PHYS 225 HW 3

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1. a) i:

$$\begin{bmatrix} ct \\ x \end{bmatrix} = \gamma \cdot \begin{bmatrix} 1 & \beta \\ \beta & 1 \end{bmatrix} \begin{bmatrix} ct' \\ x' \end{bmatrix}$$

putting the spacetime cordinate of the west clapping people and the car sycronized at origin. Then the space time event of the east people clapping in ground frame would be $\begin{pmatrix} 0 \\ L \end{pmatrix}$. According to the car's frame, the east people is moving at speed $-0.8c$ on x diection. Thus, $\gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} = \frac{5}{3}$, $\beta = \frac{v}{c} = -0.8$. Then the event in the car's reference frame is given by:

$$\begin{bmatrix} \frac{5}{3} & \frac{4}{3} \\ \frac{4}{3} & \frac{5}{3} \end{bmatrix} \begin{bmatrix} 0 \\ L \end{bmatrix} = \begin{bmatrix} \frac{4}{3}L \\ \frac{5}{3}L \end{bmatrix}$$

As $ct = \frac{4}{3}L, t = \frac{4L}{3c}$. In the car's reference frame, after $\frac{4L}{3c}$ time, its position will be at $\frac{4L}{3c} \times \frac{4}{5}c = \frac{16}{15}L$. Thus, the space time event of car passing the tree is: $\begin{pmatrix} \frac{4L}{3c} \\ 0 \end{pmatrix}$ Then in the ground reference frame the event happens at:

$$\begin{bmatrix} \frac{5}{3} & \frac{4}{3} \\ \frac{4}{3} & \frac{5}{3} \end{bmatrix} \begin{bmatrix} \frac{4L}{3} \\ 0 \end{bmatrix} = \begin{pmatrix} \frac{20}{9}L \\ \frac{16}{9}L \end{pmatrix}$$

Thus, the tree should be $\frac{16}{9}L$ east from the west people.

ii: Setting the speed of car as kc, then $\gamma = \frac{1}{\sqrt{1-k^2}}$ and $\beta = k$. Thus, the space time event that the east people claps is still $\begin{bmatrix} 0 \\ L \end{bmatrix}$. Then the space time even in car's frame is:

$$\begin{bmatrix} \frac{1}{\sqrt{1-k^2}} & \frac{k}{\sqrt{1-k^2}} \\ \frac{k}{\sqrt{1-k^2}} & \frac{1}{\sqrt{1-k^2}} \end{bmatrix} \begin{bmatrix} 0 \\ L \end{bmatrix} = \begin{pmatrix} \frac{k}{\sqrt{1-k^2}}L \\ \frac{1}{\sqrt{1-k^2}}L \end{pmatrix}$$

The tree event will be at:

$$\begin{bmatrix} \frac{1}{\sqrt{1-k^2}} & \frac{k}{\sqrt{1-k^2}} \\ \frac{k}{\sqrt{1-k^2}} & \frac{1}{\sqrt{1-k^2}} \end{bmatrix} \begin{bmatrix} \frac{k}{\sqrt{1-k^2}}L \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{k}{1-k^2}L \\ \frac{k^2}{1-k^2}L \end{bmatrix}$$

set $\frac{k^2}{1-k^2}L = L$, $k = \frac{1}{\sqrt{2}} \approx 70.7\%$.
Thus the velocity should be $\frac{c}{\sqrt{2}}$

b)

2. $\gamma_1 = \frac{1}{\sqrt{1-v_1^2/c^2}}$, $\gamma_2 = \frac{1}{\sqrt{1-v_2^2/c^2}}$, $\beta_1 = \frac{v_1}{c}$, $\beta_1 = \frac{v_1}{c}$.

$$\begin{bmatrix} \frac{1}{\sqrt{1-v_1^2/c^2}} & \frac{v_1}{\sqrt{c^2-v_1^2}} \\ \frac{v_1}{\sqrt{c^2-v_1^2}} & \frac{1}{\sqrt{1-v_1^2/c^2}} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{1-v_2^2/c^2}} & \frac{v_2}{\sqrt{c^2-v_2^2}} \\ \frac{v_2}{\sqrt{c^2-v_2^2}} & \frac{1}{\sqrt{1-v_2^2/c^2}} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{\sqrt{(1-v_1^2/c^2)(1-v_2^2/c^2)}} + \frac{v_1 v_2}{\sqrt{(c^2-v_1^2)(c^2-v_2^2)}} & \frac{v_2}{\sqrt{(1-v_1^2/c^2)(c^2-v_2^2)}} + \frac{v_1}{\sqrt{(1-v_2^2/c^2)(c^2-v_1^2)}} \\ \cdots & \cdots \end{bmatrix}$$

Thus,

$$\beta = \gamma\beta/\gamma$$
$$= \frac{v_2\sqrt{(1-v_2^2/c^2)(c^2-v_1^2)} + v_1\sqrt{(1-v_1^2/c^2)(c^2-v_2^2)}}{\sqrt{(c^2-v_1^2)(c^2-v_2^2)} + v_1 v_2\sqrt{(1-v_1^2/c^2)(1-v_2^2/c^2)}}$$

Thus, $v = \beta \times c$

$$= \frac{v_2\sqrt{(1-v_2^2/c^2)(c^2-v_1^2)} + v_1\sqrt{(1-v_1^2/c^2)(c^2-v_2^2)}}{\sqrt{(c^2-v_1^2)(c^2-v_2^2)} + v_1 v_2\sqrt{(1-v_1^2/c^2)(1-v_2^2/c^2)}} \times c$$
$$= \frac{v_2\sqrt{(c^2-v_2^2)(c^2-v_1^2)} + v_1\sqrt{(c^2-v_1^2)(c^2-v_2^2)}}{\sqrt{(c^2-v_1^2)(c^2-v_2^2)} + v_1 v_2\sqrt{(1-v_1^2/c^2)(1-v_2^2/c^2)}}$$
$$= \frac{\sqrt{(c^2-v_2^2)(c^2-v_1^2)}(v_1+v_2)}{\sqrt{(c^2-v_1^2)(c^2-v_2^2)} + v_1 v_2\sqrt{(1-v_1^2/c^2)(1-v_2^2/c^2)}}$$
$$= \frac{v_1+v_2}{1+v_1 v_2\sqrt{\frac{(1-v_1^2/c^2)(1-v_2^2/c^2)}{(c^2-v_2^2)(c^2-v_1^2)}}}$$
$$= \frac{v_1+v_2}{1+v_1 v_2\sqrt{\frac{(1-v_1^2/c^2)(1-v_2^2/c^2)}{(1-v_2^2/c^2)c^2(1-v_1^2/c^2)c^2}}}$$
$$= \frac{v_1+v_2}{1+v_1 v_2/c^2}$$