MATH 416H Lecture 2 Note

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Vector Space

Def

1. Close under these 2 calcualtions

   a)

$$\mathbb{R} \times V \longrightarrow V \text{ Scaler Multiplication} \tag{1}$$
$$(\lambda, \overrightarrow{v}) \longmapsto \lambda \overrightarrow{v} \tag{2}$$

   b)

$$V \times V \longrightarrow V \text{ Vector Addition} \tag{3}$$
$$(\overrightarrow{v_1}, \overrightarrow{v_2}) \longmapsto \overrightarrow{v_1} + \overrightarrow{v_2} \tag{4}$$

2. Holds following 8 property

   a. + is communitive, $\overrightarrow{x} + \overrightarrow{y} = \overrightarrow{y} + \overrightarrow{x}$
   b. + is associative, $\overrightarrow{x} + (\overrightarrow{y} + \overrightarrow{z}) = (\overrightarrow{x} + \overrightarrow{y}) + \overrightarrow{z}$
   c. $\exists \overrightarrow{0} \in V$ such that $\forall \overrightarrow{v} \in V,\ \overrightarrow{0} + \overrightarrow{v} = \overrightarrow{v}$
   d. $\forall \overrightarrow{v} \in V, \exists - \overrightarrow{v}$
   e. $\forall v \in V,\ 1 \times \overrightarrow{v} = \overrightarrow{v}$
   f. $\forall \lambda, \mu \in \mathbb{R},\ \lambda(\mu \cdot \overrightarrow{v}) = (\lambda\mu) \cdot \overrightarrow{v}$
   g. $\forall \lambda \in \mathbb{R},\ \lambda(\overrightarrow{v} + \overrightarrow{w}) = \lambda\overrightarrow{v} + \lambda\overrightarrow{w}$
   h. $\forall \lambda, \mu \in \mathbb{R},\ (\lambda + \mu)\overrightarrow{v} = \lambda\overrightarrow{v} + \mu\overrightarrow{v}$

Any set that have all properties above is called a vector space and the elements are called vectors.

Examples

1. $\forall n \geq 0,\ \mathbb{R}^n$ is a vector space, note that $\mathbb{R}^0 = \{0\}$
2. $\mathbb{R}[x]$ is a set of all polynomials having one variable $x \in \mathbb{R}$, such as: $\mathbb{R}[x] = \{a_0 + a_1x_1 + \cdots + a_nx_n|_{a_i\in\mathbb{R}}^{n\geq 0}\}$. Note it is a infinitly large verctor space.
3. $C[0,1] \equiv C^0[0,1] = \{f : [0,1] \to \mathbb{R}|f \text{ continuous}\}$ where C^0 means the function is continuous.
4. WRONG EXAMPLE: $v \in \left\{ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{R}^2 \,|x_1, x_2 \geq 0 \right\}$ which is the 1 quadrant with axises, while it is not close under scaler Multiplication.

Properties

lemma 2.1

For any vector space V there is a unique $\overrightarrow{0}$

Prof

if $\exists \overrightarrow{0}' \neq \overrightarrow{0}$ then:

$$\overrightarrow{0}' = \overrightarrow{0} + \overrightarrow{0}' = \overrightarrow{0}$$

lemma 2.2

If V is a vector space, then $\forall \overrightarrow{v} \in V,\ 0 \cdot v = \overrightarrow{0}$

Prof

$$\begin{aligned}
0 \cdot \overrightarrow{v} &= (0+0) \cdot \overrightarrow{v} = 0 \cdot \overrightarrow{v} + 0 \cdot \overrightarrow{v} \\
\forall \overrightarrow{w} &\in V,\ \exists - \overrightarrow{w} \text{ that } \overrightarrow{w} - \overrightarrow{w} = 0 \\
0 \cdot \overrightarrow{v} - 0 \cdot \overrightarrow{v} &= 0 \cdot \overrightarrow{v} + 0 \cdot \overrightarrow{v} - 0 \cdot \overrightarrow{v} \\
\overrightarrow{0} &= 0 \cdot \overrightarrow{v}
\end{aligned}$$

lemma 2.3

For any vector space V

1. Additive inverse is unique
2. $-1 \cdot \overrightarrow{v} = -\overrightarrow{v}$

Prof

1. Suppose $\overrightarrow{u} + \overrightarrow{v} = 0$

$$\begin{aligned} -\overrightarrow{v} &= \overrightarrow{0} + (-\overrightarrow{v}) = (\overrightarrow{u} + \overrightarrow{v}) - \overrightarrow{v} \\ &= \overrightarrow{u} + \overrightarrow{0} \end{aligned}$$

2.

$$\begin{aligned} -1 \cdot \overrightarrow{v} + \overrightarrow{v} &= -1 \cdot \overrightarrow{v} + 1 \cdot \overrightarrow{v} \\ &= (1-1) \cdot \overrightarrow{v} \\ &= \overrightarrow{0} \end{aligned}$$

Subspace

Definition

Let V be a real($\mathbb{R}$) vector space, a nonempty subset W of V is a subsapce of V if W is also a vector space with definition of the 2 operations same with V.

Properties

1. $\overrightarrow{0} \in W$
2. $\forall \overrightarrow{w} \in W, -\overrightarrow{w} \in W$

(Both of the properties are esay to prof)

Linear Maps between Vector Space

Definition

Define a map $T : V \longrightarrow W$, if $\forall \overrightarrow{v_1}, \overrightarrow{v_2} \in V,\ \lambda_1, \lambda_2 \in \mathbb{R}$, if
$T(\lambda_1 \overrightarrow{v}_1 + \lambda_2 \overrightarrow{v}_2) = \lambda_1 T(\overrightarrow{v_1}) + \lambda_2 T(\overrightarrow{v_2})$
Then T is a linear map between vector spaces.