

MATH 416H HW 2

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1. No it is no longer a vector space, as:

$$\begin{aligned}\exists \lambda, \mu \in \mathbb{R}, (\lambda + \mu) \vec{v} &= ((\lambda + \mu)^2 \times v_1, \dots, (\lambda + \mu)^2 \times v_n)^T \\ &= ((\lambda^2 + \mu^2 + 2\lambda\mu) \times v_1, \dots, (\lambda^2 + \mu^2 + 2\lambda\mu) \times v_n)^T \\ &\neq ((\lambda^2 + \mu^2) \times v_1, \dots, (\lambda^2 + \mu^2) \times v_n)^T \\ &= \lambda \vec{v} + \mu \vec{v}\end{aligned}$$

Thus, as it is not a vector space.

2. No it is not a vector space, as:

$$\begin{aligned}\exists \lambda, \mu \in \mathbb{R}, (\lambda + \mu) \vec{v} &= a^{\lambda+\mu} \\ &= a^\lambda \times a^\mu \neq a^\lambda + a^\mu \\ &= \lambda \vec{v} + \mu \vec{v}\end{aligned}$$

For example: $\lambda = 2, \mu = 1, a = 2$, as $2^{(1+2)} = 8 \neq 6 = 2^1 + 2^2$ Thus, it is not a vector space.

3. No it is not in the span. Suppose it is then $\exists a_1, a_2 \in \mathbb{R}$ such that $a_1(3, 4, 2)^T + a_2(1, 3, 3)^T = (-1, 2, 3)^T$ which is:

$$\begin{cases} 3a_1 + a_2 &= -1 \\ 4a_1 + 3a_2 &= 2 \\ 2a_1 + 3a_2 &= 3 \end{cases}$$

Deriving it will result in:

$$\begin{cases} 3a_1 + a_2 &= -1 \\ -5a_1 + 0a_2 &= 5 \\ -7a_1 + 0a_2 &= 6 \end{cases} \rightarrow \begin{cases} a_1 = -1 \\ a_1 = -\frac{6}{7} \end{cases} \quad (1)$$

Which raises a contradiction and thus, such a_1, a_2 does not exist thus, it is not in the span.

4. as v is in span of the subspace, then: $\exists a_1, \dots, a_n, a_i \in \mathbb{R}$ that $v = a_1 \vec{v}_1 + \dots + a_n \vec{v}_n$ Thus, $-1v + a_1 \vec{v}_1 + \dots + a_n \vec{v}_n = 0$ thus, the set is linear dependent.

5. As U, W are subspace of V , then $\vec{0} \in U, W$ thus $\vec{0} \in U \cap W$. Also, consider that $x, y \in U \cap W$, then $x, y \in W$ and $x, y \in U$. Thus, $x + y \in U$ as U is a subspace, similarly, $x + y \in W$ therefore. $x + y \in U \cap W$ similarly, $\forall \lambda \in \mathbb{R}, \lambda x \in U$, and $\lambda x \in W$, therefore, the addition and multiplication are closed in $U \cap W$ following V while the intersection contains the zero vector making it not empty, thus, the intersection is still a subspace.
6. a) As proved in class, any set of vector S that spans V will have a element number n that is larger or equal to any set that is linear independent. and for any vector space of dimension m , it has a basis consisting of m elements. And any additional vector in the vector space will be linear dependent with this base. Thus, basis is the largest linear independent subsets in a vector space. Therefore, a subset of a vector space that spans the vector space must have elements that is larger or equal to the number of elements in a base. Thus
- b) As V have a dimension of n , then exists a base $\{v_1, \dots, v_n\} \subseteq V$ and spans V with n linearly independent vectors v_1, \dots, v_n . Thus, for any other vector, $\forall v \in V, \exists \lambda_1, \dots, \lambda_n \in F$ that $v = \lambda_1 v_1 + \dots + \lambda_n v_n$ and $\exists \mu_1, \dots, \mu_n \in F$ that $-v = \mu_1 v_1 + \dots + \mu_n v_n$. Thus, $v + \mu_1 v_1 + \dots + \mu_n v_n = 0$ and it is linear dependent with the set of basis. Thus, the maximum numebr of elements is n .
7. $s' \in \text{span } S$, thus, $\exists \lambda_i \in F$ that $s' = \sum_i \lambda_i s'_i$ where $s'_i \in S$. Also, as $S' \subset S$, there is $\forall s'_i \in S', s'_i \in S$, thus, as s' can be expressed in $s' = \sum_i \lambda_i s'_i, s' \in \text{span } S$, thus $\text{span } S' \subset \text{span } S$
8. As $v \in \text{span } S, \exists \lambda_i \in F, \forall s_i \in S$ that $v = \sum_i \lambda_i s_i \forall w \in \text{span } (S \cup \{v\}), \exists \lambda_i, \lambda'_i, \mu \in F$, that $w = \sum_i \lambda_i s_i + \mu v = \sum_i \lambda_i s_i + \mu \sum_i \lambda'_i s_i = \sum_i (\lambda_i + \mu \lambda'_i) s_i, (\lambda_i + \mu \lambda'_i) \in F$ thus, set $\lambda_i + \mu \lambda'_i = \alpha_i$, then $w = \sum_i \alpha_i s_i$ Thus, $\text{span } S = \text{span } (S \cup \{v\})$
9. $\forall \lambda_i, \mu_i, a, b \in F$:

$$\begin{aligned}
F(a\lambda_1 + b\mu_1, \dots, a\lambda_n + b\mu_n) &= a\lambda_1 v_1 + b\mu_1 v_1 + \dots + a\lambda_n v_n + b\mu_n v_n \\
&= a\lambda_1 v_1 + \dots + a\lambda_n v_n + b\mu_1 v_1 + b\mu_n v_n \\
&= a(\lambda_1 v_1 + \dots + \lambda_n v_n) + b(\mu_1 v_1 + \mu_n v_n) \\
&= aF(\lambda_1, \dots, \lambda_n) + bF(\mu_1, \dots, \mu_n)
\end{aligned}$$

Therefore it is a linear map.