Assignment 1

COL 352
Introduction to Automata & Theory of Computation

Problem 1

Consider the following two definitions of the strings of balanced parentheses:

A. A string w is balanced iff
(i) w has an equal number of "(" and ")"
(ii) Any prefix of w has at least as many "(" as ")"

B. A string w is balanced iff
(i) ϵ is balanced.
(ii) If w is balanced, so is (w).
(iii) If w and x are balanced then so is $w \cdot x$
(iv) Nothing else is balanced.

Show that the above definitions A and B are equivalent.

Solution : To show $A \iff B$,

I) Claim : $A \Longrightarrow B$

Proof : By induction on the length of the string w

Basis : For $|w| = 0$, using B (i), w is balanced

Induction Hypothesis : Let string $w \in A$ then for all strings smaller than w, $A \Longrightarrow B$

Induction Step:

Case 1 - w does not have any prefix with equal number of "(" and ")" :

Then w is of the form (x) for some string x.

Since x has equal number of "("and ")", and its prefixes have at least as many "(" as ")", so $x \in A$

From IH, $x \in B$

From B (ii), $w \in B$

Case 2 - w has a prefix x with equal number of "(" and ")"

Then w is of the form $x.y$ for some shortest prefix x and some $y \neq \epsilon$.

Since x and w have equal number of "(" and ")", hence y will also have equal number of "(" and ")".

y has at least as many "(" as ")" because, if some prefix of y has less "(" then w will have less "(".

But w is balanced. So $x, y \in A$

From IH, $x, y \in B$

From B (iii), $w \in B$

II) Claim : $B \Longrightarrow A$

Proof : By induction on the length of the string w

Basis : For $|w| = 0$, using A (i) & (ii), w is balanced

Induction Hypothesis : Let string $w \in B$ then for all strings smaller than w, $B \Longrightarrow A$

Induction Step:

Case 1 - w is balanced according to B (ii), i.e. $w = (x)$:

By IH, x is balanced by A, i.e. x has equal number o f "(" and ")" and so is (x)

By IH, x has atleast as many "(" as ")", and so has (x)

Thus, $w \in A$

Case 2 - w is balanced according to B (iii), i.e. $w = x.y$

By IH, x and y are balanced by A, so x and y have equal

number of "(" and ")", hence $x.y$ has equal number of "(" and ")".

If some prefix z of w has less "(" than ")", then there are 2 cases :

(a) z completely lies in x - By IH, any prefix of x cannot have fewer "(" than ")", a contradiction.

(b) z extends to y - Since x has equal number of "(" and ")" and z has fewer "(" than ")", so prefix of y has fewer "(". But by IH, prefix of y cannot have less "(" and ")", a contradiction.

Thus, $w \in A$

From I and II, $A \iff B$, hence proved

Problem 2

Show that the Principle of Mathematical Induction and the Principle of Complete induction are equivalent. Hint: Express them rigorously as sentences in first order logic.

Solution : Let A denote Principle of Mathematical Induction and B denote Principle of Complete Induction. To show $A \iff B$,

Case I : $A \implies B$

Proof : Let $Q(K) = P(1)\&P(2)\&P(3)...\&P(k)$, be the preposition that denotes whether $P(i) = true \forall i \leq K$

$Q(K) \implies Q(K+1)$ from premise (Invoking Math Induction on $Q(k)$ —(1)

$Q(K) \implies P(K)$ from definition of $Q(K)$ —(2)

Invoking math induction on $P(K), P(K) \implies P(K+1)$ —(3)

Using (2) and (3) and applying transitive law we get, $Q(K) \implies P(K+1)$

Case II : $B \implies A$

Proof : From the premise, $((P(1)\&P(2)\&P(3)...\&P(k)) \implies P(k+1)$

$\implies (Q(k-1)\&P(k)) \implies P(k+1)$

$\implies \neg(Q(k-1)\&P(k)) \vee P(k+1)$

$\implies \neg Q(k-1) \vee \neg P(k) \vee P(k+1)$

$\implies \neg Q(k-1) \vee (P(k) \implies P(k+1))$

So, to prove mathematical induction holds we prove that proposition $Q(k-1)$ is true.

Let us assume on the contrary that $Q(k-1)$ is false

$\implies \exists i, P(i) = false, i \leq (k-1)$

$\implies \exists i', P(i') = false, i' \leq i$

The sequence of i's forms a strictly decreasing sequence and will terminate at $i = 1$, i.e. $P(1) = false$.

This contradicts the hypothesis of complete induction, hence $P(K) \implies P(K+1)$

Hence mathematical induction holds.

Problem 3

Show two distinct bijective mappings between integers and rationals.

Solution :

Bijection 1 - For $i \in \mathbb{N}$, define $T(i) = \begin{cases} \frac{-i}{2}, & \text{if i is even;} \\ \frac{i+1}{2}, & \text{if i is odd.} \end{cases}$ Then T is a bijection from $\mathbb{N}$ to $\mathbb{Z} \setminus \{0\}$

If $n = \prod_k p_k^{n_k}$

define $f : \mathbb{N} \to \mathbb{Q}^+$ by $f(n) = \prod_k p_k^{T(n_k)}$

Claim : f is a bijection

I. f is one-one

If $f(m) = f(n)$, we have two rational numbers which are equal and thus have the same fractional representation $\frac{p}{q}$ with $p, q \in \mathbb{N}$. Writing the fraction as a prime factorization, we get a product like $\prod_k p_k^{y_k}$, where y_k are positive or negative integers. We map the y_k to the inverse of the T transformation to get the n_k and thus the n. Hence function f is one-one.

II. f is onto

Let $\frac{p}{q} = \prod_k p_k^{y_k}$, where y_k are positive or negative integers, be any rational number.

Choose $n_k = T^{-1}(y_k)$

If $n = \prod_k p_k^{n_k}$, then we have $f(n) = \prod_k p_k^{T(n_k)} = \prod_k p_k^{y_k} = \frac{p}{q}$

Therefore there exists a natural number n for each given rational number $\frac{p}{q}$ such that $f(n) = \frac{p}{q}$.

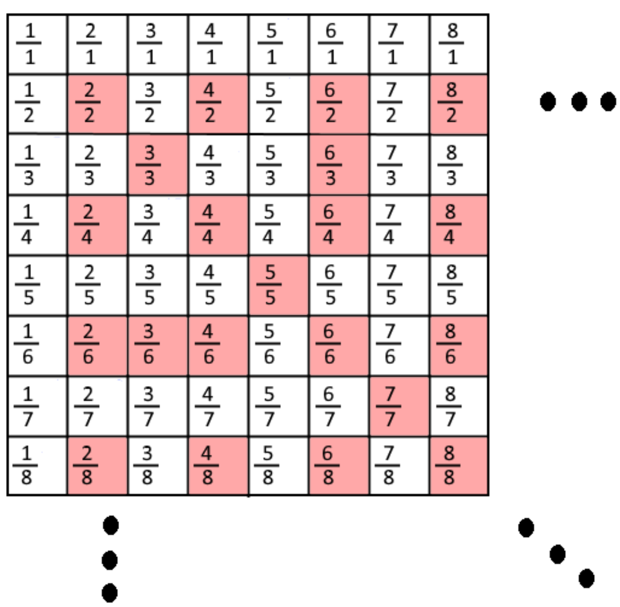
Hence $f : \mathbb{N} \to \mathbb{Q}^+$ given by $f(n) = \prod_k p_k^{T(n_k)}$ is a bijection.

Define $g : \mathbb{Z} \to \mathbb{Q}$ such that

$$g(n) = \begin{cases} f(n), & \text{if } n > 0; \\ f(-n), & \text{if } n < 0; \\ 0, & \text{if } n = 0. \end{cases}$$

Since f is bijection so g is also a bijection.

Bijection 2 - The set $\mathbb{Q}^+$ of positive rational numbers can be written as follows



We map natural numbers diagonally on this array as follows

$\frac{1}{1}$ 1	$\frac{2}{1}$ 2	$\frac{3}{1}$ 4	$\frac{4}{1}$ 6	$\frac{5}{1}$ 10	$\frac{6}{1}$ 12	$\frac{7}{1}$ 18	$\frac{8}{1}$ 22
$\frac{1}{2}$ 3	$\frac{2}{2}$	$\frac{3}{2}$ 7	$\frac{4}{2}$	$\frac{5}{2}$ 13	$\frac{6}{2}$	$\frac{7}{2}$ 23	$\frac{8}{2}$
$\frac{1}{3}$ 5	$\frac{2}{3}$ 8	$\frac{3}{3}$	$\frac{4}{3}$ 14	$\frac{5}{3}$ 19	$\frac{6}{3}$	$\frac{7}{3}$	$\frac{8}{3}$
$\frac{1}{4}$ 9	$\frac{2}{4}$	$\frac{3}{4}$ 15	$\frac{4}{4}$	$\frac{5}{4}$ 24	$\frac{6}{4}$	$\frac{7}{4}$	$\frac{8}{4}$
$\frac{1}{5}$ 11	$\frac{2}{5}$ 16	$\frac{3}{5}$ 20	$\frac{4}{5}$ 25	$\frac{5}{5}$	$\frac{6}{5}$	$\frac{7}{5}$	$\frac{8}{5}$
$\frac{1}{6}$ 17	$\frac{2}{6}$	$\frac{3}{6}$	$\frac{4}{6}$	$\frac{5}{6}$	$\frac{6}{6}$	$\frac{7}{6}$	$\frac{8}{6}$
$\frac{1}{7}$ 21	$\frac{2}{7}$ 26	$\frac{3}{7}$	$\frac{4}{7}$	$\frac{5}{7}$	$\frac{6}{7}$	$\frac{7}{7}$	$\frac{8}{7}$
$\frac{1}{8}$ 27	$\frac{2}{8}$	$\frac{3}{8}$	$\frac{4}{8}$	$\frac{5}{8}$	$\frac{6}{8}$	$\frac{7}{8}$	$\frac{8}{8}$

Numbers in blue are natural numbers.

Let f denote this bijection of natural numbers to set of positive rational numbers.

Note that this array contains each natural number exactly once and each positive rational number is image of some natural number so the array given above represents a bijection from natural numbers to positive rational numbers.

Define $g : \mathbb{Z} \to \mathbb{Q}$ such that $g(n) = \begin{cases} f(n), & \text{if } n > 0; \\ f(-n), & \text{if } n < 0; \\ 0, & \text{if } n = 0. \end{cases}$

Since f is bijection, g is also a bijection.

Problem 4

A relation $\leq_{\#}$ is defined as follows - If A, B are sets, then $A \leq_{\#} B$ iff there exists a 1-1 mapping $f : A \to B$ and an onto mapping $g : B \to A$. $If f : A \to B$ is a bijection then, $A \leq_{\#} B$.

What can you say about the pairs of sets

(i) integers and rationals (ii) Reals in [0, 1] and reals in $(10, 100)$ (open interval)

The Bernstein-Schroeder theorem says that If $A \leq_{\#} B$ and $B \leq_{\#} A$ then $A =_{\#} B$.

Solution :

(i) Using the bijection found between Rationals and Integers in Problem 4, by the second clause $\mathbb{Z} =_{\#} \mathbb{R}$

(ii) Let $f(x) : [0, 1] \to (10, 100)$ be defined as : $f(x) = \begin{cases} x + 11 & if x \in (0, 1) \\ 12 & if x = 0 \\ 13 & if x = 1 \end{cases}$

For a one-one function, if $f(x) = f(y)$ then $x = y$

Case 1 - $x = 0, y = 1$

$f(x) = 12, f(y) = 13,\ f(x) \neq f(y)$

Case 2 - $x = 0, y \in (0,1)$

$f(x) = 12, f(y) \in (11,12),\ f(x) \cap f(y) = \phi$

Case 3 - $x = 1, y \in (0,1)$

$f(x) = 13, f(y) \in (11,12),\ f(x) \cap f(y) = \phi$

Therefore f(x) is a one-one function

Let $g(x) : (10,100) \to [0,1]$ be defined as: $g(x) = \begin{cases} x - 20 & if x \in [20,30] \\ 0 & otherwise \end{cases}$

Let $y = x - 20$ for $x \in [20,30] \implies x = y + 20,\ \forall y \in [0,1]$

$\implies x \in [0 + 20, 0 + 30] \implies x \in [20,30]$

Thus $g(x)$ is onto

Since $f(x) : [0,1] \to (10,100)$ is one-one and $g(x) : (10,100) \to [0,1]$ is onto, therefore $[0,1] <_{\#} (10,100)$

Let $f(x) : (10,100) \to [0,1]$ be defined as: $f(x) = \frac{x-10}{90}$

Let $f(x_1) = f(x_2)$

$\implies \frac{x_1 - 10}{90} = \frac{x_2 - 10}{90}$

$\implies x_1 = x_2$

Therefore $f(x)$ is one-one

Let $g(x) : [0,1] \to (10,100)$ be defined as: $g(x) = \begin{cases} 90x + 10 & if x \in (0,1) \\ 20 \quad if x \in \{0,1\} \end{cases}$

Let $y = 90x + 10$ for $x \in (0,1)$

$\implies x = \frac{y-10}{90},\ \forall y \in (10,100), x \in (\frac{10-10}{90}, \frac{100-10}{90})$

$\implies x \in (0,1)$ Thus, $g(x)$ is onto

Since $f(x) : (10,100) \to [0,1]$ is one-one and $g(x) : [0,1] \to (10,100)$ is onto, therefore $(10,100) <_{\#} [0,1]$

Thus by equation 4 and 7 and Bernstein-Schroeder theorem, $[0,1] =_{\#} (10,100)$