Basis of Eigen vectors

Eigenvectors of an $n \times n$ matrix A may form a basis for R^n.
We can represent any x in R^n uniquely as a linear combination of the eigen vectors $x_1, x_2 \ldots x_n$ say,

$$x = c_1 x_1 + c_2 x_2 + \cdots + c_n x_n$$

and denoting the corresponding eigenvalues of the matrix A by $\lambda_1, \lambda_2 \ldots \lambda_n$, we have,

$$Ax_j = \lambda_j x_j$$ so that we obtain,

$$y = Ax \quad (\text{by transformation})$$
$$= A(c_1 x_1 + c_2 x_2 + \cdots + c_n x_n)$$
$$= c_1 Ax_1 + c_2 Ax_2 + \cdots + c_n Ax_n.$$
$$= c_1 \lambda_1 x_1 + c_2 \lambda_2 x_2 + \cdots + c_n \lambda_n x_n.$$

ie we have decomposed the complicated action of A on an arbitrary vector x into a sum of simple actions (multiplication by scalars) on the eigenvectors of A. This is the point of an eigenbasis.

Now if the n eigenvalues are all different, we do obtain a basis.

Theorem

Basis of Eigenvectors

If an $n \times n$ matrix A has n distinct eigenvalues, then A has a basis of eigenvectors $x_1, x_2 \ldots x_n$ for R^n.

Diagonalization of a matrix

If an $n \times n$ matrix A has a basis of eigenvectors, then

$\boxed{D = B^{-1}AB}$, where D is a diagonal matrix whose diagonal elements are the eigen values of A. Matrix B is called modal matrix

whose elements are eigen vectors as column vectors.
(Each column is an Eigen vector corresponding to each Eigen value).

The process of converting the given square matrix A into D is called diagonalization of the matrix.

Prob. Diagonalize the matrix,

$$A = \begin{bmatrix} 1 & 6 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \text{ and hence find } A^4.$$

Steps for diagonalization of a matrix

1. find the Eigen values of A
2. find Eigen vectors.
3. Construct matrix B
4. find B^{-1}
5. find $B^{-1}AB$.

Note
We use diagonalization to find Power of A.
ie To calculate $A^n = B D^n B^{-1}$

In this problem, $A = \begin{bmatrix} 1 & 6 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

$A - \lambda I = 0.$

$$\begin{vmatrix} 1-\lambda & 6 & 1 \\ 1 & 2-\lambda & 0 \\ 0 & 0 & 3-\lambda \end{vmatrix} = 0.$$

$(1-\lambda)(2-\lambda)(3-\lambda) - 6(3-\lambda) = 0$

ie $(3-\lambda)(\lambda^2 - 3\lambda - 4) = 0.$

$(3-\lambda)((\lambda-4)(\lambda+1)) = 0.$

ie $\lambda = 3, \lambda = 4, \lambda = -1$

when $\lambda = -1$

$$\begin{bmatrix} 2 & 6 & 1 \\ 1 & 3 & 0 \\ 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0.$$

$$2x_1 + 6x_2 + x_3 = 0$$
$$x_1 + 3x_2 = 0$$
$$4x_3 = 0.$$

$$x_3 = 0.$$

$x_1 = -3x_2$ put $x_2 = k$

$$x_1 = -3k.$$
$$x_2 = k$$
$$x_3 = 0.$$

Eigen vector is $k\begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix}$

when $\lambda = 3$,

$$\begin{bmatrix} -2 & 6 & 1 \\ 1 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$-2x_1 + 6x_2 + x_3 = 0$$

$$x_1 - x_2 = 0.$$

$$x_1 = x_2$$

put $x_3 = k$

$$-2x_1 + 6x_1 + k = 0.$$

$$4x_1 = -k$$

$x_1 = -k/4$; $x_2 = -k/4$, $x_3 = k$.

The Eigen vector is $k\begin{bmatrix} -1/4 \\ -1/4 \\ 1 \end{bmatrix}$ is $\begin{bmatrix} 1 \\ 1 \\ -4 \end{bmatrix}$

When $\lambda = 4$,

$$\begin{bmatrix} -3 & 6 & 1 \\ 1 & -2 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$-3x_1 + 6x_2 + x_3 = 0$$

$$x_1 - 2x_2 = 0$$

$$-x_3 = 0$$

ie $x_3 = 0$

put $x_1 = k$

$$k - 2x_2 = 0$$

$$2x_2 = k$$

$$x_2 = k/2$$

The eigen vector is $\begin{bmatrix} k \\ k/2 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$

$$B = \begin{bmatrix} 2 & -3 & 1 \\ 1 & 1 & 1 \\ 0 & 0 & -4 \end{bmatrix}$$

$$B^{-1} = \begin{bmatrix} 0.2 & 0.6 & 0.2 \\ -0.2 & 0.4 & 0.05 \\ 0 & 0 & -0.25 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 6 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

$$D = B^{-1}AB =$$

$$\begin{bmatrix} 0.2 & 0.6 & 0.2 \\ -0.2 & 0.4 & 0.05 \\ 0 & 0 & -0.25 \end{bmatrix} \begin{bmatrix} 1 & 6 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 2 & -3 & 1 \\ 1 & 1 & 1 \\ 0 & 0 & -4 \end{bmatrix}$$

$$D = \begin{bmatrix} 4 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

$$A^4 = B D^4 B^{-1}$$

$$D^4 = \begin{bmatrix} 256 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 81 \end{bmatrix}$$

$$\cancel{A^4 = B D^4 B^{-1}}$$

$$= \cancel{\begin{bmatrix} 2 & -3 & 1 \\ 1 & 1 & 1 \\ 0 & 0 & -4 \end{bmatrix} \begin{bmatrix} 4 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 3 \end{bmatrix}}$$

$$= \cancel{\begin{bmatrix} 0.2 & 0.6 & 0.2 \\ -0.2 & 0.4 & 0.05 \\ 0 & 0 & -0.25 \end{bmatrix}}$$

$$A^4 = \begin{bmatrix} 2 & -3 & 1 \\ 1 & 1 & 1 \\ 0 & 0 & -4 \end{bmatrix} \times \begin{bmatrix} 256 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 81 \end{bmatrix} \begin{bmatrix} 0.2 & 0.6 & 0.2 \\ -0.2 & 0.4 & 0.05 \\ 0 & 0 & -0.25 \end{bmatrix}$$

$$= \begin{bmatrix} 103 & 306 & 82 \\ 51 & 154 & 31 \\ 0 & 0 & 81 \end{bmatrix}$$

2) Diagonalize the matrix $\begin{bmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \\ 2 & 2 & 3 \end{bmatrix}$ and find A^8

(3) Diagonalize the matrix $\begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix}$

$|A - \lambda I| = 0$

$$\Rightarrow \begin{vmatrix} 2-\lambda & 0 & 1 \\ 0 & 2-\lambda & 0 \\ 1 & 0 & 2-\lambda \end{vmatrix} = 0$$

$$(2-\lambda)(2-\lambda)(2-\lambda) - 0 + 1(0-(2-\lambda)) = 0.$$

$$\Rightarrow (2-\lambda)^3 - (2-\lambda) = 0$$

$$(2-\lambda)^2\left[(2-\lambda)^2 - 1\right] = 0$$

$$(2-\lambda)\left[4 - 4\lambda + \lambda^2 - 1\right] = 0.$$

$$(2-\lambda)(\lambda-3)(\lambda-1) = 0$$

$$\lambda = 1, 2, 3.$$

when $\lambda = 1$

$$(A - \lambda I) = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

Quadratic forms

A quadratic form Q in the components $x_1, x_2 \cdots x_n$ of a vector x is a sum of n^2 terms, namely

$$Q = x^T A x = \sum_{j=1}^{n} \sum_{k=1}^{n} a_{jk} x_j x_k$$

$$= a_{11} x_1^2 + a_{12} x_1 x_2 + \cdots + a_{1n} x_1 x_n$$
$$+ a_{21} x_2 x_1 + a_{22} x_2^2 + \cdots + a_{2n} x_2 x_n$$
$$+ \cdots$$
$$+ a_{n1} x_n x_1 + a_{n2} x_n x_2 + \cdots + a_{nn} x_n^2.$$

$A = [a_{jk}]$ is called the coefficient matrix of the form.

Quadratic form, ~~symmetric coefficient matrix~~

Let $x^T A x = [x_1 \, x_2] \begin{bmatrix} 3 & 4 \\ 6 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 3x_1^2 + 4x_1x_2 + 6x_2x_1 + 2x_2^2$

$= 3x_1^2 + 10x_1x_2 + 2x_2^2.$

is the required quadratic form of the matrix

$$\begin{bmatrix} 3 & 4 \\ 6 & 2 \end{bmatrix}$$

Reduction of quadratic form into canonical form or principal axis form.

(principal axis theorem)

If a real quadratic form can be expressed as a sum or difference of the squares of new variables by means of any real, non singular linear transformation, then the latter quadratic expression is called canonical form of the given quadratic form. (principal axis) for eg: $ax_1^2 + 2bx_1x_2 + bx_2^2$ is a quadratic form

By applying certain linear transformation we can express it in the form $\lambda_1 y_1^2 + \lambda_2 y_2^2$ which is Called the Canonical form or Principal axis form.

Procedure for reducing quadratic form into Canonical form

(i) From the given quadratic form construct symmetric matrix A.

(ii) Find the eigen values & eigen vectors. If A is a square matrix of order 3, it has 3 eigen values say $\lambda_1, \lambda_2, \lambda_3$ and its corresponding eigen vectors are x_1, x_2 & x_3.

(iii) Prove that Eigen vectors are orthogonal.

ie P.T. $x_1 x_2 = 0$, $x_2 x_3 = 0$ & $x_3 x_1 = 0$

(iv) Take normalised form of each eigen vector.

(v) Construct the normalised model matrix B.

(vi) Find D (diagonal matrix) $D = B^{-1}AB$

(vii) The given quadratic form x^TAX should be reduced to y^TDY where $y^T = [y_1\ y_2\ y_3]$

$$[x^TAX = y^TDY]$$

$$D = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$$

So the R.H.S. will be of the form

$\lambda_1 y_1^2 + \lambda_2 y_2^2 + \dots + \lambda_3 y_3^2$