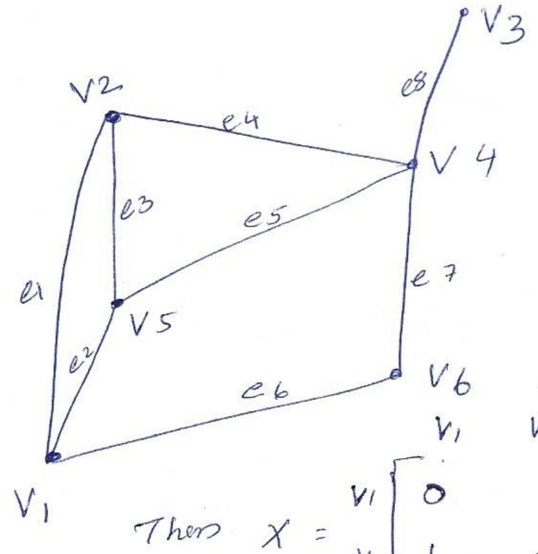
Representation of graphs - Matrix Representation

To write programs that process and manipulate graphs the graphs must be stored i.e. represented in computer memory. The graphs can be represented using adjacency matrices and incidence matrices, which in computer memory can be stored as a 2D-array.

Adjacency matrix

Let G be a graph with n vertices $(n > 0)$, then the adjacency matrix of G is an $n \times n$ matrix is $X = [x_{ij}]$ where

= the no. of edges from V_i to V_j



Then
$$X = \begin{array}{c} \\ V_1 \\ V_2 \\ V_3 \\ V_4 \\ V_5 \\ V_6 \end{array} \begin{array}{c} \begin{array}{cccccc} V_1 & V_2 & V_3 & V_4 & V_5 & V_6 \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 \end{bmatrix} \end{array}$$

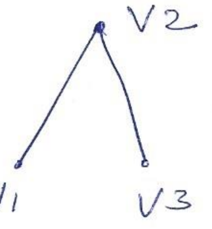
(A simple graph & its adjacency matrix.)

- Given any square symmetric matrix Q of order n, then we can construct a graph G of n vertices ~~and no edges~~ such that

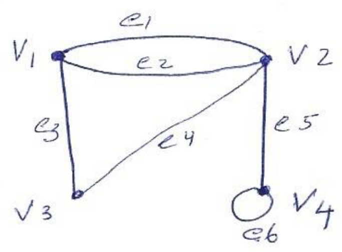
G is the adjacency matrix G.

$$
Q = \begin{array}{c|ccc} & V_1 & V_2 & V_3 \\ \hline V_1 & 0 & 1 & 0 \\ V_2 & 1 & 0 & 1 \\ V_3 & 0 & 1 & 0 \end{array}
$$

G =



Prob. (1) Consider the graph,



find the adjacency ~~graph~~ matrix of the above graph.

$$
\begin{array}{c|cccc} & V_1 & V_2 & V_3 & V_4 \\ \hline V_1 & 0 & 2 & 1 & 0 \\ V_2 & 2 & 0 & 1 & 1 \\ V_3 & 1 & 1 & 0 & 0 \\ V_4 & 0 & 1 & 0 & 1 \end{array}
$$

Prob (2) For the given 5 × 5 adjacency matrix, draw the corresponding graph.

$$
\begin{array}{c|ccccc} & V_1 & V_2 & V_3 & V_4 & V_5 \\ \hline V_1 & 1 & 0 & 1 & 1 & 0 \\ V_2 & 0 & 0 & 2 & 0 & 1 \\ V_3 & 1 & 2 & 0 & 0 & 0 \\ V_4 & 1 & 0 & 0 & 1 & 1 \\ V_5 & 0 & 1 & 0 & 1 & 0 \end{array}
$$

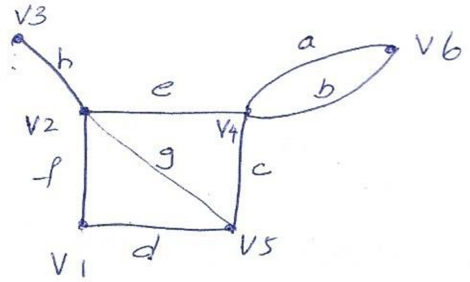
The graph is,

Incidence matrices

Let G be a graph with n vertices $v_1, v_2 \ldots v_n$ where $n \geq 0$ and m edges $e_1, e_2 \ldots e_m$. The incidence matrix I_G with respect to the ordering $v_1, v_2 \ldots v_n$ of n vertices and m edges $e_1, e_2 \ldots e_m$ is an $n \times m$ matrix $[a_{ij}]$ such that

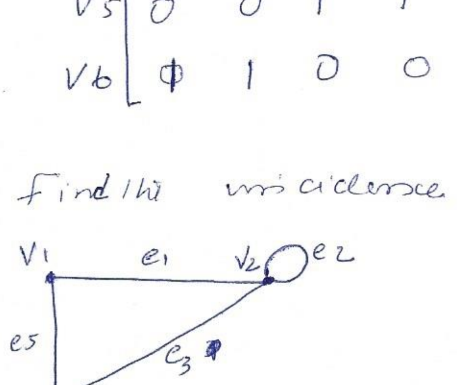
$$a_{ij} \begin{cases} 0 & \text{if } v_i \text{ is not an end vertex of } e_j \\ 1 & \text{if } v_i \text{ is an end vertex of } e_j\text{, but } e_j \text{ is not a loop.} \\ 2 & \text{if } e_j \text{ is a loop at } v_i \end{cases}$$

Prob. ① Find the incidence matrix of the following graph,



$$\begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \end{array}
\begin{array}{c} \begin{array}{cccccccc} a & b & c & d & e & f & g & h \end{array} \\
\begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 \\ \phi & 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

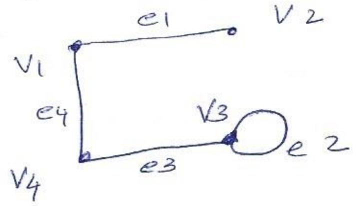
② Find the incidence matrix of the following graph,



$$\begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \end{array}
\begin{array}{c} \begin{array}{ccccc} e_1 & e_2 & e_3 & e_4 & e_5 \end{array} \\
\begin{bmatrix} 1 & 0 & 0 & 0 & 1 \\ 1 & 2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \end{array}$$

Problems

(1) Find the adjacency matrix & incidence matrix of the following graph.



(2) Draw the graph of G represented by the given adjacency matrix.

$$G = \begin{bmatrix} 0 & 2 & 2 & 0 \\ 2 & 0 & 0 & 1 \\ 2 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix}$$

Hamiltonian graphs

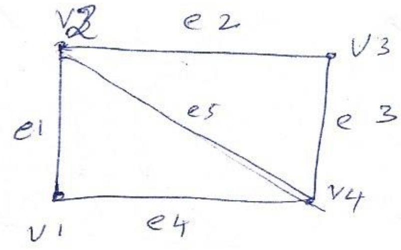
A circuit in a graph G that contains each vertex in G exactly once, except for the starting and ending vertex that appears twice is known as Hamiltonian cycle. A graph G is said to be Hamiltonian, if it contains a hamiltonian ~~Circuit~~ cycle.

Hamiltonian path.

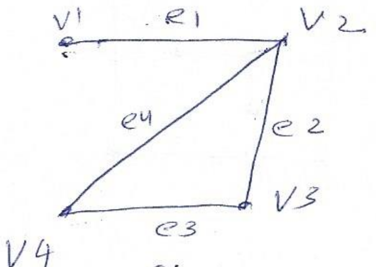
In a simple graph, a path that includes every vertex exactly once is called a Hamiltonian path.

is a Hamiltonian path is a spanning graph.

Q: which of the following graphs, is Hamiltonian cycle.



(a)



(b)

fig (a) has a Hamiltonian cycle given by,

$V_1 e_1 V_2 e_2 V_3 e_3 V_4 e_4 V_1$ (here the edge e_5 is not used)

fig (b) Does not contain Hamiltonian cycle, since every cycle containing every vertex must contain e_1 twice. But the graph does have a Hamiltonian path $V_1 - V_2 - V_3 - V_4$.

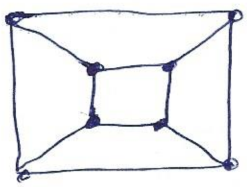
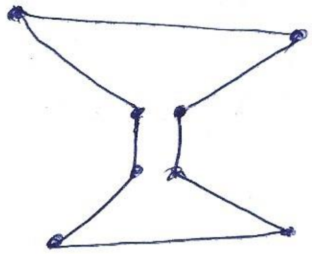
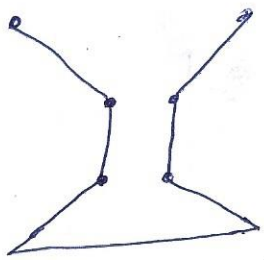
<u>Note</u> - A hamiltonian circuit contains every vertex exactly once, except for the first & last, but may skip edges.

- An Euler line uses every edge exactly once, but the vertices may be repeated.

- A Hamiltonian circuit in a graph of n vertices has exactly n edges.

- A Hamiltonian path is obtained by removing an edge from a Hamiltonian circuit.

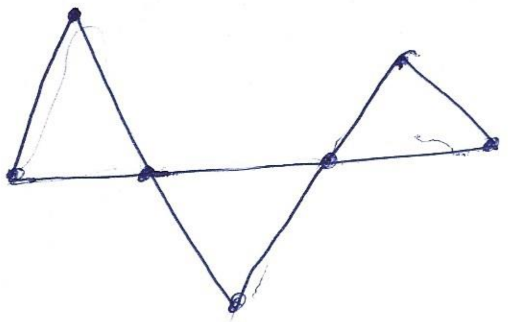
- The length of a Hamiltonian path in a graph of n vertices is $(n-1)$.

G:

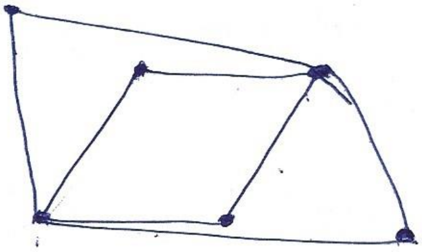
Graph G with 8 vertices | Hamiltonian Circuit in G. | Hamiltonian path in G.

The above graph G is Hamiltonian, but not Eulerian.



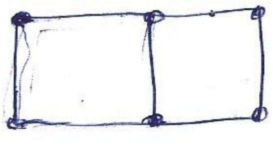
The above graph with 7 vertices is an Euler graph, but it is not Hamiltonian.

Prob ① Draw a graph with 6 vertices which is Eulerian but not Hamiltonian.



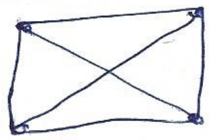
G:

② Draw a graph with 6 vertices which is Hamiltonian but not Eulerian.

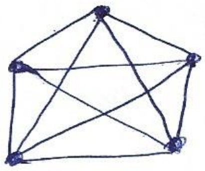


A Complete graph K_n where $n \geq 3$ is a Hamiltonian graph.

K_3

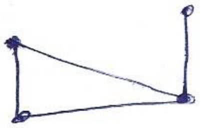


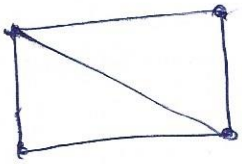
K_4

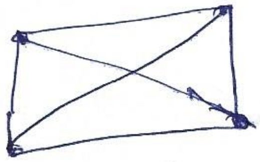


K_5

G_1 :  - non Hamiltonian graph.

G_2  - non Hamiltonian graph.

G_3  Hamiltonian graph.

G_4  Hamiltonian graph.

<u>Note</u>

Given a simple graph G with n vertices, we can construct step by step simple super graphs of G and then we get a complete graph. Thus starting from a graph G, which is not Hamiltonian, we can arrive at a Hamiltonian graph K_n. At some stage during this procedure, a non-Hamiltonian graph changes to a Hamiltonian graph.

Since all super graphs of a Hamiltonian graph are also Hamiltonian, once we get a Hamiltonian graph, all subsequent super graphs will also be Hamiltonian graphs.

Cut vertex, Cut set & Bridge

A cut vertex of a connected graph G is a vertex whose removal increases the number of components.

ii If V is a cut vertex of a connected graph G, then G - V is disconnected.

Similarly, an edge whose removal produces a graph with more connected components, then the original graph is called a cut edge or bridge.

Fleury's algorithm for constructing Euler's Circuit

Let $G = (V, E)$ be a connected graph with each vertex of even degree.

Step I

Select an edge e_1 that is not a bridge in G. Let it's vertices be V_1, V_2. Let π be specified by $V\pi : V_1 V_2$ and $E\pi : e_1$. Remove e_1 from E and let G_1 be the resulting subgraph of G.

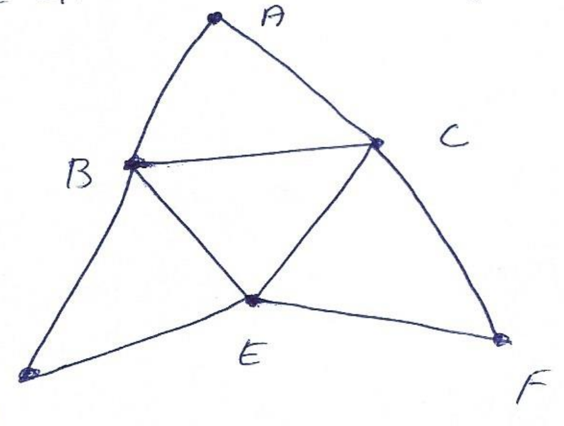
Step II – Suppose that $V\pi : V_1, V_2 \cdots V_k$ and

$E\pi : e_1, e_2 \cdots e_{k-1}$ have been constructed so far, and that all of these edges and any resulting isolated vertices have been removed from V and E to form G_{k-1}. Since V_k has even degree, and e_{k-1} ends there, there must be an edge e_k in G_{k-1}, that also has V_k as a vertex.

If there is more than one such edge select one that is not a bridge for G_{k-1}. Denote the vertex of e_k other than V_k by V_{k+1} and extend $V\pi$ and $E\pi$ to $V\pi : V_1, V_2 \cdots V_k, V_{k+1}$ and $E\pi : e_1, e_2 \cdots e_{k-1}, e_k$. Then delete e_k and any isolated vertices from G_{k-1} to form G_k.

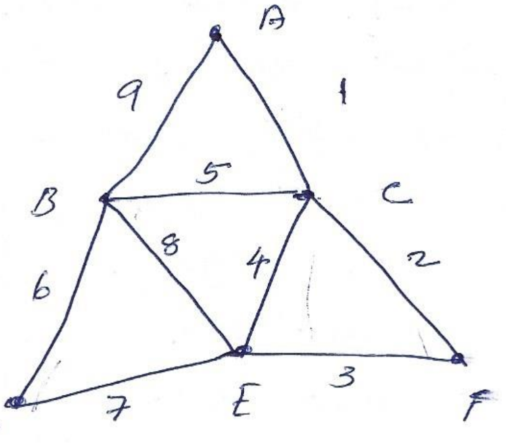
Step 3 - Repeat step 2 until no edge remains in E.

Prob(1) Apply fleury's algorithm and construct an Euler circuit for the following Graph



According to step I, we may begin anywhere. Let us arbitrarily choose A.

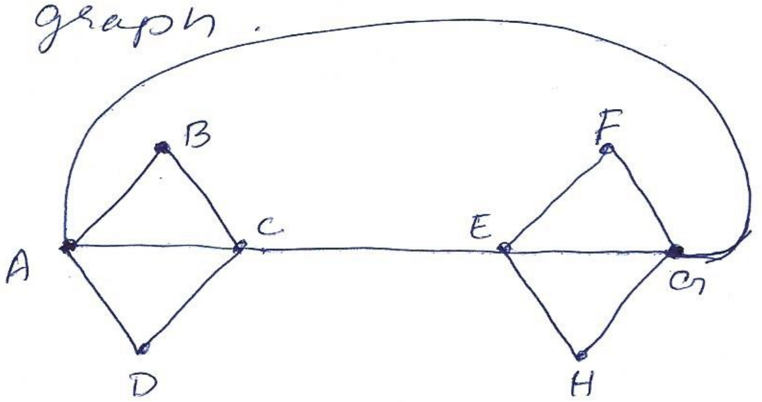
Summarize the results of applying step 2 repeatedly.

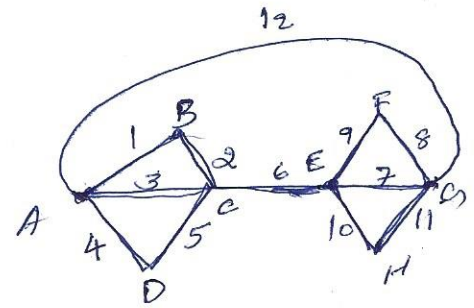


Current path	Next edge	Reasoning
π A	{A C}	No edge from A is a bridge. choose any one
π A C	{C F}	No edge from C is a bridge. Choose any one.
π A C F	{F E}	only one edge from F remaining.
π A C F E	{E C}	No edge from E is a bridge, Choose any one
π A C F E C	{C, B}	Only one edge from C remaining.
π A C F E C B	{B, D)	No edge from B is a bridge, Choose any one.
π A C F E C B D	{D, E}	Only one edge from D remaining.
π A C F E C B D E	{E, B}	Only one edge from E remaining
π A C F E C B D E B	{B, A}	Only one edge from B remaining

π A C F E C B D E B A is the Euler's circuit.

Prob(2) Apply fleury's algorithm and construct an Euler's circuit for the following graph.





Current Path	Next edge	Reasoning
Π A	{A, B}	No edge from A is a bridge. choose any one.
Π A, B	{B, C}	Only one edge from B remaining.
Π A B C	{C, A}	No edge from C is a bridge, choose any one.
Π A B C A	{A, D}	No edge from A is a bridge, choose any one.
Π A B C A D	{D, C}	only one edge from D remaining.
Π A B C A D C	{C, E}	Only one edge from C remaining.
Π A B C A D C E	{E, G}	No edge from E is a bridge, Choose any one
Π A B C A D C E G	{G, F}	{A G} is a bridge, Choose {G, F} or {G, H}
Π A B C A D C E G F	{F E}	only one edge remaining.
Π A B C A D C E G F E	{E H}	Only one edge from E remains.
Π A B C A D C E G F E H	{H, G}	only one edge from H remaining.
Π. A B C A D C E G F E H G	{G, A}	only one edge from G remains.

So Π A B C A D C E G F E H G A ~~which is~~ is the Euler's Circuit