MATH 461 Homework 3

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Part 1

3.57
a) $P(2S) = 2 \times p(1-p)$
b) $P(3I) = 3 \times p^2(1-p)$
c) $P = \dfrac{2p^2(1-p)}{3p^2(1-p)} = \dfrac{2}{3}$

3.59
a) $P = p^4$
b) $P = p^3(1-p)$
c) $1 - p^4$ as something else appear in first 4 toses which means at least a T is produced

3.64
a) $P_1 = P(C) = p$
b) $P_2 = P(C) = p^2 + 0.5 \times 2p(1-p)$

$P_2 - P_1 = p^2 + p(1-p) - p = 0$, thus both strategy shall have same probability of wining.

3.66
a) $P = P(C1C2C \cup C3C4C5) = (p_1p_2 + p_3p_3 - p_1p_2p_3p_4)p_5$
b)

$$\begin{aligned}
P(E) &= P(C_1C_4 \cup C_2C_5 \cup C_3C_1C_5 \cup C_3C_2C_4) \\
&= P\left[C_3^c(C_1C_4 \cup C_2C_5) \cup C_3(C_1C_4 \cup C_2C_5 \cup C_1C_5 \cup C_2C_4)\right] \\
&= P(C_3^c)P(C_1C_4 \cup C_2C_5) + P(C_3)P(C_1C_4 \cup C_2C_5 \cup C_1C_5 \cup C_2C_4) \\
&= p_1p_4 + p_2p_5 + p_3(p_1p_5 + p_2p_4) - (p_1p_2p_3p_4 + p_1p_2p_3p_5 \\
&\quad + p_1p_3p_4p_5 + p_2p_3p_4p_5) + 2p_1p_2p_3p_4p_5
\end{aligned}$$

3.78
a) $P = 2 \times p^3(1-p) + 2 \times p(1-p)^3$
b) $P = \frac{p^2}{p^2+(1-p)^2}$

3.81 $P = \dfrac{0.55^{15}}{0.45^{15} + 0.55^{15}} = 95.30\%$

3.83
a) $P = 0.5 \times \frac{4}{6} + 0.5 \times \frac{2}{6} = \frac{3}{6} = \frac{1}{2}$

b) $P = \frac{3}{5}$

c) $P = \frac{4}{5}$

3.84 a) $P(A) = \frac{1}{3} \sum_{i=1}^{\infty} \left(\frac{2}{3}\right)^{3(i-1)} = \frac{1}{3} \times \frac{-1}{\frac{8}{27} - 1} = \frac{9}{19}$

$P(B) = \frac{2}{9} \sum_{i=1}^{\infty} \left(\frac{2}{3}\right)^{3(i-1)} = \frac{2}{9} \times \frac{-1}{\frac{8}{27} - 1} = \frac{6}{19}$

$P(C) = \frac{19 - 9 - 6}{19} = \frac{4}{19}$

b) $P(A) = P(A1) + P(A2) + P(A3) = \frac{7}{15}$

$P(B) = P(B1) + P(B2) + P(A3) = \frac{68}{165}$

$P(C) = 1 - P(A) - P(B) = \frac{4}{33}$

Part 2

4.1 $\binom{3}{2} = 6$ types of X.

X=4: $P(X = 4) = \frac{4}{14}\frac{3}{13} = \frac{6}{91} \approx 6.59\%$

X=2: $P(X = 2) = \frac{4}{14}\frac{2}{13} + \frac{2}{14}\frac{4}{13} = \frac{\binom{4}{1}\binom{2}{1}}{\binom{14}{2}} = \frac{8}{91} \approx 8.79\%$

X=1: $P(X = 1) = \frac{\binom{4}{1}\binom{8}{1}}{\binom{14}{2}} = \frac{32}{91} \approx 35.16\%$

X=0: $P(X = 0) = \frac{\binom{2}{2}}{\binom{14}{2}} = \frac{1}{91} =\approx 1.10\%$

X=-1: $P(X = -1) = \frac{\binom{8}{1}\binom{2}{1}}{\binom{14}{2}} = \frac{16}{91} \approx 17.58\%$

X=-2: $P(X = -2) = \frac{\binom{8}{2}}{\binom{14}{2}} = \frac{28}{91} =\approx 30.77\%$

4.4

$X = 1$: $P(X = 1) = \frac{\binom{5}{1}9!}{10!} = \frac{1}{2} = 50\%$

$X = 2$: $P(X = 2) = \frac{\binom{5}{1}\binom{5}{1}8!}{10!} = \frac{5}{18} \approx 27.78\%$

$X = 3$: $P(X = 3) = \frac{2!\times\binom{5}{2}\binom{5}{1}\times 7!}{10!} = \frac{5}{36} \approx 13.89\%$

$X = 4$: $P(X = 4) = \frac{3!\times\binom{5}{3}\binom{5}{1}\times 6!}{10!} = \frac{5}{84} \approx 5.95\%$

$X = 5$: $P(X = 5) = \frac{4!\times\binom{5}{4}\binom{5}{1}\times 5!}{10!} = \frac{5}{252} \approx 1.98\%$

$X = 6$: $P(X = 6) = \frac{5!\times\binom{5}{5}\binom{5}{1}\times 4!}{10!} = \frac{1}{252} \approx 0.40\%$

$X > 6$: $P(X > 6) = 0$

4.5 $X \in \{y | y = (n - 2t), t \in \mathbb{Z}, (n - 2t) \geqslant 0\}$

Part 3

4.13 There are total of 5 possible variants of X's value.

$$\begin{aligned}
P(X=0) &= 0.7 \times 0.4 = 0.28 \\
P(X=500) &= 0.5 \times (0.3 \times 0.4 + 0.7 \times 0.6) = 0.27 \\
P(X=1000) &= 0.5 \times (0.3 \times 0.4 + 0.7 \times 0.6) + 0.5^2 \times 0.3 \times 0.6 = 0.315 \\
P(X=1500) &= 0.3 \times 0.6(0.5^2) \times 2! = 0.09 \\
P(X=2000) &= 0.3 \times 0.6(0.5^2) = 0.045
\end{aligned}$$

4.14

$$\begin{aligned}
P(X=4) &= \tfrac{1}{5} = 20\% \\
P(X=3) &= \tfrac{3!}{5!} = \tfrac{1}{20} = 5\% \\
P(X=2) &= \tfrac{1 \times \binom{3}{2} \times 2!}{5!} + \tfrac{1 \times \binom{2}{2} \times 2! \times 2!}{5!} = \tfrac{1}{12} \approx 8.33\% \\
P(X=1) &= \tfrac{1 \times \binom{3}{1} \times 1 \times 2!}{5!} + \tfrac{1 \times \binom{2}{1}\binom{2}{1} \times 2!}{5!} + \tfrac{1 \times 1 \times \binom{3}{1} \times 2!}{5!} = \tfrac{1}{6} \approx 16.67\% \\
P(X=0) &= \tfrac{\binom{4}{1} \times 3!}{5!} + \tfrac{\binom{3}{1} \times 3!}{5!} + \tfrac{\binom{2}{1} \times 3!}{5!} + \tfrac{3!}{5!} = 0.5 = 50\%
\end{aligned}$$

Part 4

4.17

a)

$$\begin{aligned}
P(X=1) &= \tfrac{1}{2} + \tfrac{0}{4} - \tfrac{1}{4} = \tfrac{1}{4} \\
P(X=2) &= \tfrac{11}{12} - \tfrac{1}{2} - \tfrac{1}{4} = \tfrac{1}{6} \\
P(X=3) &= 1 - \tfrac{11}{12} = \tfrac{1}{12}
\end{aligned}$$

b)

$$P(E) = P(\frac{3}{2}) - P(\frac{1}{2}) = \left(\frac{1}{2} + \frac{\frac{1}{2}}{4}\right) - \frac{\frac{1}{2}}{4} = \frac{1}{2}$$

4.19

$$\begin{cases}
\frac{1}{2} & x = 0 \\
\frac{1}{10} & x = 1 \\
\frac{1}{5} & x = 2 \\
\frac{1}{10} & x = 3 \\
\frac{1}{10} & x = 3.5
\end{cases}$$