Recurrence relation

Counting Problems are solved by recursive technique. The Procedure for finding the terms of a sequence in a recursive manner is called recurrence relation.

Defn A recurrence relation is an equation that recursively defines a sequence where the next term is a function of the previous ~~one~~ terms.

ii Recurrence relation is that it expresses the general term of an unknown sequence as a known function of its earlier terms.

A recurrence relation for the sequence $a_0, a_1, \ldots a_n$ is an equation that relates a_n to some of its previous terms $a_0, a_1 \ldots a_{n-1}$

Solving a recurrence relation

The process of finding an expression for the terms of a sequence from its recurrence relation is called solving RR.

Definition - The solution of a RR is an explicit formula for the general term a_n of the sequence $a_0, a_1, \ldots a_{n-1}, a_n$ satisfying the RR.

The expressions for permutations combinations and Partitions developed are the most fundamental tools for counting the elements of finite sets. Some of the Combinatorial problems that cannot be solved by fundamental tools can be solved by finding relationships called recurrence relations.

A R.R. uses prior values in a sequence to compute the current value.

A Recurrence algorithm means an algorithm in terms of itself where it is in terms of previous values.

The inductive steps of mathematical induction assume ~~that~~ the truth of prior instances of the statement to prove the truth of the current stmt.

A Recurrence relation is a Procedure for finding the terms of a sequence in a recursive manner.

A R R for the sequence $\{a_n\}$ is an equation that expresses a_n, in terms of one or more of the previous terms of the sequence namely $a_0, a_1, \ldots a_{n-1}$ for all integers n with $n \geq n_0$, where n_0 is a non negative integer.

A Sequence is called a Solution of a R.R. if its terms satisfy the recurrence relation.

A RR defines a sequence by giving the n'th value in terms of constants of its Predecessors.

Order & Degree of Recurrence Relation

The order of a recurrence relation is defined to be the difference b/w the highest and the lowest subscripts of the dependent variables (a_r or y_k) appearing in the relation.

eg: ① The equation $a_r - 3a_{r-1} + 2a_{r-2} = 0$ is a recurrence relation of order 2, since

$$r-(r-2) = r-r+2 = \underline{2}$$

② The equation $2y_r + 3y_{r-1} = 0$ is a first order recurrence relation since

$$r-(r-1) = r-r+1 = \underline{1}$$

Degree - The degree of a recurrence relation is defined to be the highest Power of a_r occuring in the relation.

eg. The recurrence relation

$$a_r^4 + 2a_{r-1}^2 + 18a_{r-2}^3 + a_{r-3} = 0.$$

has degree 4 as the highest power of ~~a_r~~ a_r ~~a~~ is 4

Linear Homogenous & Non Homogeneous Recurrence Relation.

A recurrence relation is called linear recurrence relation if its degree is one.

eg: $C_0 a_r + C_1 a_{r-1} + \cdots + C_k a_{r-k} = f(r)$

where $C_0, C_1, \ldots C_k$ are constants, $C_k \neq 0$ is called linear recurrence relation of order k with constant coefficients.

Linear Homogenous Recurrence relation.

A recurrence relation is said to be linear Homogenous relation of order k

when $C_0 a_r + C_1 a_{r-1} + \cdots + C_k a_{r-k} = 0$.

and when

$C_0 a_r + C_1 a_{r-1} + \cdots + C_k a_{r-k} = f(r)$ and $f(r) \neq 0$, is called linear Non-Homogenous relation.

eg: $a_{r+2} - 6a_{r+1} - a_r = 3r + 2$ is a linear non-homogenous equation of order 2.

First order linear recurrence relation

Each term of a sequence is a linear function of earlier terms in the sequence.

The equation $a_{n+1} = 3a_n$, $n \geq 0$, is a recurrence relation with constant coefficients.

The general ~~term~~ form of such an equation can be written $a_{n+1} = d a_n$, $n \geq 0$ where d is a constant.

The unique solution of the recurrence relation $a_{n+1} = d a_n$, $n \geq 1$ and $a_0 = A$ is given by $\boxed{a_n = A d^n}$

1. Solve the recurrence relation

$$a_n = 7 a_{n-1} \text{ where } n \geq 1 \text{ \& } a_2 = 98.$$

$$a_n = 7^n a_0$$

$$\cancel{a_n = (7)} \quad a_2 = a_0 (7)^2$$

$$\therefore 98 = a_0 \times 49$$

$$\underline{a_0 = 2}$$

2. A person invests \$1000 at 12% interest compounded anually. If ~~an~~ A_n represents the amount at the end of n years, find a recurrence relation and initial conditions that define the sequence $\{a_n\}$.

Solution

After each year, the amount is amount + interest.

Thus $A_n = A_{n-1} + (0.12) A_{n-1}$

$$A_n = A_{n-1} (1 + 0.12) = 1.12 A_{n-1}, \text{ where } n \geq 1 \quad \text{——— (1)}$$

To apply recurrence relation for $n = 1$, we need to know the value of A_0 which is given as the beginning amount $A_0 = 1000$ — (2)

From eqn's (1) & (2)

we can compute the value of A_n as,

$$A_n = (1.12)^n A_0 = (1.12)^n \, 1000$$

The second order linear Homogenous recurrence relation with constant coefficients

Defn - A second order linear homogeneous recurrence relation with constant coefficients is a recurrence relation of the form,

$$a_n = A a_{n-1} + B a_{n-2}$$ for all integers $n \geq$ some fixed integers where A & B are fixed real nos. with $B \neq 0$.

A recurrence relation of order n needs $n-1$ initial terms to define it completely

fibonacci: $S_n = S_{n-1} + S_{n-2}$ is a linear of order 2

$S_n = 2S_n - S_{n-1}$ is linear of order 2.

$S_n = S_{n-1} + 1$ is not homogeneous.

Consider the following recurrence relations

(i) $a_n = 3a_{n-1} + a_{n-2}$

(ii) $a_n = 3a_{n-1} + 5$

(iii) $a_n = 3a_{n-1} + a_{n-2} \cdot a_{n-3}$

(iv) $a_n = 3a_{n-1} + a_{n-2} + \sqrt{2}\, a_{n-3}$

(v) $a_n = 3a_{n-1} + n a_{n-2}$

The recurrence relations, (i), (ii) (iii) & (iv) are recurrence relations with constant coefficients. The recurrence relation (v) $a_n = 3a_{n-1} + n a_{n-2}$ is not a relation with constant coefficients.

(i) is a linear homogeneous recurrence relation of order 2.

(ii) is not a homogeneous recurrence relation because of the constant term 5.

(iii) is not a linear recurrence relation because it contains $a_{n-2} \cdot a_{n-3}$, is the product of terms a_{n-2} and a_{n-3}.

(iv) is a linear homogeneous recurrence relation of order 3.

Defn

A sequence $s_0, s_1, s_2, \ldots s_n \ldots$ is said to satisfy a linear homogeneous recurrence relation

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + c_3 a_{n-3} + \cdots + c_k a_{n-k} \quad k \neq 0$$

of order k with constant coefficients if

$$s_n = c_1 s_{n-1} + c_2 s_{n-2} + \cdots + c_3 s_{n-3} + \cdots + c_k s_{n-k}.$$

Defn

If a sequence $s_0, s_1, s_2 \ldots s_n \ldots$ satisfies a linear homogeneous recurrence relation, then the sequence $s_0, s_1, \ldots s_n \ldots$ is also called a solution of that recurrence relation.

for eg: Consider the recurrence relation $a_n = 3a_{n-1}$. This is a linear homogeneous recurrence relation of order 1.

Let t be a non zero number and suppose $a_n = t^n$ for all $n \geq 0$.

Then $a_n = 3a_{n-1}$ implies that $t^n = 3t^{n-1}$.

There for $t = 3$. Thus $a_n = 3^n$.

Hence the sequence $1, 3, 3^2, 3^3 \ldots 3^n$ is a solution of the recurrence relation $a_n = 3a_{n-1}$

Second order eg.

Consider the following recurrence relation of a sequence $a_0, a_1, a_2 \ldots a_n \ldots$ of nos, with $a_0 = 3$ & $a_1 = 11$

$$a_n = 7a_{n-1} - 12a_{n-2} \quad \text{---} (1)$$

This is a linear homogeneous recurrence relation of order 2 with constant coefficients.

(1) can be written as,

$$a_n - 7a_{n-1} + 12a_{n-2} = 0.$$

Substitute $a_n = r^n$ where r is a non zero number, ~~this that~~ to obtain

$$r^n - 7r^{n-1} + 12r^{n-2} = 0.$$

This implies,

$$r^{n-2}(r^2 - 7r + 12) = 0.$$

So the eqn $r^2 - 7r + 12 = 0$.

This is called the characteristic equation of the recurrence relation. we determine the roots of this equation,

$r^2 - 7r + 12 = (r-4)(r-3)$

is the roots of the characteristic equation are,

$r = 4$ & $r = 3$.

For the roots of the characteristic equations, there are 3 cases,

Case 1 $\quad b^2 - 4ac > 0$

The there are two distinct real roots r_1 & r_2

and $a_n = Ar_1^n + Br_2^n$ where A & B are constants.

Case 2 $\quad b^2 - 4ac = 0$ Then there is one real root

$r = r_1 = r_2$.

and $a_n = (A + Bn) r^n$ where A & B are constants.

Case 3 $\quad b^2 - 4ac < 0$

Then r_1 & r_2 are complex conjugate,

Then $r_1 = r(\cos\alpha + i\sin\alpha)$

$r_2 = r(\cos\alpha - i\sin\alpha)$

$a_n = r^n(A\cos n\theta + B\sin n\theta)$

For the above problem, $b^2 - 4ac = 49 - 4 \times 1 \times 12$

$= 1 > 0$

$b^2 - 4ac > 0$, so the roots are distinct.

$r = 4$ & $r = 3$.

$a_n = Ar_1^n + Br_2^n = A4^n + B3^n$

when $n = 0$,

$a_0 = A4^0 + B3^0 = A + B = 3 \quad \text{——} (1)$

$a_1 = A\,4^1 + B\,3^1 = 4A + 3B = 11$ — (2)

ie

$A + B = 3$ — (1)

$4A + 3B = 11$ — (2)

To find A & B,

multiply (1) by (4)

$4A + 4B = 12$ — (3)

(3) − (2) →

~~$3B = 12$~~

~~$B = 12/3 = 4$~~

$B = 1$

$A + 1 = 3$

$A = 2$

Then $a_n = 2 \times 4^n + 3^n, \; n \geq 0$

(8) Solve the recurrence relation,

$a_n + a_{n-1} - 6a_{n-2} = 0$

where $n \geq 2$ and $a_0 = -1$, $a_1 = 8$.

Put $a_n = r^n$ where r is a non zero number

$r^n + r^{n-1} - 6r^{n-2} = 0$

ie $r^{n-2}\left[r^2 + r - 6\right) = 0$

ie $r^2 + r - 6 = 0$ $\qquad b^2 - 4ac = 1^2 - 4 \times 1 \times -6$

$= 1 + 10 = 11$

ie $(r+3)(r-2) = 0$ $\qquad b^2 - 4ac > 0$

$r_1 = -3$ & $r_2 = 2$ $\qquad$ So the roots are distinct

$a_n = A r_1^n + B r_2^n = A(-3)^n + B(2)^n$

when $n = 0$,

$$a_0 = A \times (-3)^0 + B(2)^0 = A + B$$

$\therefore A + B = -1$ — (1)

when $n = 1$,

$$a_1 = A \times (-3)^1 + B(2)^1 = -3A + 2B$$

$\therefore 8 = -3A + 2B$

$A + B = -1$ — (1)

$\therefore -3A + 2B = 8$ — (2)

(1) $\times 3 \rightarrow$ $3A + 3B = -3$ — (3)

(2)+(3) $\rightarrow$ $5B = 5$

$B = 1$

$A + 1 = -1$

$A = -2$

$$a_n = -2 \times (-3)^n + 1 \times (2)^n$$

$$= 2^n - 2(-3)^n$$

Pro(3) find an explicit formula for the following, linear homogeneous recurrence relation.

$$a_n = -4a_{n-1} - 3a_{n-2} \quad n \geq 2$$

with initial conditions $a_0 = 4$ & $a_1 = 8$.

Put $a_n = r^n$ where r is a non zero number.

$$r^n = -4r^{n-1} - 3r^{n-2}$$

$$r^n + 4r^{n-1} + 3r^{n-2} = 0$$

$$r^{n-2}[r^2 + 4r + 3] = 0.$$

$b^2 - 4ac$

$16 - 4 \times 1 \times 3$

$= 4 > 0$

The roots are distinct.

$\boxed{\text{Ans } a_n = 10(-1)^n + (-6)(-3)^n}$ (11)

(4) find an explicit formula for the following linear homogeneous recurrence relation,

$$a_n = 6a_{n-1} - 9a_{n-2}, \quad n \geq 2$$

which initial condition $a_0 = 4$ & $a_1 = 9$

$a_n = (4-n)3^n$

(5) Find an explicit formula for the following linear homogeneous recurrence relation.

$$3a_n = 7a_{n-1} - 2a_{n-2} \quad n \geq 1$$

which initial conditions $a_0 = -2$ & $a_1 = 1$

(6) Solve the recurrence relation

$$f_n = f_{n-1} + f_{n-2}, \quad n \geq 3.$$

with initial conditions $f_1 = 1$, & $f_2 = 1$