The Ramsey model

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Agenda

Consumption theory

- Two-period model recap
- Marginal propensities to consume
- Extension to arbitrary periods

The Ramsey model

- Derivation
- Steady state
- Dynamics
- Welfare

Two-period model recap

Preferences are given by

$$U = u(c_0) + \beta u(c_1)$$

- Individuals live for two periods
- Future utility is discounted at rate β

Dynamic budget constraints

$$\begin{aligned} c_0 + a_1 &= y_0 \\ c_1 &= y_1 + (1+r)a_1 \end{aligned}$$

- Individuals receive endowments (income) each period
- Income can be reallocated across periods using saving/borrowing

Two-period model solution

Euler equation

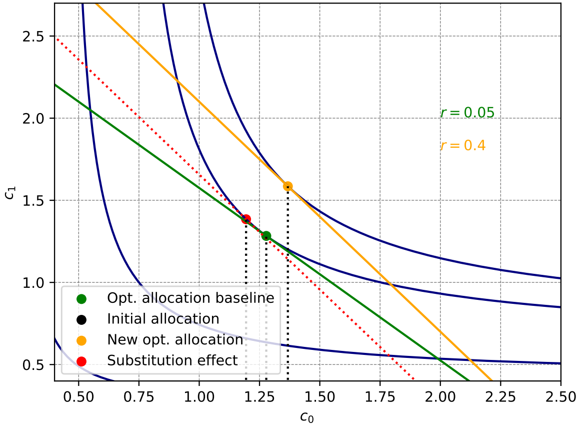
$$u'(c_0) = \beta(1+r)u'(c_1)$$

- Tradeoff between today and tomorrow is governed by β and r
- Present value of consumption = present value of endowment

c_0 with CRRA utility

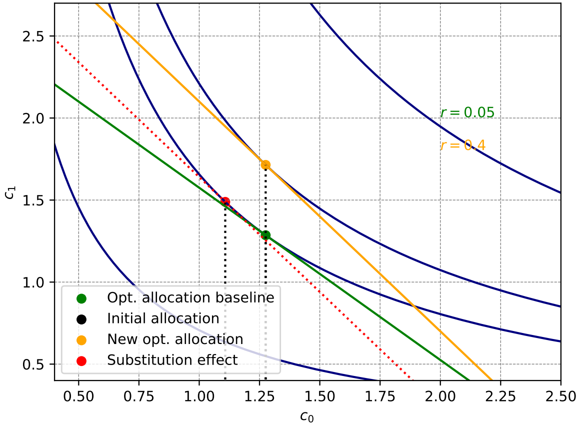
$$c_0^{-\sigma} = \beta(1+r)c_1^{-\sigma} \text{ and } c_1 = y_1 + (1+r)(y_0 - c_0)$$
$$\implies c_0\left(1 + \frac{(\beta(1+r))^{\frac{1}{\sigma}}}{1+r}\right) = \frac{y_1}{1+r} + y_0$$

Income and substitution effects



- Substitution effect: $\partial c_0 / \partial r < 0$ because of higher return
- Income effect: $\partial c_0 / \partial r > 0$ because agent is richer

Income and substitution effects



- With log-utility, the two effects exactly cancel
- Interest rate changes have no impact on consumption in period 0

Multiple periods

Two-periods is not enough

- Individuals have longer horizons
- Macroeconomic phenomena may happen on long time scales
- The Solow model has an infinite horizon

Agents with **very** long life-spans

- Intergenerational altruism
- Time-invariant survival probability
- Mathematical simplicity

Finite horizon

Arbitrary (but finite) horizon

Preferences are given by

$$U = u(c_0) + \beta u(c_1) + \beta^2 u(c_2) + \cdots = \sum_{t=0}^{T} \beta^t u(c_t)$$

- Future utility is discounted at rate β

Dynamic budget constraints

$$\begin{aligned} c_0 + a_1 &= y_0 + (1 + r_0)a_{-1} \\ c_1 + a_2 &= y_1 + (1 + r_1)a_0 \\ c_2 + a_3 &= y_2 + (1 + r_2)a_1 \\ &\dots \\ c_t + a_{t+1} &= y_t + (1 + r_t)a_t ; a_{T+1} \geq 0 \end{aligned}$$

- a_{-1} is given
- cannot die with debt (a_{T+1})

Solution method

Problem is given by

$$\max_{c_t, a_{t+1} \forall t} U = \sum_{t=0}^{T} \beta^t u(c_t)$$

$$\text{subject to} \quad c_t + a_{t+1} = y_t + (1 + r_t)a_t; a_T \geq 0; a_0 = 0$$

Lagrangian

$$\mathcal{L} = \sum_{t=0}^{T} \beta^t u(c_t) + \sum_{t=0}^{T} \lambda_t \left[y_t + (1 + r_t)a_t - c_t - a_{t+1} \right] + \mu a_{T+1}$$

- T utilities and budget constraints to optimize
- Final period a_{T+1} must be chosen separately (although trivial)

Optimization

Zoom in to period t (highly questionable notation)

$$\mathcal{L} = \cdots + \beta^t u(c_t) + \cdots + \lambda_t \left[y_t + (1+r_t)a_t - c_t - a_{t+1}\right] \\ + \lambda_{t+1}\left[y_{t+1} + (1+r_{t+1})a_{t+1} - c_{t+1} - a_{t+2}\right]$$

- The asset choice today affects tomorrow's budget!
- a_{t+1} links periods t and $t+1$ – appears twice!

First order conditions $\implies$ Euler equation

$$\frac{\partial \mathcal{L}}{\partial c_t}: \quad \beta^t u'(c_t) = \lambda_t$$

$$\frac{\partial \mathcal{L}}{\partial a_{t+1}}: \quad \lambda_t = (1+r_{t+1})\lambda_{t+1}$$

$$\implies u'(c_t) = \beta(1+r_{t+1})u'(c_{t+1})$$

- Euler equation exactly the same as in 2-period model
- Also, agents will always choose $a_{T+1} = 0$ due to $u'(c) > 0$

Infinite horizon

Infinite horizon

Problem is given by

$$\max_{c_t, a_{t+1} \forall t} U = \sum_{t=0}^{\infty} \beta^t u(c_t)$$

$$\text{subject to} \quad c_t + a_{t+1} = y_t + (1 + r_t)a_t$$

$$\lim_{T \to \infty} \left(\prod_{t=0}^{T} \frac{1}{1 + r_t} \right) a_{T+1} \geq 0; a_0 = 0$$

- New addition: No-Ponzi game condition $\implies$ cannot accumulate infinite debt (this is a constraint we impose)
- The present value of "final period" savings must be positive
- Importance becomes clear when using intertemporal budget constraint

Two ways of obtaining an Euler equation

Optimizing using the intertemporal budget constraint

$$\mathcal{L} = \sum_{t=0}^{\infty} \beta^t u(c_t) + \lambda \left(\prod_{s=0}^{t} \frac{1}{1+r_s} \right) (y_t - c_t)$$
$$\frac{\partial L}{\partial c_t} : \beta^t u'(c_t) = \lambda \left(\prod_{s=0}^{t} \frac{1}{1+r_s} \right)$$

- Combinding with $t+1$ condition yields $u'(c_t) = \beta(1 + r_{t+1})u'(c_{t+1})$

Optimizing using the dynamic budget constraints

$$\mathcal{L} = \sum_{t=0}^{T} \beta^t u(c_t) + \sum_{t=0}^{T} \lambda_t \left[y_t + (1+r_t)a_t - c_t - a_{t+1} \right]$$

$$\frac{\partial \mathcal{L}}{\partial c_t} : \qquad \beta^t u'(c_t) = \lambda_t$$
$$\frac{\partial \mathcal{L}}{\partial a_{t+1}} : \qquad \lambda_t = (1+r)\lambda_{t+1}$$

- As before, combining the two yields $u'(c_t) = \beta(1 + r_{t+1})u'(c_{t+1})$

The intertemporal budget constraint

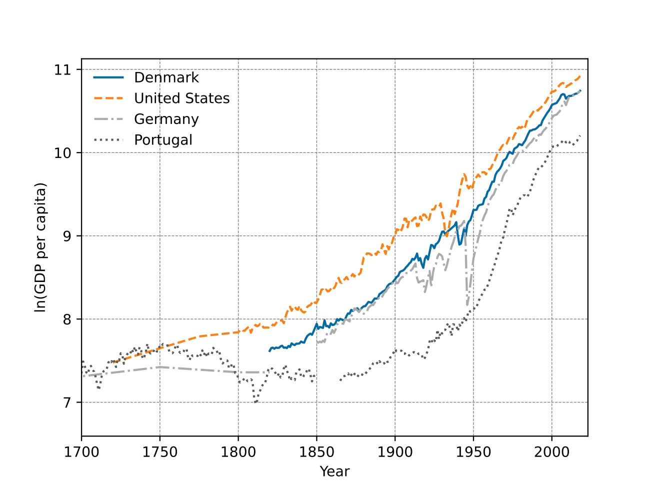
No-Ponzi pins down the path of c_t (by not allowing infinite consumption)

$$
\begin{aligned}
\frac{1}{1+r_0}a_1 &= \frac{1}{1+r_0}(y_0 - c_0) + a_0 \\
\frac{1}{1+r_1}a_2 &= \frac{1}{1+r_1}(y_1 - c_1) + a_1 \\
\implies \frac{1}{1+r_1}a_2 &= \frac{1}{1+r_1}(y_1 - c_1) + (y_0 - c_0) + (1+r_0)a_0 \\
\implies \frac{1}{1+r_1}\frac{1}{1+r_0}a_2 &= \frac{1}{1+r_1}\frac{1}{1+r_0}(y_1 - c_1) + \frac{1}{1+r_1}(y_0 - c_0) + a_0 \\
&\vdots \\
\implies \underbrace{\left(\prod_{t=0}^{T}\frac{1}{1+r_t}\right)a_{T+1}}_{\text{No-Ponzi as } T\to\infty} &= \sum_{t=0}^{T}\left(\prod_{s=0}^{t}\frac{1}{1+r_s}\right)(y_t - c_t)
\end{aligned}
$$

- Just as in 2 periods: PV of consumption equals PV of income

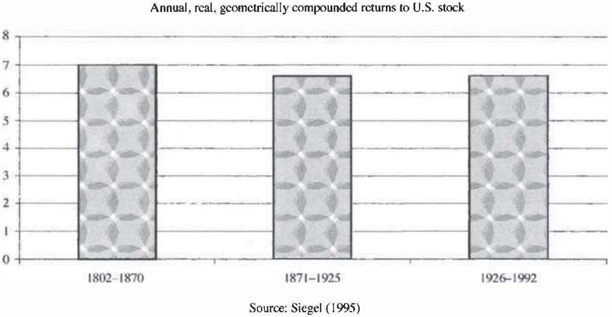
The Ramsey model

Constant growth rate



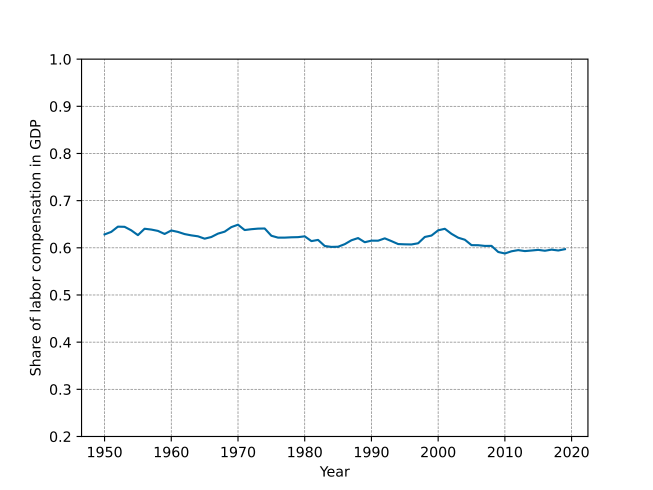
- US long-run growth has been constant for 200 years

Constant return on capital



- The return on capital has been constant for 200 years

Constant factor shares



- The labor income share is constant (Source: Fred)

Constant capital output ratio

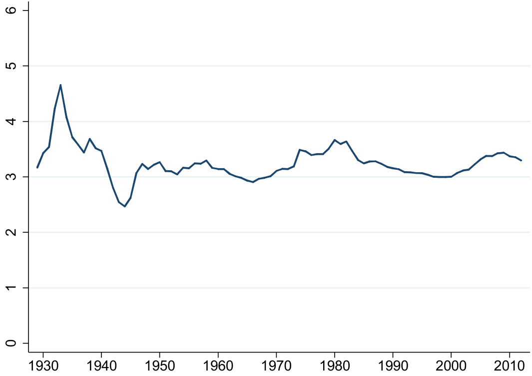


Figure: **Capital-Output ratio in the U.S.**

Source: NIPA table 1.1. The figure plots the ratio between fixed capital and consumer durables relative to the GDP.

- The capital-output ratio in the US hovers around 3 Danish data

Importance of microfoundations

Quick Solow-model recap

$$k_{t+1} = i_t + (1-\delta)k_t$$
$$y_t = F(k_t, l_t)$$
$$y_t = c_t + i_t = c_t + s y_t \implies c_t = (1-s)y_t$$

- The savings rate is not microfounded, just a parameter

Next steps:

- Combine optimal consumption choice with Solow to get Ramsey
- Add firm sector and asset market clearing
- Assume representative agent (infinitely many identical HHs)
- Assume labor is supplied inelastically at $l_t = 1 \forall t$

Firms

Representative firm

$$\max_{K_t, L_t} AK_t^{\alpha} L_t^{1-\alpha} - r_t K_t - w_t L_t$$

- Assume Cobb-Douglas specification for production function
- Production function $F(.)$ turns K and L into Y
- Firms are perfectly competitive
- They take prices r_t, w_t as given

First order conditions

$$f'_k(k_t, 1) = r_t \text{ where } k_t = K_t / L_t$$
$$f'_l(k_t, 1) = w_t$$

Labor income share is constant (prove it!)

$$\frac{w_t L_t}{AK_t^{\alpha} L_t^{1-\alpha}} = 1 - \alpha$$

Households optimality per capita

Modified first order conditions from before

$$u'(c_t) = \beta(1 + r_{t+1} - \delta)u'(c_{t+1})$$
$$c_t + a_{t+1} = w_t + (1 + r_t - \delta)a_t + \underbrace{\Pi_t}_{\text{Profits}}$$

- Households earn wage w_t (similar to endowment, since labor is supplied inelastically)
- Households lend to the firms at rate r_t
- δ depreciates each period

Transversality conditions

$$\lim_{T\to\infty} \beta^T u'(c_T)a_{T+1} = 0; \quad \lim_{T\to\infty} \beta^T u'(c_T)k_{T+1} = 0$$

- Not the same as no-Ponzi condition! No-Ponzi prevents too much debt (constraint)
- Transversality prevents too much capital/assets $\implies$ different sign (optimality)

General Equilibrium

- General equilibrium implies that all markets clear & agents optimize

Market clearing

Capital and labor market clearing

$$l_t = 1$$
$$k_t = a_t$$

Resource constraint (same as in the Solow model)

$$\begin{aligned} c_t + a_{t+1} &= w_t + (1 + r_t - \delta)a_t + \Pi_t \\ c_t + k_{t+1} &= (1-\delta)k_t + r_t k_t + w_t + f(k_t) - r_t k_t - w_t \\ k_{t+1} &= (1-\delta)k_t + f(k_t) - c_t \end{aligned}$$

Capital tomorrow (k_{t+1}) is

- Leftover capital from today (after depreciation)
- Production less consumption

Laws of motion

Ramsey

$$k_{t+1} = (1-\delta)k_t + f(k_t) - c_t$$
$$u'(c_t) = \beta(1 + r_{t+1} - \delta)u'(c_{t+1})$$

Solow

$$k_{t+1} = (1-\delta)k_t + f(k_t) - c_t$$
$$c_t = (1-s)f(k_t)$$

- The savings rate s is now endogenized through optimally chosen consumption
- One fewer ad-hoc parameter

Steady state I

In the model's steady state, c and k must be constant

Steady state of capital accumulation

$$\begin{aligned} k &= (1-\delta)k + f(k) - c \\ \implies c &= f(k) - \delta k \end{aligned}$$

- For any k, there is a specific level of c that keeps k constant

Steady state in the Euler equation

$$\begin{aligned} u'(c) &= \beta(1+r-\delta)u'(c) \\ u'(c) &= \beta(1+f'(k)-\delta)u'(c) \\ f'(k) &= \frac{1}{\beta} + \delta - 1 \end{aligned}$$

- Consumption remains constant at a specific level of capital k
- c is not pinned down from this

Steady state II

Steady state of the model

$$f'(k) = \frac{1}{\beta} + \delta - 1$$
$$c = f(k) - \delta k$$

- Both equations must hold at the same time $\implies$ intersection

Assume functional forms

$$F(K) = K^{\alpha} L^{1-\alpha} \implies k^{\alpha} \text{ if } L = 1 \qquad \text{(Cobb-Douglas production)}$$
$$u(c) = \frac{c^{1-\sigma} - 1}{1 - \sigma} \qquad \text{(CRRA utility)}$$

Unique steady state value of capital

$$k = \left(\frac{\alpha}{\frac{1}{\beta} + \delta - 1} \right)^{\frac{1}{1-\alpha}}$$

Dynamics

How do we get to the steady state?

- Does every starting point converge to the steady state?
- How fast is the speed of convergence?
- What are the dynamics of the model far away from steady state?

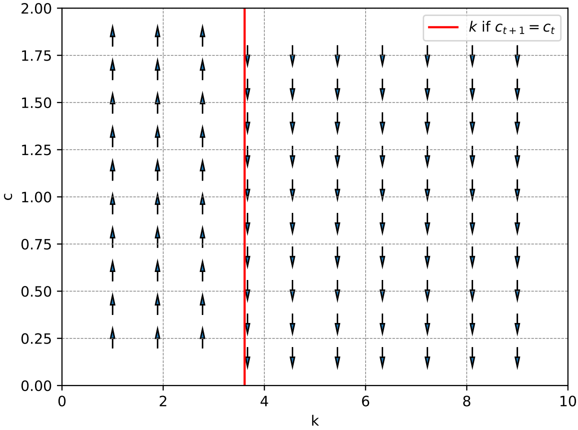
Phase diagram

- Describe the dynamics graphically

Plug in some numbers!

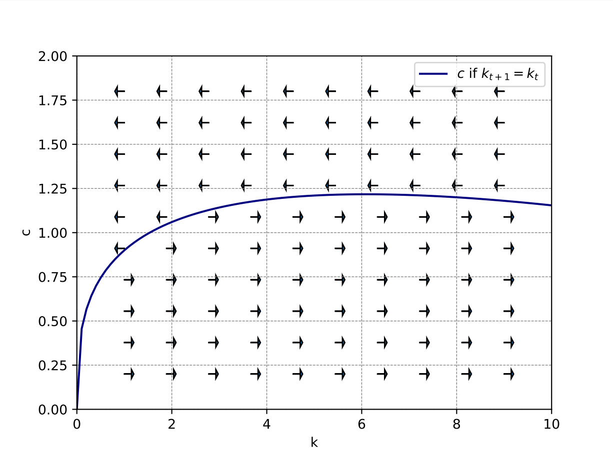
- $\beta = 0.96$
- $\delta = 0.1$
- $\alpha = 0.3$
- $\sigma = 1$

Dynamics along c dimension



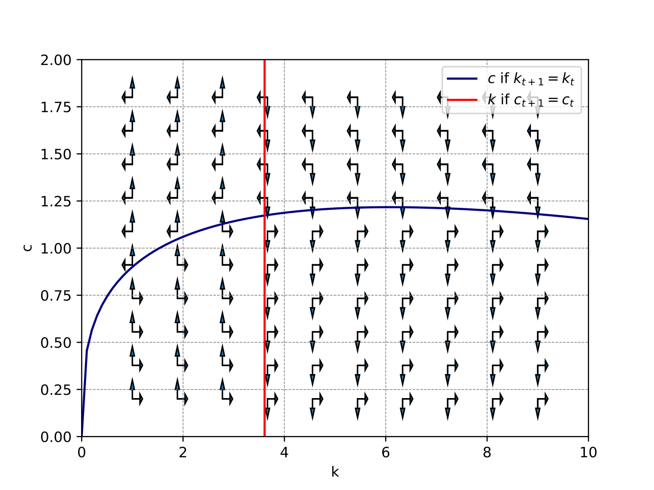
- $u'(c_t) = \beta(1 + r_t - \delta)u'(c_{t+1})$
- High $k_t \rightarrow$ low marginal product $f'(k_t) \rightarrow$ low $r_t \rightarrow c_t \uparrow$ today

Dynamics along k dimension



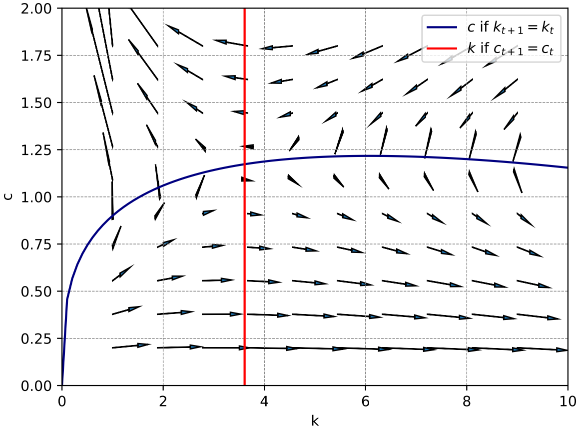
- $k_{t+1} = (1 - \delta)k_t + f(k_t) - c_t$
- High c_t → little left to invest in k_{t+1} → capital falls
- $f(k)$ is concave → as k grows, output grows by less and less

Full dynamics – Phase diagram



- Steady state where the two lines cross
- c^* and k^* define the balanced growth path (after transition)

Full dynamics – Phase diagram



- Steady state where the two lines cross
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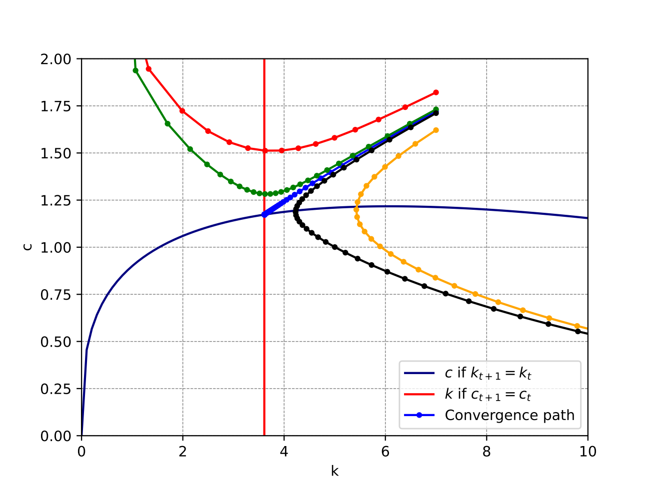
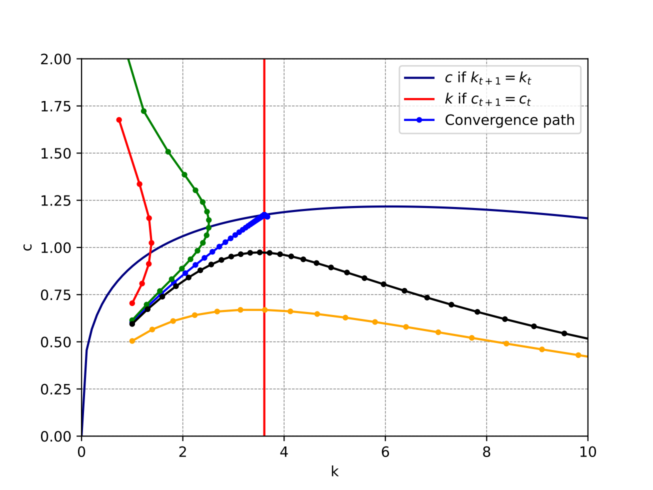
Dynamics towards convergence



- Only very specific starting conditions converge

Dynamics towards convergence



- For each k_0, only one value of c_0 leads to convergence

Solving the Ramsey model

For each k_0, only one c_0 is eligible!

- Remember the no-ponzi condition: too much debt (c) is not permitted (top left)
- Remember the transversality condition: too much wealth (k) is not permitted (bottom right)

$$\sum_{t=0}^{T} \left(\prod_{s=0}^{t} \frac{1}{1 - \delta + r_s} \right) (w_t - c_t) = 0$$

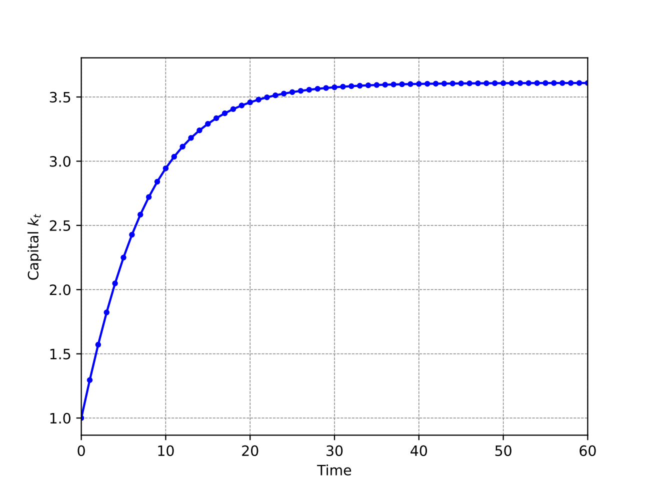
- There is no analytical solution for c_0

Solution

- Guess c_0, iterate forward using Euler eq. and resource constr.
- Narrow search c_0 using, e.g., bisection search

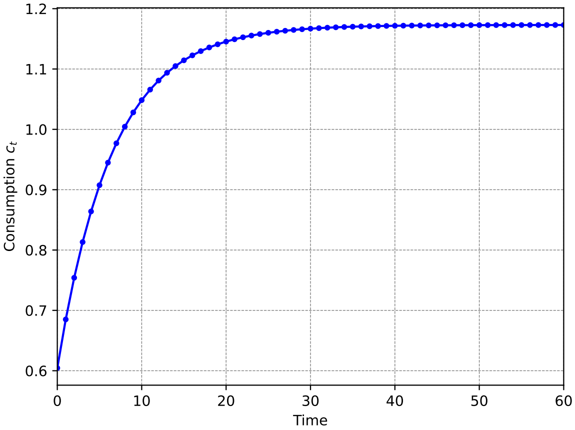
Convergence paths

Capital



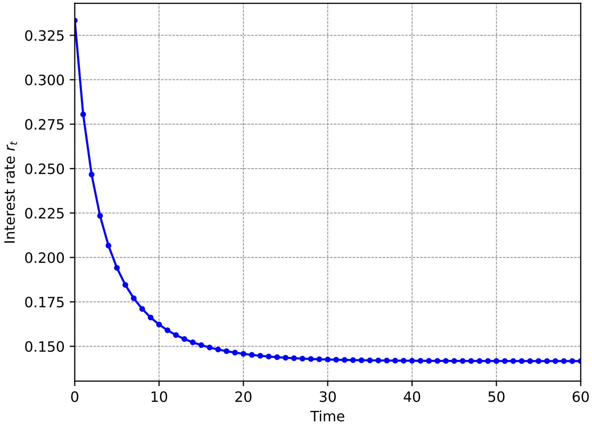
Convergence paths

Consumption



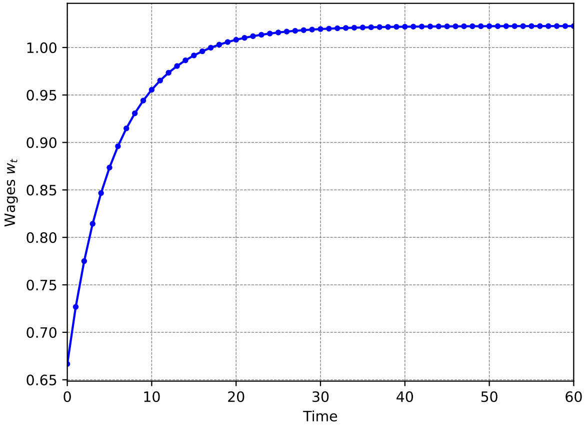
Convergence paths

Rental rate of capital



Convergence paths

Wage

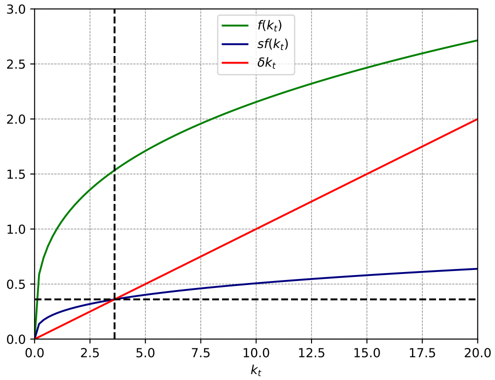


Solow comparison

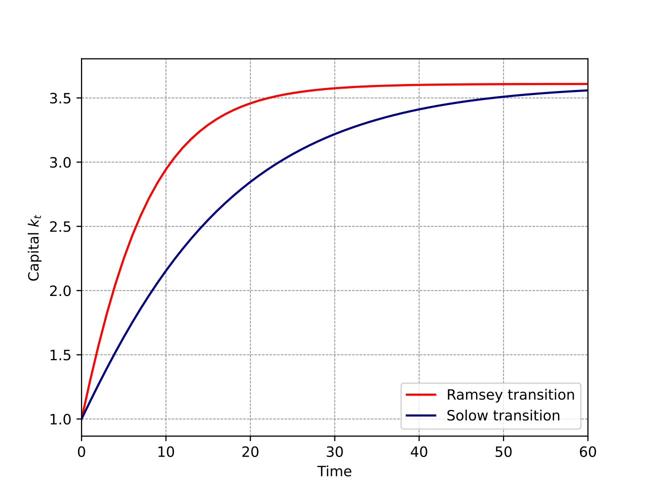
Law of motion

$$k_{t+1} = (1-\delta)k_t + sf(k_t) \implies k_{ss} = \frac{s}{\delta}k_{ss}^{\alpha} \implies k_{ss} = \left(\frac{s}{\delta}\right)^{\frac{1}{1-\alpha}}$$

- s set to match Ramsey gives the same steady state

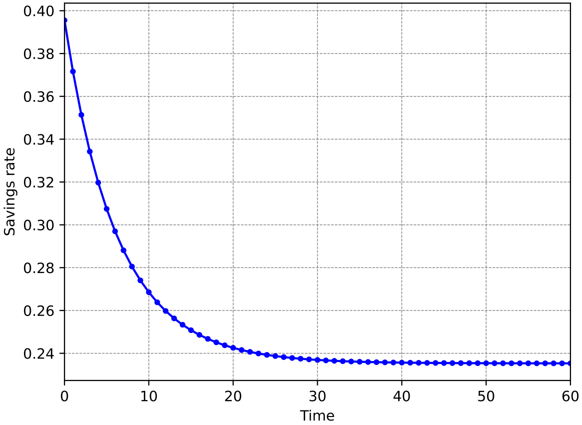


Transition comparison



- Ramsey model converges much faster. Why?

Savings rate is endogenous



- Savings rate in Ramsey is very high initially (Why?)

Welfare in the Solow model

Maximize steady state consumption subject to the resource constraint

$$\max_k f(k) - \delta k \implies f'(k^{gr}) = \delta$$

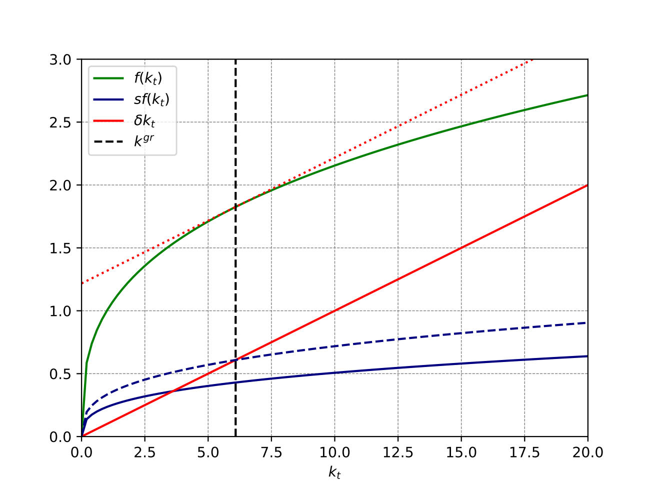
- The maximum possible consumption is attained when $\delta = \text{MPK}$

Optimal savings rate

$$k^{gr} = \left(\frac{s^{gr}}{\delta}\right)^{\frac{1}{1-\alpha}} \implies s^{gr} = \delta(k^{gr})^{1-\alpha}$$

- In Solow's model, the savings rate is an exogenous parameter
- If policy can manipulate s, it can attain optimal consumption

Golden rule capital accumulation



- Largest distance between $sf(k^{gr})$ and $f(k^{gr})$, given δ slope

Welfare in the Ramsey model I

Planner's problem

$$\max_{c_t, k_{t+1} \forall t} U = \sum_{t=0}^{\infty} \beta^t u(c_t)$$

subject to $k_{t+1} = (1-\delta)k_t + f(k_t) - c_t; \quad k_{t+1} > 0; \quad k_0$ given

- Similar to the household's optimization problem HH problem
- Note: resource constraint, not budget constraint
- w_t and r_t not taken as given

Solution

$$u'(c_t) = \beta(1 + r_{t+1} - \delta)u'(c_{t+1})$$
$$k_{t+1} = (1-\delta)k_t + f(k_t) - c_t$$

Welfare in the Ramsey model II

Modified golden rule

$$u'(c_t) = \beta(1 + r_{t+1} - \delta)u'(c_{t+1})$$
$$\implies f'(k^{ss}) = \delta + \frac{1}{\beta} - 1$$

- β enters welfare maximizing capital level
- Preferences matter

Comparison

$$f'(k^{ss}) = \delta + \frac{1}{\beta} - 1 > \delta = f'(k^{gr})$$

- Optimal capital in Ramsey always below Solow
- Return on capital (to consumers) must equal MPK
- Not necessary in Solow (since s is exogenous)

Welfare theorem

First welfare theorem

- Economy is perfectly competitive
- No inefficiencies, frictions or externalities
- The utility function exhibits the usual properties

$\Longrightarrow$ The equilibrium is Pareto efficient!

Growth model?

Baseline model without growth

$$F(K_t, L_t) = AK_t^\alpha L_t^{1-\alpha}$$

$$K_{t+1} = K_t(1-\delta) + F(K_t, L_t) - C_t$$

$$U = \sum_{t=0}^{\infty} \beta^t u(C_t) \quad \text{s.t.} \quad C_t + B_{t+1} = B_t(1 - \delta + R_t) + L_t w_t$$

$$w_t = F'_L(K_t, L_t); \quad r_t = F'_K(K_t, L_t)$$

- We set $L_t = 1$, which simplified things
- What if there is population growth in the economy?
- What if there is technological progress?
- Can this model be used to understand growth in steady state?

Normalization I

Population growth

- The model accomodates population growth
- Assume that L_t implies that the whole labor force works full time
- Assume the labor force growth at $L_{t+1} = (1+n)L_t$
- Need to assume new per-worker utility function $u(c_t)$

Ramsey model normalized for population growth

$$\frac{F(K_t, L_t)}{L_t} = \qquad f(k_t) = Ak_t^\alpha$$

$$\frac{L_{t+1}}{L_t} k_{t+1} = \qquad (1+n)k_{t+1} = k_t(1-\delta) + f(k_t) - c_t$$

$$U = \sum_{t=0}^{\infty} (\beta(1+n))^t u(c_t) \qquad \text{s.t.} \qquad c_t + (1+n)b_{t+1} = b_t(1-\delta+r_t) + w_t$$

Normalization II

Technological progress

- The model also accomodates (exogenous) technological progress
- Assume that $A_{t+1} = (1+\gamma)A_t$
- Normalize the model by dividing by $A_t L_t$
- Express everything in efficiency units, reformulate $Y_t = K_t^{\alpha}[\hat{A}_t L_t]^{1-\alpha}$
- This only works with the preferences we have assumed!

Ramsey model normalized for population and technological growth

$$\frac{F(K_t, L_t)}{A_t L_t} = f(k_t) = k_t^{\alpha}$$

$$\frac{L_{t+1} A_{t+1}}{L_t A_t} k_{t+1} = (1+n)(1+\gamma)k_{t+1} = k_t(1-\delta) + f(k_t) - \frac{c_t}{(1+\gamma)^t}$$

$$U = \sum_{t=0}^{\infty} (\widetilde{\beta}(1+n))^t u(c_t) \quad \text{s.t.} \quad c_t + (1+n)b_{t+1} = b_t(1-\delta+r_t) + w_t$$

Appendix

BC math I

Backwards substitution Back

$$a_{t+1} = y_t + (1+r_t)a_t - \left[\beta^t(1+r)^t\right]^{\frac{1}{\sigma}} c_0$$

$$a_{t+1} = y_t + (1+r)\left(y_{t-1} + (1+r)a_{t-1} - \left[\beta^{t-1}(1+r)^{t-1}\right]^{\frac{1}{\sigma}} c_0\right) - \left[\beta^t(1+r)^t\right]^{\frac{1}{\sigma}} c_0$$

$$a_{t+1} = \sum_{i=0}^{t}(1+r)^i y_{t-i} + (1+r)^t a_0 - \sum_{i=0}^{t}(1+r)^i \left[\beta^{t-i}(1+r)^{t-i}\right]^{\frac{1}{\sigma}} c_0$$

$$a_{t+1} = (1+r)^t\left[\sum_{i=0}^{t}(1+r)^{i-t} y_{t-i} - \sum_{i=0}^{t}(1+r)^{i-t}\left[\beta^{t-i}(1+r)^{t-i}\right]^{\frac{1}{\sigma}} c_0\right]$$

$$0 = \sum_{i=0}^{T}(1+r)^{i-T} y_{T-i} - \sum_{i=0}^{T}(1+r)^{i-T}\left[\beta^{T-i}(1+r)^{T-i}\right]^{\frac{1}{\sigma}} c_0$$

$$0 = \sum_{i=0}^{T}\frac{1}{(1+r)^{T-i}} y_{T-i} - \sum_{i=0}^{T}\frac{1}{(1+r)^{T-i}}\left[\beta^{T-i}(1+r)^{T-i}\right]^{\frac{1}{\sigma}} c_0$$

BC math II

Flip counting of sums to make it easier on the eyes Back

$$
\begin{aligned}
0 &= \sum_{i=0}^{T} \frac{1}{(1+r)^i} y_i - \sum_{i=0}^{T} \frac{1}{(1+r)^i} \left[\beta^i (1+r)^i\right]^{\frac{1}{\sigma}} c_0 \\
\sum_{i=0}^{T} \frac{1}{(1+r)^i} \left[\beta^i (1+r)^i\right]^{\frac{1}{\sigma}} c_0 &= \sum_{i=0}^{T} \frac{1}{(1+r)^i} y_i \\
c_0 &= \frac{\sum_{i=0}^{T} \frac{1}{(1+r)^i} y_i}{\sum_{i=0}^{T} \frac{\left[\beta^i (1+r)^i\right]^{\frac{1}{\sigma}}}{(1+r)^i}}
\end{aligned}
$$

Marginal propensity to consume

How much of an additional dollar would people spend/save in period 0?
Use result from before!

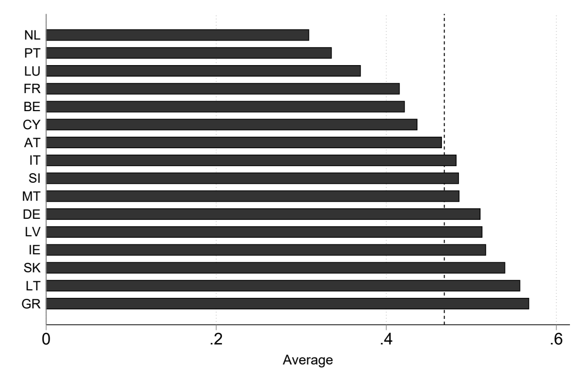
$$c_0 = \frac{\frac{y_1}{1+r} + y_0}{\left(1 + \frac{(\beta(1+r))^{\frac{1}{\sigma}}}{1+r}\right)}$$

Marginal propensity to consume

$$\frac{\partial c_0}{\partial y_0} = \frac{1}{\left(1 + \frac{(\beta(1+r))^{\frac{1}{\sigma}}}{1+r}\right)}$$

- Consumption is exactly proportional to "permanent income"
- With CRRA, the proportionality factor is constant

MPC in the data (Almgren et al, 2022)



- MPC out of unexpected lottery win equal to monthly HH income

Imposing CRRA utility

Period 0 consumption (assuming constant r and using $a_{T+1} = 0$) Algebra

$$c_0 = \frac{\sum_{i=0}^{T} \frac{1}{(1+r)^i} y_i}{\sum_{i=0}^{T} \frac{\left[\beta^i (1+r)^i\right]^{\frac{1}{\sigma}}}{(1+r)^i}}$$

Permanent income

- Consumption depends on a weighted average of all future income
- Only small shares of income increases are consumed

Marginal Propensity to consume

$$\frac{\partial c_0}{\partial y_0} = \frac{1}{\sum_{i=0}^{T} \frac{\left[\beta^i (1+r)^i\right]^{\frac{1}{\sigma}}}{(1+r)^i}}$$

Marginal propensity to consume

c_0 consumption (similar algebra as with T)

$$c_0 = \lim_{T \to 0} \frac{\sum_{i=0}^{T} \frac{1}{(1+r)^i} y_i}{\sum_{i=0}^{T} \frac{[\beta^i (1+r)^i]^{\frac{1}{\sigma}}}{(1+r)^i}} = \left(1 - \frac{[\beta(1+r)]^{\frac{1}{\sigma}}}{1+r}\right) \sum_{i=0}^{\infty} \frac{1}{(1+r)^i} y_i$$

Marginal Propensity to Consume

$$\frac{\partial c_0}{\partial y_0} = \left(1 - \frac{[\beta(1+r)]^{\frac{1}{\sigma}}}{1+r}\right)$$

- Apply "reasonable calibration" ($\sigma = 1$, $\beta = 0.9$)
- MPC $= 0.1 \implies$ agent spends only 10% of transitory income gain
- Unrealistically low, MPCs are closer to 30% in the data