THE NUMBER OF REAL ZEROS OF A RANDOM TRIGONOMETRIC POLYNOMIAL

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1. This paper, in which we calculate the probable number of zeros of a certain trigonometric polynomial whose coefficients are random, is based on part of the thesis which I submitted for the degree of Ph.D. of the University of London; and I am grateful to my supervisor, Professor A. C. Offord, for suggesting this very interesting problem and for his subsequent help and encouragement.

It is convenient to begin by seeing how the random coefficients mentioned above can be replaced by functions of a parameter t $(0 \leqslant t \leqslant 1)$, which we are able to do using the idea of independent functions developed by Kac (4). Suppose that $g_1(t), g_2(t), \ldots, g_n(t)$ are n Lebesgue-measurable functions defined for $0 \leqslant t \leqslant 1$, and let†

$$G_r = \{t\colon \alpha_r \leqslant g_r(t) \leqslant \beta_r\} \quad (r = 1, 2, \ldots, n).$$

If

$$\left|\bigcap_1^n G_r\right| = \prod_1^n |G_r|$$

for every choice of the α_r and β_r, the n functions $g_r(t)$ are said to be *independent*. The functions of an infinite sequence are independent if the same is true of those of every finite subsequence. It follows from Kac's paper that independent functions can be linked with probability theory by interpreting $|G_r|$ as the probability that a random variable g_r satisfies the inequality $\alpha_r \leqslant g_r \leqslant \beta_r$. In particular, the mean of the random variable equals the mean of the function, i.e. its integral over $(0, 1)$, and independent functions correspond to independent random variables.

In the work that follows we need a sequence of independent functions which will represent a sequence of independent random variables which are normally distributed, each having the distribution function

$$G(x) = \frac{1}{\sqrt{(2\pi)}}\int_{-\infty}^{x} e^{-\frac{1}{2}t^2}\,dt.$$

† $\{t\colon P\}$ is the set of points t which have the property P. $|G|$ is the measure of the set G.

Steinhaus (6) has constructed a sequence of continuous functions $\chi_n(t)$ $(n = 1, 2, \ldots)$ which are independent and which have the properties

(i) $$0 \leqslant \chi_n(t) \leqslant 1,$$

(ii) $$|\{t : \chi_n(t) \in A\}| = |A|,$$

where A is any measurable set contained in the interval $0 \leqslant t \leqslant 1$. We can then obtain the sequence of functions we are looking for by defining

$$g_n(t) = g(\chi_n(t)) \quad (n = 1, 2, \ldots),$$

where $g(x)$ is the inverse of $G(x)$. It can be shown that since g is continuous and the χ_n are continuous and independent, the g_n are independent; they are clearly continuous. The sequence $g_1(t)$, $g_2(t)$, ... thus represents a sequence of independent random variables each having the normal distribution function $G(x)$.

The trigonometric polynomial to be studied here is

$$\varphi(\theta) = \varphi(\theta, t) = \sum_{n=1}^{N} g_n(t) \cos n\theta.$$

As we have seen, either we may suppose that we have a polynomial with random coefficients and talk about the probability that it has a certain property, or we may suppose that we have a family of polynomials (defined by the parameter t) and speak of the *measure* of the set of polynomials that have the property (i.e. the measure of the corresponding set of values of t). We shall prove the following

THEOREM. *In the interval $0 \leqslant \theta \leqslant 2\pi$ all save a certain exceptional set of the functions $\varphi(\theta, t)$ have*†

$$\frac{2}{\sqrt{3}} N + O(N^{11/13} (\log N)^{3/13})$$

zeros when N is large. The measure of the exceptional set does not exceed $(\log N)^{-1}$.

The basic idea of the proof is as follows. If we wish to estimate the number of zeros in a certain interval I, we divide I into a suitably large number of equal subintervals and consider the number of changes of sign of $\varphi(\theta)$ at the end-points of these subintervals. In work which culminates in Lemmas 16 and 17 we calculate the mean number of changes of sign and we show that the standard deviation of this number is small, from which it follows that the probability is high that the actual number of changes of sign is nearly equal to its mean value. The proof is completed

† Throughout this paper, O refers to N tending to infinity unless otherwise stated; and the constant implied by it is independent of θ or any other variable.

by showing that the number of changes of sign corresponds closely to the number of zeros, which are to be counted according to their multiplicity.

As far as this description goes, our method resembles that used by Erdös and Offord (2) to estimate the likely number of real zeros of the polynomial

$$f_n(x) = 1+\varepsilon_1 x+\ldots+\varepsilon_n x^n,$$

where the ε_r are independent and take the two values $+1$ and -1 with equal probability. The details, however, differ considerably; in particular, we use quite different ideas to relate the number of changes of sign to the number of zeros, and in so doing we obtain *en passant* some interesting information about the multiplicity of the zeros which I hope to elaborate in a later paper.

It is interesting to compare our result with that obtained by Erdös and Offord. Since $\varphi(\theta)$ is an algebraic polynomial of degree N in $\cos\theta$, it can have at most $2N$ zeros in the interval $(0, 2\pi)$, and our theorem shows therefore that $\varphi(\theta)$ will most likely have a large number of real zeros; a proportion $1/\sqrt{3}$ of the greatest possible number in fact. By contrast, $f_n(x)$ has on the whole only a proportion $2/(\pi n \log n)$ of its zeros real. That this difference is not brought about simply by the different probability distributions of the two sets of coefficients follows from some results obtained by Littlewood and Offord ((5) Theorems 1 and 2) which show that the likely number of real zeros of an algebraic polynomial is much the same, whether the coefficients are normally distributed, as here, or have the 'E-distribution' as in Erdös and Offord.

2. To study the changes of sign of $\varphi(\theta)$, we introduce what graphical considerations prompt us to call *single* and *double crossovers*. Let (a, b) be any interval. If $\varphi(a)\varphi(b) \leqslant 0$ we shall say that $\varphi(\theta)$ has a single crossover (s.c.o.) in (a, b). If, however, $\varphi(a)\varphi(b) \geqslant 0$ and $\varphi(a)\varphi(\frac{1}{2}(a+b))\varphi(b) \leqslant 0$ we shall say that $\varphi(\theta)$ has a double crossover (d.c.o.) in (a, b). Clearly, the existence of a single crossover implies that $\varphi(\theta)$ has at least one zero in the interval (possibly at the end-points) and the existence of a double crossover implies that there are at least two zeros in the interval. Thus†

$$\Pr(\text{s.c.o.}) \leqslant \Pr(\text{at least one zero})$$

and

$$\Pr(\text{d.c.o.}) \leqslant \Pr(\text{at least two zeros}).$$

We shall want to know the probabilities that $\varphi(\theta)$ has (i) a single crossover in a given interval, (ii) a double crossover in a given interval, and (iii) simultaneous single crossovers in each of two given intervals.

† $\Pr(E)$ denotes the probability that the event E takes place.

For a given value of θ, $\varphi(\theta)$ is a linear combination of the g_n, and we accordingly need the joint probability distribution of 2, 3, and 4 such sums. We therefore start our proof in earnest by considering this problem.

Let $X_1, X_2, \ldots, X_k$ be k random variables given by

$$X_r = a_1^{(r)}g_1 + a_2^{(r)}g_2 + \ldots + a_N^{(r)}g_N,$$

the $a_s^{(r)}$ being constants. The characteristic function† of the k-dimensional random vector $(X_1, X_2, \ldots, X_k)$ is

$$f(t_1, t_2, \ldots, t_k) = \exp\left(-\frac{1}{2}\sum_{r,s=1}^{k} \gamma_{rs} t_r t_k\right),$$

where

$$\gamma_{rs} = \sum_{n=1}^{N} a_n^{(r)} a_n^{(s)}.$$

To see this, consider the point $(g_n, g_n, \ldots, g_n)$ in Euclidean space, R_k, of k dimensions. If h is small enough this will lie outside the interval defined by

$$(x_r, x_r + h) \quad (r = 1, 2, \ldots, k)$$

unless $x_1 = x_2 = \ldots = x_k$. When the x_r have the common value x, the probability that the point lies in the interval is

$$\frac{1}{\sqrt{(2\pi)}}\int_x^{x+h} e^{-\frac{1}{2}t^2}\,dt.$$

So if $F(x_1, \ldots, x_k)$ is the distribution function of $(g_n, g_n, \ldots, g_n)$ its characteristic function is

$$\begin{aligned}\int_{R_k} &\exp\{i(t_1x_1 + \ldots + t_kx_k)\}\,dF \\ &= \frac{1}{\sqrt{(2\pi)}}\int_{-\infty}^{\infty} \exp\{i(t_1 + \ldots + t_k)x - \tfrac{1}{2}x^2\}\,dx \\ &= \exp\{-\tfrac{1}{2}(t_1 + \ldots + t_k)^2\}.\end{aligned}$$

Hence the characteristic function of the vector

$$(a_n^{(1)}g_n, a_n^{(2)}g_n, \ldots, a_n^{(k)}g_n) \quad \text{is} \quad \exp\{-\tfrac{1}{2}(a_n^{(1)}t_1 + \ldots + a_n^{(k)}t_k)^2\}.$$

Now the characteristic function of a sum of independent random vectors is the product of their several characteristic functions; and

$$(X_1, \ldots, X_k) = \sum_{n=1}^{N} (a_n^{(1)}g_n, \ldots, a_n^{(k)}g_n).$$

† The properties of characteristic functions and distribution functions that are required here may be found, for example, in Cramér (1).

Therefore the characteristic function of $(X_1, \ldots, X_k)$ is

$$\exp\left\{-\frac{1}{2}\sum_{n=1}^{N}(a_n^{(1)}t_1 + \ldots + a_n^{(k)}t_k)^2\right\},$$

which reduces to the stated result.

Note. Throughout this paper f will be used exclusively for the characteristic function of a random vector of this sort, in 1, 2, 3, or 4 dimensions as indicated by the context.

We next introduce a number of difference operators. Although in practice we shall operate with them only on functions f, they can be defined for any function, ψ, say.

For a function of one variable, we define

$$\Delta\psi(t) = \psi(t) - \psi(-t).$$

For functions of several variables, we define

$$\Delta_1\psi(t_1, t_2) = \psi(t_1, t_2) - \psi(-t_1, t_2),$$

$$\Delta_{12}\psi(t_1, t_2) = \Delta_1(\Delta_2\psi),$$

and so on. We also operate with Δ upon functions of several variables, and we define it in this case by

$$\Delta\psi(t_1, t_2, \ldots, t_k) = \Delta_{2\ldots k}\psi(t_1, \ldots, t_k).$$

It is not difficult to see that for the characteristic function f introduced above,

$$\Delta_{12\ldots k}f(t_1, t_2, \ldots, t_k) = \begin{cases} 2\Delta f(t_1, \ldots, t_k) & (k \text{ even}), \\ 0 & (k \text{ odd}). \end{cases}$$

It is convenient to let

$$K_r(h) = (1 - e^{-it_r h})/it_r,$$

dropping the suffix r when only one variable t occurs.

LEMMA 1. *Let $\psi(t)$ be continuous, of bounded variation in a neighbourhood of the origin, and sufficiently well behaved at infinity to secure the absolute convergence of the integrals that follow. Suppose that $\Delta\psi(t) = O(t)$ as $t \to 0$. If*

$$P(h) = \int_{-\infty}^{\infty} K(h)\psi(t)\,dt$$

then

$$P(\infty) = \lim_{h\to\infty} P(h)$$

$$= \int_0^{\infty} \frac{\Delta\psi}{it}\,dt + \pi\psi(0)$$

and

$$P(-\infty) = \int_0^\infty \frac{\Delta\psi}{it}\,dt - \pi\psi(0).$$

Proof. By a simple rearrangement,

$$P(h) = \int_0^\infty \frac{\Delta\psi}{it}\,dt - \frac{1}{i}\int_0^\infty (\cos th)\frac{\Delta\psi}{t}\,dt + \int_0^\infty \frac{\sin th}{t}\{\psi(t)+\psi(-t)\}\,dt.$$

As $h \to \infty$ the second integral on the right tends to zero and the third tends to $\pi\psi(0)$. The argument as $h \to -\infty$ is similar.

REMARK. We may take ψ to be a characteristic function $f(t_1, \ldots, t_k)$ and t to be one of the variables t_r. In this case we replace Δ by Δ_r and $\psi(0)$ by $f(\ldots, t_{r-1}, 0, t_{r+1}, \ldots)$.

3. Let $\{a_n\}$, $\{b_n\}$, $\{c_n\}$, and $\{d_n\}$ be four sets of constants, and let

$$\mathscr{A} = \sum_1^N a_n g_n, \quad \mathscr{B} = \sum_1^N b_n g_n, \quad \mathscr{C} = \sum_1^N c_n g_n, \quad \mathscr{D} = \sum_1^N d_n g_n.$$

It is convenient to think of these as the sums $X_1, \ldots, X_4$ introduced in § 2, and to write

$$\gamma_{11} = a^2 = \sum_1^N a_n^2, \quad \text{etc.},$$

and

$$\gamma_{12} = \sum_1^N a_n b_n, \quad \text{etc.}$$

LEMMA 2. *If $f(t_1, t_2)$ is the characteristic function of the random vector $(\mathscr{A}, \mathscr{B})$ then*

$$\iint_0^\infty \frac{\Delta_{12} f(t_1, t_2)}{t_1 t_2}\,dt_1 dt_2 = -2\pi \sin^{-1}\frac{\gamma_{12}}{ab}.$$

Proof. If the integral is denoted by E then

$$E = 2\iint_0^\infty \frac{f(t_1, t_2) - f(t_1, -t_2)}{t_1 t_2}\,dt_1 dt_2,$$

where

$$f(t_1, t_2) = \exp\{-\tfrac{1}{2}(a^2 t_1^2 + 2\gamma_{12} t_1 t_2 + b^2 t_2^2)\}.$$

The subsequent calculation may be left to the reader.

LEMMA 3.

$$\Pr(\mathscr{A} \geqslant 0, \mathscr{B} \leqslant 0, \mathscr{C} \geqslant 0)$$
$$= \frac{1}{4\pi}\left\{\sin^{-1}\sqrt{\left(1-\frac{\gamma_{23}^2}{b^2c^2}\right)} - \sin^{-1}\sqrt{\left(1-\frac{\gamma_{13}^2}{a^2c^2}\right)} + \sin^{-1}\sqrt{\left(1-\frac{\gamma_{12}^2}{a^2b^2}\right)}\right\}.$$

Proof. Let

$$P(p, q, r) = \Pr(0 \leqslant \mathscr{A} \leqslant p, -q \leqslant \mathscr{B} \leqslant 0, 0 \leqslant \mathscr{C} \leqslant r).$$

Then (Cramér (1) Theorem 9a, p. 105)

$$P(p,q,r) = -\frac{1}{8\pi^3}\iiint_{-\infty}^{\infty} K_1(p)K_2(q)K_3(r)f(t_1,t_2,t_3)\,dt_1dt_2dt_3,$$

where $f(t_1,t_2,t_3)$ is the characteristic function of $(\mathscr{A},\mathscr{B},\mathscr{C})$.

We require $P(\infty,\infty,\infty)$, and we obtain this by letting p, q, r tend to infinity in succession, using Lemma 1 to evaluate each limit in turn. This process gives

$$\begin{aligned}P(\infty,\infty,\infty) = &-\frac{1}{8\pi^3}\iiint_0^\infty \frac{\Delta_{123}f(t_1,t_2,t_3)}{i^3t_1t_2t_3}\,dt_1dt_2dt_3\\ &-\frac{1}{8\pi^2}\iint_0^\infty \frac{\Delta_{23}f(0,t_2,t_3)}{i^2t_2t_3}\,dt_2dt_3\\ &+\frac{1}{8\pi^2}\iint_0^\infty \frac{\Delta_{13}f(t_1,0,t_3)}{i^2t_1t_3}\,dt_1dt_3+\frac{1}{8\pi}\int_0^\infty \frac{\Delta_3f(0,0,t_3)}{it_3}\,dt_3\\ &-\frac{1}{8\pi^2}\iint_0^\infty \frac{\Delta_{12}f(t_1,t_2,0)}{i^2t_1t_2}\,dt_1dt_2-\frac{1}{8\pi}\int_0^\infty \frac{\Delta_2f(0,t_2,0)}{it_2}\,dt_2\\ &+\frac{1}{8\pi}\int_0^\infty \frac{\Delta_1f(t_1,0,0)}{it_1}\,dt_1+\tfrac{1}{8}f(0,0,0).\end{aligned}$$

In the triple integral and in each of the single integrals the integrand is identically zero. Also $f(t_1,t_2,0)$ is the characteristic function of $(\mathscr{A},\mathscr{B})$—and similarly for $f(t_1,0,t_3)$ and $f(0,t_2,t_3)$—so we can evaluate the double integrals with the help of Lemma 2. Therefore, since $f(0,0,0) = 1$,

$$P(\infty,\infty,\infty) = \frac{1}{8}-\frac{1}{4\pi}\left\{\sin^{-1}\frac{\gamma_{23}}{bc}-\sin^{-1}\frac{\gamma_{13}}{ac}+\sin^{-1}\frac{\gamma_{12}}{ab}\right\}.$$

This is equivalent to the stated result because

$$\tfrac{1}{2}\pi-\sin^{-1}x = \sin^{-1}\sqrt{(1-x^2)}.$$

COROLLARY.

$\Pr(\mathscr{A}\mathscr{B}\mathscr{C} \leqslant 0, \mathscr{A}\mathscr{C} \geqslant 0)$

$$= \frac{1}{2\pi}\left\{\sin^{-1}\sqrt{\left(1-\frac{\gamma_{23}^2}{b^2c^2}\right)}-\sin^{-1}\sqrt{\left(1-\frac{\gamma_{13}^2}{a^2c^2}\right)}+\sin^{-1}\sqrt{\left(1-\frac{\gamma_{12}^2}{a^2b^2}\right)}\right\}.$$

Proof. This is immediate; for each g_n is distributed symmetrically about zero,

$$\Pr(\mathscr{A} \leqslant 0, \mathscr{B} \geqslant 0, \mathscr{C} \leqslant 0) = \Pr(\mathscr{A} \geqslant 0, \mathscr{B} \leqslant 0, \mathscr{C} \geqslant 0),$$

and the above probability is the sum of these two.

Each $g_n = g_n(t)$ is a continuous function of t $(0 \leqslant t \leqslant 1)$, and hence so are $\mathscr{A}$, $\mathscr{B}$, $\mathscr{C}$, $\mathscr{D}$. Therefore the functions $\mu_1(t)$ and $\mu_2(t)$, defined by

$$\mu_1(t) = \begin{cases} 1 & (\mathscr{A}\mathscr{B} \leqslant 0), \\ 0 & (\mathscr{A}\mathscr{B} > 0) \end{cases}$$

and

$$\mu_2(t) = \begin{cases} 1 & (\mathscr{C}\mathscr{D} \leqslant 0), \\ 0 & (\mathscr{C}\mathscr{D} > 0), \end{cases}$$

are measurable. Our immediate aim is to evaluate the integrals of $\mu_1(t)$, $\mu_2(t)$, and $\mu_1(t)\mu_2(t)$ respectively, taken over the interval $(0, 1)$. These integrals are the measures of the sets in which their integrands are unity, and are hence

$$\Pr(\mathscr{A}\mathscr{B} \leqslant 0), \quad \Pr(\mathscr{C}\mathscr{D} \leqslant 0), \quad \Pr(\mathscr{A}\mathscr{B} \leqslant 0, \mathscr{C}\mathscr{D} \leqslant 0),$$

respectively.

LEMMA 4.

$$\int_0^1 \mu_1(t)\,dt = \frac{1}{\pi}\sin^{-1}\sqrt{\left(1-\frac{\gamma_{12}^2}{a^2b^2}\right)}, \quad \int_0^1 \mu_2(t)\,dt = \frac{1}{\pi}\sin^{-1}\sqrt{\left(1-\frac{\gamma_{34}^2}{c^2d^2}\right)}.$$

Proof. From the foregoing remarks,

$$\begin{aligned} \int_0^1 \mu_1(t)\,dt &= \Pr(\mathscr{A} \geqslant 0, \mathscr{B} \leqslant 0) + \Pr(\mathscr{A} \leqslant 0, \mathscr{B} \geqslant 0) \\ &= 2\Pr(\mathscr{A} \geqslant 0, \mathscr{B} \leqslant 0). \end{aligned} \tag{3.1}$$

Let $f(t_1, t_2)$ be the characteristic function of $(\mathscr{A}, \mathscr{B})$, and let

$$P(p, q) = \Pr(0 \leqslant \mathscr{A} \leqslant p, -q \leqslant \mathscr{B} \leqslant 0).$$

Then

$$P(p, q) = -\frac{1}{4\pi^2}\int_{-\infty}^{\infty} K_1(-p)\,dt_1 \int_{-\infty}^{\infty} K_2(q) f(t_1, t_2)\,dt_2.$$

If we use Lemma 1 to evaluate the limits as q, p tend successively to infinity, we find that

$$\begin{aligned} P(\infty, \infty) = &-\frac{1}{4\pi^2}\iint_0^\infty \frac{\Delta_{12} f(t_1, t_2)}{i^2 t_1 t_2}\,dt_1 dt_2 \\ &+ \frac{1}{4\pi}\int_0^\infty \frac{\Delta_2 f(0, t_2)}{it_2}\,dt_2 - \frac{1}{4\pi}\int_0^\infty \frac{\Delta_1 f(t_1, 0)}{it_1}\,dt_1 \\ &+ \tfrac{1}{4} f(0, 0). \end{aligned} \tag{3.2}$$

But

$$f(t_1, t_2) = \exp\{-\tfrac{1}{2}(a^2 t_1^2 + 2\gamma_{12} t_1 t_2 + b^2 t_2^2)\},$$

and so obviously

$$\Delta_1 f(t_1, 0) = \Delta_2 f(0, t_2) = 0.$$

Since $f(0,0) = 1$ we have, from (3.2),

$$P(\infty,\infty) = \frac{1}{4} + \frac{1}{4\pi^2}\iint_0^\infty \frac{\Delta_{12} f(t_1,t_2)}{t_1 t_2}\, dt_1 dt_2$$

$$= \frac{1}{2\pi}\sin^{-1}\sqrt{\left(1-\frac{\gamma_{12}^2}{a^2 b^2}\right)}$$

by Lemma 2. But this is $\Pr(\mathscr{A} \geqslant 0, \mathscr{B} \leqslant 0)$, and so the result follows from (3.1).

The value of the other integral is found in a similar way.

LEMMA 5. *If $f(t_1,t_2,t_3,t_4)$ is the characteristic function of $(\mathscr{A},\mathscr{B},\mathscr{C},\mathscr{D})$ then*

$$\int_0^1 \mu_1(t)\mu_2(t)\, dt = -\frac{1}{4} + \frac{1}{2\pi}\sin^{-1}\sqrt{\left(1-\frac{\gamma_{12}^2}{a^2b^2}\right)} + \frac{1}{2\pi}\sin^{-1}\sqrt{\left(1-\frac{\gamma_{34}^2}{c^2d^2}\right)}$$

$$+\frac{1}{2\pi^4}\iiiint_0^\infty \frac{\Delta f(t_1,t_2,t_3,t_4)}{t_1t_2t_3t_4}\, dt_1 dt_2 dt_3 dt_4.$$

Proof. As we have seen, this integral is $\Pr(\mathscr{A}\mathscr{B} \leqslant 0, \mathscr{C}\mathscr{D} \leqslant 0)$. If p, q, r, s are all positive then

$$\Pr(0 \leqslant \mathscr{A} \leqslant p,\ -q \leqslant \mathscr{B} \leqslant 0,\ 0 \leqslant \mathscr{C} \leqslant r,\ -s \leqslant \mathscr{D} \leqslant 0) \tag{3.3}$$

is

$$P_1(p,q,r,s) = \frac{1}{16\pi^4}\int_{-\infty}^{\infty} K_1(-p)\, dt_1 \int_{-\infty}^{\infty} K_2(q)\, dt_2$$

$$\times \int_{-\infty}^{\infty} K_3(r)\, dt_3 \int_{-\infty}^{\infty} K_4(-s) f(t_1,t_2,t_3,t_4)\, dt_4. \tag{3.4}$$

We now use (3.2) to find the limit of (3.4) as r, s tend to infinity.

$$P_1(p,q,\infty,\infty) = \frac{1}{16\pi^4}\int_{-\infty}^{\infty} K_1(-p)\, dt_1 \int_{-\infty}^{\infty} K_2(q)\, dt_2$$

$$\times\left\{\iint_0^\infty \frac{\Delta_{34} f(t_1,t_2,t_3,t_4)}{i^2 t_3 t_4}\, dt_3\, dt_4 - \pi\int_0^\infty \frac{\Delta_3 f(t_1,t_2,t_3,0)}{it_3}\, dt_3\right.$$

$$\left. + \pi\int_0^\infty \frac{\Delta_4 f(t_1,t_2,0,t_4)}{it_4}\, dt_4 - \pi^2 f(t_1,t_2,0,0)\right\}.$$

This expression gives us $\Pr(0 \leqslant \mathscr{A} \leqslant p, -q \leqslant \mathscr{B} \leqslant 0, \mathscr{C} \leqslant 0, \mathscr{D} \leqslant 0)$. There is a similar expression, $P_2(p,q,\infty,\infty)$ say, for

$$\Pr(0 \leqslant \mathscr{A} \leqslant p, -q \leqslant \mathscr{B} \leqslant 0, \mathscr{C} \geqslant 0, \mathscr{D} \leqslant 0),$$

and it is easily seen from Lemma 1 that it differs from P_1 only in the signs that precede the single integrals in the brackets $\{\ \}$. Therefore

$$\begin{aligned}\Pr(0 \leqslant \mathscr{A} \leqslant p, -q \leqslant \mathscr{B} \leqslant 0, \mathscr{C}\mathscr{D} \leqslant 0)\\ &= P_1(p,q,\infty,\infty)+P_2(p,q,\infty,\infty)\\ &= -\frac{1}{8\pi^4}\int_{-\infty}^{\infty} K_1(-p)\,dt_1 \int_{-\infty}^{\infty} K_2(q) F(t_1,t_2)\,dt_2, \qquad (3.5)\end{aligned}$$

where

$$F(t_1,t_2) = \iint_0^\infty \frac{\Delta_{34} f(t_1,t_2,t_3,t_4)}{t_3 t_4}\,dt_3 dt_4 + \pi^2 f(t_1,t_2,0,0).$$

In (3.3), (3.4), and (3.5) we pass from an expression for

$$\Pr(\ldots, 0 \leqslant \mathscr{C} \leqslant r, -s \leqslant \mathscr{D} \leqslant 0)$$

to one for $\Pr(\ldots, \mathscr{C}\mathscr{D} \leqslant 0)$. In a similar way, we can start with (3.5), an expression for $\Pr(0 \leqslant \mathscr{A} \leqslant p, -q \leqslant \mathscr{B} \leqslant 0, \ldots)$, and form one for $\Pr(\mathscr{A}\mathscr{B} \leqslant 0, \ldots)$. The form of the latter may be seen by comparing (3.3) and (3.4) with (3.5) and applying the same metamorphosis, *mutatis mutandis*, to (3.5) itself. We have

$$\begin{aligned}\Pr(\mathscr{A}\mathscr{B} \leqslant 0, \mathscr{C}\mathscr{D} \leqslant 0) &= \frac{1}{4\pi^4}\iint_0^\infty \frac{\Delta_{12} F(t_1,t_2)}{t_1 t_2}\,dt_1 dt_2 + \tfrac{1}{4}F(0,0)\\ &= \frac{1}{4\pi^4}\iiiint_0^\infty \frac{\Delta_{1234} f(t_1,t_2,t_3,t_4)}{t_1 t_2 t_3 t_4}\,dt_1 dt_2 dt_3 dt_4\\ &\quad + \frac{1}{4\pi^2}\iint_0^\infty \frac{\Delta_{34} f(0,0,t_3,t_4)}{t_3 t_4}\,dt_3 dt_4\\ &\quad + \frac{1}{4\pi^2}\iint_0^\infty \frac{\Delta_{12} f(t_1,t_2,0,0)}{t_1 t_2}\,dt_1 dt_2 + \tfrac{1}{4} f(0,0,0,0).\end{aligned}$$

But $f(t_1,t_2,0,0)$ is the characteristic function of $(\mathscr{A},\mathscr{B})$, and $f(0,0,t_3,t_4)$ is that of $(\mathscr{C},\mathscr{D})$, and so the double integrals can be evaluated with the help of Lemma 2. Furthermore, $\Delta_{1234} = 2\Delta$ and $f(0,0,0,0) = 1$, and the result then follows.

4. In order to estimate the zeros of $\varphi(\theta)$ we divide them into two groups, (i) those lying in the intervals $(0,\varepsilon)$, $(\pi-\varepsilon, \pi+\varepsilon)$, and $(2\pi-\varepsilon, 2\pi)$, and (ii) those lying in the intervals $(\varepsilon, \pi-\varepsilon)$ and $(\pi+\varepsilon, 2\pi-\varepsilon)$. The zeros (i), which, it so happens, are negligible, will be estimated in § 10 by an easy application of Jensen's theorem. Those zeros which make a significant contribution to the final result are of type (ii), and their number is found by the method whose outlines we have already indicated.

The choice of ε is important. So that the type-(i) zeros *are* few in number (thus making a crude estimate adequate) it must not be too large; but if it is too small, some of the calculations concerning the type-(ii) zeros become rather complicated. It is convenient to take

$$1/\varepsilon \asymp N^{6/13}(\log N)^{4/13},$$

where by $\lambda(N) \asymp \nu(N)$ we mean that $\lambda = O(\nu)$ and $\nu = O(\lambda)$ (Hardy (**3**) 2).

We shall write

$$s(\theta) = 1+2\sum_{n=1}^{N} \cos 2n\theta = \frac{\sin(2N+1)\theta}{\sin\theta}.$$

It is obvious that, for $\nu = 0, 1, 2, \ldots,$

$$s^{(\nu)}(\theta) = O(N^{\nu+1}) \tag{4.1}$$

as $N \to \infty$, uniformly in θ for all θ.

Stricter inequalities can be obtained by restricting θ to the intervals $\varepsilon \leqslant \theta \leqslant \pi-\varepsilon$ and $\pi+\varepsilon \leqslant \theta \leqslant 2\pi-\varepsilon$. For since $|s(\theta)| \leqslant (\sin\varepsilon)^{-1}$, we then have

$$s(\theta) = O(1/\varepsilon) = O(N^{7/13}). \tag{4.2}$$

Further,

$$\begin{aligned} s'(\theta) &= (2N+1)\cos(2N+1)\theta \operatorname{cosec}\theta - s(\theta)\cot\theta \\ &= O(N^{20/13}), \end{aligned} \tag{4.3}$$

and similarly

$$s''(\theta) = O(N^{33/13}). \tag{4.4}$$

These relations are all uniform in θ, i.e. the constants implied by the O's are independent of θ.

5. We can now start applying the results of § 3 to the study of $\varphi(\theta)$. Suppose that $(\theta, \theta+\delta)$ is any interval not overlapping the ε-neighbourhoods of 0, π, and 2π, and let

$$\mathscr{A} = \varphi(\theta), \quad \mathscr{B} = \varphi(\theta+\tfrac{1}{2}\delta), \quad \mathscr{C} = \varphi(\theta+\delta).$$

Then the Corollary to Lemma 3, which gives $\Pr(\mathscr{A}\mathscr{B}\mathscr{C} \leqslant 0, \mathscr{A}\mathscr{C} \geqslant 0)$, provides us with the probability that φ has a double crossover in the interval $(\theta, \theta+\delta)$.

In the notation introduced at the beginning of § 3, we have

$$\left.\begin{aligned} a^2 &= \sum_{n=1}^{N} \cos^2 n\theta = \tfrac{1}{4}\{2N-1+s(\theta)\}, \\ b^2 &= \sum_{n=1}^{N} \cos^2 n(\theta+\tfrac{1}{2}\delta) = \tfrac{1}{4}\{2N-1+s(\theta+\tfrac{1}{2}\delta)\}, \\ c^2 &= \sum_{n=1}^{N} \cos^2 n(\theta+\delta) = \tfrac{1}{4}\{2N-1+s(\theta+\delta)\}, \\ \gamma_{13} &= \sum_{n=1}^{N} \cos n\theta \cos n(\theta+\delta) \\ &= \tfrac{1}{4}\{s(\theta+\tfrac{1}{2}\delta)+s(\tfrac{1}{2}\delta)-2\}, \\ \gamma_{23} &= \sum_{n=1}^{N} \cos n(\theta+\tfrac{1}{2}\delta)\cos n(\theta+\delta) \\ &= \tfrac{1}{4}\{s(\theta+\tfrac{3}{4}\delta)+s(\tfrac{1}{4}\delta)-2\}. \end{aligned}\right\} \tag{5.1}$$

These formulae enable us to apply Taylor's theorem to expand

$$\sqrt{(a^2c^2-\gamma_{13}{}^3)}/ac$$

and similar expressions in powers of δ. Suppose that

$$a^2c^2-\gamma_{13}{}^2 = \alpha_0+\alpha_1\delta+\tfrac{1}{2}\alpha_2\delta^2+\tfrac{1}{6}\alpha_3\delta^3+\tfrac{1}{24}\alpha_4\delta^4.$$

Straightforward calculations give

$$\begin{aligned} \alpha_0 &= \alpha_1 = 0, \\ \alpha_2 &= \tfrac{1}{8}a^2\{s''(\theta)-s''(0)\}-\tfrac{1}{32}\{s'(\theta)\}^2, \\ \alpha_3 &= \tfrac{3}{16}a^2s'''(\theta)-\tfrac{3}{64}s'(\theta)\{s''(\theta)+s''(0)\}. \end{aligned}$$

Lagrange's form for the remainder term gives

$$\alpha_4 = \left[\frac{d^4}{d\delta^4}(a^2c^2-\gamma_{13}{}^3)\right]_{\delta=\delta_1},$$

where $0 < \delta_1 < \delta$, and it is not difficult to show, using (4.1) and (5.1), that

$$\alpha_4 = O(N^6).$$

Therefore

$$\sqrt{(a^2c^2-\gamma_{13}{}^2)} = \delta\sqrt{(\tfrac{1}{2}\alpha_2)}\left\{1+\frac{\alpha_3\delta}{3\alpha_2}+O\left(\frac{N^6\delta^2}{\alpha_2}\right)\right\}^{1/2}. \tag{5.2}$$

Since the interval $(\theta, \theta+\delta)$ lies clear of the ε-neighbourhoods of 0, π, 2π, it follows from (4.1), (4.2), (4.3), (4.4), and (5.1) that

$$\left.\begin{aligned} a^2 &\asymp N, \\ \alpha_2 &\asymp N^4, \\ \alpha_3 &= O(N^5). \end{aligned}\right\} \tag{5.3}$$

Hence in the brackets on the right-hand side of (5.2), the coefficients of δ and δ^2 are $O(N)$ and $O(N^2)$ respectively. So if we impose the condition that $\delta = o(N^{-1})$ and apply the binomial expansion to (5.2), we find that

$$\sqrt{(a^2c^2-\gamma_{13}^2)} = \delta\sqrt{(\tfrac{1}{2}\alpha_2)}\left\{1+\frac{\alpha_3\delta}{6\alpha_2}+O(N^2\delta^2)\right\}.$$

We can further show that

$$\frac{1}{c} = \frac{1}{a}\left\{1-\frac{\delta}{8a^2}s'(\theta)+O(N^2\delta^2)\right\},$$

whence

$$\sqrt{\left(1-\frac{\gamma_{13}^2}{a^2c^2}\right)} = \frac{\delta}{a^2}\sqrt{(\tfrac{1}{2}\alpha_2)}\left\{1+\left(\frac{\alpha_3}{6\alpha_2}-\frac{s'(\theta)}{8a^2}\right)\delta+O(N^2\delta^2)\right\}. \tag{5.4}$$

If we replace δ by $\frac{1}{2}\delta$ in (5.4) we obtain

$$\sqrt{\left(1-\frac{\gamma_{12}^2}{a^2b^2}\right)} = \frac{\delta}{2a^2}\sqrt{(\tfrac{1}{2}\alpha_2)}\left\{1+\left(\frac{\alpha_3}{6\alpha_2}-\frac{s'(\theta)}{8a^2}\right)\frac{\delta}{2}+O(N^2\delta^2)\right\}; \tag{5.5}$$

and by an argument similar to that used to obtain (5.4) we can show that

$$\sqrt{\left(1-\frac{\gamma_{23}^2}{b^2c^2}\right)} = \frac{\delta}{2a^2}\sqrt{(\tfrac{1}{2}\alpha_2)}\left\{1+\left(\frac{\alpha_3}{4\alpha_2}-\frac{3s'(\theta)}{16a^2}\right)\delta+O(N^2\delta^2)\right\}. \tag{5.6}$$

We are now in a position to prove

LEMMA 6. *If the interval $(\theta, \theta+\delta)$ lies outside the ε-neighbourhoods of 0, π, and 2π, and if $\delta = o(N^{-1})$, the probability that $\varphi(\theta)$ has a double cross-over in this interval is $O(N^3\delta^3)$.*

Proof. As $x \to 0$,

$$\sin^{-1}x = x+O(x^3).$$

But we have seen that $\alpha_2 \asymp N^4$ and $a^2 \asymp N$, and hence from (5.4), (5.5), and (5.6), $\sqrt{(1-\gamma_{13}^2/a^2c^2)}$ and the other two similar expressions are each $O(N\delta)$ and therefore $o(1)$ as $N \to \infty$. Therefore

$$\sin^{-1}\sqrt{\left(1-\frac{\gamma_{13}^2}{a^2c^2}\right)} = \frac{\delta}{a^2}\sqrt{(\tfrac{1}{2}\alpha_2)}\left\{1+\left(\frac{\alpha_3}{6\alpha_2}-\frac{s'(\theta)}{8a^2}\right)\delta\right\}+O(N^3\delta^3),$$

$$\sin^{-1}\sqrt{\left(1-\frac{\gamma_{12}^2}{a^2b^2}\right)} = \frac{\delta}{2a^2}\sqrt{(\tfrac{1}{2}\alpha_2)}\left\{1+\left(\frac{\alpha_3}{6\alpha_2}-\frac{s'(\theta)}{8a^2}\right)\frac{\delta}{2}\right\}+O(N^3\delta^3),$$

and

$$\sin^{-1}\sqrt{\left(1-\frac{\gamma_{23}^2}{b^2c^2}\right)} = \frac{\delta}{2a^2}\sqrt{(\tfrac{1}{2}\alpha_2)}\left\{1+\left(\frac{\alpha_3}{4\alpha_2}-\frac{3s'(\theta)}{16a^2}\right)\delta\right\}+O(N^3\delta^3).$$

Now with the meanings given to $\mathscr{A}$, $\mathscr{B}$, and $\mathscr{C}$ at the beginning of this section, the probability of the double crossover is $\Pr(\mathscr{A}\mathscr{B}\mathscr{C} \leqslant 0, \mathscr{A}\mathscr{C} \geqslant 0)$. This is given by the Corollary to Lemma 3, and the required result then follows.

LEMMA 7. *If the interval $(\theta, \theta+\delta)$ lies outside the ε-neighbourhoods of 0, π, 2π, if $\mathscr{A}$, $\mathscr{B}$, $\mathscr{C}$ have their current meanings, if*

$$D = |\gamma_{rs}| = \begin{vmatrix} a^2 & \gamma_{12}, & \gamma_{13} \\ \gamma_{12}, & b^2, & \gamma_{23} \\ \gamma_{13}, & \gamma_{23}, & c^2 \end{vmatrix},$$

and if $\delta = o(N^{-1})$, then

$$\sum_{r,s=1}^{3} \frac{D_{rs}}{D} = O\left(\frac{1}{N}\right)$$

as $N \to \infty$, D_{rs} being the cofactor of γ_{rs} in D.

Proof. Denote $s'(\theta)$, $s''(\theta)$, ... by s_1, s_2, ..., and $s'(0)$, $s''(0)$, ... by σ_1, σ_2, ... If we apply Taylor's theorem to (5.1) we find that

$$b^2 = a^2 + \frac{\delta s_1}{8} + \frac{\delta^2 s_2}{32} + \frac{\delta^3 s_3}{192} + \frac{\delta^4 s_4}{1536} + O(N^6\delta^5),$$

$$c^2 = a^2 + \frac{\delta s_1}{4} + \frac{\delta^2 s_2}{8} + \frac{\delta^3 s_3}{24} + \frac{\delta^4 s_4}{96} + O(N^6\delta^5),$$

$$\gamma_{23} = a^2 + \frac{3\delta s_1}{16} + \frac{\delta^2}{128}(9s_2+\sigma_2) + \frac{27\delta^3 s_3}{1536} + \frac{\delta^4}{16.1536}(81s_4+\sigma_4) + O(N^6\delta^5),$$

$$\gamma_{13} = a^2 + \frac{\delta s_1}{8} + \frac{\delta^2}{32}(s_2+\sigma_2) + \frac{\delta^3 s_3}{192} + \frac{\delta^4}{1536}(s_4+\sigma_4) + O(N^6\delta^5),$$

$$\gamma_{12} = a^2 + \frac{\delta s_1}{16} + \frac{\delta^2}{128}(s_2+\sigma_2) + \frac{\delta^3 s_3}{1536} + \frac{\delta^4}{16.1536}(s_4+\sigma_4) + O(N^6\delta^5).$$

From this,

$$D_{11} = \tfrac{1}{8}\alpha_2\delta^2 + \tfrac{1}{16}\alpha_3\delta^3 + \lambda_{11}\delta^4 + O(N^7\delta^5),$$

$$D_{12} = -\tfrac{1}{4}\alpha_2\delta^2 - \tfrac{5}{16}\alpha_3\delta^3 + \lambda_{12}\delta^4 + O(N^7\delta^5),$$

$$D_{13} = \tfrac{1}{8}\alpha_2\delta^2 + \tfrac{1}{24}\alpha_3\delta^3 + \lambda_{13}\delta^4 + O(N^7\delta^5).$$

(α_2 and α_3 were introduced just before (5.2). The λ are certain functions of the s and σ whose exact forms are not required.)

Consider the λ. Since $s_r = o(N^{r+1})$ outside the ε-neighbourhoods of 0, π, 2π, and since

$$a^2 = \tfrac{1}{2}N + o(N), \quad \sigma_{2r} = \frac{(-)^r}{2r+1}(2N)^{2r+1} + O(N^{2r}),$$

it is easily seen that the λ are dominated by the terms involving $a^2\sigma_4$

and σ_2^2. We find in fact that

$$\lambda_{11}+\lambda_{12}+\lambda_{13} = \frac{\sigma_2^2}{32.128}+\frac{17a^2\sigma_4}{16.1536}+o(N^6)$$

$$= KN^6+o(N^6),$$

where K is a non-zero constant. We thus have

$$D_{11}+D_{12}+D_{13} = \{KN^6+o(N^6)\}\delta^4+O(N^7\delta^5)$$

$$= \{KN^6+o(N^6)\}\delta^4$$

if $\delta = o(N^{-1})$.

Now $D = a^2D_{11}+\gamma_{12}D_{12}+\gamma_{13}D_{13}$, and since

$$\gamma_{12} = a^2+\tfrac{1}{16}\delta s_1+O(N^3\delta^2)$$

and

$$\gamma_{13} = a^2+\tfrac{1}{8}\delta s_1+O(N^3\delta^2)$$

we have

$$D = a^2(D_{11}+D_{12}+D_{13})+O(N^6\delta^4)$$

$$= \{\tfrac{1}{2}KN^7+o(N^7)\}\delta^4.$$

Therefore $(D_{11}+D_{12}+D_{13})/D = O(N^{-1})$.

We can also show, in much the same way as above, that $D_{21}+D_{22}+D_{23}$ and $D_{31}+D_{32}+D_{33}$ are each $O(N^6\delta^4)$, and so the conclusion follows.

LEMMA 8. *Let $P(k) = \Pr(\mathscr{A} \geqslant k, \mathscr{B} \leqslant k, \mathscr{C} \geqslant k)$, where $\mathscr{A}$, $\mathscr{B}$, $\mathscr{C}$ have their usual meaning. If $k > 0$ and $\delta = o(N^{-1})$, then*

$$P(k) \leqslant Be^{k^4}\delta^{3/2}$$

when N is large, B depending only upon N.

Proof. By standard probability theory (see, for example, Cramér (**1**) Theorem 32),

$$P(k) = \frac{1}{\sqrt{(8\pi^3D)}}\iiint_S \exp\left(-\frac{1}{2}\sum_{r,s=1}^{3}\frac{D_{rs}}{D}x_rx_s\right)dx_1dx_2dx_3,$$

where S is the region ($x_1 \geqslant k$, $x_2 \leqslant k$, $x_3 \geqslant k$) and D is the determinant of Lemma 7. Hence, by a simple transformation,

$$P(k) = \frac{1}{\sqrt{(8\pi^3D)}}\iiint_T \exp\left\{-\frac{1}{2}\sum_{r,s=1}^{3}\frac{D_{rs}}{D}(x_r+k)(x_s+k)\right\}dx_1dx_2dx_3,$$

where T is the region ($x_1 \geqslant 0$, $x_2 \leqslant 0$, $x_3 \geqslant 0$). Now

$$\sum D_{rs}(x_r+k)(x_s+k) = \tfrac{1}{2}\sum D_{rs}x_rx_s-k^2\sum D_{rs}+\tfrac{1}{2}\sum D_{rs}(x_r+2k)(x_s+2k);$$

and using Schwarz's inequality, with Lemma 7 to deal with the term $-k^2 \sum D_{rs}$, we have†

$$P^2(k) \leqslant \frac{\exp(Ak^2/N)}{8\pi^3 D} \iiint_T \exp\left(-\tfrac{1}{2}\sum \frac{D_{rs}}{D} x_r x_s\right) dx_1 dx_2 dx_3$$

$$\times \iiint_T \exp\left\{-\tfrac{1}{2}\sum \frac{D_{rs}}{D}(x_r+2k)(x_s+2k)\right\} dx_1 dx_2 dx_3$$

$$\leqslant \frac{\exp(Ak^2/N)}{\sqrt{(8\pi^3 D)}} P(0) \iiint_{-\infty}^{\infty} \exp\left\{-\tfrac{1}{2}\sum \frac{D_{rs}}{D}(x_r+2k)(x_3+2k)\right\} dx_1 dx_2 dx_3$$

$$= \exp(Ak^2/N) P(0).$$

But, by Lemma 6, $P(0) = O(N^3\delta^3)$, and the result follows.

6. We have already remarked that the probability that $\varphi(\theta)$ has at least two zeros in a given interval is at least equal to the probability that it has a double crossover in that interval; but it is also possible, as we shall see in this section, to use the double crossover idea to set an *upper* bound to this probability. The situation is complicated if there are any zeros of even order, when the graph of $y = \varphi(\theta)$ touches the θ-axis without actually crossing it, and so as a preliminary step we show that zeros of this sort may be ignored.

LEMMA 9. *Suppose that an interval $\mathscr{I}$ on the θ-axis is bisected, and each of these halves bisected, and so on indefinitely. If there is no double crossover in any of the infinite sequence of intervals so obtained, the curve $y = \varphi(\theta)$ can cross the θ-axis at most once in $\mathscr{I}$. It may in addition touch the axis.*

Proof. The only property of $\varphi(\theta)$ that we use here is its continuity.

By hypothesis, $\varphi(\theta)$ does not have a double crossover in $\mathscr{I}$ itself. Suppose first then that $\varphi \geqslant 0$ at the two end-points of $\mathscr{I}$ and also at its mid-point. (We cannot of course have $\varphi = 0$ at all three points since this would give a double crossover immediately.) Then it is easy to see that if there is never to be a double crossover we must have $\varphi \geqslant 0$ at all later points of bisection. But these points are dense in $\mathscr{I}$, and φ is continuous, and therefore $\varphi \geqslant 0$ throughout $\mathscr{I}$—its graph never crosses the θ-axis at all.

The only case that differs essentially from the one above is as follows. We let the first bisection divide $\mathscr{I}$ into intervals i_1 and i_2, and we suppose that $\varphi \geqslant 0$ at the end-points of i_1, and that $\varphi \leqslant 0$ at that end-point of i_2 which is also an end-point of $\mathscr{I}$, again excluding the possibility that $\varphi = 0$ at all three points. As in the first case, the graph of φ cannot

† A is the absolute constant implied by the O of Lemma 7.

actually cross the θ-axis in i_1; if it does so at all, it must be in i_2. But a diagram will quickly show that i_2 is in the same position as $\mathscr{I}$ itself: when i_2 is bisected, the graph of φ can cross the θ-axis in only one of the halves so generated. And so on; the intervals (taken as closed) in which the graph may cross the axis contract about a unique point, and this is the only point where the graph of φ can cross the θ-axis.

COROLLARY. *If the curve $y = \varphi(\theta)$ cuts the θ-axis at least twice in $\mathscr{I}$, there must be a double crossover in at least one of the subintervals obtained by repeatedly bisecting $\mathscr{I}$. More generally, if the curve cuts the line $y = k$ at least twice in $\mathscr{I}$, $\varphi(\theta)$ must have a double crossover relative to the line $y = k$ in at least one of the subintervals of $\mathscr{I}$.*

LEMMA 10. *The probability that $\varphi(\theta)$ has any even-order zeros in either of the intervals $(\varepsilon, \pi-\varepsilon)$, $(\pi+\varepsilon, 2\pi-\varepsilon)$ is zero.*

Proof. Let the two intervals be divided into equal subintervals each of length $\delta = o(N^{-1})$. The number of these δ-intervals is finite, so it is sufficient to prove that there is a zero chance of an even-order zero occurring in any given one of the δ-intervals. We take one of these as the $\mathscr{I}$ of Lemma 9. If there is an even-order zero here, the curve $y = \varphi(\theta)$ touches the θ-axis either from above or else from below, and we consider just the first case.

If $\varphi(\theta)$ has a zero of even order in $\mathscr{I}$, then for all sufficiently small positive values of k the curve $y = \varphi(\theta)$ must cross the line $y = k$ at least twice; and so by the Corollary to Lemma 9, if $\mathscr{I}$ is repeatedly bisected $\varphi(\theta)$ must have a double crossover relative to $y = k$ in at least one of the intervals so generated. Therefore

$$\Pr(\text{even-order zero}) \leqslant \Pr(\text{at least one d.c.o. } re\ y = k).$$

This is true for all small k, and it is therefore sufficient to show that the right-hand side of this inequality tends to zero as $k \to 0$.

Since we are assuming that $y = \varphi(\theta)$ touches the θ-axis from above, one of the double crossovers that we have just shown to exist will be of the sort where $\varphi \geqslant k$ at the ends and $0 \leqslant \varphi \leqslant k$ at the middle of the interval concerned. Let $P(k, \theta, \delta)$ be the probability of this sort of double crossover occurring in the interval $(\theta, \theta+\delta)$. Clearly $P(k, \theta, \delta) \leqslant P(k)$, where $P(k)$ was defined in Lemma 8, and so if we suppose that $k < 1$, we have by that lemma

$$P(k, \theta, \delta) \leqslant B\delta^{3/2}, \tag{6.1}$$

where B depends only upon N and is independent of k and θ.

At the first bisection of $\mathscr{I}$ two intervals are generated, at the second bisection four, and so on. We can arrange these blocks of intervals as a sequence of intervals whose left-hand end-points are θ_1, θ_2, ..., say. The nth block, created at the nth bisection, contains 2^n intervals each of length $2^{-n}\delta$, and for any one of these

$$P(k, \theta_\nu, \delta_\nu) \leqslant B(2^{-n}\delta)^{3/2}$$

by (6.1). Adding these 2^n probabilities, we have

$$\sum_\nu P(k, \theta_\nu, \delta_\nu) \leqslant B2^{-\frac{1}{2}n}\delta^{3/2}.$$

Hence, summing over *all* the intervals θ_ν,

$$\begin{aligned}\sum_{\nu=1}^{\infty} P(k, \theta_\nu, \delta_\nu) &\leqslant B\sum_{\nu=1}^{\infty} \delta_\nu^{3/2} \\ &\leqslant B\delta^{3/2}\sum_{n=1}^{\infty} 2^{-\frac{1}{2}n}.\end{aligned}$$

Thus the infinite series of probabilities converges. Now

$$\Pr(\text{at least 1 d.c.o. } re\ y = k) \leqslant \sum_{\nu=1}^{\infty} P(k, \theta_\nu, \delta_\nu),$$

and so

$$\begin{aligned}\Pr(\text{even-order zero}) &\leqslant \lim_{k\to 0}\sum_{\nu=1}^{\infty} P(k, \theta_\nu, \delta_\nu) \\ &= \sum_{\nu=1}^{\infty}\lim_{k\to\infty} P(k, \theta_\nu, \delta_\nu)\end{aligned}$$

by Tannery's theorem, which applies since the B of (6.1) is independent of k. Finally, for any k, θ, and δ,

$$\begin{aligned}P(k, \theta, \delta) &= \Pr(\mathscr{A} \geqslant k, 0 \leqslant \mathscr{B} \leqslant k, \mathscr{C} \geqslant k) \\ &\leqslant \Pr(0 \leqslant \mathscr{B} \leqslant k);\end{aligned}$$

and since $\mathscr{B}$ is normally distributed its distribution function is continuous. Hence $P(k, \theta, \delta) \to 0$ as $k \to 0$, and the proof is complete.

LEMMA 11. *Provided that the interval $\mathscr{I}$, of length $\delta = o(N^{-1})$, lies outside the ε-neighbourhoods of 0, π, and 2π, the probability that $\varphi(\theta)$ has at least two zeros (counted according to their multiplicity) in $\mathscr{I}$ is $O(N^3\delta^3)$.*

Proof. Let $\mathscr{I}$ be repeatedly bisected as in Lemma 9. If none of the subintervals thus generated has a double crossover associated with it, that lemma shows that $\varphi(\theta)$ has at most one odd-order zero in $\mathscr{I}$; it may, in addition, have any number of zeros of even order. Therefore

$$\begin{aligned}\Pr(\text{no d.c.o.}) &\leqslant \Pr(\text{at most 1 odd-order zero}) + \Pr(\text{even-order zeros}), \\ &= \Pr(\text{at most 1 odd-order zero}) \qquad (6.2)\end{aligned}$$

by Lemma 10. Hence, from (6.2),

$$\begin{aligned}\Pr(\text{at least 1 d.c.o.}) &\geqslant \Pr(\text{at least 2 odd-order zeros})\\ &= \Pr(\text{at least 2 odd-order zeros})\\ &\qquad + \Pr(\text{even-order zeros})\end{aligned}$$

by Lemma 10 again. Thus

$$\Pr(\text{at least 1 d.c.o.}) \geqslant \Pr(\text{at least 2 zeros}), \tag{6.3}$$

where the zeros are counted according to their multiplicity.

Now at the nth bisection of $\mathscr{I}$, 2^n subintervals are generated, each of length $2^{-n}\delta$, and by Lemma 6 the probability that a given one of these has a double crossover associated with it is $O(N^3(2^{-n}\delta)^3)$. Therefore the probability that there is a double crossover associated with at least one of these 2^n intervals does not exceed $AN^3\delta^3/4^n$, where A is some absolute constant. Finally, the probability that there is a double crossover associated with at least one of the whole infinite sequence of subintervals does not exceed

$$AN^3\delta^3 \sum_0^\infty 4^{-n} = O(N^3\delta^3).$$

The lemma now follows immediately from (6.3).

Again let $\mathscr{I}$ be any interval of length $\delta = o(N^{-1})$ which does not overlap any of the ε-neighbourhoods of 0, π, and 2π. We define $N(t)$ to be the number of zeros that $\varphi(\theta, t)$ has in $\mathscr{I}$, the zeros being counted according to their multiplicity; and we further define

$$N^*(t) = \begin{cases} N(t) & \text{if} \quad N(t) \geqslant 2,\\ 0 & \text{otherwise.}\end{cases}$$

LEMMA 12.

$$\int_0^1 N^*(t)\,dt = O(N^3\delta^3).$$

Proof. Let q_k be the probability that $\varphi(\theta)$ has at least k zeros in $\mathscr{I}$. Suppose that $k > 2$, and let p be an integer such that

$$2^p < k \leqslant 2^{p+1}.$$

Now if $\varphi(\theta)$ has at least k zeros in $\mathscr{I}$, and if $\mathscr{I}$ is divided into 2^p equal parts, at least one of these parts must contain two or more zeros. By Lemma 11, the probability that this will be the case does not exceed

$$A2^p\{N^3(2^{-p}\delta)^3\} < AN^3\delta^3k^{-2},$$

and so

$$q_k < AN^3\delta^3k^{-2}.$$

Let p_k be the probability that $\varphi(\theta)$ has exactly k zeros in $\mathscr{I}$, so that $p_k = q_k - q_{k+1}$. The integral we require is the mean of a random variable which takes the values 2, 3, ... with probabilities $p_2, p_3, \ldots$, and therefore

$$\int_0^1 N^*(t)\,dt = \sum_{k=2}^{\infty} kp_k = \sum_{k=2}^{\infty} q_k$$

$$< AN^3\delta^3 \sum_{k=2}^{\infty} k^{-2}$$

$$= AN^3\delta^3.$$

This is the desired result.

7. We come now to the third calculation mentioned at the beginning of § 2. We divide the interval $(\varepsilon, \pi-\varepsilon)$ into equal subintervals of length δ, and estimate the probability that $\varphi(\theta)$ shall simultaneously have a single crossover in each of a given pair of these δ-intervals. If the intervals are close enough, a very rough calculation is sufficient; but when the intervals are widely spaced, in a sense that we shall define in a moment, a more careful technique is needed which we develop in §§ 7, 8. We shall find in the end what we might have suspected at the beginning, that the probabilities associated with widely spaced intervals are almost independent.

Let

$$\lambda = \lambda(N) = N^{14/39}(\log N)^{-7/13}, \tag{7.1}$$

which is increasing for large enough values of N. The total number of δ-intervals into which $(\varepsilon, \pi-\varepsilon)$ is divided is obviously $(\pi-2\varepsilon)/\delta$; we choose δ so that

$$(\pi-2\varepsilon)/\delta = [N\lambda^{3/7}],$$

where the square brackets indicate that the integral part is to be taken. Then

$$\delta \asymp (N\lambda^{3/7})^{-1} = o(N^{-1}).$$

We also write $K = N^{1/3}\lambda$, so that

$$N\delta \asymp \lambda^{-3/7}, \quad K\delta \asymp \lambda^{4/7}N^{-2/3}.$$

Let $(\theta, \theta+\delta)$ and $(\theta+\tau, \theta+\tau+\delta)$ be two of the δ-intervals into which $(\varepsilon, \pi-\varepsilon)$ is divided so that $\tau > K\delta$ (this is the sense in which they are to be 'widely spaced'), and let

$$\mathscr{A} = \varphi(\theta), \quad \mathscr{B} = \varphi(\theta+\delta), \quad \mathscr{C} = \varphi(\theta+\tau), \quad \mathscr{D} = \varphi(\theta+\tau+\delta).$$

From (5.1),

$$\begin{aligned}\gamma_{13} &= \tfrac{1}{4}\{s(\theta+\tfrac{1}{2}\tau)+s(\tfrac{1}{2}\tau)-2\}\\ &= O(\varepsilon^{-1}) = O(K^{-1}\delta^{-1})\end{aligned}$$

from the value of ε specified in § 4. We have seen in (5.3) that $a^2 \asymp N$, and the same is true of c^2; hence $ac \asymp N$, which gives

$$\gamma_{13}/ac = O(N^{-1}K^{-1}\delta^{-1}) = O(N^{-1/3}\lambda^{-4/7}). \tag{7.2}$$

Similarly, γ_{14}/ad, γ_{23}/bc, and γ_{24}/bd all satisfy the inequality (7.2), for they are all associated with pairs of widely spaced points.

In § 5, $\varphi(\theta+\delta)$ was denoted by $\mathscr{C}$, but here we use $\mathscr{B}$ for this purpose. Allowing for this change of notation, it follows from (5.4) that

$$\begin{aligned}1-\frac{\gamma_{12}^2}{a^2b^2} &= \frac{\alpha_2\delta^2}{2a^4}\{1+O(N\delta)\}\\ &\asymp N^2\delta^2 \asymp \lambda^{-6/7}.\end{aligned} \tag{7.3}$$

Just as $\mathscr{A}$ and $\mathscr{B}$ relate to a pair of points whose distance apart is only δ, so do $\mathscr{C}$ and $\mathscr{D}$. Therefore

$$1-\frac{\gamma_{34}^2}{c^2d^2} \asymp \lambda^{-6/7}. \tag{7.4}$$

8. Let $\mathscr{A}$, $\mathscr{B}$, $\mathscr{C}$, $\mathscr{D}$ have the same meanings as in § 7, the δ-intervals with which they are associated being widely spaced and contained in the interval $(\varepsilon, \pi-\varepsilon)$. The characteristic function of the random vector $(\mathscr{A}, \mathscr{B}, \mathscr{C}, \mathscr{D})$ is

$$\begin{aligned}f(t_1,t_2,t_3,t_4) = \exp\{-\tfrac{1}{2}(a^2t_1^2+b^2t_2^2+c^2t_3^2 &\\ +d^2t_4^2+2\gamma_{12}t_1t_2+2\gamma_{13}t_1t_3+2\gamma_{14}t_1t_4 &\\ +2\gamma_{23}t_2t_3+2\gamma_{24}t_2t_4+2\gamma_{34}t_3t_4)\},&\end{aligned}$$

and it is convenient to write

$$\begin{aligned}\psi(t_1,t_2,t_3,t_4) = \exp\{-\tfrac{1}{2}(t_1^2+t_2^2+t_3^2+t_4^2 &\\ +2\gamma_1t_1t_2+2\gamma_2t_1t_3+2\gamma_3t_1t_4+2\gamma_4t_2t_3 &\\ +2\gamma_5t_2t_4+2\gamma_6t_3t_4)\},&\end{aligned}$$

where

$$\left.\begin{aligned}\gamma_1 = \frac{\gamma_{12}}{ab}, \quad \gamma_2 = \frac{\gamma_{13}}{ac}, \quad \gamma_3 = \frac{\gamma_{14}}{ad},\\ \gamma_4 = \frac{\gamma_{23}}{bc}, \quad \gamma_5 = \frac{\gamma_{24}}{bd}, \quad \gamma_6 = \frac{\gamma_{34}}{cd},\end{aligned}\right\} \tag{8.1}$$

and, further, to let ψ^* denote ψ with γ_1, γ_2, γ_4 replaced by zero.

LEMMA 13.

$$\iiiint_0^\infty \frac{\Delta\psi^*(t_1,t_2,t_3,t_4)}{t_1t_2t_3t_4}\,dt_1dt_2dt_3dt_4 = O(N^{-2/3}\lambda^{-8/7}).$$

Proof. Let $I(u)$ denote this integral with γ_6 replaced by u, so that it is $I(\gamma_6)$ that is required. Now when $u = 0$, $\Delta = \Delta_{234}$ operates on an even function of t_3, which makes the integrand identically zero. Hence $I(0) = 0$, and so

$$I(\gamma_6) = \int_0^{\gamma_6} I'(u)\,du.$$

Now

$$\frac{d}{du}\iint_0^\infty \frac{\Delta_{34}\exp(-ut_3t_4-\gamma_3t_1t_4-\gamma_5t_2t_4)}{t_3t_4}\,dt_3dt_4$$

$$= -\iint_{-\infty}^\infty \exp(-ut_3t_4-\gamma_3t_1t_4-\gamma_5t_2t_4)\,dt_3dt_4,$$

and so

$$I'(u) = -\iint_0^\infty \exp\{-\tfrac{1}{2}(t_1^2+t_2^2)\}\frac{dt_1dt_2}{t_1t_2}$$

$$\times\Delta_2\iint_{-\infty}^\infty \exp\{-\tfrac{1}{2}(t_3^2+t_4^2+2\gamma_3t_1t_4+2\gamma_5t_2t_4+2ut_3t_4)\}\,dt_3dt_4$$

$$= -\surd(2\pi)\iint_0^\infty \exp\{-\tfrac{1}{2}(t_1^2+t_2^2)\}\frac{dt_1dt_2}{t_1t_2}$$

$$\times\Delta_2\int_{-\infty}^\infty \exp\{-\tfrac{1}{2}t_3^2+\tfrac{1}{2}(\gamma_3t_1+\gamma_5t_2+ut_3)^2\}\,dt_3.$$

But

$$\int_{-\infty}^\infty \exp\{-\tfrac{1}{2}t_3^2+\tfrac{1}{2}(\gamma_3t_1+\gamma_5t_2+ut_3)^2\}\,dt_3$$

$$= \sqrt{\left(\frac{2\pi}{1-u^2}\right)}\exp\left\{\frac{(\gamma_3t_1+\gamma_5t_2)^2}{2(1-u^2)}\right\},$$

and therefore

$$I'(u) = -\frac{4\pi}{\surd(1-u^2)}\iint_0^\infty \exp\left\{-\frac{1}{2}\,\frac{(1-\gamma_3^2)t_1^2+(1-\gamma_5^2)t_2^2}{1-u^2}\right\}\sinh\left(\frac{\gamma_3\gamma_5t_1t_2}{1-u^2}\right)\frac{dt_1dt_2}{t_1t_2}$$

$$= -\frac{2\pi^2}{\surd(1-u^2)}\sin^{-1}\frac{\gamma_3\gamma_5}{\surd\{(1-\gamma_3^2)(1-\gamma_5^2)\}}.$$

By Cauchy's inequality, $|\gamma_{34}| \leqslant cd$, and so $|\gamma_6| \leqslant 1$ from (8.1). Therefore

$$|I(\gamma_6)| \leqslant .2\pi^2\left|\sin^{-1}\frac{\gamma_3\gamma_5}{\surd\{(1-\gamma_3^2)(1-\gamma_5^2)\}}\right|\int_0^1(1-u^2)^{-1/2}\,du,$$

and the result now follows, since, by (7.2) and (8.1), γ_3 and γ_5 are each $O(N^{-1/3}\lambda^{-4/7})$ which is itself $o(1)$.

LEMMA 14.

$$\frac{1}{2\pi^4}\iiiint_0^\infty \frac{\Delta f(t_1,t_2,t_3,t_4)}{t_1t_2t_3t_4}\,dt_1dt_2dt_3dt_4$$

$$= \frac{1}{4}-\frac{1}{2\pi}\left\{\sin^{-1}\sqrt{\left(1-\frac{\gamma_{12}^2}{a^2b^2}\right)}+\sin^{-1}\sqrt{\left(1-\frac{\gamma_{34}^2}{c^2d^2}\right)}\right\}$$

$$+\frac{1}{\pi^2}\sin^{-1}\sqrt{\left(1-\frac{\gamma_{12}^2}{a^2b^2}\right)}\sin^{-1}\sqrt{\left(1-\frac{\gamma_{34}^2}{c^2d^2}\right)}+O(\lambda^{1/7}N^{-2/3}).$$

Proof. An obvious change of variable shows that this result is equivalent to

$$\frac{1}{2\pi^4}\iiiint_0^\infty \frac{\Delta\psi(t_1,t_2,t_3,t_4)}{t_1t_2t_3t_4}\,dt_1dt_2dt_3dt_4$$

$$= \frac{1}{4}-\frac{1}{2\pi}\{\sin^{-1}\sqrt{(1-\gamma_1^2)}+\sin^{-1}\sqrt{(1-\gamma_6^2)}\}$$

$$+\frac{1}{\pi^2}\sin^{-1}\sqrt{(1-\gamma_1^2)}\sin^{-1}\sqrt{(1-\gamma_6^2)}+O(\lambda^{1/7}N^{-2/3}). \tag{8.2}$$

Let $\bar{\psi}$ denote ψ with γ_1 and γ_6 replaced by $\gamma_1 u$ and $\gamma_6 v$ respectively, and let

$$I(u,v) = \iiiint_0^\infty \frac{\Delta\bar{\psi}}{t_1t_2t_3t_4}\,dt_1dt_2dt_3dt_4. \tag{8.3}$$

Then

$$\frac{\partial^2 I}{\partial u\partial v} = \tfrac{1}{2}\gamma_1\gamma_6\iiiint_{-\infty}^\infty \bar{\psi}\,dt_1dt_2dt_3dt_4$$

$$= 2\pi^2\gamma_1\gamma_6 D^{-1/2},$$

where

$$D = \begin{vmatrix} 1 & \gamma_1 u & \gamma_2 & \gamma_3 \\ \gamma_1 u & 1 & \gamma_4 & \gamma_5 \\ \gamma_2 & \gamma_4 & 1 & \gamma_6 v \\ \gamma_3 & \gamma_5 & \gamma_6 v & 1 \end{vmatrix}.$$

If we now integrate over the square defined by $0 \leqslant u \leqslant 1$, $0 \leqslant v \leqslant 1$, we find that

$$I(1,1)-I(1,0)-I(0,1)+I(0,0) = 2\pi^2\gamma_1\gamma_6\iint_0^1 D^{-1/2}\,dudv. \tag{8.4}$$

The determinant D can be expanded in Laplace's form to give

$$D = (1-\gamma_1^2u^2)(1-\gamma_6^2v^2)+\eta,$$

where η consists of five terms, a typical one being $(\gamma_1\gamma_4 u-\gamma_2)(\gamma_2-\gamma_3\gamma_6 v)$. From (7.2), the sentence which follows it, and (8.1), γ_2, γ_3, γ_4, and γ_5 are each $O(N^{-1/3}\lambda^{-4/7})$, and so, since $0 \leqslant u \leqslant 1$, $0 \leqslant v \leqslant 1$,

$$\eta = O(N^{-2/3}\lambda^{-8/7}).$$

Thus

$$D = (1+k-\gamma_1^2u^2)(1-\gamma_6^2v^2),$$

where

$$k = k(u,v) = \frac{\eta}{1-\gamma_6^2v^2} = O(N^{-2/3}\lambda^{-2/7})$$

since $1-\gamma_6^2 \asymp \lambda^{-6/7}$ by (7.4). Therefore

$$\iint_0^1 D^{-1/2}\,dudv = \iint_0^1 \{(1+k-\gamma_1^2u^2)(1-\gamma_6^2v^2)\}^{-1/2}\,dudv.$$

In this last integral, k is a bounded continuous function of u, v, the bounds being $O(N^{-2/3}\lambda^{-2/7})$. Hence there must be a k of the same order of magnitude and independent of u, v which gives the integral the same value as before. Using this constant k, we have

$$\iint_0^1 D^{-1/2}\,dudv = (\gamma_1\gamma_6)^{-1}\sin^{-1}\gamma_6\sin^{-1}\{\gamma_1(1+k)^{-1/2}\}. \tag{8.5}$$

When we remember that $1-\gamma_6^2 \asymp \lambda^{-6/7}$ and that $k = O(N^{-2/3}\lambda^{-6/7})$, it is easy to show that

$$\sin^{-1}\{\gamma_1(1+k)^{-1/2}\} = \sin^{-1}\gamma_1+O(\lambda^{1/7}N^{-2/3}),$$

and it then follows by (8.4) and (8.5) that

$$\begin{aligned} &I(1,1)-I(1,0)-I(0,1)+I(0,0)\\ &\quad= \tfrac{1}{2}\pi^4-\pi^3\{\sin^{-1}\surd(1-\gamma_1^2)+\sin^{-1}\surd(1-\gamma_6^2)\}\\ &\qquad+2\pi^2\sin^{-1}\surd(1-\gamma_1^2)\sin^{-1}\surd(1-\gamma_6^2)+O(\lambda^{1/7}N^{-2/3}). \end{aligned} \tag{8.6}$$

It is clear from (8.2) and (8.3) that the integral we require is given by $I(1,1)$, and it is also clear from (8.6) that we shall obtain the desired value for it if we can show that $I(1,0)$, $I(0,1)$, and $I(0,0)$ are all $O(\lambda^{1/7}N^{-2/3})$. These three quantities may be regarded as special cases of $I(1,1)$ where either or both of γ_1 and γ_6 are zero, and we deal with them by modifying the previous argument, the crux of which is the removal of $t_1t_2t_3t_4$ in the denominator of the integrand in (8.2) by suitable differentiation.

Let us assume that γ_2, γ_3, γ_4, γ_5 are all $O(N^{-1/3}\lambda^{-4/7})$ as before, and that either or both of γ_1, γ_6 are zero. Let $J(u,v)$ denote the integral in (8.2) with γ_2, γ_5 replaced by $\gamma_2 u$, $\gamma_5 v$ respectively. As before,

$$\frac{\partial^2 J}{\partial u\partial v} = 2\pi^2\gamma_2\gamma_5 D^{-1/2}, \tag{8.7}$$

where this time

$$D = \begin{vmatrix} 1 & \gamma_1 & \gamma_2 u & \gamma_3 \\ \gamma_1 & 1 & \gamma_4 & \gamma_5 v \\ \gamma_2 u & \gamma_4 & 1 & \gamma_6 \\ \gamma_3 & \gamma_5 v & \gamma_6 & 1 \end{vmatrix}.$$

Now

$$D = 1+\gamma_1^2+\gamma_6^2+\gamma_1^2\gamma_6^2+\eta,$$

where η consists of twelve terms, each one being a product of u, v, and the γ's. Clearly each term of η contains at least one of $\gamma_2, \gamma_3, \gamma_4, \gamma_5$, each of which satisfies $|\gamma| \leqslant 1$ as may be seen by applying Cauchy's inequality to (8.1). If, therefore, $0 \leqslant u \leqslant 1$ and $0 \leqslant v \leqslant 1$, it follows that

$$\eta = O(N^{-1/3}\lambda^{-4/7}) = o(1).$$

If *both* γ_1 and γ_6 are zero, we have immediately

$$D = 1+o(1).$$

If, on the other hand, only one of γ_1, γ_6 is zero ($\gamma_6 = 0$, say), then by (7.3) and (8.1),

$$1-\gamma_1^2 = O(\lambda^{-6/7}) = o(1),$$

and so

$$D = 2+o(1).$$

In either case, if N is large enough $D^{-1/2}$ is bounded for $0 \leqslant u \leqslant 1$, $0 \leqslant v \leqslant 1$, thus giving

$$\iint_0^1 D^{-1/2}\,du\,dv = O(1).$$

Hence, from (8.7),

$$\begin{aligned} J(1,1)-J(1,0)-J(0,1)+J(0,0) &= 2\pi^2\gamma_2\gamma_5\iint_0^1 D^{-1/2}\,du\,dv \\ &= O(N^{-2/3}\lambda^{-8/7}). \qquad (8.8) \end{aligned}$$

We now indicate the dependence of the integral in (8.2) on the γ's by writing it as $I(\gamma_1,\gamma_2,\gamma_3,\gamma_4,\gamma_5,\gamma_6)$, or $I(123456)$ for brevity; we write $I(0,\gamma_2,\ldots)$ as $I(02\ldots)$, etc. In (8.6) we have found an expression for

$$I(123456)-I(023456)-I(123450)+I(023450). \qquad (8.9)$$

To achieve our aim, the evaluation of $I(123456)$, we have seen that it is sufficient to show that the last three terms of (8.9) are each $O(N^{-2/3}\lambda^{-8/7})$.

Consider $I(023456)$. In (8.8) we have shown that

$$I(023456)-I(003456)-I(023406)+I(003406) = O(N^{-2/3}\lambda^{-8/7}). \qquad (8.10)$$

To obtain this, we applied our differentiation technique to γ_2 and γ_5, but we could equally well have applied it to γ_3 and γ_4; for they are both $O(N^{-1/3}\lambda^{-4/7})$ like γ_2 and γ_5, and the differentiation would still have removed the $t_1t_2t_3t_4$ in the denominator of the integrand. Thus with (r, s) being $(2, 5)$ or $(3, 4)$ we have

$$I(\ldots r\ldots s\ldots)-I(\ldots 0\ldots s\ldots)-I(\ldots r\ldots 0\ldots)+I(\ldots 0\ldots 0\ldots) = O(N^{-2/3}\lambda^{-8/7}). \tag{8.11}$$

Now apply (8.11) with $r = 3$, $s = 4$ to the term $I(003456)$ in (8.10). We find that

$$I(003456)-I(000456)-I(003056)+I(000056) = O(N^{-2/3}\lambda^{-8/7}). \tag{8.12}$$

In $I(000456)$, ψ contains t_1 in even powers only, and so $\Delta\psi = 0$, giving $I(000456) = 0$. For a similar reason $I(000056) = 0$. $I(003056)$ is the integral that was the subject of Lemma 13 and is therefore $O(N^{-2/3}\lambda^{-8/7})$. It follows from (8.12) that $I(003456) = O(N^{-2/3}\lambda^{-8/7})$. Similarly, we can show that the last two terms on the right of (8.10) are of the same order of magnitude, so that $I(023456) = O(N^{-2/3}\lambda^{-8/7})$. The same sort of argument can be used on the last two terms of (8.9), so that the lemma follows from (8.6) and (8.9).

LEMMA 15.

$$\int_0^1 \mu_1(t)\mu_2(t)\,dt - \int_0^1 \mu_1(t)\,dt \int_0^1 \mu_2(t)\,dt = O(N^{-2/3}\lambda^{1/7}).$$

Proof. This is an immediate consequence of Lemmas 4, 5, and 14.

9. In this section we shall calculate the number of zeros of $\varphi(\theta)$ in the intervals $(\varepsilon, \pi-\varepsilon)$ and $(\pi+\varepsilon, 2\pi-\varepsilon)$. It is sufficient to consider just the first interval, since

$$\varphi(\theta+\pi) = \sum_1^N (-1)^n g_n \cos n\theta,$$

and g_n and $-g_n$ have the same distribution function.

We divide $(\varepsilon, \pi-\varepsilon)$ into intervals i_ν each of length δ (defined in §7 to be $\asymp N^{-1}\lambda^{-3/7}$), and with each i_ν we associate a random function $\mu_\nu(t)$ defined as follows:

$$\mu_\nu(t) = \begin{cases} 1 & \text{if } \varphi(\theta, t) \text{ has a s.c.o. in } i_\nu, \\ 0 & \text{otherwise.} \end{cases}$$

Thus the $\mu_1(t)$ and $\mu_2(t)$ of the preceding section are a typical pair selected arbitrarily from the set $\{\mu_\nu\}$. Let

$$m_\nu = \int_0^1 \mu_\nu(t)\,dt.$$

LEMMA 16.

$$\sum_\nu m_\nu = \frac{N}{\sqrt{3}}\{1+O(\lambda^{-3/7})\}.$$

Proof. From Lemma 4,

$$m_1 = \frac{1}{\pi}\sin^{-1}\sqrt{\left(1-\frac{\gamma_{12}^2}{a^2b^2}\right)}, \tag{9.1}$$

and from (7.3),

$$\sqrt{\left(1-\frac{\gamma_{12}^2}{a^2b^2}\right)} = \delta\sqrt{\left(\frac{\alpha_2}{2a^4}\right)}\{1+O(N\delta)\}. \tag{9.2}$$

The calculations in § 5 show that

$$\frac{\alpha_2}{2a^4} = \frac{s''(\theta)-s''(0)}{16a^2} - \frac{1}{64}\left\{\frac{s'(\theta)}{a^2}\right\}^2, \tag{9.3}$$

and (5.1) and (4.2) give

$$a^2 = \tfrac{1}{2}N+O(N^{7/13}).$$

But

$$s(\theta) = 1+2\sum_1^N \cos 2n\theta,$$

and so

$$s''(0) = -\tfrac{8}{3}N^3+O(N^2);$$

moreover we have shown in § 4 that if $\varepsilon \leqslant \theta \leqslant \pi-\varepsilon$,

$$s'(\theta) = O(N^{20/13}), \quad s''(\theta) = O(N^{33/13}),$$

and therefore from (9.3) we deduce that

$$\frac{\alpha_2}{2a^4} = \tfrac{1}{3}N^2+O(N^{20/13}).$$

Hence, making use of (7.1), we find that

$$\sqrt{\left(\frac{\alpha_2}{2a^4}\right)} = \frac{N}{\sqrt{3}}\{1+O(\lambda^{-3/7})\}. \tag{9.4}$$

We pointed out in § 8 that $N\delta \asymp \lambda^{-3/7}$, and therefore from (9.1), (9.2) and (9.4),

$$m_1 = \frac{N\delta}{\pi\sqrt{3}}\{1+O(\lambda^{-3/7})\}. \tag{9.5}$$

This is true of all the other m_ν, and so

$$\sum_\nu m_\nu = \frac{N}{\pi\sqrt{3}}\{1+O(\lambda^{-3/7})\}\sum_\nu \delta;$$

but

$$\sum_\nu \delta = \pi-2\varepsilon = \pi+O(\lambda^{-3/7}),$$

and the lemma is therefore proved.

LEMMA 17.
$$\int_0^1 \left\{\sum_\nu (\mu_\nu(t)-m_\nu)\right\}^2 dt = O(N^{4/3}\lambda).$$

Proof. We have
$$\begin{aligned}\int_0^1 \left\{\sum_\nu (\mu_\nu(t)-m_\nu)\right\}^2 dt &= \int_0^1 \left(\sum_\nu \mu_\nu(t)\right)^2 dt - \left(\sum_\nu m_\nu\right)^2 \\ &= \sum_{\nu,n} \int_0^1 \{\mu_\nu(t)\mu_n(t) - m_\nu m_n\}\, dt \\ &= \Sigma_1 + \Sigma_2, \quad \text{say,}\end{aligned}$$
where Σ_1 contains those terms for which $|\nu-n| > K$ and Σ_2 those for which $|\nu-n| \leqslant K$. (K was defined in § 7 as $K = N^{1/3}\lambda$.)

If we define $\mu_\nu(t) = 0$ when $\nu \leqslant 0$, we may write
$$\begin{aligned}\Sigma_2 &= \sum_\nu \sum_{n=\nu-k}^{\nu+k} \int_0^1 \{\mu_\nu(t)\mu_n(t) - m_\nu m_n\}\, dt \\ &\leqslant \sum_\nu \sum_{n=\nu-k}^{\nu+k} \int_0^1 \mu_\nu(t)\mu_n(t)\, dt, \quad \text{since} \quad m_\nu \geqslant 0, \\ &\leqslant \sum_\nu \sum_{n=\nu-k}^{\nu+k} \sqrt{(m_\nu m_n)}\end{aligned}$$
by Schwarz's inequality, remembering that $\mu^2 = \mu$ since μ takes only the values 0, 1. Therefore
$$\begin{aligned}\Sigma_2 &\leqslant \sum_\nu (2K+1) \max_n m_n \\ &\leqslant \frac{\pi}{\delta}(2K+1)\max_n m_n\end{aligned}$$
since the number of δ-intervals does not exceed π/δ. From (9.5), which holds for all the m_ν, $m_\nu = O(N\delta)$; and this gives
$$\Sigma_2 = O(NK) = O(N^{4/3}\lambda).$$

In Σ_1, $|\nu-n| > K$ and so the pairs of δ-intervals which contribute to Σ_1 are widely spaced in the sense of § 7, which means that the analysis of § 8 may be applied to them. Consequently, by Lemma 15, each term of Σ_1 satisfies
$$\int_0^1 \mu_\nu(t)\mu_n(t)\, dt - m_\nu m_n = O(\lambda^{1/7}N^{-2/3});$$
and since the number of terms in Σ_1 certainly does not exceed $(\pi/\delta)^2$, which is $O(N^2\lambda^{6/7})$, it follows that $\Sigma_1 = O(N^{4/3}\lambda)$.

When these inequalities for Σ_1 and Σ_2 are combined, the proof of the lemma is complete.

In § 6 we defined a random function $N^*(t)$ to be associated with any given interval. We now do this for all the δ-intervals i_ν into which $(\varepsilon, \pi - \varepsilon)$ has been divided. Let $N_\nu(t)$ be the number of zeros (counted by their multiplicity) of $\varphi(\theta, t)$ in i_ν, and let

$$N_\nu^*(t) = \begin{cases} N_\nu(t) & \text{if } N_\nu(t) \geqslant 2, \\ 0 & \text{otherwise.} \end{cases}$$

Clearly

$$0 \leqslant N_\nu(t) - \mu_\nu(t) \leqslant N_\nu^*(t),$$

and so from Lemma 12 (remembering that the number of terms in the sum on the left does not exceed π/δ),

$$\begin{aligned} \int_0^1 \sum_\nu \{N_\nu(t) - \mu_\nu(t)\}\, dt &= O(N^3\delta^2) \\ &= O(N\lambda^{-6/7}). \end{aligned} \tag{9.6}$$

The total number of zeros of $\varphi(\theta, t)$ in the interval $(\varepsilon, \pi - \varepsilon)$ is $\sum_\nu N_\nu(t)$, and so we now prove

LEMMA 18. *Except for a set of values of t of measure not exceeding* $(4 \log N)^{-1}$,

$$\sum_\nu N_\nu(t) = \frac{N}{\sqrt{3}} + O(N^{11/13}(\log N)^{3/13}).$$

Proof. It follows from Lemma 17 that outside a set of t of measure at most $(8 \log N)^{-1}$,

$$\left\{\sum_\nu (\mu_\nu(t) - m_\nu)\right\}^2 = O(N^{4/3}\lambda \log N),$$

i.e.

$$\sum_\nu (\mu_\nu(t) - m_\nu) = O(N^{2/3}(\lambda \log N)^{1/2});$$

and therefore, using the value of $\sum_\nu m_\nu$ given by Lemma 16, we find that

$$\sum_\nu \mu_\nu(t) = \frac{N}{\sqrt{3}} + O(N\lambda^{-3/7}) + O(N^{2/3}(\lambda \log N)^{1/2}) \tag{9.7}$$

outside the exceptional set of t. But (9.6) shows that outside another exceptional set also of measure not exceeding $(8 \log N)^{-1}$,

$$\sum_\nu N_\nu(t) = \sum_\nu \mu_\nu(t) + O(N\lambda^{-6/7} \log N),$$

and when this is combined with (9.7) we can see that outside an exceptional set of t of measure at most $(4 \log N)^{-1}$,

$$\sum_\nu N_\nu(t) = \frac{N}{\sqrt{3}} + O(N\lambda^{-3/7}) + O(N^{2/3}(\lambda \log N)^{1/2}) + O(N\lambda^{-6/7} \log N).$$

As λ varies, the first two error terms on the right of this equation vary in opposite directions, but if we give λ the value assigned to it in (7.1) these two terms become equal—each being $O(N^{11/13}(\log N)^{3/13})$—and the third error term becomes negligible by comparison—$O(N^{9/13}(\log N)^{6/13})$ in fact.

The lemma is now proved. Further, as we have remarked, the interval $(\pi+\varepsilon, 2\pi-\varepsilon)$ can be treated in the same way, and so it follows that in the two intervals $(\varepsilon, \pi-\varepsilon)$ and $(\pi+\varepsilon, 2\pi-\varepsilon)$ all the functions $\varphi(\theta, t)$ outside an exceptional set of measure at most $(2\log N)^{-1}$ have

$$\frac{2N}{\sqrt{3}}+O(N^{11/13}(\log N)^{3/13}) \tag{9.8}$$

zeros.

10. We conclude by showing that outside a small exceptional set of values of t, $\varphi(\theta, t)$ has a negligible number of zeros in the intervals $(0, \varepsilon)$, $(\pi-\varepsilon, \pi+\varepsilon)$, and $(2\pi-\varepsilon, 2\pi)$. By periodicity, the number of zeros in $(0, \varepsilon)$ and $(2\pi-\varepsilon, 2\pi)$ is the same as the number in $(-\varepsilon, \varepsilon)$, and so we shall confine our discussion to this last interval; the interval $(\pi-\varepsilon, \pi+\varepsilon)$ can be treated in exactly the same way to give the same result.

We apply Jensen's theorem (Titchmarsh (7) 125) to the function of complex argument

$$\varphi(z) = \varphi(z, t) = \sum_1^N g_n(t)\cos nz.$$

The number of real zeros between $\pm\varepsilon$ does not exceed the number in the circle $|z| \leqslant \varepsilon$. Let $n(r) = n(r, t)$ be the number of zeros of $\varphi(z, t)$ in $|z| \leqslant r$. By Jensen's theorem,

$$\begin{aligned}\int_\varepsilon^{2\varepsilon} n(r)\frac{dr}{r} &\leqslant \int_0^{2\varepsilon} n(r)\frac{dr}{r}\\ &= \frac{1}{2\pi}\int_0^{2\pi}\log\left|\frac{\varphi(2\varepsilon e^{i\theta})}{\varphi(0)}\right|d\theta,\end{aligned}$$

assuming that $\varphi(0) \neq 0$, from which

$$n(\varepsilon)\log 2 \leqslant \frac{1}{2\pi}\int_0^{2\pi}\log\left|\frac{\varphi(2\varepsilon e^{i\theta})}{\varphi(0)}\right|d\theta. \tag{10.1}$$

Now $|\cos(2n\varepsilon e^{i\theta})| \leqslant 2e^{2n\varepsilon}$, and so

$$|\varphi(2\varepsilon e^{i\theta})| \leqslant 2Ne^{2N\varepsilon}\max_{1\leqslant n\leqslant N}|g_n|. \tag{10.2}$$

But the distribution function of $|g_n|$ is easily seen to be

$$F(x) = \begin{cases}\sqrt{(2/\pi)}\int_0^x \exp(-\tfrac{1}{2}t^2)\,dt & (x \geqslant 0),\\ 0 & (x < 0),\end{cases}$$

and therefore, since the g_n are independent,

$$\begin{aligned}\Pr(\max|g_n| \leqslant N) &= \{F(N)\}^N \\ &= \left\{1-\sqrt{\left(\frac{2}{\pi}\right)}\int_N^\infty e^{-\frac{1}{2}t^2}\,dt\right\}^N \\ &\geqslant 1-N\sqrt{\left(\frac{2}{\pi}\right)}\int_N^\infty e^{-\frac{1}{2}t^2}\,dt \\ &> 1-e^{-\frac{1}{2}N^2}.\end{aligned}$$

We thus see from (10.2) that outside a set of values of t of measure not exceeding $\exp(-\frac{1}{2}N^2)$,

$$|\varphi(2\varepsilon e^{i\theta},t)| \leqslant 2N^2e^{2N\varepsilon} \tag{10.3}$$

for all θ.

By standard probability theory, the distribution function of

$$|\varphi(0,t)| = \left|\sum_1^N g_n(t)\right|$$

is

$$G(x) = \begin{cases}\sqrt{\left(\dfrac{2}{\pi N^2}\right)}\displaystyle\int_0^x \exp\left(-\frac{t^2}{2N^2}\right)dt & (x \geqslant 0),\\ 0 & (x < 0),\end{cases}$$

from which we can see that $|\varphi(0,t)| \geqslant 1$ except in a set of values of t of measure

$$\sqrt{\left(\frac{2}{\pi N^2}\right)}\int_0^1 \exp\left(-\frac{t^2}{2N^2}\right)dt < \left(\frac{2}{\pi N^2}\right).$$

When we combine this with (10.3) we find that, for all θ,

$$\left|\frac{\varphi(2\varepsilon e^{i\theta},t)}{\varphi(0)}\right| \leqslant 2N^2e^{2N\varepsilon} \tag{10.4}$$

outside an exceptional set of values of t whose measure does not exceed

$$\exp(-\tfrac{1}{2}N^2)+\sqrt{(2/\pi N^2)},$$

which itself does not exceed $(4\log N)^{-1}$ if N is large enough. From (10.1) and (10.4) we can see that outside the exceptional set,

$$\begin{aligned}n(\varepsilon,t) &\leqslant 1+(2\log N+2N\varepsilon)/\log 2 \\ &= O(N^{7/13})\end{aligned}$$

from § 4. This gives an upper bound for the number of zeros of $\varphi(\theta,t)$ in the interval $(-\varepsilon,\varepsilon)$ which is much smaller than the error term which occurs in (9.8).

We shall obviously obtain a similar result for the number of zeros in $(\pi-\varepsilon,\pi+\varepsilon)$, and so when we add these to those given by (9.8) and add together all the exceptional sets—one of measure at most $(2\log N)^{-1}$ and two of measures at most $(4\log N)^{-1}$ each—we arrive at the result stated in our theorem.

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