The overlapping generations model

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Agenda

The overlapping generations model

- Two-period model lived agents, young and old
- Consumption/savings problem
- Dynamics of the economy
- Welfare

Government debt and social security

- The usefulness of government debt
- Fully funded pension systems
- Pay-as-you-go pension systems

The neoclassical growth model

Ramsey

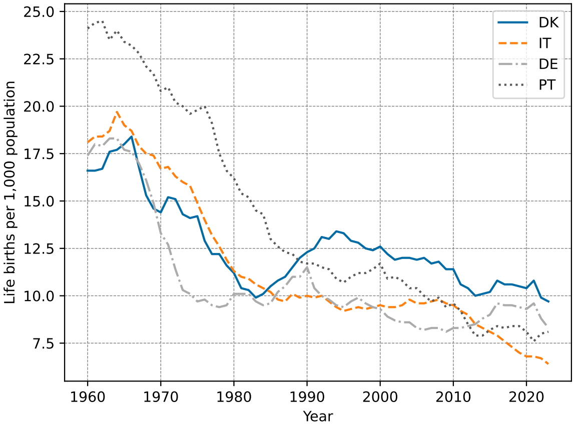
- Everything is efficient
- Infinitely lived agent(s)
- Ricardian equivalence holds
- Perfect credit markets $\rightarrow$ perfect insurance

Main issues

- People don't live forever
- Ageing has big impacts on labor productivity and income
- Retirement savings are a large piece of government spending
- Many issues span generations (e.g., climate change)

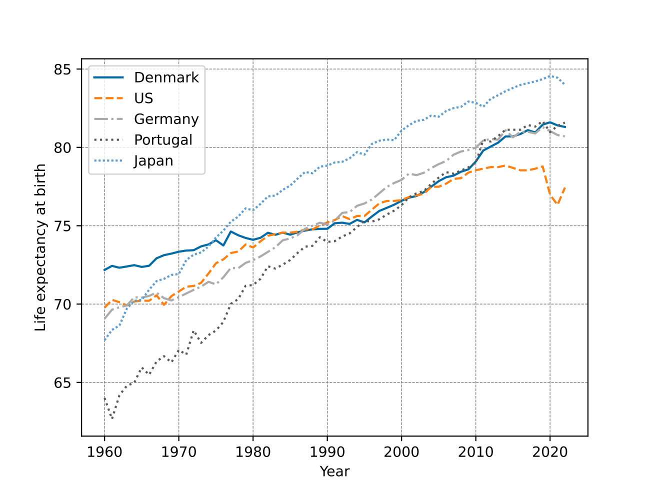
$\Longrightarrow$ need a model with multiple generations

Demographics



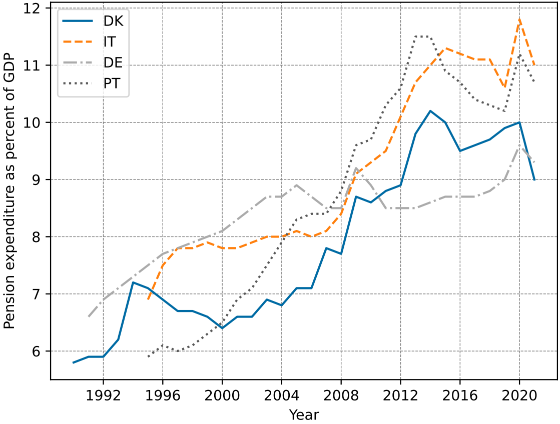
- Population growth has been decreasing Eurostat

Demographics



- Life expectancy is rising Fred

Demographics



- Governments are spending more on pensions Eurostat

Demographics

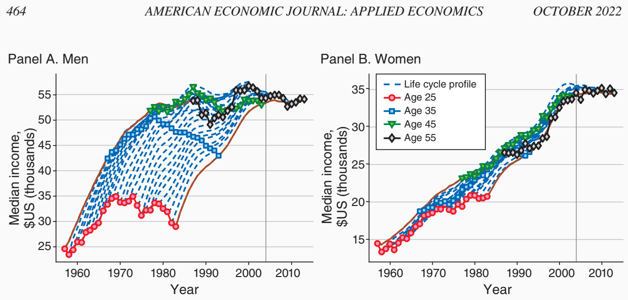


FIGURE 5. AGE PROFILES OF MEDIAN EARNINGS BY COHORT

- Many economic variables have a lifecycle profile (Guvenen et al, 2022)

The overlapping generations model

Peter A. Diamond

- Combines Samuelson's OLG endowment economy with production function
- Tractable model that allows saving for old age
- 2010 Nobel prize winner for labor market model

Deceptively simple

- Point is not to accurately predict savings
- Instead: make deep statement about welfare

Peter A. Diamond

Prize share: 1/3

The model

The model

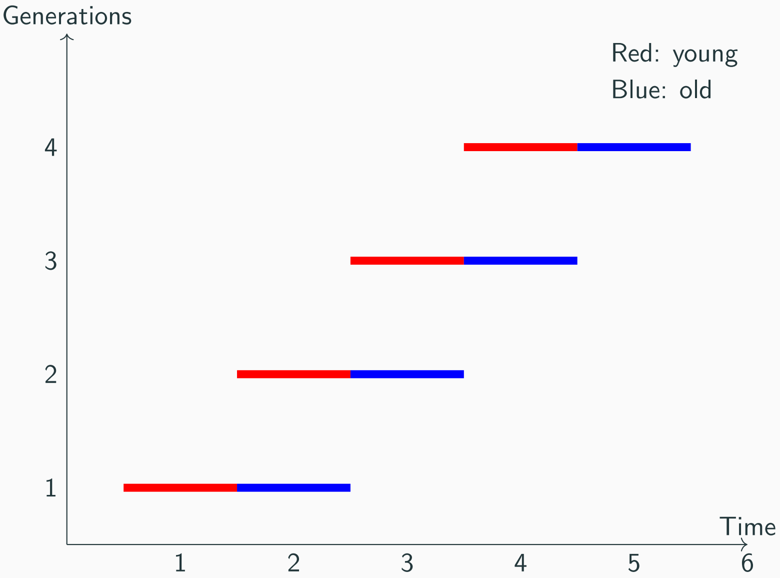
Model assumptions

- Time is discrete
- Agents in the economy live for 2 periods
- Each generation, $L_{t+1} = (1+n)L_t$ individuals are alive
- Two generations are alive: L_t young and $L_t/(1+n)$ old
- Agents only work when young, inelastically
- Per-capita $\neq$ per-worker!
- Agents derive utility from consumption only when alive
- Agents discount the future at rate β

Timing each period

- Firms produce, pay out wages and interest rates
- Consumers consume and save

The OLG model graphically



Households

Lifetime utility

$$\max_{c,a} U = u(c_{1,t}) + \beta u(c_{2,t+1})$$

- Very similar to two period model

Budget constraint

$$\begin{aligned} c_{1,t} + a_{t+1} &= w_t \\ c_{2,t+1} &= (1 + r_{t+1})a_{t+1} \\ \implies c_{1,t} + \frac{c_{2,t+1}}{1 + r_{t+1}} &= w_t \end{aligned}$$

- No inheritance or dynasties
- Young need to save some of their wage
- r_t is the economy-wide interest rate

Household problem

Lagrange

$$\mathcal{L} = u(c_{1,t}) + \beta u(c_{2,t+1}) + \lambda \left[c_{1,t} + \frac{c_{2,t+1}}{1 + r_t} - w_t \right]$$

Optimality conditions

$$\begin{aligned} u'(c_{1,t}) &= \beta(1 + r_{t+1}) u'(c_{2,t+1}) \\ c_{1,t} + \frac{c_{2,t+1}}{1 + r_{t+1}} &= w_t \end{aligned}$$

- Euler equation
- Trade off utility today for discounted utility tomorrow
- In optimum, discounted marginal utilities equalize
- PV of consumption = PV of income (as before)

Firms

Firm problem

$$\max_{K_t, L_t} \Pi = F(K_t, L_t) - K_t r_t - L_t w_t$$

- Assume $F(\cdot)$ exhibits constant returns to scale
- Competitive factor markets
- Firms maximize profits
- Abstract from depreciation for today

Cobb-Douglas

$$F(K_t, L_t) = K_t^{\alpha} L_t^{1-\alpha}$$

- Attractive properties discussed before: e.g., balanced growth

$\implies$ usual expressions for $r = \alpha k^{\alpha-1}$ and $w = (1-\alpha)k^{\alpha}$ per worker

Individual savings decisions

Optimal savings pinned down by

$$u'(w_t - a_{t+1}) = \beta(1 + r_{t+1})u'((1 + r_{t+1})a_{t+1})$$

- Substitution effect: $\frac{\partial a_{t+1}}{\partial r_{t+1}} > 0$ because of higher return
- Income effect: $\frac{\partial a_{t+1}}{\partial r_{t+1}} < 0$ because agent is richer

Special case: closed form solution for savings with CRRA preferences and $\sigma = 1$ (prove it)

$$a_{t+1} = \frac{\beta}{1 + \beta} w_t$$

- Substitution and income effects cancel, $(1 + r_{t+1})$ cancels
- General case: a_{t+1} defined by wage and interest: $a_{t+1}(w_t, r_{t+t})$

Aggregation

Total savings

$$S_t = a_{t+1} L_t$$

- At the beginning of the period, the young save
- a represents savings per worker → multiply by #young $= L_t$

Capital stock for next period

$$K_{t+1} = \underbrace{S_t}_{\text{Savings of young}} - \underbrace{(1-\delta)K_t}_{\text{Old consume all capital}} + \underbrace{(1-\delta)K_t}_{\text{Undepreciated capital}}$$

$$K_{t+1} = S_t$$

$$\frac{K_{t+1}}{L_{t+1}} \frac{L_{t+1}}{L_t} = (1+n)k_{t+1} = s_t$$

$\Longrightarrow$ capital per-worker

General Equilibrium

- Tractable result using special functional form

GE in the OLG model

Assumed

- Production: $Y = K^{\alpha} L^{1-\alpha}$
- Utility: $u(c) = \ln(c)$

Factor prices

$$r_t = \alpha k_t^{\alpha-1}$$
$$w_t = (1-\alpha) k_t^{\alpha}$$

Aggregate savings and capital $\quad (s_t = (1+n)k_{t+1})$

$$s_t = \frac{\beta}{1+\beta} w_t$$
$$(1+n)k_{t+1} = (1-\alpha)\frac{\beta}{1+\beta} k_t^{\alpha}$$

- Steady state: $k^* = \left(\frac{1-\alpha}{1+n}\frac{\beta}{1+\beta}\right)^{\frac{1}{1-\alpha}} \implies$ per-worker variables constant

Consumption

Consumption in steady state

$$c_0 = \frac{1}{1+\beta} w \qquad \text{(c of young)}$$

$$c_1 = (1+r) \frac{\beta}{1+\beta} w \qquad \text{(c of old)}$$

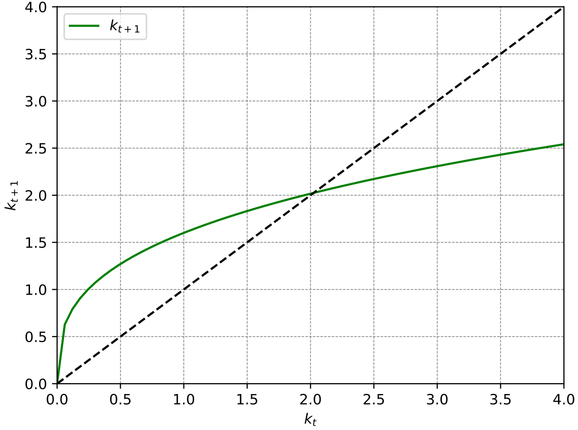
Total consumption

$$c = c_1 + c_0 = \frac{1}{1+\beta}(1-\alpha)k^{\alpha} + \frac{(1+\alpha k^{\alpha-1})}{1+n} \frac{\beta}{1+\beta}(1-\alpha)k^{\alpha}$$

$$c = k^{\alpha} - nk$$

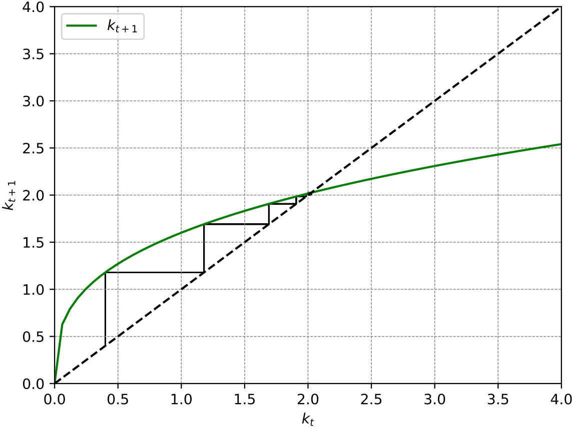
- Market clearing holds Algebra

Dynamics of the OLG economy



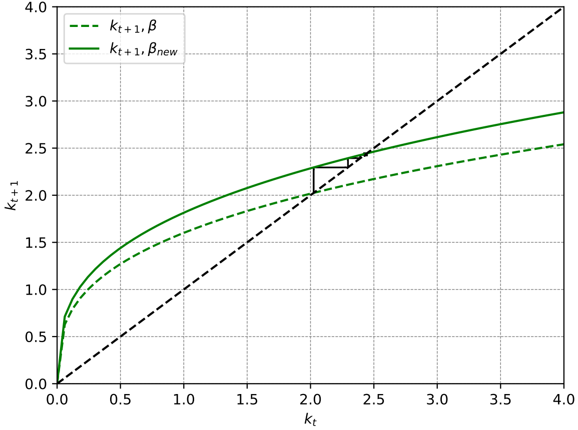
- This OLG model always converges, much like Solow

Dynamics of the OLG economy



- Dynamics can be read off of the graph (special case!)

A rise in β



- More patience→ higher savings → higher capital

Welfare

The welfare in the economy

Ramsey model

- Welfare = utility of the representative household
- Planner can maximize

OLG model

- Infinity of households, different HHs alive at same time
- No obvious way to weight their respective utilities
- Discount future welfare? Count everyone equally? Young vs old?
- Decentralized equilibrium unlikely to maximize arbitrary welfare function

$\Longrightarrow$ Difficult to evaluate whether welfare is maximized

But can investigate weaker concept: Pareto optimality?

Golden Rule in OLG

Maximizing steady state consumption

- The resource constraint in the economy (with $\delta = 0$) is

$$
\begin{aligned}
(1+n)k_{t+1} &= f(k_t) + k_t - c_t \\
\implies c &= f(k) - nk \\
\max(c): \quad n &= f'(k_{gr})
\end{aligned}
$$

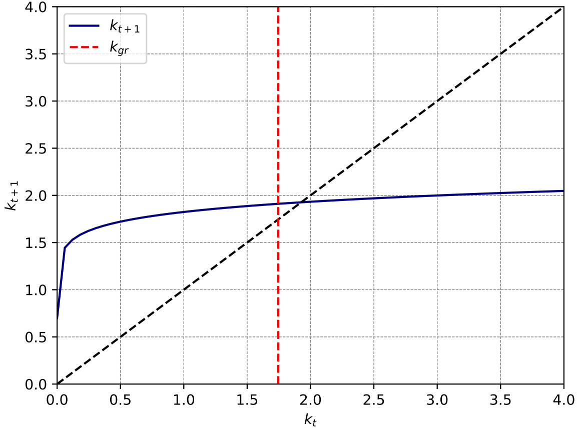
- At $n = f'(k_{gr})$, steady state consumption is maximized

Competitive equilibrium: $k^* = \left(\frac{1-\alpha}{1+n}\frac{\beta}{1+\beta}\right)^{\frac{1}{1-\alpha}}$

$$f'(k^*) = \alpha k^{*(\alpha-1)} = (1+n)\frac{\alpha}{1-\alpha}\frac{1+\beta}{\beta} \gtrless n$$

- It's possible for k^* to be above or below k_{gr}

Capital relative to the golden rule level

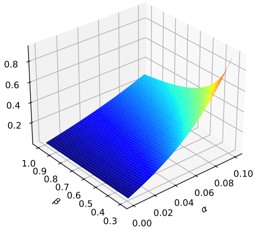


- $k^* > k_{gr} \implies f'(k^*) < n$

Competitive equilibrium capital vs golden rule

If competitive equilibrium capital "too high" depends on parameters

$$k^* > k_{gr}$$
$$f'(k^*) < f'(k_{gr})$$
$$(1+n)\frac{\alpha}{1-\alpha}\frac{1+\beta}{\beta} < n$$
$$\frac{\frac{\alpha}{1-\alpha}\frac{1+\beta}{\beta}}{1-\frac{\alpha}{1-\alpha}\frac{1+\beta}{\beta}} < n$$



- If β or n are high or α is low, $f'(k^*)$ is low and k^* is high
  $\rightarrow$ individuals are saving too much, relative to k_{gr}

Planner's intervention

Can the planner make everyone better off if $k^* > k_{gr}$

- In some period 0, increase consumption and lower capital s.t. $k = k_{gr}$
- Consumption rises in the first period (dissaving)
- Consumption is also higher in all future periods $(k_{gr} < k^* \implies c_{gr} > c^*)$

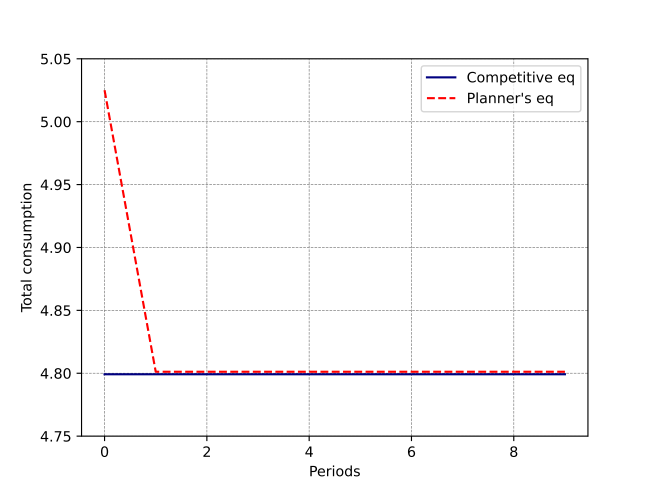
$\implies$ The planner can increase aggregate consumption

What's going on?

- The planner is not constrained by the savings technology!
- Competitive economy: work when young, then save
- Planner: Transfer some of the young's income to the old

  The young are better off because they receive more c when old

Consumption after the planner's intervention



- Eat a lot of capital in $t = 0$, $k \downarrow$, save less and eat more after

Planner's intervention

Planner intuition

- Institute a system in which **every young generation** saves a little less in k but gives some labor income to the contemporary old
- k falls, $f'(k)$ rises
- For each old agent, there are $(1+n)$ young
- Give up x today, receive $(1+n)x > (1+f'(k))x$ tomorrow
- Every young agent gets a return of $(1+n)$ on this deal

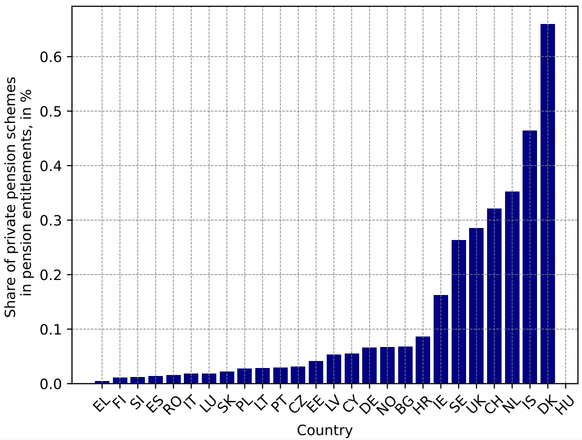
Why is this impossible in competitive markets?

- Young cannot make a contract with future unborn young
- The planner has no such limitation
- Scheme only works because there are infinite generations

  Similar logic to Hilbert's hotel see here

Government debt and social security

Private pensions across the EU



- Large variety in private vs public pensions across EU

Government intervention

Restoring efficiency

- The previous result suggests that welfare can be improved
- Needed: long-lived planner with ability to redistribute

  $\Longrightarrow$ Government!

What exactly should the government do?

- Spend across generations?
- Explicit transfers?
- Hold debt?

Household problem with a government

Lifetime utility

$$\max_{c,a} U = u(c_{1,t}) + \beta u(c_{2,t+1})$$

$$\text{subject to } c_{1,t} + a_{t+1} = w_t - \tau_t$$

$$c_{2,t+1} = (1 + r_{t+1})a_{t+1}$$

Optimality (analogous derivation to before)

$$s_t = \frac{\beta}{1+\beta}(w_t - \tau_t); \quad c_{1,t} = (1 - s_t)w_t = \frac{1}{1+\beta}(w_t - \tau_t)$$

- Ricardian equivalence breaks in the OLG model
- Only the young pay taxes $\implies$ their consumption adjusts

Household problem with a government

Market clearing with government debt

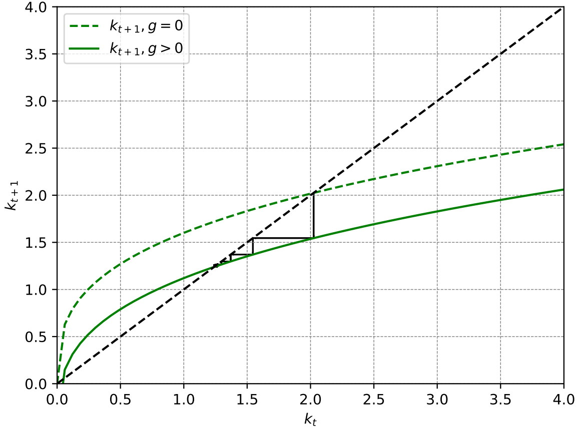
- Assume the government spends $g > 0$ each period from now
- $\tau_t = g_t$

$$
\begin{aligned}
s_t &= \frac{\beta}{1+\beta}\left(w_t - \tau_t\right) \\
(1+n)k_{t+1} &= \frac{\beta}{1+\beta}\left(w_t - g_t\right) \\
k_{t+1} &= \frac{1}{1+n}\frac{\beta}{1+\beta}\left((1-\alpha)k_t^{\alpha} - g_t\right)
\end{aligned}
$$

Implications of government spending (temporary or permanent)

- The law of motion for capital changes
- Only the young pay taxes $\implies$ their consumption adjusts
- In Ramsey: c falls, k stays constant; here?

OLG economy with government spending



- Capital falls as $g > 0$

Decreasing capital explanation

Ramsey model

- Level of capital was pinned down by the Euler equation
- Once c_0 adjusted, consumption was smoothed forever

OLG model

- As the government spends money, consumption falls
- Households want to lower c_0 and c_1 to smooth
- Only way to lower c_1 is to save less $\rightarrow k \downarrow$

$\Longrightarrow$ wasteful spending is worse in OLG model, since $\Delta c > \Delta g$

- $g \uparrow \rightarrow k \downarrow \rightarrow y \downarrow \rightarrow c = y - nk - g \downarrow\downarrow$ (from resource constraint)

Government debt

What if the government holds debt but $g = 0$?

- Hold debt and roll over each period
- Tax households to make up for the difference
- Benefit from population growth → more taxpayers tomorrow
- Assume debt is a constant fraction of capital stock: $\gamma k_t = b_t$

Government budget constraint

$$\begin{aligned} L_t \tau_t + B_{t+1} &= (1 + r_t) B_t \\ \implies \tau_t + (1+n) b_{t+1} &= (1 + r_t) b_t \end{aligned}$$

Market clearing? Split savings between bonds and capital

$$(k_{t+1} + b_{t+1})(1+n) = s_t = \frac{\beta}{1+\beta}(w_t - \tau_t)$$

Equilibrium with government debt

Combining from previous slide:

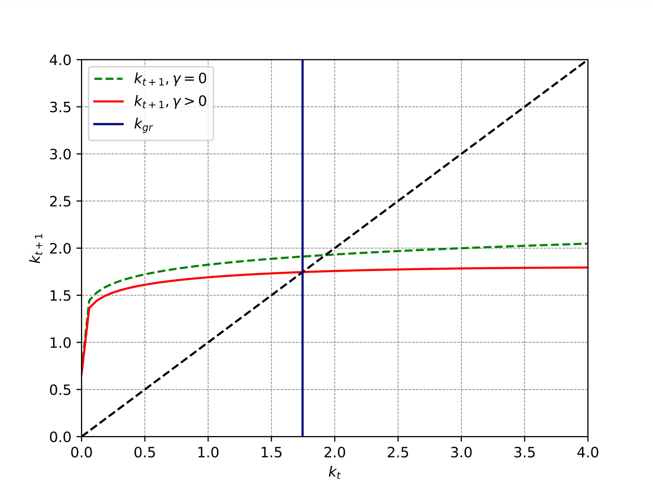
$$k_{t+1} = \frac{1}{1+n}\frac{\beta}{1+\beta}\left(w_t - ((1+r_t)b_t - (1+n)b_{t+1})\right) - b_{t+1}$$

In steady state Algebra

$$k_{debt} = \left[\beta \frac{(1-\alpha) - \gamma\alpha}{(1+\gamma)(1+n)(1+\beta) - \beta\gamma n}\right]^{\frac{1}{1-\alpha}}$$

- If γ is chosen just right, can replicate $f'(k) = n$ Proof
- Government can bring about Pareto improvement (if $f'(k^*) < n$)
- Long-lived debt has higher return on saving through redistribution
- Doesn't work in Ramsey, agent internalized population growth

Equilibrium with government debt



- If k^* is too high, government debt lowers it, ideally to k_{gr}; $c\uparrow$

Important notes

Caveats

- This result relies on ridiculous parameter values
- In the Figure: $\alpha = 1/12$ and $n = 0.25$
- Policy only leads to pareto improvement if $k^* > k_{gr}$

Hope

- The result can be recovered in more complicated models
- More complicated utility and risky income
- But: impossible to solve by hand

Social security I: Fully funded

Fully funded pension system

Individuals are forced to save

- Mandatory contribution out of paycheck
- Sometimes matched by the employer
- Invested in stock market (potentially favorable tax treatment)
- Accessible upon retirement

Reasons

- Individuals don't save enough for retirement
- Planning horizons too short
- Behavioral arguments

Fully funded pension system in the model

Government taxes the young (who can still save themselves)

$$c_{1,t} = w_t - s_t - d_t$$

Money is invested

$$b_{t+1} = (1 + r_{t+1})d_t$$

Individuals receive pension income upon retirement

$$\begin{aligned} c_{2,t+1} &= (1 + r_{t+1})s_t + b_{t+1} \\ &= (1 + r_{t+1})(d_t + s_t) \end{aligned}$$

- Government invests money at the same interest rate as households
- Only difference: forced contribution
- Lump sum contributions → no impact on Euler equation (show!)

Fully funded pension system solution

Euler equation leads to previous solution

$$
\begin{aligned}
u'(c_{1,t}) &= \beta(1+r_{t+1})u'(c_{2,t+1}) \\
u'(w_t - (s_t + d_t)) &= \beta(1+r_{t+1})u'((1+r_{t+1})(s_t + d_t)) \\
(s_t + d_t) &= \frac{\beta}{1+\beta}w_t \qquad \text{impose CRRA with } \sigma = 1
\end{aligned}
$$

- Fully funded contributions have same return as normal savings
- Forced savings crowd out individual savings 1-for-1
- No effect on capital accumulation or consumption

$$
k_{t+1} = \frac{(s_t + d_t)}{1+n} \implies k^* = \left(\frac{1-\alpha}{1+n}\frac{\beta}{1+\beta}\right)^{\frac{1}{1-\alpha}}
$$

Fully funded system doesn't take advantage of population growth!

Social security II: Pay-as-you-go

Pay-as-you-go pension systems

Redistribution from the young to the current old

$$b_t = (1 + n)d_t$$

- For every benefit to the old (b_t) there are $(1 + n)$ to pay it

New budget constraints

$$c_{1,t} + s_t = w_t - d_t$$
$$c_{t,t+1} = (1 + r_{t+1})s_t + (1 + n)d_{t+1}$$

- This takes advantage of population growth
- Can it implement the planner's solution if $k^* > k_{gr}$?

PAYG optimality conditions

Euler equation

$$
\begin{aligned}
u'(c_{1,t}) &= \beta(1+r_{t+1})u'(c_{2,t+1}) \\
u'(w_t-(s_t+d_t)) &= \beta(1+r_{t+1})u'((1+r_{t+1})s_t+(1+n)d_{t+1})
\end{aligned}
$$

Assume $\sigma = 1$

$$
\begin{aligned}
\frac{1}{w_t-(s_t+d_t)} &= \beta(1+r_{t+1})\frac{1}{(1+r_{t+1})s_t+(1+n)d_{t+1}} \\
(1+r_{t+1})s_t+(1+n)d_{t+1} &= \beta(1+r_{t+1})(w_t-s_t-d_t) \\
(1+\beta)(1+r_{t+1})s_t &= \beta(1+r_{t+1})(w_t-d_t)-(1+n)d_{t+1} \\
s_t &= \frac{\beta}{1+\beta}(w_t-d_t)-\frac{1+n}{1+r_{t+1}}d_{t+1}
\end{aligned}
$$

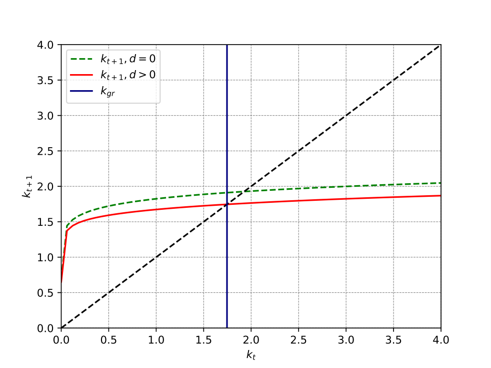
- As before, s_t is a function of income and transfers/taxes
- If $d = 0$, we recover the undistorted solution

Capital with PAYG

Market clearing condition

$$k_{t+1} = \frac{1}{1+n}\frac{\beta}{1+\beta}(w_t - d_t) - \frac{1+n}{1+r_{t+1}}d_{t+1}$$

Steady state (unsolvable by hand without further assumptions)



Pension systems and debt

Crucial insight

- Redistribution can tap an additional resource: population growth
- If population grows fast, makes sense to "borrow from the young"
- This breaks down if there are fewer young

Forced savings

- Government policy in many countries
- Useless with rational agents, need behavioral explanation
- Limited foresight, incorrect expectations about life-expectancy, etc

Final thoughts

The OLG model

- There is a role for the government after all (under certain assumptions)
- A long-lived government can improve the economy by exploiting the return from population growth
- The government cannot force people to save in capital (same return)
- Holding debt can have the same effect as a PAYG system

Next time

- Labor Supply
- Risk

Appendix

Market clearing algebra Back

$$
\begin{aligned}
c &= \frac{1-\alpha}{1+\beta}k^{\alpha}\left(1+\beta\frac{(1+\alpha k^{\alpha-1})}{1+n}\right) \\
&= \frac{1-\alpha}{1+\beta}k^{\alpha}\left(1+\frac{\beta}{1+n}+\beta\frac{\alpha k^{\alpha-1}}{1+n}\right) \\
&= \frac{1-\alpha}{1+\beta}k^{\alpha}\left(1+\frac{\beta}{1+n}+\frac{\alpha(1+\beta)}{1-\alpha}\right) \\
&= \frac{1-\alpha}{1+\beta}k^{\alpha}+\frac{1-\alpha}{1+\beta}\frac{\beta}{1+n}k^{\alpha}+\alpha k^{\alpha} \\
&= k\frac{1+n}{\beta}+k+\alpha k^{\alpha}(+nk-nk) \\
&= k\left(\frac{1+n}{\beta}+1+n\right)+\alpha k^{\alpha}-nk = k(1+n)\left(\frac{1+\beta}{\beta}\right)+\alpha k^{\alpha}-nk \\
&= (1-\alpha)k^{\alpha-1}k+\alpha k^{\alpha}-nk \\
&= k^{\alpha}-nk \qquad \text{at} \qquad k=\left(\frac{1-\alpha}{1+n}\frac{\beta}{1+\beta}\right)^{\frac{1}{1-\alpha}}
\end{aligned}
$$

Algebra for government debt

Back

$$
\begin{aligned}
k &= \frac{1}{1+n}\frac{\beta}{1+\beta}\left(w - (r-n)\gamma k)\right) - \gamma k \\
[1+\gamma]\,k &= \frac{1}{1+n}\frac{\beta}{1+\beta}\Big((1-\alpha)Ak^{\alpha} - ((\alpha)Ak^{\alpha-1} - n)\gamma k \\
&= \frac{1}{1+n}\frac{\beta}{1+\beta}\left((1-\alpha)Ak^{\alpha} - (\gamma\alpha Ak^{\alpha} - \gamma nk)\right) \\
&= \beta\frac{(1-\alpha)-\gamma\alpha}{(1+n)(1+\beta)}Ak^{\alpha} + \frac{\beta\gamma n}{(1+n)(1+\beta)}k \\
\left[1+\gamma - \frac{\beta\gamma n}{(1+n)(1+\beta)}\right]k &= \beta\frac{(1-\alpha)-\gamma\alpha}{(1+n)(1+\beta)}Ak^{\alpha} \\
\left[\frac{(1+\gamma)(1+n)(1+\beta)-\beta\gamma n}{(1+n)(1+\beta)}\right]k &= \beta\frac{(1-\alpha)-\gamma\alpha}{(1+n)(1+\beta)}Ak^{\alpha} \\
k &= \beta\frac{(1-\alpha)-\gamma\alpha}{(1+\gamma)(1+n)(1+\beta)-\beta\gamma n}Ak^{\alpha}
\end{aligned}
$$

Algebra for steady state and golden rule

Back

$$k_{debt} = \left[A\beta \frac{(1-\alpha) - \gamma\alpha}{(1+\gamma)(1+n)(1+\beta) - \beta\gamma n} \right]^{\frac{1}{1-\alpha}}$$

For golden rule:

$$k_{debt} = \left[A\beta \frac{(1-\alpha) - \gamma\alpha}{(1+\gamma)(1+n)(1+\beta) - \beta\gamma n} \right]^{\frac{1}{1-\alpha}} = \left(A\frac{\alpha}{n} \right)^{\frac{1}{1-\alpha}} = k_{gr}$$

$$n\beta\left[(1-\alpha) - \gamma\alpha\right] = \alpha\left[(1+\gamma)(1+n)(1+\beta) - \beta\gamma n\right]$$

$$n\beta(1-\alpha) - n\beta\gamma\alpha = \alpha(1+n)(1+\beta) + \alpha(1+n)(1+\beta)\gamma - \beta\gamma n\alpha$$

$$n\beta(1-\alpha) = \alpha(1+n)(1+\beta) + \alpha(1+n)(1+\beta)\gamma$$

$$\frac{n\beta(1-\alpha)}{\alpha(1+n)(1+\beta)} = (1+\gamma)$$