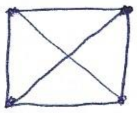
Planar Graphs

A graph is said to be planar if it can be drawn in a plane so that no two edges intersect except only at the common vertex.

A graph that cannot be drawn on a plane without a cross over (the points of intersection are called cross overs) between its edges is called non-planar.



fig(1)

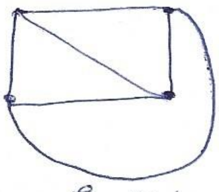


fig (2)

The graph in fig(1) can be redrawn as in fig(2) such that no two edges intersect. So the graph in fig(2) is a planar graph.

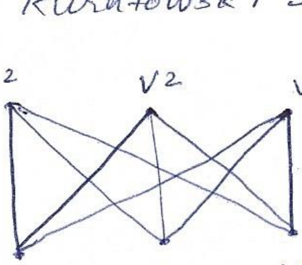
Kuratowski's two graphs

There are 2-specific non-planar graphs which are of fundamental importance.

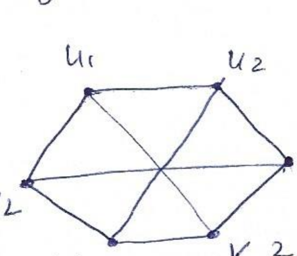
Kuratowski's first graph is the complete graph K_5 of 5 vertices and second graph is the bipartite graph $K(3,3)$.

Theorem

Kuratowski's second graph $K(3,3)$ is non planar.



G: fig(1)



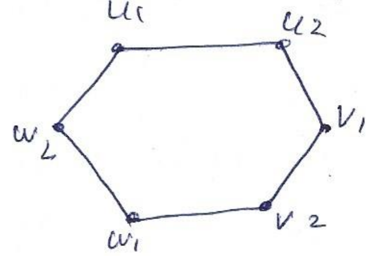
Redrawn G fig(2)

Proof. Map construction method.

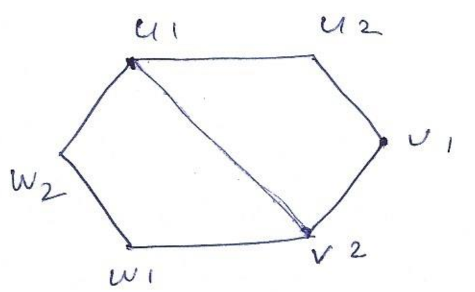
Kuratowski's second graph $K(3,3)$ in fig(1) can be redrawn as in fig(2)

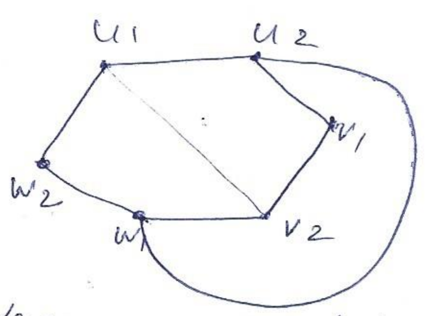
Let the vertices of $K(3,3)$ be $u_1, u_2, v_1, v_2, w_1, w_2$.

Draw the hexagon with 3 vertices taken in order



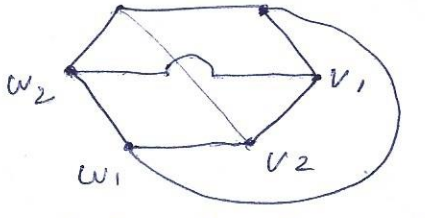
Since u_1 is to be connected to v_2 by an edge, this edge may be drawn inside or outside the hexagon. Suppose we draw this edge inside the hexagon.





Now we have to draw an edge from u_2 to w_1, Since this edge cannot be drawn inside the hexagon, without crossing an edge, we draw it outside the hexagon. Then the edge connecting v_1 to w_2 cannot be drawn inside or outside the hexagon without crossing other edges.

Thus,

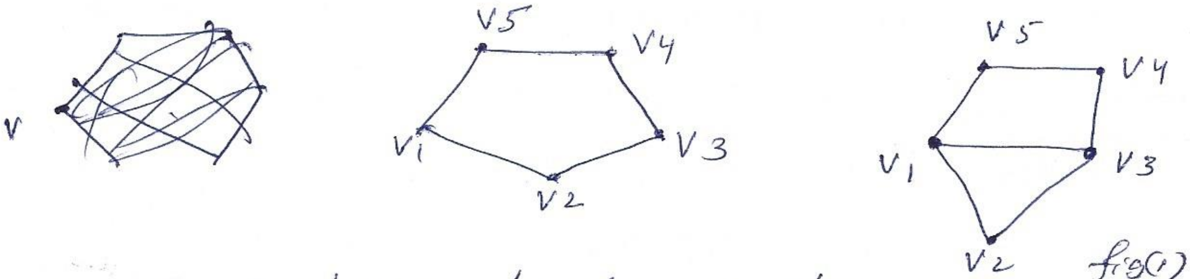


$\therefore$ The graph $K(3,3)$ cannot be embedded in a plane. So $K(3,3)$ is non planar.

Theorem

A complete graph of 5 vertices is non planar.

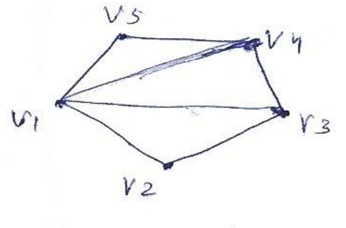
Let the 5-vertices in the complete graph K_5 be V_1, V_2, V_3, V_4 & V_5. Draw the Pentagon joining the vertices taken in order.



fig(1)

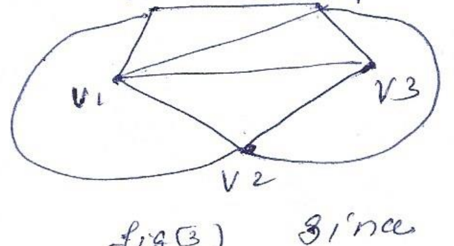
Now the pentagon divides the plane into two regions one inside and the other outside the pentagon. Since V_1 is to be connected to V_3 by means of an edge, this edge may be drawn inside or outside the pentagon.

Suppose we draw the edge inside the Pentagon. Now we have to draw an edge from V_1 to V_4. This edge can be drawn inside the Pentagon without crossing any other edge.

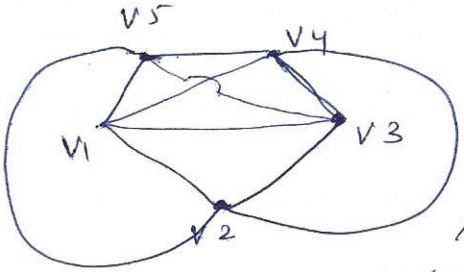


fig(2)

Then we have to draw an edge from V_2 to V_5 and another from V_2 to V_4



fig(3) Since neither of these edges can be drawn inside the pentagon without crossing other edges, we draw both the edges outside the pentagon.



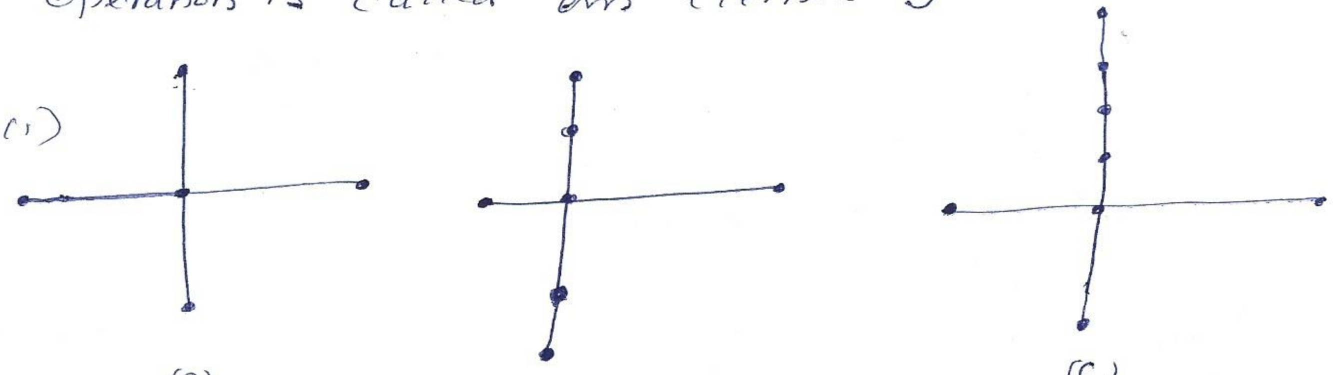
Now the edges connecting V3 to V5 cannot be drawn inside or outside the pentagon without crossing other edges.

∴ The complete graph K5 cannot be embedded in a plane so that no two edges intersect. Hence the complete graph K5 of 5 vertices is nonplanar.

Homeomorphic graphs

Two graphs are said to be homeomorphic if both can be obtained from the same graph by inserting new vertices of degree 2 into it's edges or by merger of edges in series. Such an operation is called an elementary subdivision.

fig (1)



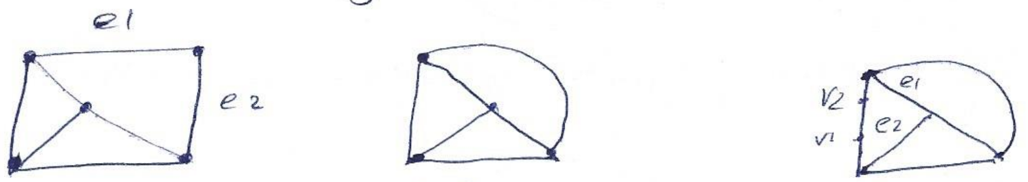
The graphs (b) & (c) are Homeomorphic since each can be obtained from (a) by inserting vertices of degree two (creating edges in serie)

fig (2)

(4) merging e1 & e2

In fig(2) graphs (a) & b are Homeomorphic because graph (b) can be obtained from (a) by merging two edges e_1 & e_2 in series.

fig (3)



(a)

(b)

merging e_1 & e_2

(c)

Inserting vertices V_1 & V_2 (ie creating edge in series)

In fig(3) the graphs (a) (b) & (c) are Homeomorphic to one another.
The graph (b) is obtained by merging edges e_1 & e_2 in (a) and the graph in (c) is obtained by creating two edges in series by inserting V_1 & V_2

<u>Note</u>

A graph can be Homeomorphic to a proper subgraph of itself, but a graph cannot be isomorphic to a proper subgraph of itself.

<u>Kuratowski's Theorem</u>

A graph is planar if and only if it does not contains any subgraph homeomorphic to K_5 or K_{33} (Kuratowski's graphs).

Properties common to Kuratowski's graphs

(ie for K5 & K33)

1. Both are regular graphs. (ie vertices with equal degree)
2. Both are non-planar
3. Removal of one edge or one vertex makes them planar graphs.
4. Kuratowski's first graph (K5) is non planar with smallest no. of vertices and second graph K(3, 3) is non-planar with smallest no. of edges.

Thus both are simplest non-planar graphs.

Regions (faces)

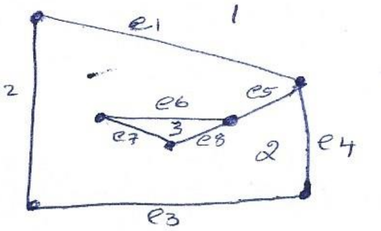
A plane representation of a graph divides the plane into regions or faces.

A region is usually bounded by a set of edges of the graph.

A region is not defined in a non-planar graph. The portion of the plane lying outside a graph embedded in a plane is called an unbounded region or infinite region.

- The set of edges which bound a region of a planar graph is called its boundaries.
- If the area of the region is finite, then the region is called a finite region.

- If the region is infinite it is called an infinite region.
- A planar graph has only one infinite region.



The above graph has 3 regions.
- 2 are finite & one is infinite.

The infinite region is characterized by the set {e1, e2, e3, e4} and hence has degree 4

The region 3 is characterized by {e6, e7, e8} has degree 3 and region 2 has degree 9 (because one edge is encountered twice, one on each side.)

Euler's formula

The basic result about the planar graph is known as Euler's formula.

Thm

If a connected planar graph G has n vertices, e edges and r regions, then $n - e + r = 2$

Proof

Let us prove the theorem, by induction on e, number of edges of G.

Basis of induction — If $e = 0$, then G must have just one vertex ie $n = 1$ and one infinite region, ie $r = 1$. Then $n - e + r = 1 - 0 + 1 = 2$

If $e = 1$, then the number of vertices of G is either 1 or 2, (1 vertex if the edge is a loop)

If the no. of vertices is 1, then there are 2 regions & if the no. of vertices is 2 then there is one region.

$n=1$ $\quad$ $n=2$

(Connected plane graph with one edge).

In the case of loop, $n-e+r = 1-1+2 = 2$

In the case of non loop, $n-e+r = 2-1+1 = 2$

Hence the result is true.

<u>Induction hypothesis</u>

Now suppose that the result is true for any connected plane graph G with $e-1$ edges.

<u>Induction step</u>: We add one new edge K to G to form a connected supergraph of G which is denoted by $G+K$.

There are the following 3 possibilities.

(i) K is a loop, in which case a new region bounded by the loop is created but the number of vertices remains unchanged.

(ii) K joins two distinct vertices of G, in which case one of the region of G is split into two, so that the number of regions is increased by 1, but the number vertices remains unchanged.

(iii) K is incident with only one vertex of G on which case another vertex must be added, increasing the number of vertices by one, but leaving the number of regions unchanged.

In case(i) $n-e+r = n'-(e'+1)+(r'+1) = n'-e'+r'$

In case(ii) $n-e+r = n'-(e'+1)+(r'+1) = n'-e'+r'$

In case(iii) $n-e+r = (n'+1)-(e'+1)+r' = n'-e'+r'$

But by our induction hypothesis.

$n'-e'+r' = 2$. Thus in each case

$n-e+r = 2$.

Now any plane connected graph with e edges is of the form $G+k$, for some connected graph G with $(e-1)$ edges and a new edge k. Hence by mathematical induction the formula is true for all plane graphs. Hence the theorem is proved.

Corollary. If a plane graph has k components, then $n-e+r = k+1$

Proof

Let $G_1, G_2, G_3, \dots G_i \dots G_k$ be the k components of the disconnected graph G.

Suppose the component G_i has n_i vertices, e_i edges and r_i' bounded regions excluding the infinite region.

Then $\Sigma r_i' = r-1$

Now for each component G_i we have Euler's formula,

$n_i - e_i + [r_i' + 1] = 2$.

Taking sum ~~ov~~ for $i = 1, 2 \dots k$,

$$\sum_{i=1}^{k} n_i - \sum_{i=1}^{k} e_i + \sum_{i=1}^{k} r_i' + k = 2k$$

i.e. $n - e + \cancel{r} - 1 + k = 2k$

i.e. $n - e + \cancel{r} = k + 1$

Corollary

In any simple connected plane graph with r regions, n vertices and e edges $(e \geq 2)$

(1) $2e \geq 3r$

(2) $e \leq 3n - 6$.

Proof.

(1) Let G be a simple connected planar graph. The boundary of each ~~face~~ region contains at least 3 edges. Therefore the sum of the no. of edges on the boundary of r regions

i.e. $\geq 3\cancel{r}$ — (1)

Since each edge belongs to 2 regions, the sum of the no. of edges equals $2e$ — (2)

From (1) & (2)

$$2e \geq 3\cancel{r}$$

(2) By Euler's formula,

$$r = e - n + 2.$$

But we have $2e \geq 3\cancel{r}$,

i.e. $2e \geq 3(e - n + 2)$

i.e. $2e \geq 3e - 3n + 6$

i.e. $3n - 6 \geq e$

i.e. $e \leq 3n - 6$

Corollary

If a simple planar graph has no triangles then $e \leq 2n-4$

Proof.

Since the graph has no triangles, all the regions are bounded by at least 4 edges.

$$\therefore 2e \geq 4r \quad \text{---(1)}$$

By Eulers formula,

$$r = e - n + 2$$

Substituting in (1)

$$2e \geq 4(e-n+2)$$

$$2e \geq 4e - 4n + 8$$

$$\text{i.e. } 4n - 8 \geq 2e$$

$$2n - 4 \geq e.$$

$$\text{i.e. } e \leq 2n-4$$

All the above 3 conditions are only necessary conditions for planarity of a connected graph.

So a connected graph which does not satisfy the condition $e \leq 3n-6$ is certainly non-planar.

eg: Consider Kuratowski's first graph, K_5. Here $n = 5$.

Then $e = 5C_2 = \frac{5!}{2!3!} = 10.$

$3n-6 = 3\times5-6 = 15-6 = 9$

$e = 10$

i.e. $10 > 9$

$e > 3n-6$

i.e. $e \nleq 3n-6$.

$\therefore$ The condition $e \leq 3n-6$ is not satisfied and hence K_5 is nonplanar.

eg: (2) Consider Kuratowski's 2nd graph K_{33}

In K_{33} there are no triangles.

$n = 6 = 3+3$.

$e = 3\times3 = 9$

$2n-4 = 2\times6-4 = 12-4 = 8$.

$e = 9$

$9 > 8$

$e > 2n-4$

i.e. $e \nleq 2n-4$.

The condition $e \leq 2n-4$ is not satisfied

Hence K_{33} is non planar.

Note: From the formula $e \leq 3n-6$ we get the maximum no. of edges and minimum no. of vertices in a planar graph.

1. What is the minimum no. of vertices necessary for a connected graph with 6 edges to be planar.

$e \leq 3n - 6$

But $e = 6$.

$\Rightarrow 6 + 6 \leq 3n$

$\Rightarrow 12 \leq 3n$

$\Rightarrow 4 \leq n$

$\Rightarrow n \geq 4$

$\therefore$ The planar graph should contain at least <u>4 vertices.</u>

2. What is the maximum no. of edges possible in a planar graph with 8 vertices.

$e \leq 3n - 6$

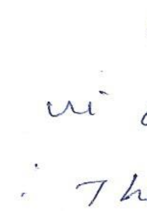
$\Rightarrow e \leq 3 \times 8 - 6$

$e \leq 24 - 6 = 18$

$\Rightarrow e \leq 18$.

$\therefore$ The maximum no. of edges that can be drawn is <u>18</u>

4. Count the no. of vertices, no. of edges and no. of regions of each planar graph & verify Euler's formula.

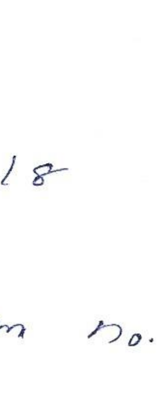


$n = 4$

$e = 6$

$r = 4$

$n - e + r = 4 - 6 + 4 = 2$



$n = 6$

$e = 9$

$r = 5$

$n - e + r = 6 - 9 + 5$
$= \underline{2}$

$n = 5$

$e = 10$

$r = 7$

$n - e + r$
$= 5 - 10 + 7$
$= \underline{2}$