

MATH 416H HW 11

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1. Consider the base: $\{\frac{1}{0!}x^0, \frac{1}{1!}x^1, \frac{1}{2!}x^2 \dots, \frac{1}{n!}x^n\}$,

$$\text{Then, } A = [T]_{\mathcal{B}\mathcal{B}} = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & 1 \\ 0 & 0 & 0 & \dots & 0 & 0 \end{bmatrix}$$

Which is in jordan normal form

2. Notice that $\begin{pmatrix} 1 & 3 \\ 3 & 1 \end{pmatrix}$ is diagonalizable. Therefore, $USDS^{-1}U^{-1}$ is a diagonal matrix. Then, diagonalize $\begin{pmatrix} 1 & 3 \\ 3 & 1 \end{pmatrix}$. There is $\lambda_1 = 4, \lambda_2 = -2$ and $v_1 = (s, -s)^T, v_2 = (s, s)^T$ Then the equation becomes:

$$U \begin{bmatrix} s & s \\ s & -s \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{2s} & \frac{1}{2s} \\ \frac{1}{2s} & -\frac{1}{2s} \end{bmatrix} U^*$$

Consider the specific case where $US = I, U^*S^{-1} = I$ while setting the imaginary part to zero, we have $UU = I, US = I, US^{-1} = I$, then $S = S^{-1}$, we have $\begin{bmatrix} \frac{1}{2s} & \frac{1}{2s} \\ \frac{1}{2s} & -\frac{1}{2s} \end{bmatrix} = \begin{bmatrix} s & s \\ s & -s \end{bmatrix}$, then $s = \frac{\sqrt{2}}{2}$, then $U = S^{-1} =$

$$\begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \end{bmatrix} = U^*.$$

3. a) Since the standard basis of \mathbb{C}^n is orthonormal, $\langle e_i, e_j \rangle = \delta_j^i$ and $\varphi e_i = b_i$, then We have $\langle \varphi(e_i), \varphi(e_j) \rangle = \langle b_i, b_j \rangle = \delta_j^i = \langle e_i, e_j \rangle$. Thus the map is unitary.
- b) The transformation that sends $\{e_1, e_2, \dots, e_n\}$ into \mathcal{B}' is Unitary with similar proof done in a). And for a Unitary map we can write out a matrix, note $\varphi = [\varphi] = \mathcal{T}$ that $[\varphi] = \mathcal{T}$ is the matrix representation. Then its inverse would be \mathcal{T}^* which is also unitary as:

$\mathcal{T}^* (\mathcal{T}^*)^* = \mathcal{T}^* \mathcal{T} = I$. Then consider $\varphi' \circ \varphi$ ($\varphi' = \mathcal{T}'$):

$$\begin{aligned}\varphi' \circ \varphi &= \mathcal{T}' \mathcal{T} = T \\ TT^* &= \mathcal{T}' \mathcal{T} (\mathcal{T}' \mathcal{T})^* \\ &= \mathcal{T}' \mathcal{T} \mathcal{T}^* \mathcal{T}'^* \\ &= I\end{aligned}$$

Therefore, $\varphi' \circ \varphi$ is unitary. And such transformation transforms from \mathcal{B} to \mathcal{B}' . Which is $[\text{id}_V]_{\mathcal{B}, \mathcal{B}'}$

4.

Forward: For any Hermitian matrix A that is positive definite, for all $v \in V$ $\langle v, Av \rangle > 0$ then for all eigenvectors v_i There is:

$$\begin{aligned}Av_i &= \lambda_i v_i \\ \langle v_i, Av_i \rangle &= \langle v_i, \lambda_i v_i \rangle \\ &= \overline{\lambda_i} \langle v_i, v_i \rangle \\ &= \overline{\lambda_i} \|v_i\|^2\end{aligned}$$

For any non-zero vector v_i , $\|v_i\|^2 > 0$, then for $\langle v_i, Av_i \rangle > 0$, $\overline{\lambda_i} \|v_i\|^2 > 0$ Thus $\lambda_i > 0$ also as it is a hermitian matrix, all of its eigenvalues are real.

Backward: Assume that $\forall \lambda_i > 0$. According to the spectral theorem, $\forall A$, $\exists U$ that is unitary that $UAU^* = D$ which is the diagonal matrix with all of A 's eigenvalues on its main diagonal. Then we have:

$$\begin{aligned}\langle v, Av \rangle &= \langle Uv, UAv \rangle \\ &= \langle UU^*v, UAU^*v \rangle \\ &= \langle Iv, Dv \rangle\end{aligned}$$

Consider a set of orthonormal basis of V , $\{b_1, \dots, b_n\}$. $v = \sum \mu_i b_i$, $\mu_i \in \mathbb{C}$, then $Dv = \sum_{i=1}^n \lambda_i \mu_i b_i$, Thus:

$$\begin{aligned}\langle v, Dv \rangle &= \sum_{i=1}^n \mu_i \overline{\lambda_i \mu_i} \|b_i\|^2 \\ &= \sum_{i=1}^n \lambda_i \|\mu_i\|^2 \|b_i\|^2\end{aligned}$$

since $\|\mu_i\|^2 > 0$, $\|b_i\|^2 > 0$ and $\forall \lambda_i > 0$ (and real as it is from a hermitian matrix), $\sum_{i=1}^n \lambda_i \|\mu_i\|^2 \|b_i\|^2 > 0$ Therefore the matrix is positive definite.

5. Since A is hermitian, then $\exists U, D$ that $A = UDU^*$ where U is unitary and D is diagonal with entries equaling to eigenvalues. Therefore, as A is positive definite, all entries of D would be positive real numbers. then write the matrix as $D_{ij} = \delta_{ij} \lambda_i$, then $\exists D'$ that $D'_{ij} = \delta_{ij} \sqrt{\lambda_i}$, then $D'D' = \Gamma$, and $\Gamma_{ij} = \sum_{k=1}^n \delta_{ik} \sqrt{\lambda_k} \delta_{kj} \sqrt{\lambda_k} = \delta_{ij} \lambda_i = D$, Therefore: $A = UD'D'U^*$ As D' only have real positive entries in its main diagonal, $D'^* = D'$, then take $S = D'U^*$, then $S^* = (D'U^*)^* = UD'^* = UD'$, Thus, $S^*S = UD'D'U^* = A$

6. a)

Forward:

Backward:

b)

c)

7. a)

b)

c)

8. a)

b)