}	q_{11110}	q_{11111}

(b) If there is no explanation of the states/DFA, at most 5/10 will be given.



Problem 2

Design NFA for the following languages

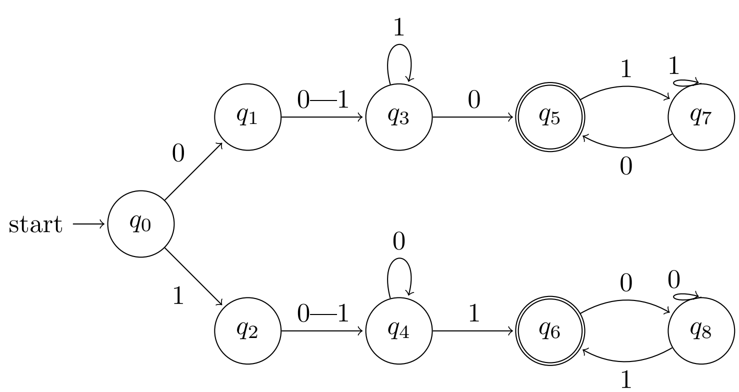
(a) The set of strings over a, b that have the same value when multiplied from left to right as from left to right. The rules of multiplication are $a \times a = b$, $b \times b = a$, $a \times b = b$, $b \times a = b$. Note that $((a \times b) \times b) = a$ and $(a \times (b \times b)) = b$, i.e. it is not associative

(b) The set of strings of the form $\{xwx^R \mid x, w$ are strings over 0,1 of non-zero length$\}$

Solution :

(a) The problem reduces to finding strings of length at least 3 having the same initial and final characters.

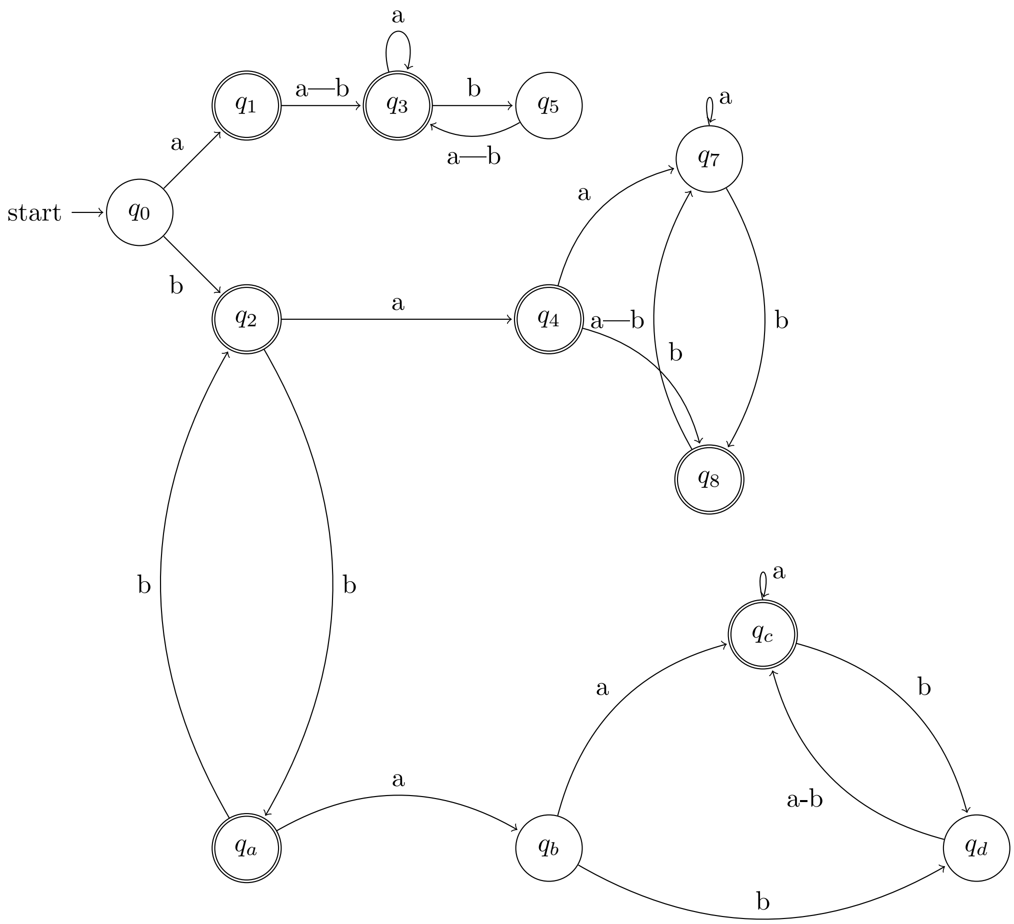
The NFA can be specified as a 5-tuple $< Q, \Sigma, s, F, \delta >$, with $Q = \{q_0, q_1, q_2, q_3, q_4, q_5, q_6, q_7, q_8\}$, $\Sigma = \{0, 1\}$, Start state $s = q_0$ and accepting states $F = \{q_5, q_6\}$



(b) The following observations capture all possible strings with the following DFA

1. The product is 'a' iff both the inputs are 'b'
2. The parity of the number of b's to the leftmost a's must be equal to number of b's to the rightmost 'a' iff both are non-empty. Let either of them be denoted by $L[a]$ and $L[a']$ respectively

3. If $L[a] = 0$ or $L[a'] = 0$, then parity of number of 'b's should be odd on the other side
4. If $L[a] = 0$ and $L[a'] = 0$, then string is palindromic and gives the same result



Problem 3

Prove or disprove $(r + s)^* = r^* + s^*$, where r, s and t are regular expressions and $r = s$ implies $L(r) = L(s)$)

Solution : Let $\Sigma = \{0, 1\}$ and $r = 0^*$, $s = 1^*$

Then, $(r + s)^* = \epsilon \cup \{0, 1\} \cup \{0, 1\}^2 \cup \dots$

and, $r^* + s^* = \epsilon \cup \{0\} \cup \{0\}^2 \cup \dots \cup \{1\} \cup \{1\}^2 \cup \dots$

Now, consider a string 01.

Clearly, $01 \in \{0, 1\}^2 \Rightarrow 01 \in (r + s)^*$, but $01 \notin r^* + s^*$

Hence, $(r + s)^* \neq r^* + s^*$

Problem 4

Which of the following are regular sets

(a) $\{0^{2^n} | \; n \geq 1\}$

(b) $\{xx^R w | \; x, w \in (0 + 1)^+\}$

Solution :

(a) Let L denote the set, then assuming that it is regular, let $n \geq 1$ denote the pumping lemma constant

Consider the string $0^{2^n} \in L = 0^p 0^q 0^r$ with $p + q + r \geq n$, $q \geq 1$ and $p + q \leq n$.

Now, from pumping lemma, $\forall i \geq 0, 0^p 0^{iq} 0^r \in L$, i.e. $p + iq + r = 2^m$, for some $m \geq 1$.

Setting $i = 2$, gives $p + 2q + r = 2^m$, and since $p + q + r = 2^n \Rightarrow 2^n + q = 2^m$, i.e. q is a power of 2.

Specifically, since n, m are integers and $q \geq 1, q = 2^n$.

But this means that $q > n$, since $n \geq 1$ and so, $p + q > n$ - a contradiction. Thus the L is not a regular set.

(b) Let L denote the set, then assuming that it is regular, let $n \geq 1$ denote the pumping lemma constant

Consider the string $0^n 1.10^n.1 \in L = 0^l 0^m 0^{n-m-l} 1.10^n.1,\ 1 \leq m < n,\ 0 < l < n$, with $x = 0^l,\ y = 0^m$

Now, from pumping lemma, $\forall i \geq 0,\ xy^iz \in L$, as $|xy| \leq n$ and $y \geq 1$

Setting $i = 2$, gives $xy^2z = 0^{n+m}1.10^n.1 \in L$

But since $n + m > n$, so $xy^2z \notin L$ - a contradiction. Thus the L is not a regular set.

Also accepted : Solutions that use the Myhill-Nerode theorem correctly

Problem 5

Is the set $\frac{1}{2}(L) = \{x |\ \exists y$ such that $|x| = |y|,\ xy \in L\}$ regular?

Solution : Since, L is a regular language, so, let, $M = (Q, \Sigma, \delta, q_0, F)$ be the DFA that accepts L.

Let $L' = \frac{1}{2}L$. We will construct a DFA $M' = (Q', \Sigma, \delta', q_0', F')$ which accepts L' and hence prove that it is a regular language.

- $Q' = Q \times 2^Q$
- Let $S \in 2^Q$. Define $prev(S) = \{q \in Q \mid \exists\ a \in \Sigma,\ q' \in S\ s.t.\ \delta(q, a) = q'\}$
- $\delta'((q, S), a) = (\delta(q, a), prev(S))$
- $q_0' = (q_0, F)$
- $F' = \{(q, S), q \in S\}$

$F_1 = prev(F)$ will give us all the states in M from where we can reach a final state of M on reading a string of length 1. Similarly, $F_n = prev(F_{n-1})$ will give us all the states in M from where we can reach a final state of M on reading a string of length n.

Let $xy \in L$ and $|x| = |y| = n$. M reaches to a state p after reading x and from p it reaches a final state f on reading y. So, $p \in F_n$. And, on reading x, M' will reach a state (p, F_n). By definition of F', it is a final state of M'. So, xy will be accepted by M'.

Now, let x be accepted by M' and $|x| = n$. So, let, M' reaches a state (p, P) where $p \in P$ on reading x. Also, $P = F_n$. So, there exists a string y, reading which we can reach from p to a final state in M. So, xy is accepted by M.

Hence, L' is the language accepted by M'.

Regarding the grading:

- If you have simply specified the DFA with no explanation, you will get at most 5/10.
- You need to prove that any string in $\frac{L}{2}$ will be accepted by the DFA **AND** any string accepted by the DFA will be contained in $\frac{L}{2}$.