**Note:** The rank of a matrix A is the maximum number of linearly independent row vectors of A.

It is the number of non-zero rows, when the system is converted to row echelon form.

**Theorem** - Row equivalent matrices have the same rank

## Solution of linear Systems - Non homogenous system

Fundamental theorem of linear systems.

(a) Existence - A linear system of m equations in 'n' unknowns $x_1, x_2 \dots x_n$

$$a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1$$
$$a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = b_2$$
$$- - - - - - - - -$$
$$a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n = b_m$$

is consistent, that is has solutions, if and only if the coefficient matrix A and the Augmented matrix $[A:B]$ have the same rank.

Here

$$(1) \quad A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & - & - & - \\ \vdots & - & - & - \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix} \text{ and } AB = \left[\begin{array}{cccc:c} a_{11} & a_{12} & \dots & a_{1n} & b_1 \\ a_{21} & a_{22} & \dots & a_{2n} & b_2 \\ - & - & - & - & \vdots \\ - & - & - & - & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} & b_m \end{array}\right].$$

(b) Uniqueness - The equation (1) has precisely one solution if and only if the common rank of $[A]$ and $[A:B]$ equals n, the number of unknowns.

(c) Infinitely many solutions - If this common rank r is less than n, the system (1) has infinitely many solutions. All of these solutions are obtained by determining r suitable unknowns ~~in terms~~ (whose submatrix of coefficients must have rank r) in terms of the remaining (n-r) unknowns, to which arbitrary values can be assigned.

(d) Gauss elimination - If solutions exists, they can all be obtained the Gauss elimination.

So to find the solution of the above system of equations,

Step I Construct the augmented matrix [A:B]. ~~By reducing it into the Echelon form using row transformations~~.

Step II

find the rank of [A:B] by reducing it to the Echelon form using row transformation.

Step III If $R[A:B] = R[A]$, then the system is consistent.

If $R[A:B] \neq R[A]$, the system is inconsistent

If $R[A:B] = R[A] =$ no. of unknowns then the system has unique solution.

If $R[A:B] = R[A] <$ no. of unknowns, the system has infinite no. of solutions.

Problems

(1) For the following system of equations, Test for consistency & if consistent solve, $2x-3y+7z=5$; $3x+y-3z=13$; $2x+19y-47z=32$

The given system of equations can be

written as,

$$\begin{bmatrix} 2 & -3 & 7 \\ 3 & 1 & -3 \\ 2 & 19 & -47 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5 \\ 13 \\ 32 \end{bmatrix}$$

Construct the augmented matrix $[A:B]$ as,

$$[A:B] = \left[\begin{array}{ccc|c} 2 & -3 & 7 & 5 \\ 3 & 1 & -3 & 13 \\ 2 & 19 & -47 & 32 \end{array}\right]$$

$R_1 \leftrightarrow R_2$ & $R_1 \to R_1 - R_2$

$$\left[\begin{array}{ccc|c} 1 & 4 & -10 & 8 \\ 2 & -3 & 7 & 5 \\ 2 & 19 & -47 & 32 \end{array}\right]$$

$R_2 \to R_2 - 2R_1$ ; $R_3 \to R_3 - 2R_1$

$$\left[\begin{array}{ccc|c} 1 & 4 & -10 & 8 \\ 0 & -11 & 27 & -11 \\ 0 & 11 & -27 & 16 \end{array}\right]$$

$R_2 \to R_2/-11$

$$\left[\begin{array}{ccc|c} 1 & 4 & -10 & 8 \\ 0 & 1 & -27/11 & 1 \\ 0 & 11 & -27 & 16 \end{array}\right]$$

$R_3 \to R_3 + -11 R_2$

$$\left[\begin{array}{ccc|c} 1 & 4 & -10 & 8 \\ 0 & 1 & -27/11 & 1 \\ 0 & 0 & 0 & 5 \end{array}\right] ; \quad \begin{array}{l} R[A:B] = 3 \\ R[A] = 2 \\ R[A:B] \neq R[A] \end{array}$$

$\therefore$ The system is inconsistent

(2) Solve

$x - y + z = 3$

$2x + z = 1$

$3x + 2y + z = 4$

$$\begin{bmatrix} 1 & -1 & 1 \\ 2 & 0 & 1 \\ 3 & 2 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 4 \end{bmatrix}$$

$$[A:B] = \left[\begin{array}{ccc|c} 1 & -1 & 1 & 3 \\ 2 & 0 & 1 & 1 \\ 3 & 2 & 1 & 4 \end{array}\right]$$

~~$R_2 \to R_2/2$~~ $R_2 \to R_2 - 2R_1$ & $R_3 \to R_3 - 3R_1$

~~1 −1 1 3~~

$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & 3 \\ 0 & 2 & -1 & -5 \\ 0 & 5 & -2 & -5 \end{array}\right]$$

$R_2 \to R_2/2$

$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & 3 \\ 0 & 1 & -1/2 & -5/2 \\ 0 & 5 & -2 & -5 \end{array}\right]$$

$R_3 \to R_3 - 5R_2$

$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & 3 \\ 0 & 1 & -1/2 & -5/2 \\ 0 & 0 & 1/2 & 15/2 \end{array}\right]$$

$R_3 \to 2R_3$

$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & 3 \\ 0 & 1 & -1/2 & -5/2 \\ 0 & 0 & 1 & 15 \end{array}\right]$$

$R[A:B] = R[A] = 3 =$ no. of unknowns, so the system has a unique solution.

$$AX = B$$

$$\begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & -1/2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ -5/2 \\ 15 \end{bmatrix}$$

$$z = 15$$

$$y - \frac{1}{2} z = -5/2$$

$$y - \frac{1}{2} \times 15 = -\frac{5}{2}$$

$$y = -\frac{5}{2} + \frac{15}{2} = \frac{10}{2} = \underline{5}$$

$$x - y + z = 3$$

$$x - 5 + 15 = 3$$

$$x = 3 - 10 = \underline{\underline{-7}}$$

3. Show that the equations, $x + 2y - z = 3$,
$3x - y + 2z = 1$
$2x - 2y + 3z = 2$
$x - y + z = -1$ are consistent and solve them.

$$\begin{bmatrix} 1 & 2 & -1 \\ 3 & -1 & 2 \\ 2 & -2 & 3 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 2 \\ -1 \end{bmatrix}$$

$$[A:B] = \left[\begin{array}{ccc|c} 1 & 2 & -1 & 3 \\ 3 & -1 & 2 & 1 \\ 2 & -2 & 3 & 2 \\ 1 & -1 & 1 & -1 \end{array}\right]$$

$R_2 \to R_2 - 3R_1$ ; $R_3 \to R_3 - 2R_1$ ; $R_4 \to R_4 - R_1$

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 3 \\ 0 & -7 & 5 & -8 \\ 0 & -6 & 5 & -4 \\ 0 & -3 & 2 & -4 \end{array}\right]$$

$R_2 \to R_2 + R_3$

$$\left[\begin{array}{cccc} 1 & 2 & -1 & 3 \\ 0 & 1 & 0 & 4 \\ 0 & -6 & 5 & -4 \\ 0 & -3 & 2 & -4 \end{array}\right]$$

$R_3 \to R_3 + 6R_2$

$R_4 \to R_4 + 3R_2$

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 3 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 5 & 20 \\ 0 & 0 & 2 & 8 \end{array}\right]$$

$R_3 \to R_3/5$

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 3 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 2 & 8 \end{array}\right]$$

$R_4 \to R_4 - 2R_3$

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 3 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Rank $[A:B]$ = Rank$[A]$ = 3 = no. of unknowns

∴ The system is consistent & ~~h~~ has unique solution

$z = 4$

$y + 0z = 4$

$y = 4$

~~$x + 2y - y$~~

$x + 2y - z = 3$

$x + 2 \times 4 - 4 = 3$

$x + 8 - 4 = 3$

$x + 4 = 3$

$x = -1$

(4) Test for consistency & if consistent solve,

$4x - 2y + 6z = 8$

$x + y - 3z = -1$

$15x - 3y + 9z = 21$

$$\begin{bmatrix} 4 & -2 & 6 \\ 1 & 1 & -3 \\ 15 & -3 & 9 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 8 \\ -1 \\ 21 \end{bmatrix}$$

$$[A : B] = \begin{bmatrix} 4 & -2 & 6 & 8 \\ 1 & 1 & -3 & -1 \\ 15 & -3 & 9 & 21 \end{bmatrix}$$

$R_1 \leftrightarrow R_2$

$$\begin{bmatrix} 1 & 1 & -3 & -1 \\ 4 & -2 & 6 & 8 \\ 15 & -3 & 9 & 21 \end{bmatrix}$$

$R_2 \to R_2 - 4R_1$ & $R_3 \to R_3 - 15R_1$

$$\begin{bmatrix} 1 & 1 & -3 & \vdots & -1 \\ 0 & -6 & 18 & \vdots & 12 \\ 0 & -18 & 54 & \vdots & 36 \end{bmatrix}$$

$R_2 \to R_2/-6$

$$\begin{bmatrix} 1 & 1 & -3 & , & -1 \\ 0 & +\cancel{6} & -3 & , & -2 \\ 0 & -18 & 54 & , & 36 \end{bmatrix} \xrightarrow{R_3 \to R_3 + 18R_2} \begin{bmatrix} 1 & 1 & -3 & -1 \\ 0 & 1 & -3 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$R[A:B] = R[A] = 2 <$ no. of unknowns

$\therefore$ The system has infinite no of solutions

$$\begin{bmatrix} 1 & 1 & -3 \\ 0 & 1 & -3 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -1 \\ -2 \\ 0 \end{bmatrix}$$

$y - 3z = -2$

$x + y - 3z = -1$

Here we have only 2 equations with 3 unknowns.
So the system has infinite no of solutions.

To get the solution, put any one of the variables $= k$, an arbitrary constant say put $z = k$

Then $y = -2 + 3k$

$x + -2 + 3k - 3k = -1$

$x - 2 = -1$

$x = -1 + 2 = 1$

$\therefore$ The solutions are $x = 1$, $y = -2 + 3k$, $z = k$

(5) For what value of $\lambda$ & $\mu$, the system of equations

$x + y + z = 6$, $x + 2y + 3z = 10$,

$x + 2y + \lambda z = \mu$, have (i) No solution

(ii) infinite no. of soln (iii) unique solution.

we have,

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 2 & \lambda \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 10 \\ \mu \end{bmatrix}$$

$[A:B]$

$$\begin{bmatrix} 1 & 1 & 1 & : & 6 \\ 1 & 2 & 3 & : & 10 \\ 1 & 2 & \lambda & : & \mu \end{bmatrix}$$

$R_2 \to R_2 - R_1$ & $R_3 \to R_3 - R_1$

$$\sim \begin{bmatrix} 1 & 1 & 1 & : & 6 \\ 0 & 1 & 2 & : & 4 \\ 0 & 1 & \lambda-1 & : & \mu-6 \end{bmatrix}$$

$R_3 \to R_3 - R_2$

$$\sim \begin{bmatrix} 0 & 1 & 1 & : & 6 \\ 0 & 1 & 2 & : & 4 \\ 0 & 0 & \lambda-3 & : & \mu-10 \end{bmatrix}$$

(i) The system has no solution if $R[A] \neq R[A:B]$

ie $\lambda = 3$ and $\mu \neq 10$

(ii) The system has infinite no. of solutions if $R[A] = R[A:B] <$ no. of unknowns.

ie $\lambda = 3$ & $\mu = 10$

(iii) The system has unique solution if

$\lambda \neq 3$ & $\mu$ = any value

H.W I Solve the system of equations
$5x + 3y + 7z = 4$
$3x + 26y + 2z = 9$
$7x + 2y + 11z = 5$

H.W II Find the values of $\lambda$ and $\mu$ so that the equations $2x+3y+5z=9$, $7x+3y-2z=8$, $2x+3y+\lambda z=\mu$ may have (i) No solution (ii) Infinite no. of solutions (iii) Unique solution.

## Solution of Linear Homogenous Systems of Equations

A linear system (1) is called homogenous if all the $b_j$'s are zero and non homogenous if one or several $b_j$'s are not zero. ie An equation of the form $AX=0$ is homogenous equation.

For homogenous system we obtain from the fundamental theorem, the following results

**Theorem.**

A Linear homogenous system,

$$a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = 0$$
$$a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = 0$$
$$\dots$$
$$a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n = 0$$

always has the trivial solution $x_1=0, x_2=0, \dots, x_n=0$. Nontrivial solutions exist if and only if rank $A<n$.

**Theorem**

A homogenous linear system with fewer equations than unknowns always has nontrivial solutions.

ie If $R[A]=$ no. of unknowns, then the system has trivial solution.

If $R[A]<$ no. of unknowns, the system has infinite no. of solutions.

1. Solve the equations,

$x_1+3x_2+2x_3=0$, $2x_1-x_2+3x_3=0$

$3x_1-5x_2+4x_3=0$, $x_1+17x_2+4x_3=0$

$$\begin{bmatrix} 1 & 3 & 2 \\ 2 & -1 & 3 \\ 3 & -5 & 4 \\ 1 & 17 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$R_2 \to R_2 - 2R_1, \quad R_3 \to R_3 - 3R_1 \quad \& \quad R_4 \to R_4 - R_1$

$$\begin{bmatrix} 1 & 3 & 2 \\ 0 & -7 & -1 \\ 0 & -14 & -2 \\ 0 & 14 & 2 \end{bmatrix}$$

$R_2 \to \frac{R_2}{-7} \quad \& \quad R_3 \to R_3 + R_4$

$$\begin{bmatrix} 1 & 3 & 2 \\ 1 & 1 & 1/7 \\ 0 & 0 & 0 \\ 0 & 14 & 2 \end{bmatrix}$$

$R_4 \to R_4 - 14R_2$

$$\sim \begin{bmatrix} 1 & 3 & 2 \\ 0 & 1 & 1/7 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$R(A) = 2$

$x_2 + \frac{x_3}{7} = 0 \quad — (1)$

$x_1 + 3x_2 + 2x_3 = 0 \quad — (2)$

Put $x_3 = k$

$x_2 = -k/7$

$x_1 + 3x_2 + 2x_3 = 0.$

$x_1 = \frac{3k}{7} - 2k = \frac{3k - 14k}{7} = \frac{-11k}{7}$

$\therefore x_1 = \frac{-11k}{7}, \quad x_2 = \frac{-k}{7}, \quad x_3 = k$

(2) Show that the system of equations,

$2x_1 - 2x_2 + x_3 = \lambda x_1$

$2x_1 - 3x_2 + 2x_3 = \lambda x_2$

$-x_1 + 2x_2 = \lambda x_3$ can posses a non trivial solution only if $\lambda = 1$ or $\lambda = -3$.

The given system of equations can be written as,

$$(2-\lambda)x_1 - 2x_2 + x_3 = 0$$
$$2x_1 - (3+\lambda)x_2 + 2x_3 = 0$$
$$-x_1 + 2x_2 - \lambda x_3 = 0.$$

This is a homogeneous equation of the form $AX = 0$. This system

This system possess non-trivial solution only when $|A| = 0$.

{when $|A| \neq 0$, rank $= 3 =$ no. of unknowns $\Rightarrow$ trivial solution}

$$\begin{vmatrix} 2-\lambda & -2 & 1 \\ 2 & -(3+\lambda) & 2 \\ -1 & 2 & -\lambda \end{vmatrix} = 0$$

$$\text{i.e. } (2-\lambda)\left[(\lambda(3+\lambda)-4)\right] + 2\left[-2\lambda+2\right] + 1\left[4+3+\lambda\right] = 0$$

$$\text{i.e. } (2-\lambda)(3\lambda+\lambda^2-4) - 4\lambda + 4 + 7 + \lambda = 0$$

$$6\lambda - 3\lambda^2 + 2\lambda^2 - \lambda^3 - 8 + 4\lambda - 4\lambda + 4 + 7 + \lambda = 0$$

$$\text{i.e. } -\lambda^3 - \lambda^2 + 7\lambda + 3 = 0.$$

H.W. Find the value of $k$, so that the equations

$$x + y + 3z = 0$$
$$4x + 3y + kz = 0$$
$$2x + y + 2z = 0$$

have a non-trivial solution.

## Linear transformation

Let the point $P(x, y)$ in a plane can be transformed into the point $P'(x', y')$ either under the reflection about the co-ordinate axis or by the rotation through an angle $\theta$. Then the co-ordinates of the point $P'$ can be represented generally as,

$$x' = a_1 x + b_1 y \quad \&$$

$$y' = a_2 x + b_2 y.$$ This can be written in matrix form as,

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

ie $Y = AX$

This $Y = AX$ is called linear transformation.

A is called the matrix of transformation.

If A is a non singular matrix, then that transformation is non-singular or regular.

If A is singular, then that transformation is called singular transformation.

## Matrix Eigen value Problem

## Eigen values & Eigen vectors

If $A$ is a square matrix of order $n$ and $\lambda$ is an unknown constant, then we can form a matrix $A - \lambda I$.

The determinant of this matrix which is equal to '0' is called the characteristic equation.

i.e. $(|A - \lambda I| = 0)$

{ Consider the linear transformation $y = Ax$.

Let us take $A$ as a scalar, say $\lambda$.

i.e. $y = \lambda x$

i.e. $Ax = \lambda x$

$$(A - \lambda I)x = 0. \qquad \text{--- (i)}$$

This system of equation has non-trivial solutions only when $|A - \lambda I| = 0$, which is called the characteristic equation. }

• $|A - \lambda I| = 0$ is called the characteristic equation of the matrix $A$.

The problem of finding nonzero $x$'s and $\lambda$'s that satisfy equation(i) is called an eigen value problem.

The values of $\lambda$ are called Eigen values and corresponding to each $\lambda$, the solution of the system of equations $(A - \lambda I)x = 0$ is called Eigen vectors

Prob(4) Find the Eigen values and Eigen vectors of the matrix $A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$

The characteristic equation is

$$|A - \lambda I| = 0.$$

$$\left| \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} \right| = 0$$

or $\begin{vmatrix} 1-\lambda & 2 \\ 2 & 1-\lambda \end{vmatrix} = 0$

or $(1-\lambda)^2 - 4 = 0.$

$1 - 2\lambda + \lambda^2 - 4 = 0$

$\lambda^2 - 2\lambda - 3 = 0$

$(\lambda - 3)(\lambda + 1) = 0$

$\lambda = 3$ or $\lambda = -1$

So Eigen values are $3, -1$

To find Eigen vectors,

Case I when $\lambda = 3$,

$$[A - \lambda I] = \begin{bmatrix} 1-3 & 2 \\ 2 & 1-3 \end{bmatrix} = \begin{bmatrix} -2 & 2 \\ 2 & -2 \end{bmatrix}$$

$R_1 \to R_1/-2$

$$\sim \begin{bmatrix} 1 & -1 \\ 2 & -2 \end{bmatrix}$$

$R_2 \to R_2 + 2R_1$

$$\sim \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$$

which is in echelon form. $AX = \lambda X$

$$\Rightarrow \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$x_1 - x_2 = 0.$

$x_1 = x_2$

Put $x_2 = a$

$x_1 = a$

Eigen vector $X = \begin{bmatrix} a \\ a \end{bmatrix} = a\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Case II

when $\lambda = -1$

$$[A - \lambda I] = \begin{bmatrix} 1-(-1) & 2 \\ 2 & 1-(-1) \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}$$

$R_1 \to R_1/2$

$$\sim \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$$

$R_2 \to R_2 - 2R_1$

$$\sim \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$$

which is in Echelon form,

$$\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$x_1 + x_2 = 0$

$x_1 = a$

$x_2 = -a$

Eigen vector $X = \begin{bmatrix} a \\ -a \end{bmatrix} = a\begin{bmatrix} 1 \\ -1 \end{bmatrix}$

Prob Find Eigen values ~~& Eigen vectors of the~~ matrix

$$\begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}$$

(16)