Partial Ordering

A relation on a set S is called Partial ordering or Partial order, if it is reflexive anti-symmetric and transitive.

ie (1) $a R a \ \forall a \in S$ (reflexive)

(2) aRb and $bRa \Rightarrow a=b$ (anti symmetric)

3. aRb and $bRc \Rightarrow aRc$ (transitive)

A set S, together with a partial order R is called a partially ordered set or Poset. It is denoted by (S, R)

$\leq$ is ~~usually~~ generally used to denote partial order relation.

ie $S(\leq)$ is used to denote the partial ordering.

eg. (1) Let R be a set of real numbers. Then the relation $\leq$ (less than or equal to) is partial ordering on R.

(2) The relation 'subset of $(\subseteq)$' is a partial ordering relation on power set of a non empty set.

Let X be a non-empty set and let $P(X)$ be its power set then $P(X)$ is the set of all subsets of X.

Let $A, B, C \in P(X)$

(1) $A \subseteq A$ for all A (reflexive)

(ii) $A \subseteq B$ & $B \subseteq A \Rightarrow A = B$ (antisymmetric)

(iii) $A \subseteq B$ & $B \subseteq C \Rightarrow A \subseteq C$ (transitivity)

Hence $\subseteq$ is a P.O.R (Partially ordered relation).

eg(3) Show that in the set of positive integers Z^+. The relation divides (/) is a poset.

Let $n, m, p \in X$

(1) $n/n \quad \forall n \in Z^+$ (reflexive)

(2) n/m and $m/n \Rightarrow n = m$ (antisymmetric)

(3) n/m & $m/p \Rightarrow n/p$ (transitive)

[but for all integers, it is not true, since $a/-a$ & $-a/a$ but they are not equal.]

4. Let $A = \{1, 2, 3\}$ show that the relation $\leq$ (less than or equal to) is a partially ordered relation on A.

n-ary relations

An n-ary relation on sets $A_1, A_2 \ldots A_n$ is a set of ordered n-tuples $\{(a_1, a_2 \ldots a_n)\}$ where a_i is an element of A_i for all i, $1 \leq i \leq n$.

Thus n-ary relation on sets $A_1, A_2 \ldots A_n$ is a subset of cartesian product $A_1 \times A_2 \times \ldots \times A_n$.

eg: Let A_1 be the a set of names, A_2 set of addresses and A_3 set of phone numbers

Then a set of 3-tuples <name, address, phone no>

Such as {(Rita, peace villa, 9966332422) (Devi, Sreenilayam, 2224445553) (udaya, Anns villa, 3332243357)}

(ternary)
is a 3-ary Relation over A_1 A_2 & A_3.

eg2) Let A_1 be a set of names. Then a set of 1-tuples such as {(Amy) (Barbara) (charles)} is a 1-ary (unary) relation over A_1

3. Let A_1, be a set of names & A_2 be a set of Marks. Then the 2-tuples such as

{(Mary, 80) (Emy 95) (Joe 90)} is a 2-ary binary ~~relat~~ relation over A_1 & A_2.

functions

A relation f from a set X to another set Y is called a function, if for every $x \in X$, there is a unique $y \in Y$, such that $(x, y) \in f$

ie A ~~total~~ function f from X to Y is an assignment of exactly one element of Y to every element of X.

ie $f(x) = y$.

If 'f' is a function from x to y, we represent it as $f: X \to Y$ or $X \xrightarrow{f} Y$

Some times the terms transformation, mapping or Correspondence are also used in the place of function.

If $y = f(x)$, x is called an argument or Pre-image and y is called the image of x under f or value of the function f at x.

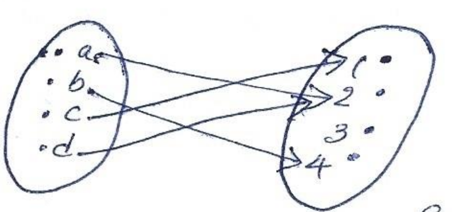
X is Called the domain of f denoted by D.f and Y is called the co-domain of f. The set Consisting of all the images of the elements of x under the function 'f' is called the range of f. It is denoted by $f(x)$

$\therefore$ Range of $f = \{f(x) / \text{for all } x \in A\}$

$$f(x) \subseteq Y \text{ (co-domain)}.$$

Representation of a function

A function can be expressed by means of a mathematical rule or formula, such as $y = x^3$ or relation matrix (since f is a relation) or a graph. when there are only few elements, f can be represented pictorically as



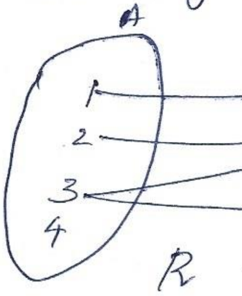
here domain $D = \{a, b, c, d\}$

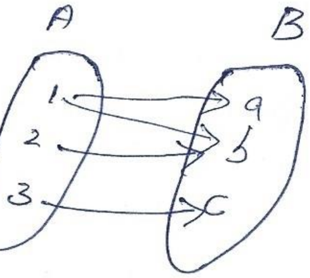
$f(a) = 2$; $f(b) = 4$

$f(c) = 1$, $f(d) = 2$.

Range $= \{1, 2, 4\}$ which is a subset of the Co-domain $\{1, 2, 3, 4\}$

The following relations are not functions.





Types of functions

(1) One-to-one (1-1) or injective function

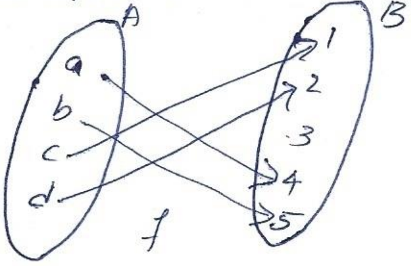
A function $f: X \to Y$ is called one-to-one (1-1) or injective if distinct elements of X are mapped into distinct elements of Y

or f is one-to-one iff

$f(x_1) \neq f(x_2)$ whenever $x_1 \neq x_2$.

or $f(x_1) = f(x_2)$ whenever $x_1 = x_2$.

eg(1)



Here the function is one-to-one.

eg(2). $f(x) = 3x - 1$ is one-to-one because

$f(x_1) = 3x_1 - 1$

$f(x_2) = 3x_2 - 1$

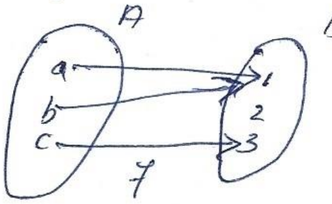
$f(x_1) = f(x_2) \Rightarrow 3x_1 - 1 = 3x_2 - 1$

$\Rightarrow x_1 = x_2$

(2) Many-to-one function

A function f from A to B is said to be many-to-one iff two or more elements of A have same image in B.

eg(1)



eg(2) Let $f(x) = x^2$; where x is any real number and $f: R \to R$ then f is many-to-one function.

i.e. for $x = 1$, $f(x) = x^2 = 1^2 = 1$

when $x = -1$; $f(x) = -1^2 = 1$

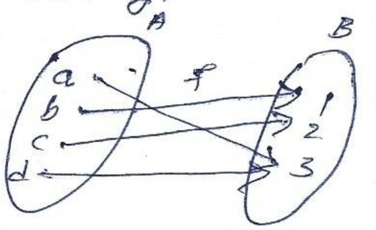
$\therefore f(1) = f(-1)$

(3) Onto or Surjective function

A function $f: X \to Y$ is called onto or surjective if Range of f is Y.

or

A function f is onto, iff for every element $y \in Y$, there is an element $x \in X$, such that $f(x) = y$.

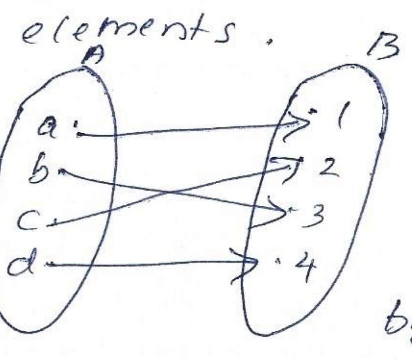


here f is an onto function.

(4) Bijective function (one-to-one onto)

A function $f: X \to Y$ is called bijective or 1-1 correspondence, if f is both one-to-one (injective) and onto (surjective).

$\&$: ie if X & Y are finite such that $f: X \to Y$ is bijective then X & Y have the same number of elements.

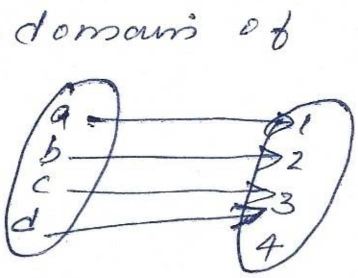


bijective function.

(5) Into function.

A function f from X to Y is called into function, iff there exists at least one element in Y which is not the image of any element in X.

ie The range of f is a proper subset of Co-domain of f.



into function.

Composition of functions

If $f: A \to B$ and $g: B \to C$ then the composition of f & g is a new function from A to C denoted by $g \circ f$, given by

$$(g \circ f)(x) = g\{f(x)\} \quad \text{for all } x \in A.$$

To find $(g \circ f)(x)$, we first find the image of x under f, and then find the image of $f(x)$ under g.

Thus the range of f is the domain of g.
$g \circ f$ (read as g of f) is also called the relative product of the functions f and g or left composition of g with f.

g: Let $A = \{1, 2, 3\}$ $B = \{a, b\}$
$C = \{r, s\}$ & $f: A \to B$ be defined by
$f(1) = a$; $f(2) = a$; $f(3) = b$ & $g: B \to C$ be defined by $g(a) = s$ & $g(b) = r$
Then $g \circ f: A \to C$ is defined by,

$$(g \circ f)(1) = g\{f(1)\} = g(a) = s$$

$$(g \circ f)(2) = g\{f(2)\} = g(a) = s$$

$$(g \circ f)(3) = g\{f(3)\} = g(b) = r$$

<u>Note</u> Composition of functions is not commutative.
i.e. $f \circ g \neq g \circ f$.

Prob (1) If $f: R \to R$ and $g: R \to R$ are defined by
$f(x) = x + 2$ for all x in R
and $g(x) = x^2$ $\forall x$ in R.
Then find $f \circ g$ & $g \circ f$ and P.T. $f \circ g \neq g \circ f$.

$(g \circ f)(x) = g(f(x)) = g(x+2)$
$= (x+2)^2 = x^2 + 4x + 4$

$(f \circ g)(x) = f(g(x)) = f(x^2) = x^2 + 2$

$x^2 + 4x + 4 \neq x^2 + 2$

$\therefore g \circ f \neq f \circ g$

(2) $f : R^+ \to R^+$ & $g : R^+ \to R^+$ are defined by the formula, $f(x) = \sqrt{x}$ & $g(x) = 3x + 1$ $\forall x \in R^+$, find $f \circ g$ & $g \circ f$.

$(f \circ g)(x) = f(g(x)) = f(3x+1) = \sqrt{3x+1}$

$(g \circ f)(x) = g(f(x)) = g(\sqrt{x}) = 3\sqrt{x} + 1$

$f \circ g \neq g \circ f$

Theorem: Associative Law of Composition of functions

Let $f : A \to B$, $g : B \to C$ and $h : C \to D$
Then P.T. $h \circ (g \circ f) = (h \circ g) \circ f$.

Proof

Since $f : A \to B$ & $g : B \to C$ & $h : C \to D$ then
$g \circ f : A \to C$ & $h \circ g : B \to D$
Since $g \circ f : A \to C$ & $h : C \to D$
$h \circ (g \circ f) : A \to D$
Also $f : A \to B$ & $h \circ g : B \to D$
$(h \circ g) \circ f : A \to D$
Thus the domain & Co-domain of $h \circ (g \circ f)$ and $(h \circ g) \circ f$ are the same.

Let $x \in A$, $y \in B$ and $z \in C$ such that $f(x) = y$, $g(y) = z$

Then $[(h \circ g) \circ f](x) = (h \circ g)[f(x)]$

$= h[g[f(x)]]$

$= h[g(y)] = h(z)$ — (1)

$(h \circ (g \circ f))(x) = h \circ [(g \circ f)(x)]$

$= h \circ (g \circ f(x)) = h(g(y))$

$= h(z)$ — (2)

from (1) & (2) we have,

$[(h \circ g) \circ f](x) = [h \circ (g \circ f)](x) \quad \forall x \in A.$

$\therefore (h \circ g) \circ f = h \circ (g \circ f)$

Theorem

Let $f: A \to B$ & $g: B \to C$ be functions,

Then (a) If f & g are injections then $g \circ f: A \to C$ is an injection.

(b) If f & g are surjections then $g \circ f$ is surjection.

(c) If f & g are bijections then $g \circ f$ is bijection.

Proof

(i) Let $a_1, a_2 \in A$.

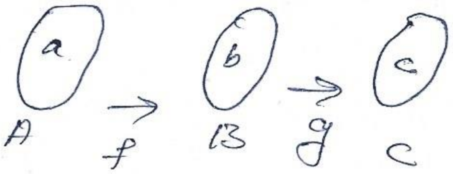
Then $(g \circ f)(a_1) = (g \circ f)\, a_2 \implies$

$g(f(a_1)) = g(f(a_2)) \implies f(a_1) = f(a_2)$ (since g is injective)

$\implies a_1 = a_2$ (since f is injective)

$\therefore g \circ f$ is injective.

Let $c \in C$



Since g is onto there is an element $b \in B$ s.

$c = g(b)$.

Since f is onto there is an element $a \in A$ s.

$b = f(a)$

Now $(g \circ f)(a) = g\{f(a)\} = g(b) = c$

This means $g \circ f : A \to C$ is onto

ie Corresponding to each element $c \in C$ there is a preimage in A.

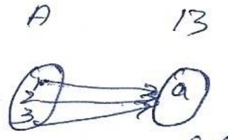
$\therefore g \circ f$ is onto.

(iii) Since $g \circ f$ is one-to-one & onto, $g \circ f$ is bijective

Constant function

A function f is said to be a constant function if its range is a singleton set.

ie



ie $f(1) = f(2) = f(3) = a$. ie $f(x) = a \ \forall x \in A$

Identity function

The function $f : A \to A$ defined by $f(x) = x$ for every $x \in A$ is called the identity of A and is ~~deno~~ denoted by I_A.

ie Every element of A assigns each element to itself.

The function I_A is one-to-one & onto.

$I_A = \{(a,a) / a \in A\}$

The composition of any function with the identity function is the function itself.

Inverse of a function

If $f: A \to B$ and $g: B \to A$, then the function g is called the inverse of the function f, if $gof = I_A$ and $fog = I_B$.

If $x \in A$ and $y \in B$ then

$(gof)(x) = I_A(x)$

i.e. $g\{f(x)\} = x$ —(1)

Also $(fog)(y) = I_B(y)$

$f(g(y)) = y$ —(2)

from (1) & (2) we have

if $y = f(x)$ then $x = g(y)$ and vice versa.

Thus the function $g: B \to A$ is called the inverse of $f: A \to B$, if $x = g(y)$ whenever $y = f(x)$

Inverse of f is denoted by f^{-1}.

Thus if f^{-1} is the inverse of f, then $x = f^{-1}(y)$, whenever $y = f(x)$

The necessary & sufficient condition for $f: A \to B$ to have the inverse $f^{-1}: B \to A$ is that f is bijective.

Property

The inverse of a function f, if exists, is unique.

Proof

If possible let g and h be the inverse of f.

Then by definition $gof = I_B$ & $fog = I_A$ —(1)

Also $hof = I_B$ and $foh = I_A$ —(2)

$h = hoI_A = ho(fog)$ [by (1)]

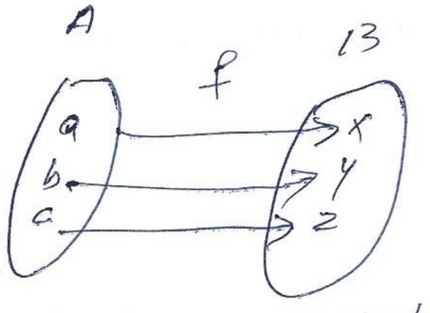
$= (hof)og$ (by associativity)

$= I_Bog$ [by (2)]

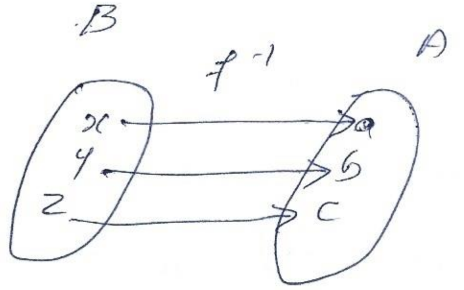
$= g$.

i.e. $h = g$: Thus the inverse of a function f, if exists, is unique.

eg.. Let the function $f: A \to B$ be defined by the following diagram



Then f is 1-1 & onto. Therefore f^{-1}, the inverse function exists.
$f^{-1}: B \to A$ is given by the following diagram,



Prob (1) Show that the functions $f(x) = x^3$ and $g(x) = x^{1/3}$ for all $x \in R$, are inverses of each other.

Since $(f \circ g)x = f(x^{1/3}) = x = Ix$ and

$(g \circ f)x = g(x^3) = x = Ix$

$\underline{f = g^{-1} \text{ or } g = f^{-1}}$

(2) Let $A = \{1, 2, 3, 4\}$ and $B = \{a, b, c, d\}$

and let $f = \{(1, a)(2, a)(3, d)(4, c)\}$

Show that f is a function & f^{-1} is not a function.

We have $f(1) = a$; $f(2) = a$, $f(3) = d$; $f(4) = c$

Since each set $f(x)$ is a single value, f is a function. Now $f^{-1} = \{(a, 1)(a, 2)(d, 3)(c, 4)\}$
f^{-1} is not a function since $f^{-1}(a) = \{1, 2\}$

(3) If f is a function from $S = \{0, 1, 2, 3, 4, 5\}$ to S, defined by $4x \bmod (6)$, write f as as a set of ordered pairs. Is the function f one-one or onto.

Solution

$f(x)$ is the remainder when $4x$ is divided by 6.

$\therefore f(0) = 0$, $f(1) = 4$ $f(2) = 2$, $f(3) = 0$, $f(4) = 4$

$f(5) = 2$.

Hence $f = \{(0, 0)(1, 4)(2, 2)(3, 0)(4, 4)(5, 2)\}$

Since $f(0) = f(3) = 0$, $f(1) = f(4) = 4$ and

$f(2) = f(5) = 2$. The function is not one-to-one.

Also the range of $f = (0, 2, 4) \neq S$.

Hence f is not onto

(4) Let $f: X \to Y$ be an invertible function and A & B are non-empty subsets of Y, then show that,

(1) $f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B)$

(2) $f^{-1}(A \cap B) = f^{-1}(A) \cap f^{-1}(B)$

Solution

(1) Let $x \in X$ then

Let $x \in f^{-1}(A \cup B) \iff f(x) \in A \cup B$

$\iff f(x) \in A$ or $f(x) \in B$

$\iff x \in f^{-1}(A)$ or $x \in f^{-1}(B)$

$\iff x \in f^{-1}(A) \cup f^{-1}(B)$

$\therefore f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B)$

(2) Let $x \in f^{-1}(A \cap B)$

$\iff f(x) \in A \cap B$

$\iff f(x) \in A$ and $f(x) \in B$

$\iff x \in f^{-1}(A)$ and $x \in f^{-1}(B)$

$\iff x \in f^{-1}(A) \cap f^{-1}(B)$

$\therefore f^{-1}(A \cap B) = f^{-1}(A) \cap f^{-1}(B)$

1. Let f & g be the functions from the set of integers such that

$f(x) = 2x + 3$ and $g(x) = 3x + 2$

Determine fog and gof. Also show that $fog \neq gof$.

2. If $A = \{1, 2, 3, 4, 5\}$; $B = \{1, 2, 3, 8, 9\}$ and the functions $f: A \to B$ and $g: A \to A$ are defined by $f = \{(1,8)(3,9)(4,3)(2,1),(5,2)\}$ and $g = \{(1,2)(3,1)(2,2)(4,3)(5,2)\}$ find fog, gof, fof and gog if they exists.

3. If $S = \{1, 2, 3, 4, 5\}$ and if the functions $f, g, h: S \to S$ are given by

$f = \{(1,2)(2,1)(3,4)(4,5)(5,3)\}$

$g = \{(1,3)(2,5)(3,1)(4,2),(5,4)\}$

$h = \{(1,2)(2,2)(3,4)(4,3)(5,1)\}$

(a) Verify whether $fog = gof$.

(b) Explain why f & g have inverses and h does not

(c) find f^{-1} & g^{-1}

(d) show that $(fog)^{-1} = g^{-1} o f^{-1} \neq f^{-1} o g^{-1}$

(4) If $f, g, h: R \to R$ are defined by $f(x) = x^3 - 4x$, $g(x) = \frac{1}{x^2+1}$ and $h(x) = x^4$ find $\{(fog)oh\}(x)$ and $\{fo(goh)\}(x)$. Also check if they are equal

(5) A function f is defined on the set of integers as follows.

$$f(x) = \begin{cases} x & \text{if } 0 \leq x < 1 \\ x+2 & \text{if } 1 \leq x < 3 \\ 4x-5 & \text{if } 3 \leq x < 5 \end{cases}$$

(i) Find the domain of the function
(ii) Find the range of the function
(iii) state whether f is one-one or many-one function.