Module 4

Linear system of equations

* One of the most important use of matrices is to solve systems of linear equations.

* The information contained in a system of linear equations can be represented by a matrix, called augmented matrix.

Augmented matrix example

Suppose we are given a system of linear equations, ie a linear system such as

$$4x_1 + 6x_2 + 9x_3 = 6$$
$$6x_1 \qquad - 2x_3 = 20$$
$$5x_1 - 8x_2 + x_3 = 10$$

- where x_1, x_2, x_3 are unknowns. Now let us write the coefficient matrix called A, by listing the coefficients of the unknowns in the position in which they appear in the linear equations.

- In the second linear equation, there is no unknown x_2, ie the coefficient of x_2 is 0 and hence in matrix A, $a_{22} = 0$.

Thus $A = \begin{bmatrix} 4 & 6 & 9 \\ 6 & 0 & -2 \\ 5 & -8 & 1 \end{bmatrix}$

Then we form another matrix,

$\bar{A} = \begin{bmatrix} 4 & 6 & 9 & 6 \\ 6 & 0 & -2 & 20 \\ 5 & -8 & 1 & 10 \end{bmatrix}$ by augmenting A

with the right sides of the linear system and call it the augmented matrix of the system.

Since we can go back and recapture the system of linear equations directly from the augmented matrix $\tilde{A}$, $\tilde{A}$ contains all the information of the system and can thus be used to solve the linear system of equations.

We are using Gauss Elimination method ~~for~~ solving linear system.

Linear system has many applications in Engg, Economics, statistics & many other areas.

Linear system, Coefficient matrix, Augmented matrix

A linear system of m equations in n unknowns $x_1, x_2, \dots x_n$ is a set of equations of the form,

$$\left.\begin{aligned} a_{11}x_1 + \dots\dots &+ a_{1n}x_n = b_1 \\ a_{21}x_1 + \dots\dots &+ a_{2n}x_n = b_2 \\ \dots\dots\dots&\dots\dots\dots \\ a_{m1}x_1 + \dots\dots &+ a_{mn}x_n = b_m \end{aligned}\right\} \quad (1)$$

The system is called linear because each variable x_j appears in the first power only, just as in the equation of a straight line.

$a_{11} \dots a_{mn}$ are given numbers, called the Coefficients of the system. $b_1, \dots b_m$ on the right are also given numbers. If all the b_j are zero, then (1) is called a homogeneous

System. If at least one b_j is not zero, then (1) is called a nonhomogeneous system.

A <u>solution</u> of (1) is a set of numbers $x_1, x_2 \ldots x_n$ that satisfies all the m equations.

A solution vector of (1) is a vector x whose components form a solution of (1). If the system (1) is homogeneous, it always has at least the trivial solution $x_1 = 0, \ldots x_n = 0$.

<u>Matrix Form of the Linear system (1).</u>

From the definition of matrix multiplication we see that the m equations of (1) may be written as a single vector equation,

$$Ax = b \quad \text{——} \quad (2)$$

where the coefficient matrix $A = [a_{jk}]$ is the $m \times n$ matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}, \quad X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \text{ and } b = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

are column vectors. We assume that the coefficients a_{jk} are not all zero, so that A is not a zero matrix.

x has n components & b has m components.

The matrix,

$$\bar{A} = \left[\begin{array}{cccc|c} a_{11} & \cdots & \cdots & a_{1n} & b_1 \\ a_{21} & \cdots & \cdots & a_{2n} & b_2 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ a_{m1} & \cdots & \cdots & a_{mn} & b_m \end{array}\right]$$ is called the

augmented matrix of the system (1)

Here the augmented matrix $\bar{A}$ determines the system (1) completely because it contains all the given numbers appearing in (1)

Gauss Elimination & Back Substitution

Consider a linear system that is in triangular form (upper triangular) such as

$$2x_1 + 5x_2 = 2$$
$$13x_2 = -26$$

(upper triangular means that all nonzero entries of the corresponding coefficient matrix lie above the diagonal)

Here we can solve the system by back substitution ie we solve the last equation for the variable $x_2 = -26/13 = -2$.

and obtain $x_1 = \frac{1}{2}(2 - 5x_2) = \frac{1}{2}(2 - 5 \times -2)$
$= \frac{1}{2}(2 + 10) = \frac{12}{2} = 6$

This gives the idea of 1st reducing a general system to triangular form.

~~Ares~~. Elementary Row operations for matrices

1, Interchange of rows / switching.

2. Addition of a constant multiple of one row to another row. (~~scaling~~)

3. Multiplication of a row by a nonzero Constant c

When these operations are performed on rows they are called elementary row operations.

* These operations are for rows not for Columns. They correspond to the following Elementary operation for Equations.

1. Interchange of two equations.
2. Addition of a constant multiple of one equation to another equation.
3. Multiplication of an equation by nonzero Constant c.

eg. for 1st elementary row operation i.e. interchange.

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

$R_2 \Longleftrightarrow R_3$

eg: of multiply a row by a number.

Suppose we want to multiply each element in the second row of matrix A by 7.

Let $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ then $A \Rightarrow \begin{bmatrix} 1 & 0 \\ 0 & 7 \end{bmatrix}$

$R_2 \rightarrow R_2 \times 7$

eg: multiply a row and add it to another row.

Assume A is a 2×2 matrix. Suppose we want to multiply each element in the first row of A by 3, and we want to add that result to the second row of A.

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 \\ 0 + 3\times 1 & 1 + 3\times 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}$$

$R_2 \to R_2 + 3R_1$

prob

Let the given system be,

$$\left.\begin{aligned} 2x_1 + 5x_2 &= 2 \\ -4x_1 + 3x_2 &= -3. \end{aligned}\right\}$$ Its augment matrix is, $\begin{bmatrix} 2 & 5 & 2 \\ -4 & 3 & -30 \end{bmatrix}$

we leave the I^{st} equation as it is.

we eliminate x_1 from the second equation, to get a triangular system. For this we add twice the first equation to the second, and we do the same operation on the rows of the augmented matrix,

This gives $-4x_1 + 4x_1 + 3x_2 + 10x_2 = -30 + 2 \times 2$

ie

$$2x_1 + 5x_2 = 2$$
$$13x_2 = -26$$

$R_2 + 2R_1 \Rightarrow$ ie $\begin{bmatrix} 2 & 5 & 2 \\ 0 & 13 & -26 \end{bmatrix}$

~~R_2+~~

This is the Gauss elimination (for 2 equations in 2 unknowns) giving the triangular form, from which back

back substitution now gives,

$x_2 = -2$ and $x_1 = 6$ — as before.

Since a linear system is completely determined by its augmented matrix, Gauss elimination can be done by merely considering the matrices, as in the above problem.

Row-equivalent systems

A linear system S_1 is called row-equivalent to a linear system S_2 if S_1 can be obtained from S_2 by row operations. This justifies Gauss elimination method.

Theorem -

> Row equivalent linear systems have the same set of solutions.

Due to this theorem, systems having the same solution sets are often called equivalent systems.

A linear system is called overdetermined if it has more equations than unknowns, determined if $m = n$ and underdetermined if it has fewer ~~solutions~~ equations than unknowns.

A system is called consistent if it has at least one solution, inconsistent if it has no solutions at all.

~~1~~ WKT, in Gauss elimination method, the unknown are eliminated by bringing the equations into an upper triangular system. The system can be solved by 3 methods.

(1) Direct elimination in ordinary format.

(2) Without partial pivoting in matrix format

(3) With partial pivoting in matrix format.

Note

The first row of the matrix A is called the pivot row and the I^{st} equation of the linear system is called pivot equation.

~~The pivot row is used to eleminate~~

The coefficient of x_1 in the ~~first~~ pivot row is called the pivot. Pivot row is used to eliminate x_1 in other rows.

In the case of matrix algorithms, a pivot entry is usually required to be at least ~~difference~~ distinct from zero and often destant from it in this case finding this element is called pivoting.

Pivoting may be followed by an interchange of rows & columns to bring the pivot to a fixed position and allow the algorithm to proceed successfully and possibly to reduce round ~~of~~ of error.

In Gauss elimination, only partial pivoting (ie interchanging rows only) is required.

eg that requires pivoting

$$\left[\begin{array}{ccc|c} 1 & -1 & 2 & 8 \\ 0 & 0 & -1 & -11 \\ 0 & 2 & -1 & -3 \end{array}\right]$$

The above system requires the interchange of rows 2 and 3 to perform elimination.

The system that result from pivoting is as follows and will allow the elimination algorithm and backwards substitution to output the solution to the system.

ie

$$\left[\begin{array}{ccc|c} 1 & -1 & 2 & 8 \\ 0 & 2 & -1 & -3 \\ 0 & 0 & -1 & -11 \end{array}\right]$$

In Gaussian elimination, it is generally desirable to choose a pivot element with large absolute value. This improves the numerical stability.

Prob(1) Solve using Gauss elimination method,

$$10x_1 + x_2 + 2x_3 = 4$$
$$x_1 + 10x_2 - x_3 = 3$$
$$2x_1 + 3x_2 + 20x_3 = 7$$

The augmented matrix is,

$$\left[\begin{array}{ccc|c} 10 & -1 & 2 & 4 \\ 1 & 10 & -1 & 3 \\ 2 & 3 & 20 & 9 \end{array}\right]$$

Here R_1 is the Pivot row.

Step 1 - Elimination of x_1

$$R_2 \to R_2 - \frac{R_1}{10}$$

$$\left[\begin{array}{ccc|c} 10 & -1 & 2 & 4 \\ 0 & \frac{101}{10} & \frac{-12}{10} & \frac{26}{10} \\ 2 & 3 & 20 & 7 \end{array}\right]$$

$$R_3 \to R_3 - \frac{2 \times R_1}{10}$$

$$\left[\begin{array}{cccc} 10 & -1 & 2 & 4 \\ 0 & \frac{101}{10} & \frac{-12}{10} & \frac{26}{10} \\ 0 & \frac{32}{10} & \frac{196}{10} & \frac{62}{10} \end{array}\right]$$

Now $\frac{101}{10} \quad \frac{-12}{10} \quad \frac{26}{10}$ is the Pivot row.

So $R_3 \to R_3 - R_2 \times \frac{32}{101}$

$$\left[\begin{array}{ccc|c} 10 & -1 & 2 & 4 \\ 0 & \frac{101}{10} & \frac{-12}{10} & \frac{26}{10} \\ 0 & 0 & \frac{20180}{1010} & \frac{5430}{1010} \end{array}\right]$$

$$\therefore \frac{20180}{1010} x_3 = \frac{5430}{1010}$$

$$x_3 = \underline{0.269}$$

using backward substitution,

$2 - \frac{20}{10}, \; 3 - \frac{-2}{10},$

$20 - \frac{4}{10}$

$\cancel{7} - \frac{8}{10}$

$0, \; \frac{32}{10}, \; \frac{196}{10}, \; \frac{62}{10}$

$\frac{32}{10} - \frac{101}{10} \times \frac{32}{101}$

$\frac{196}{10} - \frac{\cancel{32}^{16}}{101} \times \frac{-12}{\cancel{10}_{5}}$

$\frac{1964}{10} + \frac{192}{505}$

$\frac{19796 + 384}{1010} = 1010$

16
12
32
16
192

$\frac{20180}{1010}$

$\frac{62}{10} - \frac{32}{101} \times \frac{26}{10}$

$\frac{62}{10} - \frac{832}{1010}$

$\frac{6262 - 832}{1010}$

$= \frac{5430}{1010}$

$\frac{10.1x_2}{10} + \frac{-12}{10} \times 0.269 = 2.6$

$10.1x_2 + -12 \times 2.69 = 2.6$

$10.1x_2 = 2.6 + 0.3228$

$x_2 = \frac{2.9228}{10.1} = \underline{0.289}$

$10x_1 - 0.289 + 2 \times 0.269 = 4$

$10x_1 - 0.289 + 0.538 = 4$

$10x_1 + 0.249 = 4$

$10x_1 = 4 - 0.249 = 3.751$

$x_1 = \frac{3.751}{10} = \underline{0.375}$

$\therefore X = \begin{bmatrix} 0.375 \\ 0.289 \\ 0.269 \end{bmatrix}$

2. Solve using Gauss Elimination,

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 3 & 3 & 4 & 20 \\ 2 & 1 & 3 & 13 \end{array}\right]$$

$R_2 \to R_2 - 3R_1$

$$\begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & 0 & 1 & 2 \\ 2 & 1 & 3 & 13 \end{bmatrix}$$

$R_3 \Rightarrow R_3 - 2R_1$

$$\begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & 0 & 1 & 2 \\ 0 & -1 & 1 & 1 \end{bmatrix}$$

Here $a_{22} = 0$. So G E fails