at this stage because $a_{22} = 0$ the pivot element.

At any stage of GE procedure if pivotal element pivotal equation is 0 we cannot proceed further. In this case the problem can be solved by interchanging R_2 & R_3 and considering R_3 as pivotal equation.

$$R_2 \longleftrightarrow R_3$$

$$\begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & -1 & 1 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

Then using back substitution,

$x_3 = 1$

$\therefore -1 \times x_2 + x_3 = 1$

$\therefore -x_2 = -1$

$x_2 = 1$

$1 \times x_1 + x_2 + x_3 = 6$

$x_1 + 1 + 1 = 6$

$x_1 = 3.$

$\therefore$ The solution is $X = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}$

Prob 3 Solve using Gauss Elimination method with Pivoting,

$$\begin{bmatrix} 0.143 & 0.357 & 2.01 \\ -1.31 & 0.911 & 1.99 \\ 11.2 & -4.3 & -0.605 \end{bmatrix} = \begin{bmatrix} -5.17 \\ -5.46 \\ 4.42 \end{bmatrix}$$

Prob Solve the linear system, by Gauss elimination

$$x_1 - x_2 + x_3 = 0$$

$$-x_1 + x_2 - x_3 = 0$$

$$10x_2 + 25x_3 = 90$$

$$20x_1 + 10x_2 = 80.$$

Augmented matrix is,

$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ -1 & 1 & -1 & 0 \\ 0 & 10 & 25 & 90 \\ 20 & 10 & 0 & 80 \end{array}\right]$$

$R_1 \Leftrightarrow R_4$

$$\left[\begin{array}{cccc} 20 & 10 & 0 & 80 \\ -1 & 1 & -1 & 0 \\ 0 & 10 & 25 & 90 \\ 1 & -1 & 1 & 0 \end{array}\right]$$

$R_4 \to R_4 - \frac{1}{20} R_1$

$R_2 \to R_2 + R_4$

$$\left[\begin{array}{ccc|c} 20 & 10 & 0 & 80 \\ 0 & 0 & 0 & 0 \\ 0 & 10 & 25 & 90 \\ 0 & -3/2 & 1 & -4 \end{array}\right]$$

$-1 - \frac{10}{20}$

$-1 - \frac{1}{2}$

$\frac{-2-1}{2}$

$-\frac{3}{2}$

$0 - \frac{1}{20} \cdot 80$

By Partial Pivoting, interchange 2nd & 3rd row, then 3rd & 4th row so as to put 0 = 0 row at the end.

So
$$\begin{bmatrix} 20 & 10 & 0 & 80 \\ 0 & 10 & 25 & 90 \\ 0 & -3/2 & 1 & -4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$R_3 \to R_3 + \frac{3}{20} R_2$

$\cancel{R_3 \to R_3 + \frac{3}{20} R_2}$

$$\left[\begin{array}{ccc|c} 20 & 10 & 0 & 180 \\ 0 & 10 & 25 & 90 \\ 0 & 0 & 19/4 & \frac{19}{2} \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$-\frac{3}{2} \times$

$-\frac{3}{2} + \frac{3}{2}$

$\frac{3}{20} \times 10$

$1 + \frac{3}{20} \times 25$

$1 + \frac{15}{4}$

$\frac{19}{4}$

$-4 + \frac{3}{20} \times 90$

$-4 + \frac{27}{2}$

$\frac{-8+27}{2}$

$\frac{19}{2}$

Back Substituting,

$\frac{19}{4} x_3 = \frac{19}{2}$

$\frac{x_3}{2} = 1$

$x_3 = 2$

$10x_2 + 25x_3 = 90$

$10x_2 + 25 \times 2 = 90$

$10x_2 = 90 - 50 = 40$

$x_2 = \frac{40}{10} = 4$

$20x_1 + 10x_2 + 0 \cdot x_3 = 80$

$20x_1 + 10 \times 4 + 0 = 80$

$20x_1 = 80 - 40 = 40$

$x_1 = \frac{40}{20} = 2.$

$\therefore x_1 = 2;\ x_2 = 4;\ x_3 = 2$

$X = \begin{bmatrix} 2 \\ 4 \\ 2 \end{bmatrix}$

(16)

Problems

* Solve by Gauss Elimination method,

(1)
$$x + y = 2$$
$$2x + 3y = 5$$

(2)
$$x + y + z = 7$$
$$3x + 3y + 4z = 24$$
$$2x + y + z = 16$$

(3)
$$\begin{bmatrix} 0.143 & 0.357 & 2.01 \\ -1.31 & 0.911 & 1.99 \\ 11.2 & -4.3 & -0.605 \end{bmatrix} = \begin{bmatrix} -5.17 \\ -5.46 \\ 4.42 \end{bmatrix}$$

(4)
$$2x_1 - x_2 + x_3 = 1$$
$$4x_1 + x_2 - x_3 = 5$$
$$x_1 + x_2 + x_3 = 0$$

Gauss Elimination – Three possible Cases.

So far we have considered systems that have a unique solution. Gauss elimination can be applied to systems with infinitely many solutions and system with no solution.

Gauss elimination if infinitely many solutions exist

Solve the following linear system of three equations in four unknowns whose augmented matrix is,

$$\begin{bmatrix} 3 & 2 & 2 & -5 & 8 \\ 0.6 & 1.5 & 1.5 & -5.4 & 2.7 \\ 1.2 & -0.3 & -0.3 & 2.4 & 2.1 \end{bmatrix}$$

$R_2 \to -0.2 \times R_1 + R_2$
$R_3 \to -0.4 R_1 + R_3$

$$\begin{bmatrix} 3 & 2 & 2 & -5 & 8 \\ 0 & 1.1 & 1.1 & -4.4 & 1.1 \\ 0 & -1.1 & -1.1 & 4.4 & -1.1 \end{bmatrix}$$

$R_3 \to R_3 + R_2$

$$\left[\begin{array}{cccc|c} 3 & 2 & 2 & -5 & 8 \\ 0 & 1.1 & 1.1 & -4.4 & 1.1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Dividing 2nd row by 1.1 we have

$$\left[\begin{array}{cccc|c} 3 & 2 & 2 & -5 & 8 \\ 0 & 1 & 1 & -4 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

From the second row,

$$x_2 = 1 - x_3 + x_4 . \quad \text{— (1)}$$

from first row,

$$3x_1 + 2x_2 + 2x_3 - 5x_4 = 8 . \quad \text{— (2)}$$

Substituting the value of x_2 in (2) we have

$$x_1 = 2 - x_4 \quad \text{— (3)}$$

From equns (1) & (3) we ~~have~~ get the values of x_1 & x_2, Since x_3 & x_4 remains arbitrary, we have infinitely many solutions.

If we choose a value of x_3 and a value of x_4, then the corresponding values of x_1 & x_2 are uniquely determined.

<u>Gauss elimination if no solution exists</u>

<u>Prob.</u> Use Gauss elimination to solve the following System of linear equations.

$$\left[\begin{array}{ccc|c} 3 & 2 & 1 & 3 \\ 2 & 1 & 1 & 0 \\ 6 & 2 & 4 & 6 \end{array}\right]$$

Taking 6 2 4 6 as the pivot row we have

$$\begin{bmatrix} 6 & 2 & 4 & 6 \\ 2 & 1 & 1 & 0 \\ 3 & 2 & 1 & 3 \end{bmatrix}$$

Also interchanging R_2 & R_3 we have

$$\begin{bmatrix} 6 & 2 & 4 & 6 \\ 3 & 2 & 1 & 3 \\ 2 & 1 & 1 & 0 \end{bmatrix}$$

$R_2 \to R_2 - \frac{1}{2} R_1$
$R_3 \to R_3 - \frac{1}{3} R_1$

$$\begin{bmatrix} 6 & 2 & 4 & 6 \\ 0 & 1 & -1 & 0 \\ 0 & 1/3 & -1/3 & -2 \end{bmatrix}$$

$R_3 \to R_3 + \frac{-1}{3} R_2$
$$\begin{bmatrix} 6 & 2 & 4 & 6 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & -2 \end{bmatrix}$$

Now we have $0 \times x_3 = -2$

$\cancel{0 = -2}$

i.e. $0 = -2$

which is a contradiction

$\therefore$ The system has no solution

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iii) If we, apply the Gauss elimination to a linear system that that has no solution, then we will arrive at a contradiction.

Row Echelon Form and Information from it.

At the end of the Gauss Elimination, the form of the coefficient matrix, the augmented matrix, and the system itself are called the row echelon form. In it rows of zeros, if present, are the last rows, and in each non zero row, the left most non zero entry is

farther to the right than in the previous row.
For instance, in the previous problem,
the coefficient matrix and its augmented
are in row echelon form are,

$$\begin{bmatrix} 6 & 2 & 4 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \text{ and } \left[\begin{array}{ccc|c} 6 & 2 & 4 & 6 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & -2 \end{array}\right]$$

The original system of m equations in n unknowns
has augmented matrix $[A \mid b]$. This is to be
row reduced to matrix $[R \mid f]$. The two
systems $Ax = b$ and $Rx = f$ are equivalent.
If either has a solution, so does the
other and the solutions are identical.

At the end of the Gauss elimination, ~~back substitution~~,
(before the back substitution) the row echelon
form of the augmented matrix will be

$$\left[\begin{array}{cccc|c} r_{11} & r_{12} & \cdots & r_{1n} & f_1 \\ & r_{22} & \cdots & r_{2n} & f_2 \\ & & \ddots & \vdots & \vdots \\ & & r_{rr} & \cdots\, r_{rn} & f_r \\ & & & & f_{r+1} \\ & & & & \vdots \\ & & & & f_m \end{array}\right]$$

Here $r \leq m$, $r_{11} \neq 0$ and all entries in the triangle and
rectangle are zero.

The number of nonzero rows, r in the row reduced coefficient matrix R is called the rank of R and also the rank of A.

Here is the method for determining whether $Ax = b$ has solutions and what they are,

(a) <u>No Solution.</u>

If r is less than m (meaning that R actually has at least one row of all 0s) and at least one of the numbers $f_{r+1}, f_{r+2}, \dots f_m$ is not zero, then the system $Rx = f$ is inconsistent. ie no solution is possible. Therefore the system $Ax = b$ is inconsistent.

In the previous example, where $r = 2 < m = 3$ and $f_{r+1} = f_3 = -2$

If the system is consistent (either $r = m$, or $r < m$ and all the numbers $f_{r+1}, f_{r+2} \dots f_m$ are zero), then there are solutions.

(b) <u>Unique solution.</u> If the system is consistent and $r = n$, then there is exactly one solution (as in example (1) & (2)), which can be found by back substitution

(c) Infinitely many solutions. To obtain any of these solutions, choose values of $x_{r+1}, \dots x_n$ arbitrarily. Then solve the r^{th} equation for x_r (in terms of those arbitrary values) then the $(r-1)^{th}$ equation for x_{r-1} and so on up the line.

Linear Independence & Dependence of vectors

Given any set of m vectors $a_{(1)}, a_{(2)}, \dots a_{(m)}$, a linear Combination of these vectors is an expression of the form,

$$c_1 a_{(1)} + c_2 a_{(2)} + \dots + c_m a_{(m)}$$

where $c_1, c_2 \dots$ are any scalars.

Now Consider the equation,

$$c_1 a_{(1)} + c_2 a_{(2)} + \dots + c_m a_{(m)} = 0 \quad \text{——} (1)$$

Equation (1) holds if we choose all c_j's zero, because it becomes $0 = 0$. If this is the only m-tuple of scalars for which (1) holds, then our vectors $a_{(1)}, a_{(2)} \dots a_{(m)}$ are said to form a linearly independent set or they are linearly independent.

Otherwise, if (1) also holds with scalars not all zero, we call these vectors linearly dependent. This means that we can express at least one of the vectors as a linear combination of the other vectors. for eg if (1) holds with say, $c_1 \neq 0$, we can solve (1) for $a_{(1)}$

$a_{(1)} = k_2 a_2 + \dots + k_m a_m$ where $k_j = -c_j/c_1$ and some k_j's may be zero.