MATH 461 Homework 7

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5.37 a)

$$P(x < -\frac{1}{2}) + P(x > \frac{1}{2}) = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$$

b) $f(x) = \frac{1}{1-(-1)} = \frac{1}{2}$

$$\begin{aligned} P(|x| \leq y) &= P(-y \leq x \leq y) \\ &= \int_{y}^{-y} \frac{1}{2}\mathrm{d}x \\ &= y \\ \frac{\mathrm{d}}{\mathrm{d}y}P(|x| \leq y) &= 1 \end{aligned}$$

5.39

$$\begin{aligned} f(x) &= 1e^{-1\times x} \\ &= e^{-x} \\ P(X \leq x) &= \int_0^x e^{-s}\mathrm{d}s \\ &= \left.(-e^{-s})\right|_0^x \\ &= -e^{-x} - (-e^0) \\ &= 1 - e^{-x} \\ P(\log X < y) &= P(e^{\log(X)} \leq e^y) \\ &= P(X \leq e^y) \\ &= 1 - e^{-e^y} \\ \frac{\mathrm{d}}{\mathrm{d}y}P(X \leq e^y) &= \frac{\mathrm{d}}{\mathrm{d}y}(1 - e^{-e^y}) \\ &= (e^y)(e^{-e^y}) \end{aligned}$$

5.40

$$
\begin{aligned}
f(x) &= \frac{1}{1-0} = 1 \\
P(e^X \leq y) &= P(X \leq \log(y)) \\
&= \int_0^{\log(y)} 1 \text{ d}x \\
&= \log(y) \\
\frac{\text{d}}{\text{d}y} \log(y) &= \frac{1}{y}(1 < y < e)
\end{aligned}
$$

5.41

$$
\begin{aligned}
f(\theta) &= \frac{1}{\frac{\pi}{2} + \frac{\pi}{2}} \\
&= \frac{1}{\pi} \\
\int_{-\frac{\pi}{2}}^{\theta} \frac{1}{\pi} \text{d}x &= \frac{1}{\pi} \; (x)|_{-\frac{\pi}{2}}^{\theta} \\
&= \frac{\theta + \frac{\pi}{2}}{\pi} \\
P(A\sin(\theta) \leq r) &= P(\sin(\theta) \leq \frac{r}{A}) \\
(\text{for}|r| < A) &= P(\theta \leq sin^{-1}(\frac{r}{A})) \\
\frac{\text{d}}{\text{d}r} \frac{sin^{-1}(\frac{r}{A}) + \frac{\pi}{2}}{\pi} &= \frac{1}{\pi}\left(\frac{1}{\sqrt{1-\frac{r^2}{A^2}}} \times \frac{1}{A}\right) \\
&= \frac{1}{\pi\sqrt{A^2 - r^2}}(-A < r < A)
\end{aligned}
$$

6.2 a)

$$
\begin{cases}
\frac{\binom{5}{2}}{\binom{13}{2}} & \text{(Both balls are white)} \\
\frac{\binom{5}{1}\binom{8}{1}}{\binom{13}{2}} & (X_1 \text{ White, } X_2 \text{ red}) \\
\frac{\binom{8}{1}\binom{5}{1}}{\binom{13}{2}} & (X_1 \text{ red, } X_2 \text{ white}) \\
\frac{\binom{8}{2}}{\binom{13}{2}} & \text{(Both red)}
\end{cases}
$$

b)

$$\begin{cases} \frac{\binom{5}{3}}{\binom{13}{3}} & \text{(all balls are white)} \\ \frac{\binom{5}{2}\binom{8}{1}}{\binom{13}{3}} & \text{(1 red 2 white)} \\ \frac{\binom{8}{2}\binom{5}{1}}{\binom{13}{3}} & \text{(2 red 1 white)} \\ \frac{\binom{8}{3}}{\binom{13}{3}} & \text{(all red)} \end{cases}$$

6.7

$$\begin{aligned} P(X_1 = x_1, X_2 = x_2) &= P(X_1 = x_1)P(X_2 = x_2) \\ &= p(1-p)^{x_1} \times p(1-p)^{x_2} \\ &= p^2(1-p)^{x_1+x_2} \end{aligned}$$

6.8 a)

$$\begin{aligned} 1 &= \iint R^2 f dx dy \\ &= \int_0^{\infty} (\int_{-y}^{y} c(y^2 - x^2)e^{-y} dx) dy \\ &= \frac{4c}{3} \int_0^{\infty} y^3 e^{-y} dy \\ &= \frac{4c}{3} \times 6c \\ c &= \frac{1}{8} \end{aligned}$$

b)

$$\begin{aligned} f_X(x) &= \int_x^{\infty} \frac{1}{8}(y^2 - x^2)e^{-y} dy \\ &= \frac{1}{8}\left(\int_x^{\infty} y^2 e^{-y} dy - \int_x^{\infty} x^2 e^{-y} dy\right) \\ &= \frac{1}{8}\left(x^2 e^{-x} + 2xe^{-x} + 2e^{-x} - x^2 e^{-x}\right) \\ &= \frac{1}{4} e^{-|x|}(1 + |x|) \\ f_Y(y) &= \frac{1}{8} \int_{-y}^{y} (y^2 - x^2)e^{-y} dx \\ &= \frac{1}{8}\frac{4}{3} y^3 e^{-y} \\ &= \frac{1}{6} y^3 e^{-y} \end{aligned}$$

c)

$$\begin{aligned} E(X) &= \int_{-\infty}^{\infty} x f(x) dx \\ &= \int_{-\infty}^{\infty} x \frac{1}{4} e^{-|x|}(1+|x|) dx \\ &= 0 \end{aligned}$$

6.9 a)

$$\begin{aligned} \int_0^1 \int_0^2 \left(\frac{6}{7}\left(x^2 + \left(\frac{xy}{2}\right)\right)\right) dydx &= \frac{6}{7}\int_0^1 \int_0^2 \left(x^2 + \left(\frac{xy}{2}\right)\right) dydx \\ &= \frac{6}{7}\int_1^0 x^2y + \frac{xy^2}{2}\Big|_{y=0}^{y=2} dx \\ &= \frac{6}{7}\int_1^0 2x^2 + \frac{4x}{4} dx \\ &= \frac{6}{7}\frac{7}{6} \\ &= 1 \end{aligned}$$

Thus, it is a density function.

b)

$$\begin{aligned} \int_0^2 \frac{6}{7}\left(x^2 + \frac{xy}{2}\right) dy &= \frac{6}{7}\left(x^2y + \frac{xy^2}{2}\right)|_0^2 \\ &= \frac{6}{7}(2x^2 + x) \end{aligned}$$

c)

$$\begin{aligned} \int_0^1 \int_0^x \frac{6}{7}\left(x^2 + \frac{xy}{2}\right) dydx &= \int_0^1 \frac{6}{7}(x^3 + \frac{x^3}{4})dx \\ &= \frac{6}{7}\left(\frac{1^4}{4} + \frac{1^4}{16}\right) \\ &= \frac{15}{56} \end{aligned}$$

d)

$$\int_0^{0.5} \int_{0.5}^2 \frac{6}{7}\left(x^2 + \frac{xy}{2}\right) dydx = \frac{69}{80}$$

e)

$$\int_0^1 x\frac{6}{7}(2x^2 + x)dx = \frac{5}{6}$$

f)

$$\int_0^1 \frac{6}{7}(2x^2+x)dx = \frac{6}{7}(\frac{1}{3}+\frac{y}{4})$$
$$\int_0^2 y\frac{6}{7}(\frac{1}{3}+\frac{y}{4})dy = \frac{8}{7}$$

6.10 a) $e^{-(x+y)} = e^{-x}e^{-y}$ Thus, $f_X(x) = e^{-x}, f_Y(y) = e^{-y}$

$$\int_0^\infty \int_x^\infty e^{-x}e^{-y}dxdy = \frac{1}{2}$$

b)

$$\int_a^\infty e^{-x}dx = 1 - e^{-a}$$