OLG, endogenous labor, and risk

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Agenda

Wrap-up of the OLG model

- Brief recap from last time (planner)
- Public debt
- Pension systems

Endogenous labor supply

- Labor in the utility function
- Labor supply elasticity

Decision making with risk

- Consumption/saving under uncertainty

The overlapping generations model (simplified)

Household problem

$$\max_{c,a} U = \ln(c_{1,t}) + \beta \ln(c_{2,t+1})$$

$$\text{subject to } c_{1,t} + \frac{c_{2,t+1}}{1 + r_{t+1}} = w_t$$

Assumed

- Production: $Y = K^{\alpha} L^{1-\alpha}$
- Competitive markets

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Assumed

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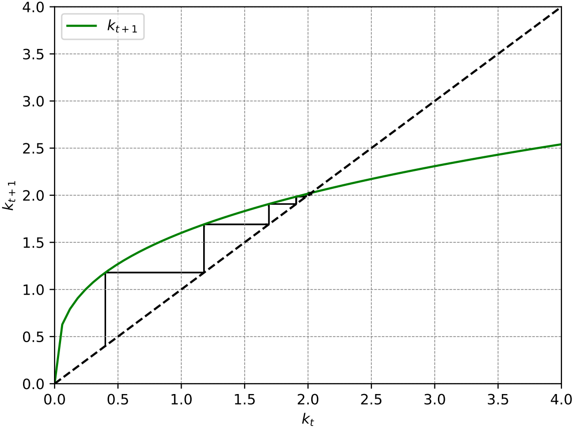
- Competitive markets

General equilibrium

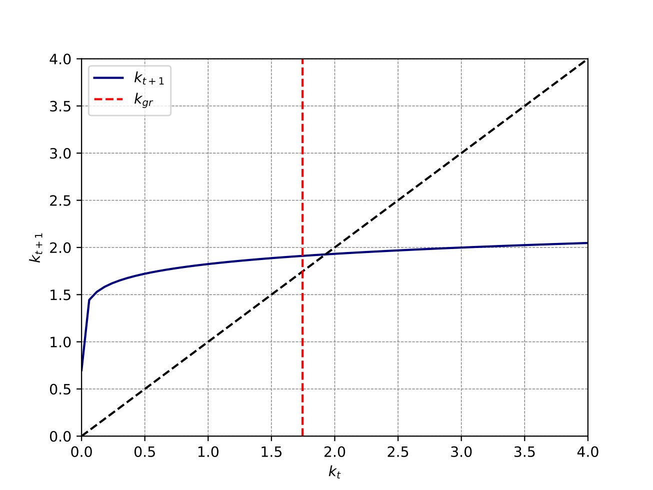
$$(1 + n)k_{t+1} = (1 - \alpha)\frac{\beta}{1 + \beta}k_t^{\alpha}$$

Dynamics of the OLG economy



- Convergence to steady state

The planner may have a job to do



- If k^* "too high" in market equilibrium, Pareto improvement possible

Planner's intervention

Planner intuition

- Institute a system in which **every young generation** saves a little less in k but gives some labor income to the contemporary old
- k falls, $f'(k)$ rises
- For each old agent, there are $(1+n)$ young
- Give up x today, receive $(1+n)x > (1+f'(k))x$ tomorrow
- Every young agent gets a return of $(1+n)$ on this deal

Important implications

- Market cannot deliver; requires infinitely many contracts with unborn
- First welfare theorem is broken! Market outcome can be improved

Caveats

- Only works if r^* is very low or n is very high

Government debt

Government debt

What if the government holds debt but $g = 0$?

- Hold debt and roll over each period
- Benefit from population growth → more taxpayers tomorrow
- Assume debt is a constant fraction of capital stock: $\gamma k_t = b_t$

Government budget constraint

$$
\begin{aligned}
L_t\tau_t + B_{t+1} &= (1+r_t)B_t \\
\implies \tau_t + (1+n)b_{t+1} &= (1+r_t)b_t
\end{aligned}
$$

Market clearing? Split savings between bonds and capital

$$
\begin{aligned}
(k_{t+1} + b_{t+1})(1+n) = s_t &= \frac{\beta}{1+\beta}(w_t - \tau_t) \\
c_{0,t} &= \frac{1}{1+\beta}(w_t - \tau_t)
\end{aligned}
$$

Equilibrium with government debt

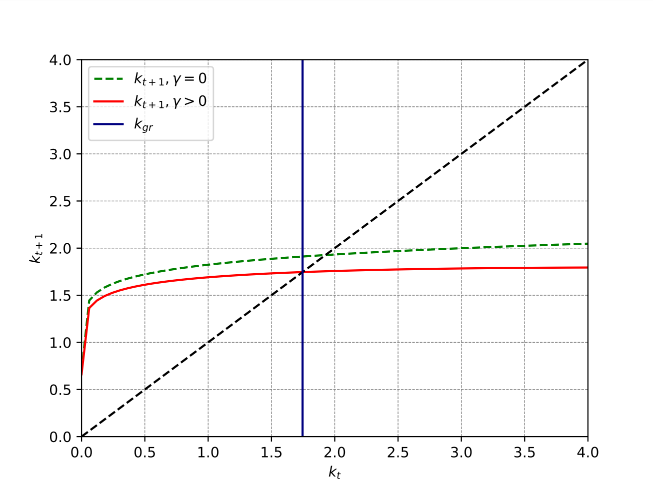
Combining from previous slide:

$$k_{t+1} = \frac{1}{1+n}\frac{\beta}{1+\beta}\left(w_t - ((1+r_t)b_t - (1+n)b_{t+1})\right) - b_{t+1}$$

In steady state Algebra

$$k_{debt} = \left[\beta\frac{(1-\alpha)-\gamma\alpha}{(1+\gamma)(1+n)(1+\beta)-\beta\gamma n}\right]^{\frac{1}{1-\alpha}}$$

- If γ is chosen just right, can replicate $f'(k) = n$ Proof
- Government can bring about Pareto improvement (if $f'(k^*) < n$)
- Long-lived debt has higher return on saving through redistribution
- Path for τ_t needs to be just right (no Ricardian equivalence)
- In steady state, $\tau = (f'(k^*) - n)b = 0$, but debt B_t grows at $(1+n)$



- If k^* is too high, government debt lowers it, ideally to k_{gr}; $c\uparrow$

Social security I: Fully funded

Fully funded pension system

Individuals are forced to save

- Mandatory contribution out of paycheck
- Sometimes matched by the employer
- Invested in stock market (potentially favorable tax treatment)
- Accessible upon retirement

Reasons

- Individuals don't save enough for retirement
- Planning horizons too short
- Behavioral arguments

Fully funded pension system in the model

Government taxes the young (who can still save themselves)

$$c_{1,t} = w_t - s_t - d_t$$

Money is invested – at the market interest rate

$$b_{t+1} = (1 + r_{t+1})d_t$$

Individuals receive pension income upon retirement

$$\begin{aligned} c_{2,t+1} &= (1 + r_{t+1})s_t + b_{t+1} \\ &= (1 + r_{t+1})(d_t + s_t) \end{aligned}$$

- Government invests money at the same interest rate as households
- Only difference: forced contribution
- Lump sum contributions → no impact on Euler equation (show!)

Fully funded pension system solution

Euler equation leads to previous solution

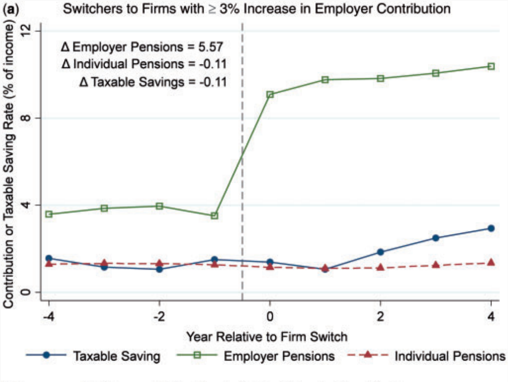
$$
\begin{aligned}
\frac{1}{c_{1,t}} &= \beta(1+r_{t+1})\frac{1}{c_{2,t+1}} \\
\frac{1}{w_t-(s_t+d_t)} &= \beta(1+r_{t+1})\frac{1}{(1+r_{t+1})(s_t+d_t)} \\
(s_t+d_t) &= \frac{\beta}{1+\beta}w_t
\end{aligned}
$$

- Total savings are unchanged
- Fully funded contributions have same return as normal savings
- Forced savings crowd out individual savings 1-for-1

$$
k_{t+1} = \frac{(s_t+d_t)}{1+n} \implies k^* = \left(\frac{1-\alpha}{1+n}\frac{\beta}{1+\beta}\right)^{\frac{1}{1-\alpha}}
$$

Fully funded system doesn't take advantage of population growth!

Forced pension contributions in Denmark



- Forced pension savings don't affect private savings (Chetty et al, 2014)

Social security II: Pay-as-you-go

Pay-as-you-go pension systems

Redistribution from the young to the current old

$$b_t = (1+n)d_t$$

- For every benefit to the old (b_t) there are $(1+n)$ to pay it

New budget constraints

$$c_{1,t} + s_t = w_t - d_t$$
$$c_{1,t+1} = (1+r_{t+1})s_t + (1+n)d_{t+1}$$

- This takes advantage of population growth
- Can it implement the planner's solution if $k^* > k_{gr}$?

PAYG optimality conditions

Euler equation unchanged! (lump-sum contributions)

$$
\begin{aligned}
u'(c_{1,t}) &= \beta(1+r_{t+1})u'(c_{2,t+1}) \\
\frac{1}{w_t-(s_t+d_t)} &= \beta(1+r_{t+1})\frac{1}{(1+r_{t+1})s_t+(1+n)d_{t+1}} \\
s_t &= \frac{\beta}{1+\beta}(w_t-d_t)-\frac{1+n}{1+r_{t+1}}d_{t+1}
\end{aligned}
$$

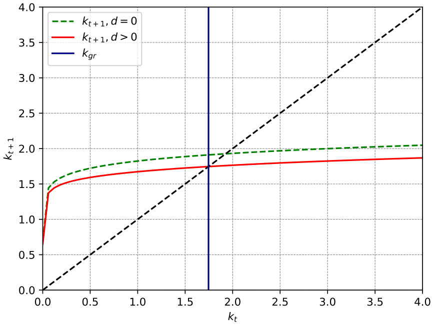
- As before, s_t is a function of income and transfers/taxes
- If n is high, save less (internalize pop. growth)
- If $d = 0$, we recover the undistorted solution

Capital with PAYG

Market clearing condition

$$k_{t+1} = \frac{1}{1+n}\frac{\beta}{1+\beta}(w_t - d_t) - \frac{1+n}{1+r_{t+1}}d_{t+1}$$

Steady state (unsolvable by hand without further assumptions)



Final thoughts on the OLG model

Crucial insight

- There is a role for the government after all (under certain assumptions)
- Redistribution can tap an additional resource: population growth
- If population grows fast, makes sense to "borrow from the young"
- This breaks down if there are fewer young
- Government debt is just as useful as a PAYG system

Forced savings

- Government policy in many countries
- Useless with rational agents, need behavioral explanation
- Limited foresight, incorrect expectations about life-expectancy, etc

Endogenous labor supply

Endogenous labor supply

Previous assumption: labor is exogenously supplied

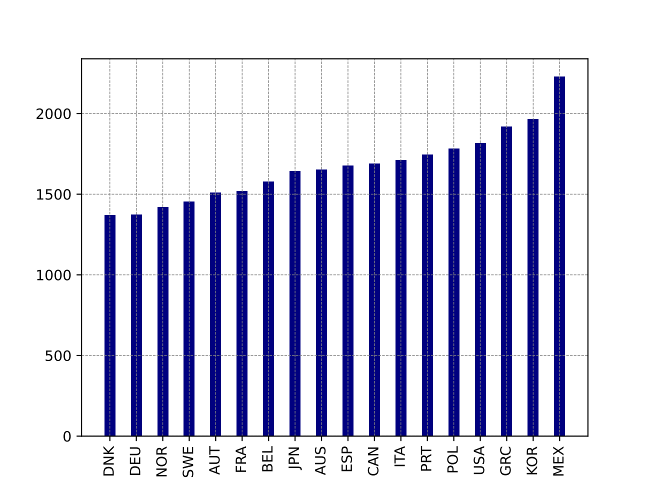
Empirical evidence

- Hours worked differ across space (why do Americans work so much?)
- Hours worked differ over time (why do we work less today?)

Need some theory of the labor market

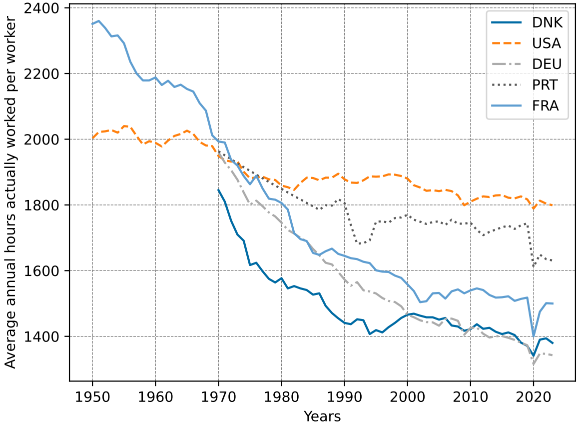
- Today: Perfectly competitive labor market
- Employment is measured in worked hours, not in jobs
- Soon: More thorough theory of unemployment

Hours worked across time and countries



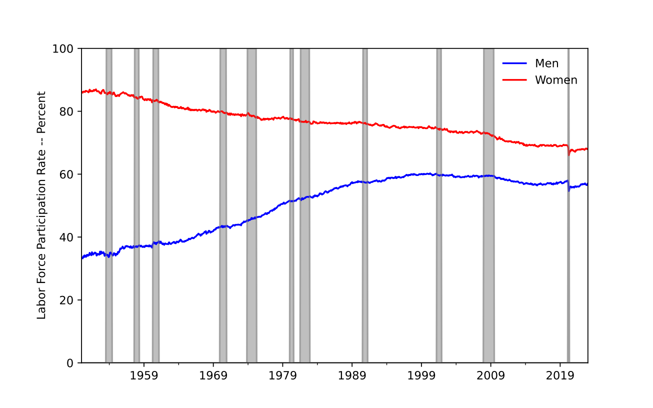
- Big differences in hours worked across countries

Hours worked across time and countries



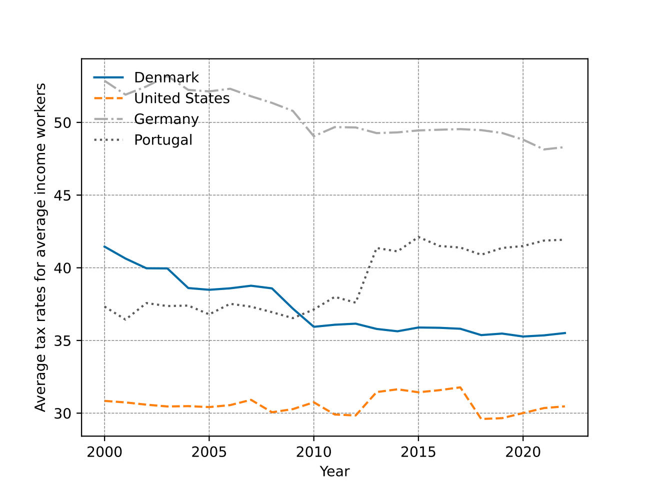
- France: from 300 8h days per year to 190

Labor force participation in the US



- Labor force participation is converging

Labor-tax rates across countries



- Average income tax rates differ across countries

A model of labor supply

New utility function to endogenize labor choice

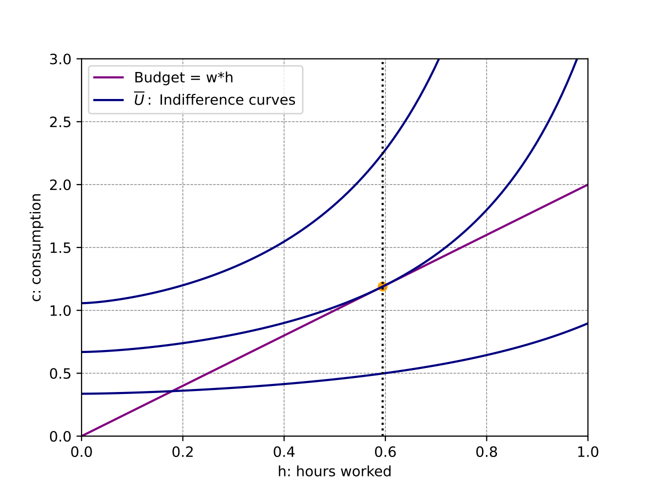
$$U = u(c) - v(h) \text{ s.t. } c = wh$$

- $u(c)$ just as before
- $v(h)$ is disutility of labor (exponentially increasing)
- Household has time budget of $1 = h + l$ (just normalization)
- Consumption and labor are separable (don't influence each other)

Optimality

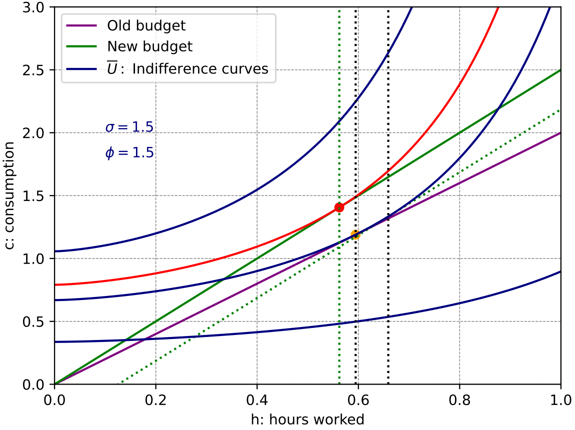
$$\frac{v'(h)}{u'(c)} = w$$

A model of labor supply – graphical



- Labor supply depends only on the wage

A model of labor supply – graphical



- $w \uparrow$ – substitution: work more, income: work less

Choosing functional form of utility

Long-run considerations

- Real wages have been rising for a long time
- Big debate: are hours decreasing or stable?
- Hours should be inelastic to wages

Short-run considerations

- Hours move a lot over the business cycle
- Wages don't move much
- Hours should be elastic to wages

Balanced growth preferences – Long-run

MaCurdy (1981) preferences

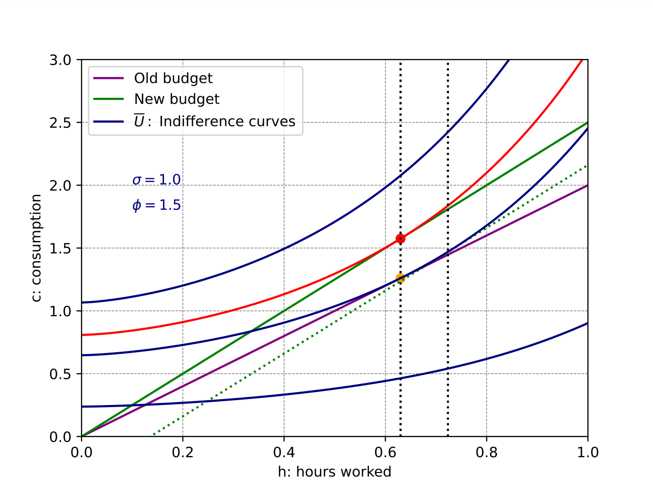
$$U = \frac{c^{1-\sigma} - 1}{1-\sigma} - B\frac{1}{1+\phi}h^{1+\phi}$$

- In a growth model, with technological progress, $K_t \to \infty$
- This could present a problem, since $K \uparrow \to F_L(K, L) = w \uparrow$
- With the calibration above, it appears that $h_t \to 0$

Solution

- With $\sigma = 1$, substitution and income effect cancel
- King-Plosser-Rebelo preferences (KPR)
- h is constant and independent of w

KPR preferences



- With $\sigma = 1$, income and substitution effect cancel, $h = \overline{h}$

The many elasticities of labor supply

The worker's problem

$$\max_{c,h} \frac{c^{1-\sigma}-1}{1-\sigma} - B\frac{1}{\phi}h^{\phi} + \lambda(wh-c)$$

Optimality conditions

$$\begin{aligned}
&[c:] && c^{-\sigma} = \lambda \\
&[h:] && Bh^{\phi} = w\lambda \\
&\Longrightarrow && \frac{Bh^{\phi}}{c^{-\sigma}} = w
\end{aligned}$$

- All three conditions are the same with intertemporal choice
- Labor choice is also called "intratemporal choice"

The many elasticities of labor supply

Marshallian elasticity

$$\frac{Bh^{\phi}}{(wh)^{-\sigma}} = w$$

$$\implies h = \left(\frac{w^{1-\sigma}}{B}\right)^{\frac{1}{\sigma+\phi}}$$

$$\varepsilon_M = \frac{\partial h}{\partial w}\frac{w}{h} = \frac{1-\sigma}{\phi+\sigma}$$

- This is the "general equilibrium" elasticity
- Total effect of w on h
- A government financing spending with taxes (lowering after tax wages) should care

The many elasticities of labor supply

$$\max_{c,h} \frac{c_t^{1-\sigma}-1}{1-\sigma} - B\frac{1}{\phi}h_t^{\phi} + \lambda \sum_{t=0}^{\infty}\left(\prod_{s=0}^{t}\frac{1}{(1+r_s)}\right)(w_t h_t - c_t)$$

Frisch elasticity – intertemporal elasticity

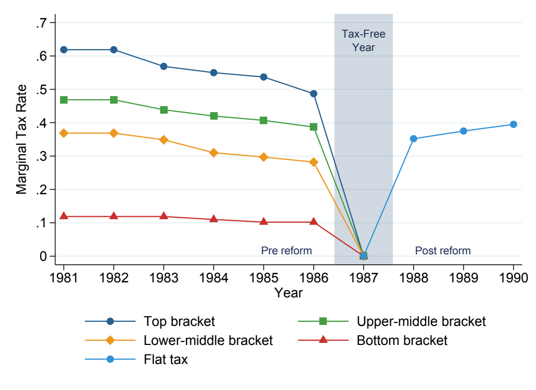
$$\frac{Bh_t^{\phi}}{\lambda} = w_t$$

$$\varepsilon_F = \frac{\partial h_t}{\partial w_t}\frac{w_t}{h_t} = \frac{1}{\phi}$$

- Hold the Lagrange multiplier on the budget constraint constant
- Fix marginal utility of consumption for a given wage change

$\implies$ $1/\phi$ governs elasticity of hours

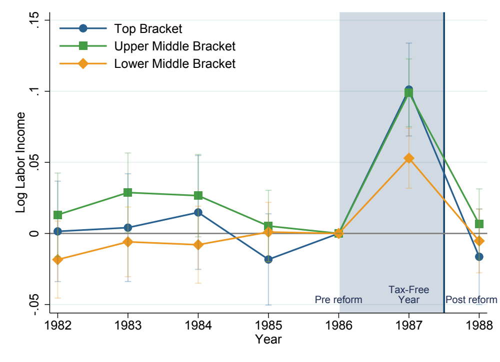
Elasticities measured in the micro data are low



(a) Marginal tax rate by tax bracket

- Income tax free year in Iceland (Sigurdsson, *forthcoming*)

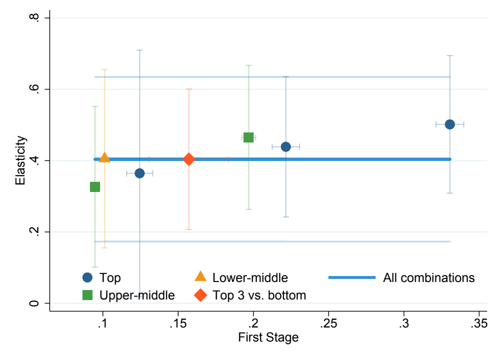
Elasticities measured in the micro data are low



(a) Top-, Upper-middle, and Lower-middle brackets vs. bottom bracket

- Labor supply rises during tax-cut year

Elasticities measured in the micro data are low



(b) Labor supply elasticity by size of tax cut (first stage)

- $1/\phi \approx 0.4$

Macroeconomics need estimates to be larger

Table 1—Micro vs. Macro Labor Supply Elasticities

		Intensive Margin	Extensive Margin	Aggregate Hours
Steady State	micro	0.33	0.26	0.59
(Hicksian)	macro	0.33	0.17	0.50
Intertemporal	micro	0.54	0.28	0.82
Substitution (Frisch)	macro	[0.54]	[2.30]	2.84

Note: Each cell shows a point estimate of the relevant elasticity based on meta analyses of existing micro and macro evidence. Micro estimates are identified from quasi-experimental studies; macro estimates are identified from cross-country variation in tax rates (steady state elasticities) and business cycle fluctuations (intertemporal substitution elasticities). The aggregate hours elasticity is the sum of the extensive and intensive elasticities. Macro studies do not always decompose intertemporal aggregate hours elasticities into extensive and intensive elasticities. Therefore, the estimates in brackets show the values implied by the macro aggregate hours elasticity if the intensive Frisch elasticity is chosen to match the micro estimate of 0.54. Sources are described in the appendix.

- To get hours to move over the cycle, $1/\phi > 2$ (Chetty et al, 2011)

Labor demand

A model of labor demand

Labor demand

- Representative firm that only uses labor in production
- Ignore capital for now

$$y = f(h); \quad f'(h) > 0, f''(h) < 0$$

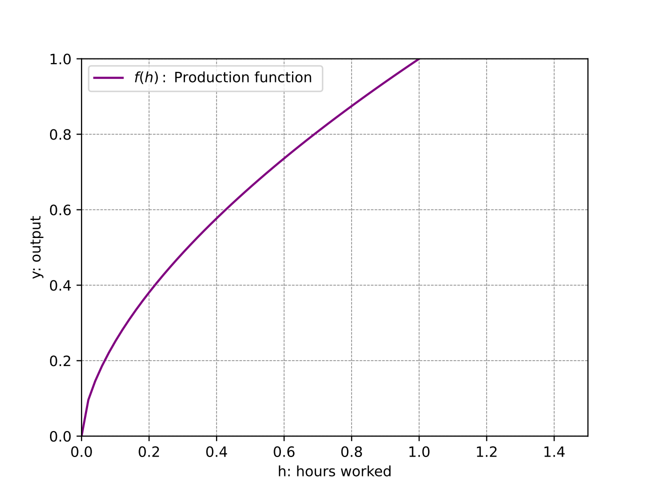
$$\text{e.g., } y = Ah^{1-\alpha}$$

Maximizing profits

$$\Pi = \max_h f(h) - wh$$

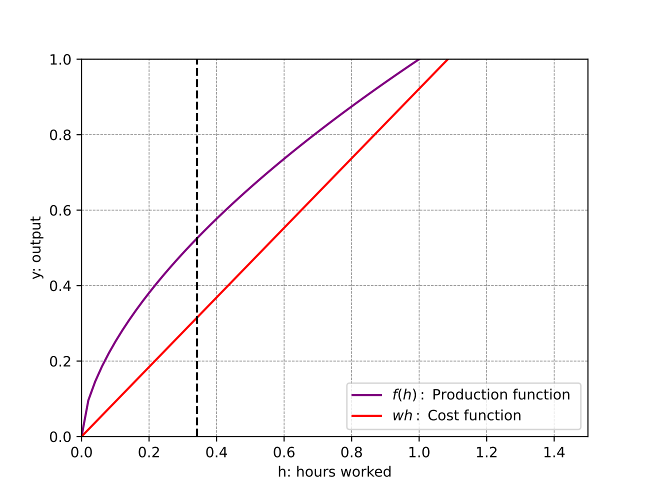
$$\implies f'(h) = (1-\alpha)Ah^{-\alpha} = w$$

A model of labor demand – graphical



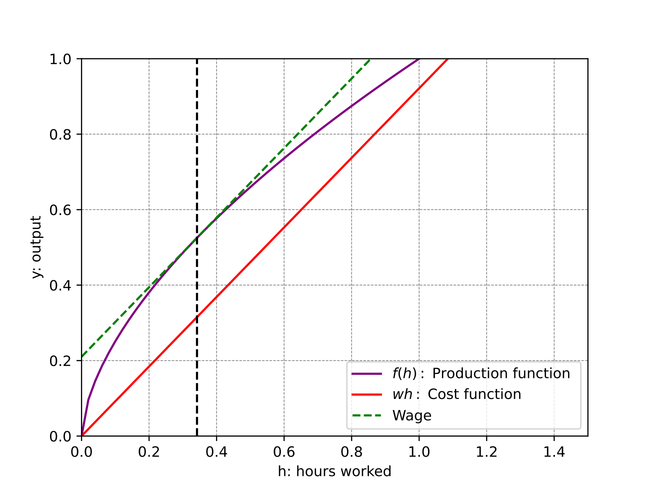
- Production function increasing in h

A model of labor demand – graphical



- $\Pi = \max_n f(h) - wn$

A model of labor demand – graphical

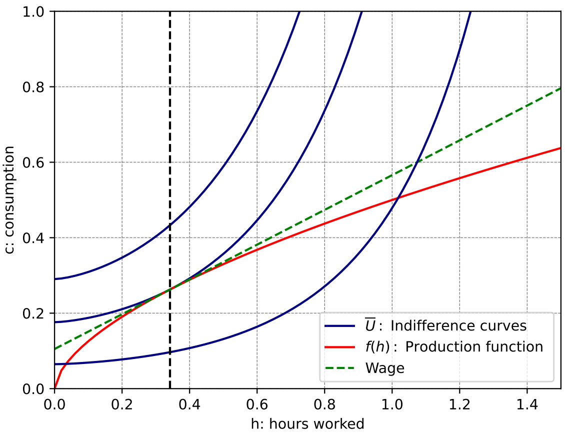


- Revenue is highest at $f'(h) = w$

General Equilibrium

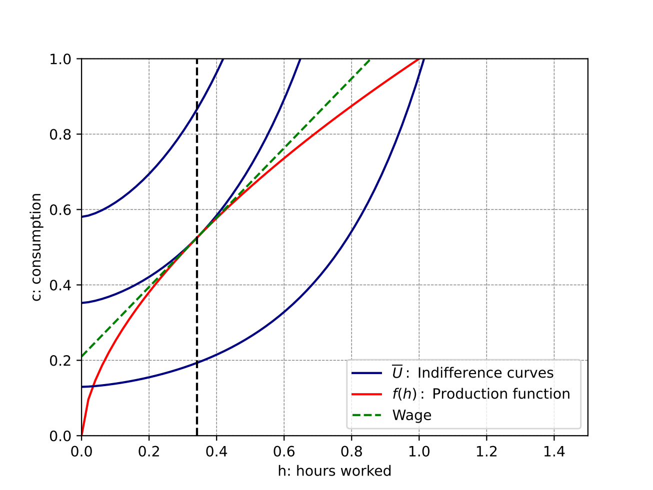
- Labor supply = labor demand: $f'(h) = \frac{v'(h)}{u'(c)}$

General equilibrium



- If utility is $\log$, wage changes do not affect labor supply

General equilibrium



- If utility is $\log$, wage changes do not affect hours worked

Risk

Uncertainty in an endowment economy

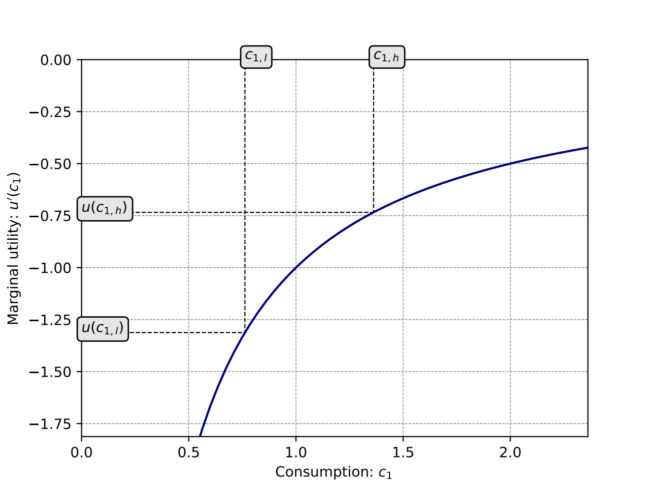
What is risk?

- **Known** uncertainty about the future
- Known unknowns
- Not modelled: "Knightian uncertainty"/unknown unknowns/black swans

Risk is a central aspect of economic decision making

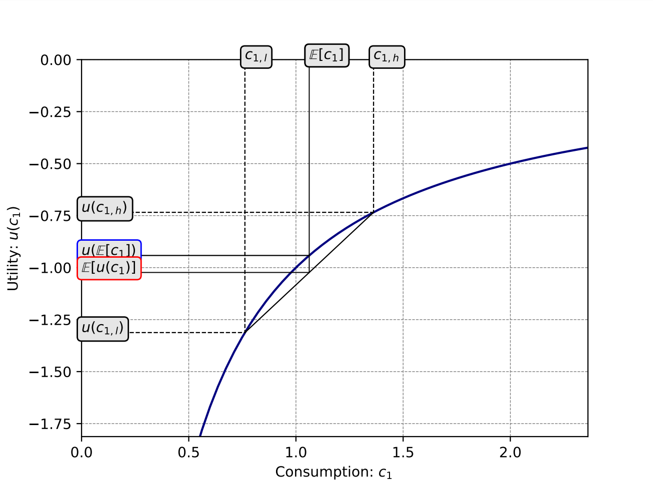
- Career choice
- Business cycle luck
- Investment strategies
- Firm-specific risk
- Health shocks

Uncertainty – graphical



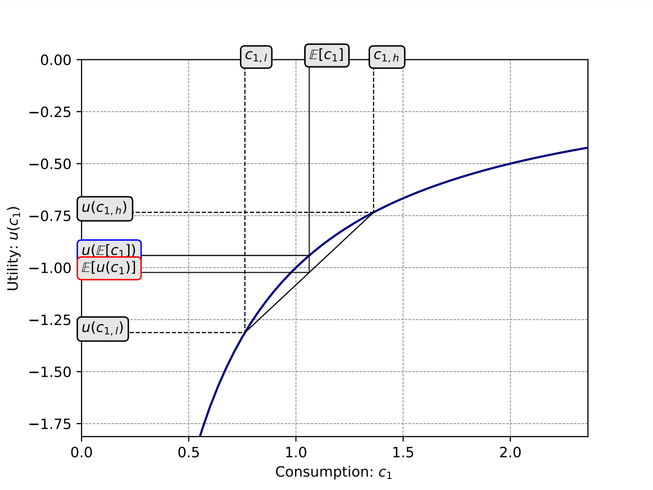
- Assume $u(c) = \frac{c^{1-\sigma}-1}{1-\sigma}$, then $u'(c) = c^{-\sigma}$
- Curvature of utility function governs effect of risk

Uncertainty – graphical



- $\mathbb{E}[u(c_1)]$ is the expected utility from a gamble
- $u(\mathbb{E}[c_1])$ is the utility of the expected outcome of a gamble

Uncertainty – graphical



- With common utility functions, $\mathbb{E}[u(c_1)] < u(\mathbb{E}[c_1])$
- Expected outcome with certainty always better than gamble

Uncertainty in an endowment economy

Two period model with risky endowment

$$\max_{c_0, c_1} \quad U = u(c_0) + \beta \mathbb{E}_0[u(c_1)] \quad s.t. \quad c_0 + \mathbb{E}_0\left[\frac{c_1}{1+r}\right] = y_0 + \mathbb{E}_0\left[\frac{y_1 + L}{1+r}\right]$$

$$Pr(L = \varepsilon) = 0.5; \quad Pr(L = -\varepsilon) = 0.5$$

Uncertainty in an endowment economy

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$$Pr(L = \varepsilon) = 0.5; \quad Pr(L = -\varepsilon) = 0.5$$

Budgets in each state of the world

$$c_{1,h} = y_1 + \varepsilon + (1+r)(y_0 - c_0) \qquad \text{(good day)}$$

$$c_{1,l} = y_1 - \varepsilon + (1+r)(y_0 - c_0) \qquad \text{(bad day)}$$

Uncertainty in an endowment economy

Two period model with risky endowment

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Budgets in each state of the world

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Consumer problem

$$\max_{c_0} u(c_0) + \beta \left[\underbrace{\frac{1}{2} u(y_1 + \varepsilon + (1+r)(y_0 - c_0)) + \frac{1}{2} u(y_1 - \varepsilon + (1+r)(y_0 - c_0))}_{\mathbb{E}[u(c_1)]} \right]$$

Uncertainty in an endowment economy

First order condition

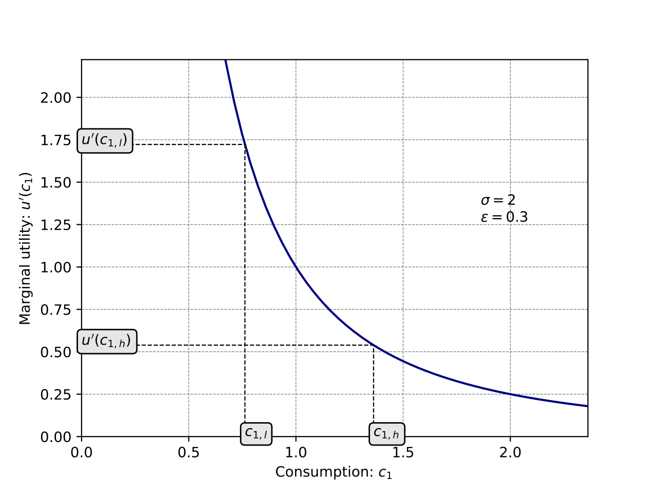
$$u'(c_0) - \beta\left[\frac{1}{2}u'(\underbrace{y_1 + \varepsilon + (1+r)(y_0 - c_0)}_{c_{1,h}})(1+r) + \frac{1}{2}u'(c_{1,l})(1+r)\right] = 0$$

$$u'(c_0) = \beta(1+r)\left[\frac{1}{2}u'(c_{1,h}) + \frac{1}{2}u'(c_{1,l})\right]$$

$$u'(c_0) = \beta(1+r) \quad \underbrace{\mathbb{E}_0\left[u'(c_1)\right]}_{\text{Let's focus on this part}}$$

- Very similar to a normal Euler equation, except for expectations
- If expected marginal utility tomorrow $\mathbb{E}_0\left[u'(c_1)\right] \uparrow$, then $c_0 \downarrow$
- How does uncertainty about tomorrow's income affect $\mathbb{E}_0\left[u'(c_1)\right]$?
- Remember: Picture flips! High marginal utility → low consumption

Uncertain marginal utility



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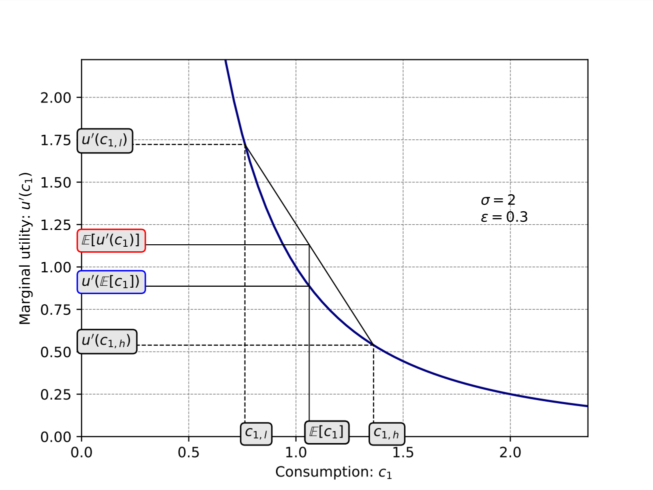
Uncertain marginal utility



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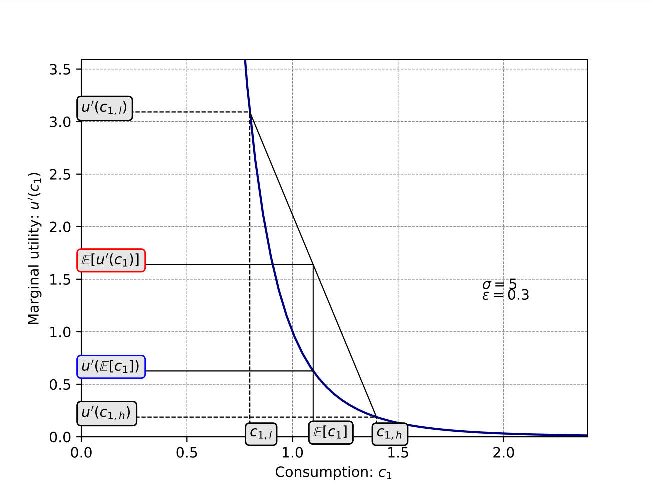
Uncertain marginal utility



- Risk aversion ($\sigma \uparrow$) worsens the tradeoff
- Expected outcome with certainty always better than gamble

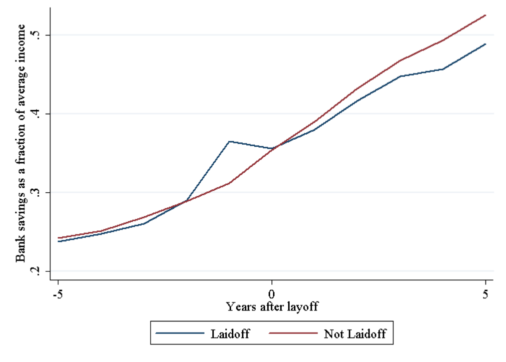
Uncertainty in the Euler equation

$$u'(c_0) = \beta(1 + r)\mathbb{E}_0 [u'(c_1)]$$

Implications of risk for consumption

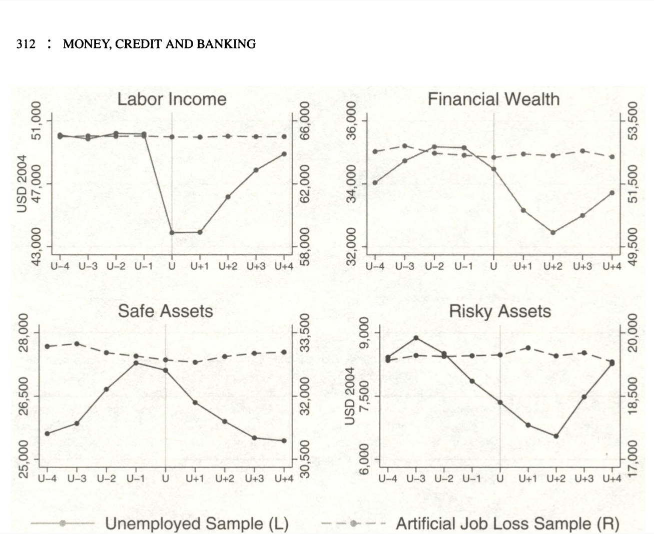
- Risk about future consumption raises marginal utility tomorrow
- Cons tomorrow becomes more valuable → "better safe than sorry"
- c_0 falls, the agent saves more
- Effect is stronger with more risk and/or more risk aversion (& borrowing constraints)

The theory in practice



- People who become unemployed save more (Denmark, Gallen, 2013)

The theory in practice



- People who become unemployed save more (Norway, Basten et al, 2016)

The theory in practice

Table 4
Regression results from estimating (10) and (11) on actual and model-generated data. The coefficients show the effect of a change in unemployment risk growth on the expenditure on non-durable goods and the purchase probability of durable goods, where the latter is normalized by the unconditional purchase probability. For the empirical regressions, the standard errors are bootstrapped, clustered at the 2-digit regional level. For the model regressions, all estimates are averages from 1,000 simulations. *, **, *** indicate that the coefficients are significant at the 10%, 5% and 1% level, respectively.

	Data			Model
	(1)	(2)	(3)	(4)
Nondurables	−0.32	−0.60***	−0.56**	−0.21***
	(0.21)	(0.23)	(0.23)	(0.07)
R^2	0.27	0.27	0.32	0.73
Durables	−2.84***	−3.38***	−2.84***	−2.81*
	(0.96)	(0.96)	(0.91)	(1.67)
R^2	0.01	0.01	0.08	0.15
Time fixed effects	Yes	Yes	Yes	-
Change in factor loadings	No	Yes	Yes	-
Household characteristics	No	No	Yes	-
Financial variables	No	No	Yes	Yes
N	5095	5095	5095	5220

- Fear of unemployment lowers consumption, esp. durables (Harmenberg & Oberg, 2021)

Macroeconomic implications

Expectations matter

- Recession news → more saving, less consumption → recession
- Agents hold more liquid assets in recessions (Graves, *forthcoming*)
- Policy can move the economy through expectations

Research

- Understanding expectations is crucial for macroeconomists

Appendix

Market clearing algebra Back

$$
\begin{aligned}
c &= \frac{1-\alpha}{1+\beta}k^{\alpha}\left(1+\beta\frac{(1+\alpha k^{\alpha-1})}{1+n}\right)\\
&= \frac{1-\alpha}{1+\beta}k^{\alpha}\left(1+\frac{\beta}{1+n}+\beta\frac{\alpha k^{\alpha-1}}{1+n}\right)\\
&= \frac{1-\alpha}{1+\beta}k^{\alpha}\left(1+\frac{\beta}{1+n}+\frac{\alpha(1+\beta)}{1-\alpha}\right)\\
&= \frac{1-\alpha}{1+\beta}k^{\alpha}+\frac{1-\alpha}{1+\beta}\frac{\beta}{1+n}k^{\alpha}+\alpha k^{\alpha}\\
&= k\frac{1+n}{\beta}+k+\alpha k^{\alpha}(+nk-nk)\\
&= k\left(\frac{1+n}{\beta}+1+n\right)+\alpha k^{\alpha}-nk = k(1+n)\left(\frac{1+\beta}{\beta}\right)+\alpha k^{\alpha}-nk\\
&= (1-\alpha)k^{\alpha-1}k+\alpha k^{\alpha}-nk\\
&= k^{\alpha}-nk \qquad \text{at} \qquad k = \left(\frac{1-\alpha}{1+n}\frac{\beta}{1+\beta}\right)^{\frac{1}{1-\alpha}}
\end{aligned}
$$

Algebra for government debt Back

$$
\begin{aligned}
k &= \frac{1}{1+n}\frac{\beta}{1+\beta}\left(w-(r-n)\gamma k)\right)-\gamma k \\
\left[1+\gamma\right]k &= \frac{1}{1+n}\frac{\beta}{1+\beta}\left((1-\alpha)Ak^{\alpha}-((\alpha)Ak^{\alpha-1}-n)\gamma k\right. \\
&= \frac{1}{1+n}\frac{\beta}{1+\beta}\left((1-\alpha)Ak^{\alpha}-(\gamma\alpha Ak^{\alpha}-\gamma nk)\right) \\
&= \beta\frac{(1-\alpha)-\gamma\alpha}{(1+n)(1+\beta)}Ak^{\alpha}+\frac{\beta\gamma n}{(1+n)(1+\beta)}k \\
\left[1+\gamma-\frac{\beta\gamma n}{(1+n)(1+\beta)}\right]k &= \beta\frac{(1-\alpha)-\gamma\alpha}{(1+n)(1+\beta)}Ak^{\alpha} \\
\left[\frac{(1+\gamma)(1+n)(1+\beta)-\beta\gamma n}{(1+n)(1+\beta)}\right]k &= \beta\frac{(1-\alpha)-\gamma\alpha}{(1+n)(1+\beta)}Ak^{\alpha} \\
k &= \beta\frac{(1-\alpha)-\gamma\alpha}{(1+\gamma)(1+n)(1+\beta)-\beta\gamma n}Ak^{\alpha}
\end{aligned}
$$

Algebra for steady state and golden rule Back

$$k_{debt} = \left[A\beta \frac{(1-\alpha) - \gamma\alpha}{(1+\gamma)(1+n)(1+\beta) - \beta\gamma n} \right]^{\frac{1}{1-\alpha}}$$

For golden rule:

$$k_{debt} = \left[A\beta \frac{(1-\alpha) - \gamma\alpha}{(1+\gamma)(1+n)(1+\beta) - \beta\gamma n} \right]^{\frac{1}{1-\alpha}} = \left(A\frac{\alpha}{n} \right)^{\frac{1}{1-\alpha}} = k_{gr}$$

$$n\beta\left[(1-\alpha) - \gamma\alpha\right] = \alpha\left[(1+\gamma)(1+n)(1+\beta) - \beta\gamma n\right]$$

$$n\beta(1-\alpha) - n\beta\gamma\alpha = \alpha(1+n)(1+\beta) + \alpha(1+n)(1+\beta)\gamma - \beta\gamma n\alpha$$

$$n\beta(1-\alpha) = \alpha(1+n)(1+\beta) + \alpha(1+n)(1+\beta)\gamma$$

$$\frac{n\beta(1-\alpha)}{\alpha(1+n)(1+\beta)} = (1+\gamma)$$