PHYS 225 HW 9

James Liu

Due:Nov 7 Edit: November 6, 2024

1. a) $f(x, y, z) = x^2 + y^2 + z^2$

Cartesian:

$$\begin{aligned} f(x, y, z) &= x^2 + y^2 + z^2 \\ \nabla f &= \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right) \\ &= (2x, 2y, 2z) \end{aligned}$$

Cylindrical:

$$\begin{aligned} f(x, y, z) &= x^2 + y^2 + z^2 \\ f(r, \theta, z) &= r^2 + z^2 \\ \nabla f &= \left(\frac{\partial f}{\partial r}, \frac{1}{r}\frac{\partial f}{\partial \theta}, \frac{\partial f}{\partial z}\right) \\ \nabla f &= (2r, 0, 2z) = 2r\hat{r} + 2z\hat{z} \\ &= 2\sqrt{x^2 + y^2} \times \frac{x\hat{x} + y\hat{y}}{\sqrt{x^2 + y^2}} + 2z\hat{z} \\ &= 2x\hat{x} + 2y\hat{y} + 2z\hat{z} \end{aligned}$$

Spherical:

$$\begin{aligned} f(x, y, z) &= x^2 + y^2 + z^2 \\ f(\rho, \theta, \phi) &= \rho^2 \\ \nabla f &= \left(\frac{\partial f}{\partial r}, \frac{1}{r}\frac{\partial f}{\partial \theta}, \frac{1}{r\sin(\phi)}\frac{\partial f}{\partial \phi}\right) \\ \nabla f &= (2\rho, 0, 0) = 2\rho\hat{\rho} \\ &= 2\sqrt{x^2 + y^2 + z^2} \times \frac{x\hat{x} + y\hat{y} + z\hat{z}}{\sqrt{x^2 + y^2 + z^2}} \\ &= 2x\hat{x} + 2y\hat{y} + 2z\hat{z} \end{aligned}$$

b) $f(x, y, z) = \sin(z)$

Cartesian:

$$\begin{aligned} f(x, y, z) &= \sin(z) \\ \nabla f &= (0, 0, \cos(z)) \\ &= \cos(z)\hat{z} \end{aligned}$$

Cylindrical:

$$\begin{aligned} f(x, y, z) &= \sin(z) \\ f(r, \theta, z) &= \sin(z) \\ \nabla f &= (0, 0, \cos(z)) = \cos(z)\hat{z} \end{aligned}$$

Spherical:

$$\begin{aligned} f(x, y, z) &= \sin(z) \\ f(\rho, \theta, \phi) &= \sin(\rho \times \cos(\theta)) \\ \nabla f &= (\cos(\rho\cos(\theta))\cos(\theta), -\cos(\rho\cos(\theta))\sin(\theta), 0) \\ &= \cos(\rho\cos(\theta))\cos(\theta)\hat{r} - \cos(\rho\cos(\theta))\sin(\phi)\hat{\theta} \\ &= \cos(z) \times \frac{z}{\sqrt{x^2 + y^2 + z^2}} \times \frac{x\hat{x} + y\hat{y} + z\hat{z}}{\sqrt{x^2 + y^2 + z^2}} \\ &- \cos(z)\frac{\sqrt{x^2 + y^2}}{\sqrt{x^2 + y^2 + z^2}} \times \frac{(x\hat{x} + y\hat{y})z - (x^2 + y^2)\hat{z}}{\sqrt{x^2 + y^2 + z^2}\sqrt{x^2 + y^2}} \\ &= \cos(z)\frac{1}{x^2 + y^2 + z^2}\left((x\hat{x} + y\hat{y} + z\hat{z})z - (x\hat{x} + y\hat{y})z + (x^2 + y^2)\hat{z}\right) \\ &= \cos(z)\frac{1}{x^2 + y^2 + z^2}\left((x^2 + y^2 + z^2)\hat{z}\right) \\ &= \cos(z)\hat{z} \end{aligned}$$

c) $f(x,y,z) = x + y + z$

Cartesian:

$$\begin{aligned} f(x,y,z) &= x + y + z \\ \nabla f &= (1,1,1) \\ &= \hat{x} + \hat{y} + \hat{z} \end{aligned}$$

Cylindrical:

$$\begin{aligned} f(x,y,z) &= x + y + z \\ f(r,\theta,z) &= r\cos(\theta) + r\sin(\theta) + z \\ \nabla f &= (\cos(\theta) + \sin(\theta), \cos(\theta) - \sin(\theta), 1) \\ &= (\cos(\theta) + \sin(\theta))\hat{\rho} + (\cos(\theta) - \sin(\theta))\hat{\theta} + \hat{z} \\ &= \left(\frac{x+y}{\sqrt{x^2+y^2}}\right)\frac{x\hat{x}+y\hat{y}}{\sqrt{x^2+y^2}} + \left(\frac{x-y}{\sqrt{x^2+y^2}}\right) \times \frac{-y\hat{x}+x\hat{y}}{\sqrt{x^2+y^2}} + \hat{z} \\ &= \frac{(x^2+xy)\hat{x} + (xy+y^2)\hat{y}}{x^2+y^2} + \frac{(y^2-xy)\hat{x} + (x^2-xy)\hat{y}}{x^2+y^2} + \hat{z} \\ &= \hat{x} + \hat{y} + \hat{z} \end{aligned}$$

Spherical:

$$
\begin{aligned}
f(x,y,z) &= x+y+z \\
f(\rho,\theta,\phi) &= \rho\sin(\theta)\cos(\phi)+\rho\sin(\theta)\sin(\phi)+\rho\cos(\theta) \\
\nabla f &= \begin{bmatrix} \sin(\theta)\cos(\phi)+\sin(\theta)\sin(\phi)+\cos(\theta) \\ \cos(\theta)\cos(\phi)+\cos(\theta)\sin(\phi)-\sin(\theta) \\ -\sin(\phi)+\cos(\phi) \end{bmatrix} \\
&= \left(\frac{\sqrt{x^2+y^2}}{\sqrt{x^2+y^2+z^2}}\times\frac{x}{\sqrt{x^2+y^2}}+\frac{\sqrt{x^2+y^2}}{\sqrt{x^2+y^2+z^2}}\times\frac{y}{\sqrt{x^2+y^2}}+\frac{z}{\sqrt{x^2+y^2+z^2}}\right)\hat{\rho} \\
&+ \left(\frac{z}{\sqrt{x^2+y^2+z^2}}\times\frac{x}{\sqrt{x^2+y^2}}+\frac{z}{\sqrt{x^2+y^2+z^2}}\times\frac{y}{\sqrt{x^2+y^2}}-\frac{\sqrt{x^2+y^2}}{\sqrt{x^2+y^2+z^2}}\right)\hat{\theta} \\
&+ \left(\frac{x-y}{\sqrt{x^2+y^2}}\right)\hat{\phi} \\
&= \left(\frac{x+y+z}{\sqrt{x^2+y^2+z^2}}\right)\frac{x\hat{x}+y\hat{y}+z\hat{z}}{\sqrt{x^2+y^2+z^2}} \\
&+ \left(\frac{zx+zy-x^2-y^2}{\sqrt{x^2+y^2+z^2}\sqrt{x^2+y^2}}\right)\frac{(x\hat{x}+y\hat{y})z-(x^2+y^2)\hat{z}}{\sqrt{x^2+y^2+z^2}\sqrt{x^2+y^2}} \\
&+ \left(\frac{x-y}{\sqrt{x^2+y^2}}\right)\frac{-y\hat{x}+x\hat{y}}{\sqrt{x^2+y^2}} \\
&= \frac{(x^2+y^2)(x+y+z)(x\hat{x}+y\hat{y}+z\hat{z})}{(x^2+y^2+z^2)(x^2+y^2)} \\
&+ \frac{(zx+zy-x^2-y^2)(zx\hat{x}+zy\hat{y}-(x^2+y^2)\hat{z})}{(x^2+y^2+z^2)(x^2+y^2)} \\
&+ \frac{(x-y)(x^2+y^2+z^2)(-y\hat{x}+x\hat{y})}{(x^2+y^2+z^2)(x^2+y^2)} \\
&= \frac{x(x^2+y^2)(x+y+z)+zx(zx+zy-x^2-y^2)-y(x-y)(x^2+y^2+z^2)}{(x^2+y^2+z^2)(x^2+y^2)}\hat{x} \\
&+ \frac{y(x^2+y^2)(x+y+z)+zy(zx+zy-x^2-y^2)+x(x-y)(x^2+y^2+z^2)}{(x^2+y^2+z^2)(x^2+y^2)}\hat{y} \\
&+ \frac{z(x^2+y^2)(x+y+z)-(x^2+y^2)(zx+zy-x^2-y^2)}{(x^2+y^2+z^2)(x^2+y^2)}\hat{z} \\
&= \frac{x^4+y^4+2x^2y^2+z^2x^2+z^2y^2}{(x^2+y^2+z^2)(x^2+y^2)}\hat{x} \\
&+ \frac{x^4+y^4+2x^2y^2+z^2x^2+z^2y^2}{(x^2+y^2+z^2)(x^2+y^2)}\hat{y} \\
&+ \frac{zx+zy+z^2-zx-zy+x^2+y^2}{x^2+y^2+z^2}\hat{z} \\
&= \hat{x}+\hat{y}+\hat{z}
\end{aligned}
$$

2. a) $v = x\hat{x} + y\hat{y} + z\hat{z}$

Cartesian:

$$\nabla \cdot v = 1 + 1 + 1 = 3$$

Cylindrical:

$$\begin{aligned}
v &= x\hat{x} + y\hat{y} + z\hat{z} \\
v &= \rho\cos(\theta)(\cos(\theta)\hat{\rho} - \sin(\theta)\hat{\theta}) + \rho\sin(\theta)(\sin(\theta)\hat{\rho} + \cos(\theta)\hat{\theta}) + z\hat{z} \\
&= \rho\hat{\rho} + 0\hat{\theta} + z\hat{z} \\
\nabla v &= \frac{1}{\rho}\frac{\partial}{\partial\rho}\rho A_\rho + \frac{1}{\rho}\frac{\partial}{\partial\theta}A_\theta + \frac{\partial}{\partial z}A_z \\
&= 2 + 0 + 1 = 3
\end{aligned}$$

Spherical:

$$\begin{aligned}
v &= x\hat{x} + y\hat{y} + z\hat{z} \\
&= \rho\sin(\theta)\cos(\phi)\hat{x} + \rho\sin(\theta)\sin(\phi)\hat{y} + \rho\cos(\theta)\hat{z} \\
&= \rho\sin^2(\theta)\cos^2(\phi)\hat{\rho} + \rho\sin(\theta)\cos(\theta)\cos^2(\phi)\hat{\theta} - \rho\sin(\theta)\cos(\phi)\sin(\phi)\hat{\phi} \\
&+ \rho\sin^2(\theta)\sin^2(\phi)\hat{\rho} + \rho\cos(\theta)\sin(\theta)\sin^2(\phi)\hat{\theta} + \rho\sin(\theta)\sin(\phi)\cos(\phi)\hat{\phi} \\
&+ \rho\cos^2(\theta)\hat{\rho} - \rho\cos(\theta)\sin(\theta)\hat{\theta} \\
&= \rho(\cos^2(\theta) + \sin^2(\theta)(\cos^2(\theta) + \sin^2(\theta)))\hat{\rho} \\
&+ \rho\sin(\theta)\cos(\theta)(\cos^2(\phi) + \sin^2(\phi) - 1)\hat{\theta} \\
&+ \rho\sin(\theta)(\sin(\phi)\cos(\phi) - \sin(\phi)\cos(\phi)) \\
&= \rho\hat{\rho} + 0\hat{\theta} + 0\hat{\phi} \\
\nabla v &= \frac{1}{\rho^2}\frac{\partial}{\partial\rho}\rho^2 A_\rho + \frac{1}{\rho\sin(\theta)}\frac{\partial}{\partial\theta}(A_\theta\sin(\theta)) + \frac{1}{\rho\sin(\theta)}\frac{\partial}{\partial\phi}A_\phi \\
&= 3 + 0 + 0 = 3
\end{aligned}$$

b) $v = \hat{\rho}$

Cartesian:

$$\begin{aligned}
v &= \frac{x\hat{x} + y\hat{y}}{\sqrt{x^2+y^2}} \\
&= \frac{x}{\sqrt{x^2+y^2}}\hat{x} + \frac{y}{\sqrt{x^2+y^2}}\hat{y} \\
\nabla v &= \frac{y^2}{(x^2+y^2)^{\frac{3}{2}}} + \frac{x^2}{(x^2+y^2)^{\frac{3}{2}}} \\
&= \frac{1}{\sqrt{x^2+y^2}}
\end{aligned}$$

Cylindrical:

$$\begin{aligned}
v &= 1\hat{\rho} + 0\hat{\theta} + 0\hat{z} \\
\nabla v &= \frac{1}{\rho} = \frac{1}{\sqrt{x^2+y^2}}
\end{aligned}$$

c) $v = \hat{\theta}$

Cartesian:

$$\begin{aligned}
v &= \frac{(x\hat{x} + y\hat{y})z - (x^2+y^2)\hat{z}}{\sqrt{x^2+y^2+z^2}\sqrt{x^2+y^2}} \\
&= \frac{xz}{\sqrt{x^2+y^2+z^2}\sqrt{x^2+y^2}}\hat{x} + \frac{yz}{\sqrt{x^2+y^2+z^2}\sqrt{x^2+y^2}}\hat{y} - \frac{(x^2+y^2)}{\sqrt{x^2+y^2+z^2}\sqrt{x^2+y^2}}\hat{z} \\
\nabla v &= -\frac{z(x^4-y^4-z^2y^2)}{(x^2+y^2+z^2)^{\frac{3}{2}}(x^2+y^2)^{\frac{3}{2}}} - \frac{z(y^4-x^4-z^2x^2)}{(x^2+y^2+z^2)^{\frac{3}{2}}(x^2+y^2)^{\frac{3}{2}}} + \frac{z^2x(x^2+y^2)}{(x^2+y^2+z^2)^{\frac{3}{2}}(x^2+y^2} \\
&= \frac{z}{\sqrt{x^2+y^2+z^2}\sqrt{x^2+y^2}}
\end{aligned}$$

Spherical:

$$\begin{aligned}
v &= \hat{\theta} \\
\nabla v &= \frac{\cos(\theta)}{r\sin(\theta)} \\
&= \frac{\frac{z}{\sqrt{x^2+y^2+z^2}}}{\sqrt{x^2+y^2+z^2}\frac{\sqrt{x^2+y^2}}{\sqrt{x^2+y^2+z^2}}} \\
&= \frac{z}{\sqrt{x^2+y^2+z^2}\sqrt{x^2+y^2}}
\end{aligned}$$

3.

$$\begin{aligned}
\partial_i v^i &= (R_i^j \partial'_j)((R^{-1})^i_k v'^k) \\
&= \partial'_j (R_i^j)(R^{-1})^i_k v'^k \\
&= \delta^j_k \partial'_j v'^k \\
&= \partial'_j v'^j
\end{aligned}$$

Thus the divergence is invarianced