# Determining Linear Independence

eg ① Two vectors $\begin{bmatrix}1\\1\end{bmatrix}$ & $\begin{bmatrix}1\\-1\end{bmatrix}$ are independent

Let us make a linear combination by $x, y$ by scalars.

∴ $c_1\begin{bmatrix}1\\1\end{bmatrix} + c_2\begin{bmatrix}1\\-1\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$

$$\begin{bmatrix}c_1\\c_1\end{bmatrix} + \begin{bmatrix}c_2\\-c_2\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$$

∴ $c_1 + c_2 = 0$

$c_1 - c_2 = 0 \implies 2c_1 = 0$.

∴ $c_1 = 0$ & $c_2 = 0$

∴ these vectors are linearly independent

If there are more vectors $v_1, v_2 \ldots v_n$, then using the scalars $c_1, c_2 \ldots c_n$, write the system as a linear combination of the scalars & vectors ~~as~~ and when solving the corresponding equations, if all $c_1, c_2 \ldots c_n$ are zero, then they are linearly independent, if some are non-zero then the system is linearly dependent.

To solve generate a matrix and get it into row echelon form.

eg: ②

$$\begin{bmatrix}1\\1\\3\\3\end{bmatrix} \begin{bmatrix}-1\\1\\-1\\-3\end{bmatrix} \begin{bmatrix}2\\5\\9\\6\end{bmatrix}$$ are linearly independent.

For this take scalars $c_1, c_2$ & $c_3$ and write the vectors as the linear combination of these scalars.

$$\begin{bmatrix}1\\1\\3\\3\end{bmatrix} \begin{bmatrix}-1\\1\\-1\\-3\end{bmatrix} \begin{bmatrix}2\\5\\9\\6\end{bmatrix} \longrightarrow$$ linear combination,

$$C_1\begin{bmatrix}1\\1\\3\\3\end{bmatrix} + C_2\begin{bmatrix}-1\\1\\-1\\-3\end{bmatrix} + C_3\begin{bmatrix}2\\5\\9\\6\end{bmatrix} = \begin{bmatrix}0\\0\\0\\0\end{bmatrix}$$

writing as a matrix equation,

$$\begin{bmatrix}1 & -1 & 2\\ 1 & 1 & 5\\ 3 & -1 & 9\\ 3 & -3 & 6\end{bmatrix}\begin{bmatrix}C_1\\C_2\\C_3\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

writing the augmented matrix as,

$$\left[\begin{array}{ccc|c}1 & -1 & 2 & 0\\ 1 & 1 & 5 & 0\\ 3 & -1 & 9 & 0\\ 3 & -3 & 6 & 0\end{array}\right]$$

$R_2 \to R_2 - R_1$ $$\begin{bmatrix}1 & -1 & 2\\ 0 & 2 & 3\\ 0 & 2 & 3\\ 3 & -3 & 6\end{bmatrix}$$

$R_3 \to R_3 - R_1$

$R_4 \to R_4 - 3R_1$

we have $$\begin{bmatrix}1 & -1 & 2\\ 0 & 2 & 3\\ 0 & 2 & 3\\ 0 & 0 & 6\end{bmatrix}$$ $R_3 \to R_3 - R_2 \Longrightarrow$ $$\begin{bmatrix}1 & -1 & 2\\ 0 & 2 & 3\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}$$

$R_3 \to R_3 - R_2$ From this we can see that some vectors can be expressed as a linear combination of other vectors. So the vectors a linearly dependent

# Rank of a Matrix

Let 'A' be an $m \times n$ matrix. It has square sub matrices of different orders. The order of the largest square sub-matrix whose determinant is non-zero is called the rank of the given matrix A.

The determinants of square sub matrices are called minors of A. If all minors of order '$r+1$' are '0', but there is atleast 1 non-zero minor of order '$r$', then '$r$' is called rank of A and it is denoted by $R(A)$.

Examples

1) Consider the matrix $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 0 & 2 & 2 \end{bmatrix}$

$$|A| = 1(6-8) - 2(4) + 3(4)$$
$$= -2 - 8 + 12$$
$$= \underline{\underline{2}} \neq 0$$

$\therefore$ rank of A $= 3$

2) $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 5 & 7 \end{bmatrix}$

$$|A| = 1(21-20) - 2(14-12) + 3(10-9) = \underline{0}$$

Take a $2 \times 2$ matrix $\begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}$

$\Delta = 3 - 4 = \underline{\underline{-1}}$ $\therefore$ rank $= 2$

## Echelon Form of a Matrix

A matrix is said to be an Echelon Matrix if it satisfies the following conditions

i) The $1^{st}$ non-zero number in any non-zero row is 1

ii) The remaining elements in that column is 0

eg:- $A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

$B = \begin{bmatrix} 1 & 2 & -1 & 3 \\ 0 & 1 & 4 & 5 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

The rank of any matrix can be computed by reducing the matrix into Echelon form using elementary row transformation. In the Echelon form of a matrix, the rank is the number of non-zero rows in the Echelon form.

The rank of a matrix A is the maximum number of linearly independent row vectors of A.

① Find the rank of the matrix by reducing it to Echelon form

$$B = \begin{bmatrix} 3 & 2 & 5 & 7 & 12 \\ 1 & 1 & 2 & 3 & 5 \\ 3 & 3 & 6 & 9 & 15 \end{bmatrix}$$

$R_1 \leftrightarrow R_2$

$$\begin{bmatrix} 1 & 1 & 2 & 3 & 5 \\ 3 & 2 & 5 & 7 & 12 \\ 3 & 3 & 6 & 9 & 15 \end{bmatrix}$$

$R_2 \to R_2 - 3R_1$ & $R_3 \to R_3 - 3R_1$

$$\begin{bmatrix} 1 & 1 & 2 & 3 & 5 \\ 0 & -1 & -1 & -2 & -3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$R_2 \to R_2/-1$

$$\begin{bmatrix} 1 & 1 & 2 & 3 & 5 \\ 0 & 1 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$R[A] = 2$$

② Find the rank of matrix $A = \begin{bmatrix} 1 & 5 & 2 \\ 4 & 5 & 3 \\ 9 & -3 & 6 \\ 7 & 5 & 4 \end{bmatrix}$

$R_2 \Rightarrow R_2 - 4R_1$ , $R_3 \rightarrow R_3 - 9R_1$ , $R_4 \Rightarrow R_4 - 7R_1$

$$\begin{bmatrix} 1 & 5 & 2 \\ 0 & -15 & -5 \\ 0 & -48 & -12 \\ 0 & -30 & -10 \end{bmatrix}$$

$R_2 \Rightarrow R_2/-15$ $\begin{bmatrix} 1 & 5 & 2 \\ 0 & 1 & 1/3 \\ 0 & -48 & -12 \\ 0 & -30 & -10 \end{bmatrix}$

$R_3 \Rightarrow R_3 + 48R_2$ , $R_4 \rightarrow R_4 + 30R_2$

$$\begin{bmatrix} 1 & 5 & 2 \\ 0 & 1 & 1/3 \\ 0 & 0 & 4 \\ 0 & 0 & 0 \end{bmatrix}$$

$R_3 \rightarrow R_3/4$

$$\begin{bmatrix} 1 & 5 & 2 \\ 0 & 1 & 1/3 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$R[A] = \underline{\underline{3}}$

3) Find the rank of $A = \begin{bmatrix} 0 & 1 & -3 & -1 \\ 1 & 0 & 1 & 1 \\ 3 & 1 & 0 & 2 \\ 1 & 1 & -2 & 0 \end{bmatrix}$

$R_1 \leftrightarrow R_2$

$$\begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -3 & -1 \\ 3 & 1 & 0 & 2 \\ 1 & 1 & -2 & 0 \end{bmatrix}$$

$R_3 \to R_3 - 3R_1$ & $R_4 \to R_4 - R_1$

$$\begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -3 & -1 \\ 0 & 1 & -3 & -1 \\ 0 & 1 & -3 & -1 \end{bmatrix}$$

$R_4 \to R_4 - R_2$ & $R_3 \to R_3 - R_2$

$$\begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -3 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\underline{\underline{R[A] = 2}}$