Fuzzy set: A fuzzy set is defined as the set of pairs of an element along with their degree of membership (i.e. A real number between 0 & 1 including 0 & 1)

eg: F = {0.7 Gora, 0.9 Somnath, 0.3 Gautam, 0.2 Rana, 0.5 Monu}

R = {0.5 Gora, 0.8 Somnath, 0.2 Gautam, 0.7 Rana, 0.6 Monu}

Complement of fuzzy set.

The complement of fuzzy set S is ~~the~~ S' with the degree of membership of an element in S' = 1 − the degree of membership of these element in S.

F' = {0.3 Gora, 0.1 Somnath, 0.7 Gautam, 0.8 Rana, 0.4 Monu}

R' = {0.5 Gora, 0.2 Somnath, 0.8 Gautam, 0.3 Rana, 0.4 Monu}

Union - The union of a fuzzy set S & T is the fuzzy set of S∪T where the degree of membership of an element in S∪T is the maximum of the degree of membership of this elements in S & T.

F ∪ R = {0.7 Gora, 0.9 Somnath, 0.3 Gautam, 0.7 Rana, 0.6 Monu}

Intersection - The intersection of a fuzzy set S & T is the fuzzy set S∩T in which the degree of members of an element in S∩T is the minimum of the degree of membership of that element in S and T.

$F \cap R = \{0.5$ Nora, 0.8 Somnath, 0.2 Gautam, 0.2 Rana, 0.5 Monu$\}$

Relations

A relation is a structure that represents the relationship of elements of a set to the elements of another set.

- The simplest way to express a relationship between elements of two sets is to use ordered pairs consisting of two related elements.

- Let A and B be 2 sets. A relation from A to B is a subset of the cartesian product A × B.

Suppose R is a relation from A to B. Then R is a set of ~~subset~~ ordered pairs (a, b) where a ∈ A & b ∈ B, where (a, b) ∈ R, we use the notation aRb read as "a is related to b by R".

If (a, b) ∉ R, it is denoted as a $\not R$ b

Domain & Range – The set $\{a \in A \mid (a, b) \in R$ for some $b \in B\}$ is called the domain of R and denoted by $Dom(R)$ or $D(R)$

The set $\{b \in B \mid (a, b) \in R$ for some $a \in A\}$ is called the range of R and denoted by $Ran(R)$ or $R(R)$

Prob (1) Let $A = \{2, 3, 4\}$; $B = \{3, 4, 5\}$. List all elements of each relation R defined below. Also write it's domain & Range.

1. $a \in A$ is related to $b \in B$ iff $a < b$

i.e. $a R b$ iff $a < b$.

$A \times B = \{(2,3)(2,4)(2,5)(3,3)(3,4)(3,5)(4,3)$
$(4,4)(4,5)\}$

R is $\{(2,3)(2,4)(2,5)(3,4)(3,5)(4,5)\}$

Domain $D(R) = \{2, 3, 4\}$
Range $R(R) = \{3, 4, 5\}$

(2) $a \in A$ is related to $b \in B$ if a & b are odd.

R is $\{(3,3)(3,5)\}$

$D(R) = \{3\}$; Range $R(R) = \{3, 5\}$

Prob (2) Let $S = \{x\ y\}$ and S^2 is the set of all words of length 2.
(i) Find the elements of S^2
(ii) The relation R on S^2 is defined by $v R w$ means that the first letter in v is the same as the first letter in w when v & w are in S^2.

write R as a set of ordered pairs.

$S^2 = \{xx, xy, yx, yy\}$

$R = \{(xx, xy)(xy, xx)(yx, yy)(yy, yx)\}$

Types of Relations

1. Universal relation.

A relation R on a set A is called a universal relation, if $R = A \times A$.

eg. If $A = \{1, 2, 3\}$ then $R = A \times A$

$= \{(1,1)(1,2)(1,3)(2,1)(2,2)(2,3)(3,1)(3,2)(3,3)\}$ is the universal relation on A.

(2) Void Relation :- A relation R on a set A is called a void relation if R is the null set.

eg: Let $A = \{3, 4, 5\}$ and R is defined as

aRb iff $A + B > 10$, Then R is a null set since no element in $A \times A$ satisfies the given condition.

(3) Identity Relation - A relation on a set A is called an identity relation if $R = \{(a,a) / a \in A\}$ and denoted by I_A.

eg. If $A = \{1, 2, 3\}$; $R = \{(1,1)(2,2)(3,3)\}$ is the identity relation on A.

(4) Inverse of a Relation - R^{-1}

If R is any relation from set A to set B the inverse of R denoted by R^{-1} is the relation from B to A which consists of those ordered pairs got by interchanging the elements of the ordered pairs in R.

$R^{-1} = \{(b, a) / (a, b) \in R\}$

eg. If $A = \{2, 3, 5\}$; $B = \{6, 8, 10\}$

and aRb if and only if $a \in A$ divides $b \in B$,

then $R = \{(2, 6)(2, 8)(2, 10)(3, 6)(5, 10)\}$

$R^{-1} = \{(6, 2)(8, 2)(10, 2)(6, 3)(10, 5)\}$

we can see that

$D(R) = R(R^{-1}) = \{2, 3, 5\}$

$R(R) = D(R^{-1}) = \{6, 8, 10\}$

Classification of relations / Properties

1. Reflexive relation - A relation R on a set A is reflexive if aRa for every $a \in A$ ie if $(a, a) \in R$ for every $a \in A$ ie every elt ie of A is related to itself.

eg (1) If $R_1 = \{(1, 1)(1, 2)(2, 2)(2, 3)(3, 3)\}$ be a relation on $A = \{1, 2, 3\}$ then R_1 is Reflexive since for every $a \in A$, $(a, a) \in R_1$

eg (2) If $R_2 = \{(1, 1)(1, 2)(2, 3)(3, 3)\}$ be a relation on $A = \{1, 2, 3\}$ then R_2 is not reflexive relation since $(2, 2) \notin R$ for $2 \in A$

eg (3) $R_3 = \{(x, y) \in R^2 / x \leq y\}$ is a reflexive relation, since $x \leq x$ for any $x \in R$.

Irreflexive relation - A relation R on a set A is irreflexive if for every $a \in A$, $(a, a) \notin R$.
~~is~~ ie There is no $a \in A$ such that aRa.

eg: Let
① $R_1 = \{(1, 2)(1, 3)(2, 1)(2, 3)\}$ on $A = \{1, 2, 3\}$
Then R_1 is irreflexive since $(x, x) \notin R_1$ for every $x \in$ ~~A~~ R.

(2) The relation $R_2 = \{(x, y) \in R^2 / x < y\}$ is an irreflexive relation since $x < x$ for no $x \in R$.

Non reflexive relation - A relation R on a set A is non-reflexive if R is neither reflexive nor irreflexive. ie if aRa is true for some a and false for other.

eg: $R = \{(1, 2)(2, 3)(2, 2)(3, 1)\}$ on $A = \{1, 2, 3\}$ is a non-reflexive relation since $2R2$ is ~~not~~ true, but $1R1$ & $3R3$ are flalse.

Symmetric Relation -

A relation R on a set A is said to be symmetric, if whenever aRb then bRa,
ie if whenever $(a, b) \in R$ then (b, a) also $\in R$.

eg ① Let $A = \{1, 2, 3\}$; $R = \{(2, 2)(2, 3)(3, 2)\}$
Then R is symmetric since both $(2, 3)$ & $(3, 2) \in R$.

A relation R on A is not symmetric if there exist $a, b \in A$ such that $(a, b) \in R$ but $(b, a) \notin R$.

(2) The relation of perpendicularity on a set of lines in a plane is symmetric.

(3) Anti symmetric – A relation R on a set A is said to be antisymmetric whenever (a, b) and $(b, a) \in R$ then $a = b$.
If there exist $a, b \in A$ such that (a, b) & $(b, a) \in R$ but $a \neq b$, then R is not antisymmetric.

e.g. The relation of divisibility on N is antisymmetric since whenever m is divisible by n and n is divisible by m, then $m = n$.

Asymmetric Relation – A relation R on a set A is asymmetric if whenever $(a, b) \in R$ then $(b, a) \notin R$ for $a \neq b$.
i.e if $aRb \Rightarrow b \not R a$.
i.e The presence of (a, b) in R excludes the presence of (b, a) in R.

(4) Transitive relation – A relation R on a set A is said to be transitive, if whenever $a R b$ and $b R c$ then $a R c$.
i.e if whenever (a, b) and $(b, c) \in R$, then $(a, c) \in R$.

e.g.(1) The relation of set inclusion on a collection of sets is transitive, since if $A \subseteq B$ & $B \subseteq C$ then $A \subseteq C$.

eg (2) Let $A = \{1, 2, 3\}$; $R = \{(1,1)(2, 2)(3,2)(2,3)(3,3)\}$
The R is transitive since $2R2$, $2R3 \Rightarrow 3R2 \in R$.
$(2,3)(3,2) \Rightarrow (2,2) \in R$

Prob. Give an example of a relation which is

(i) Reflexive and transitive but not symmetric

(ii) Symmetric and transitive but not reflexive

(iii) reflexive and symmetric but not transitive.

(iv) Reflexive and transitive but neither symmetric nor antisymmetric.

Solution

(i) $R_1 = \{(1,1)(2,2)(3,3)(1,3)\}$.
is reflexive & transitive but not symmetric since $(1,3) \in R$ but $(3,1) \notin R$.

(ii) $R_2 = \{1,1)\ (\cancel{\ })\ (1,3)(3,1)\}$
is ~~reflexive~~ symmetric and transitive, but not reflexive since $(3,3) \notin R_2$

(iii) $R_3 = \{(1,1)(2,2)(3,3)(1,2)(2,1)(2,3)(3,2)\}$
is reflexive & symmetric but not transitive since $(1,2) \in R_3$ and $(2,3) \in R_3$ but ~~not~~ $(1,3) \notin R_3$.

(iv) Let z^* be the set of all non-zero ~~transitive~~ integers and R be the relation on z^* given by $(a,b) \in R$, if a is a factor b, ie if a/b.
Since a/a for all $a \in z^*$.

a/b and $b/c \Rightarrow a/c$, hence R is reflexive and transitive.

$2/6$ but $6/2$ is not true. hence R is not symmetric.
Again $5/-5$ and $-5/5$ but $5 \neq -5$ hence R is not antisymmetric.

Equivalence Relation

A relation on a set A is called an equivalence relation if it is reflexive, symmetric and transitive.

ie R is an equivalence relation on A, if it has the following 3 properties.

(1) $(a, a) \in R$ for all $a \in A$ (reflexive)

(2) $(a, b) \in R$ implies $(b, a) \in R$ (symmetric)

(3) (a, b) and $(b, c) \in R \Rightarrow (a, c) \in R$ (transitive)

eg: Let R is the relation on the set of strings of Hindi letters such that aRb iff $l(a) = l(b)$ where $l(x)$ is the length of the string x. Show that R is an equivalence relation.

Soln Since $l(a) = l(a)$, we have aRa whenever a is a string. So R is reflexive.

Suppose aRb so that $l(a) = l(b)$ then bRa since $l(b) = l(a)$ Hence R is symmetric.

Also suppose that aRb and bRc ie $l(a) = l(b)$ and $l(b) = l(c)$

~~The~~ Hence $l(a) = l(c)$ which implies aRc. So R is transitive.

Since R is reflexive, symmetric and transitive, R is an equivalence relation.

Theorems

1. Let R & S be relations from A to B, Then show that

(a) If $R \subseteq S$ then $R^{-1} \subseteq S^{-1}$

(b) $(R \cap S)^{-1} = R^{-1} \cap S^{-1}$

✓ (c) $(R \cup S)^{-1} = R^{-1} \cup S^{-1}$

(a) Given $R \subseteq S$.

If $(a, b) \in R^{-1}$ then $(b, a) \in R$

$\Rightarrow (b, a) \in S$

$\Rightarrow (a, b) \in S^{-1}$

$\therefore (a, b) \in R^{-1} \Rightarrow (a, b) \in S^{-1}$

$\therefore R^{-1} \subseteq S^{-1}$

(b) P.T. $(R \cap S)^{-1} = R^{-1} \cap S^{-1}$

Let $(a, b) \in (R \cap S)^{-1}$

$\Rightarrow (b, a) \in R \cap S$

$\Rightarrow (b, a) \in R$ & $(b, a) \in S$

$\Rightarrow (a, b) \in R^{-1}$ & $(a, b) \in S^{-1}$

$\Rightarrow (a, b) \in R^{-1} \cap S^{-1}$

$\therefore (R \cap S)^{-1} \subseteq R^{-1} \cap S^{-1}$ — (1)

Let $(a, b) \in R^{-1} \cap S^{-1}$

$= (a, b) \in R^{-1}$ and $(a, b) \in S^{-1}$

$\Rightarrow (b, a) \in R$ and $(b, a) \in S$

$\Rightarrow (b, a) \in R \cap S \Rightarrow (a, b) \in (R \cap S)^{-1}$ — (2)

From (1) & (2),

$(R \cap S)^{-1} = R^{-1} \cap S^{-1}$

(2) If R & S are equivalence relations on a set A. Then Prove that

(a) R^{-1} is an equivalence relation.

(b) $R \cap S$ is an equivalence relation.

Proof.

Let R be an equivalence relation.

$\therefore$ R is reflexive, symmetric & transitive

(a) Let $a, b, c \in A$.

The relation R^{-1} is

(1) Reflexive, since $(a, a) \in R$ for all a

$\Rightarrow (a, a) \in R^{-1}$

$\therefore$ R^{-1} is reflexive

(2) Symmetric suppose $(a, b) \in R^{-1}$

$\Rightarrow (b, a) \in R$

$\Rightarrow (a, b) \in R$ (since R is symmetric)

$\Rightarrow (b, a) \in R^{-1}$

$\therefore$ R^{-1} is symmetric

(3) Transitive suppose $(a, b)(b, c) \in R^{-1}$

$\Rightarrow (b, a)(c, b) \in R$

$\Rightarrow (c, b)(b, a) \in R$

$\Rightarrow (c, a) \in R$ (since R is transitive)

$\Rightarrow (a, c) \in R^{-1}$

$\therefore$ R^{-1} is transitive.

$\therefore$ R^{-1} is reflexive, symmetric & transitive.

$\therefore$ R^{-1} is an equivalence relation

(3) Which of the following relations on $\{0, 1, 2, 3\}$ are equivalence relations? Find the properties of an equivalence relation that the other lack?

(a) $R_1 = \{(0,0)(1,1)(2,2)(3,3)\}$

(b) $R_2 = \{(0,0)(0,2)(2,0)(2,2)(2,3)(3,2)(3,3)\}$

(c) $R_3 = \{(0,0)(1,1)(1,2)(2,1)(2,2)(3,3)\}$

(d) $R_4 = \{(0,0)(1,1)(1,3)(2,2)(2,3)(3,1)(3,2)(3,3)\}$

(e) $R_5 = \{(0,0)(0,1)(0,2)(1,0)(1,1)(1,2)(2,0)(2,2)(3,3)\}$

(4) ~~Show that the relation $x \equiv y \pmod m$, in the set of integers where $m>1$ is an equivalen~~

(4) If R be a relation in the set of integers Z defined by $R = \{(x, y) / x \in Z, y \in Z, (x-y) \text{ is divisible by } 6\}$

Then prove that R is an equivalence relation.

Solution

Let $x \in Z$. Then $x - x = 0$ & 0 is divisible by 6.

$\therefore xRx$ for all $x \in Z$

Hence R is reflexive.

Again $xRy \Rightarrow (x-y)$ is divisible by 6

$\Rightarrow -(x-y)$ is divisible by 6

$\Rightarrow (y-x)$ is divisible by 6.

$\Rightarrow yRx$

Hence R is symmetric.

xRy and $xRz \Rightarrow (x-y)$ is divisible by 6 and $(y-z)$ is divisible by 6.

$\Rightarrow (x-y)+(y-z)$ is divisible by 6

$\Rightarrow (x-z)$ is divisible by 6.

$\Rightarrow xRz$; Hence R is transitive.

$\therefore R$ is an equivalence relation.