PHYS 225 HW 10

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1. a)

$$V = (\gamma, \gamma v_x, \gamma v_y, \gamma v_z)$$

For $V \cdot V$ in rest frame, considering $c = 1$, we have $v_x = v_y = v_z = 0$, then:

$$\begin{aligned} V \cdot V &= \gamma^2 - (0 + 0 + 0) \\ &= \frac{1}{1-0} = 1 \end{aligned}$$

For $V \cdot V$ in the frame where the particle only moves in the x direction, $v_x = |v|, v_y = v_z = 0$, Then:

$$\begin{aligned} V \cdot V &= \gamma^2 - (\gamma^2 v_x^2 + 0 + 0) \\ &= \frac{1}{1-v^2} - \frac{v^2}{1-v^2} \\ &= \frac{1-v^2}{1-v^2} = 1 \end{aligned}$$

Therefore it is indeed in variant.

b)

$$A = \left(\gamma^4 va, \gamma^4 vav_x + \gamma^2 a_x, \gamma^4 vav_y + \gamma^2 a_y, \gamma^4 vav_z + \gamma^2 a_z\right)$$

First consider a rest frame with $v = 0$ and $a = a_x$, then:

$$\begin{aligned} A \cdot A &= \gamma^8 v^2 a^2 - (\gamma^4 va\mathbf{v} + \gamma^2 \mathbf{a}) \cdot (\gamma^4 va\mathbf{v} + \gamma^2 \mathbf{a}) \\ &= 0 - (\gamma^4 a_x^2) \\ &= -a_x^2 \end{aligned}$$

Now, consider a frame where $v' = v_x'$, now calculate the respect acceleration, $a'_x = \frac{a_x}{\gamma^3}$, now:

$$\begin{aligned}
A \cdot A &= \gamma^8 v^2 a^2 - (\gamma^4 va\mathbf{v} + \gamma^2 \mathbf{a}) \cdot (\gamma^4 va\mathbf{v} + \gamma^2 \mathbf{a}) \\
&= \gamma^8 v'^2 \left(\frac{a}{\gamma^3}\right)^2 - \left(\frac{a_x}{\gamma}\left(v^2+1\right)\right)^2 \\
&= \gamma^2 v_x^2 a_x^2 - \left(\frac{a^2(\gamma^2 v^2 + 1)^2}{\gamma^2}\right) \\
&= \gamma^2 v^2 a^2 - \left(\frac{a^2\left(\frac{1+v^2}{1-v^2}\right)^2}{\gamma^2}\right) \\
&= \frac{\gamma^4 v^2 a^2 - a^2\left(\frac{1+v^2}{1-v^2}\right)^2}{\gamma^2} \\
&= a^2 \frac{\gamma^4 v^2 - \left(\frac{1+v^2}{1-v^2}\right)^2}{\gamma^2} \\
&= a^2 \frac{\left(\gamma^2 v - \left(\frac{1+v^2}{1-v^2}\right)\right)\left(\gamma^2 v + \left(\frac{1+v^2}{1-v^2}\right)\right)}{\gamma^2} \\
&= -a^2(1 + v^2 + v^4)
\end{aligned}$$

which is a bit different from the one in rest fram.

c)

$$\begin{aligned}
F &= m\left(\gamma^4 va_F, \gamma^4 a_x, \gamma^4 a_y, \gamma^4 a_z\right) \\
V &= (\gamma, \gamma v_x, \gamma v_y, \gamma v_z)
\end{aligned}$$

in the rest frame, with $v_z = \frac{1}{2}$, $F = (0, f_x, 0, 0)$, $V = (0, 0, 0, 1)$, $V \cdot F = f_x$, Now, transform everything into a moving frame by multipling Λ, We have, $V \cdot F = \gamma^2 f_x$ which is also different.

d) in rest frame, whith $\mathbf{F} = f_x$, then $F \cdot dx = f_x dx$, and in a transforming frame, $F' dx' = \gamma^2 f_x dx$

2. a)

$$\begin{aligned}
E_{k,ball} &= 3m \\
E_{ball} &= 3mc^2 + mc^2 = 4mc^2 \\
E_{ball}^2 &= p^2c^2 + m^2c^4 \\
16m^2c^4 &= p^2c^2 + m^2c^4 \\
p^2c^2 + m^2c^4 - 16m^2c^4 &= 0 \\
p^2 &= \frac{16m^2c^4 - m^2c^4}{c^2} \\
p_{ball} &= \sqrt{15}mc \\
p'_{x,ball} &= p \cdot \cos(\theta) = \left(\sqrt{15}mc\right)\cos(\theta) \\
p'_{y,ball} &= p \cdot \sin(\theta) = \left(\sqrt{15}mc\right)\sin(\theta)
\end{aligned}$$

Transform momentum into S frame.

$$\begin{aligned}
\Lambda &= \begin{bmatrix} \gamma & \gamma\beta & 0 & 0 \\ \gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \\
\mathbf{P}c &= \Lambda P' \\
&= \begin{bmatrix} \gamma & \gamma\beta & 0 & 0 \\ \gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 16m^2c^4 \\ \left(\sqrt{15}mc\right)\cos(\theta)c \\ \left(\sqrt{15}mc\right)\sin(\theta)c \\ 0 \end{bmatrix} \\
&= \begin{bmatrix} \gamma\left(4mc^2 + \beta\left(\sqrt{15}mc^2\right)\cos(\theta)\right) \\ \gamma\left(\beta 4mc^2 + \left(\sqrt{15}mc^2\right)\cos(\theta)\right) \\ \left(\sqrt{15}mc^2\right)\sin(\theta) \\ 0 \end{bmatrix} \\
p_{x,ball} &= \gamma\left(\beta 4mc + \left(\sqrt{15}mc\right)\cos(\theta)\right) \\
p_{y,ball} &= \left(\sqrt{15}mc\right)\sin(\theta)
\end{aligned}$$

Now get the velocity of the ball in the rest frame or S frame.

$$
\begin{aligned}
p_{x,ball} &= \gamma m v \\
\frac{p}{m} &= \frac{v}{\sqrt{1-\frac{v^2}{c^2}}} \\
\frac{p^2}{m^2} &= \frac{v^2}{1-\frac{v^2}{c^2}} \\
v^2 &= \frac{\left(\frac{p}{m}\right)^2}{1+\left(\frac{p}{mc}\right)^2} \\
v_{x,ball} &= \frac{\gamma\left(4\beta c+\left(\sqrt{15}c\right)\cos(\theta)\right)}{\sqrt{1+\gamma\left(4\beta+\left(\sqrt{15}\right)\cos(\theta)\right)}} \\
v_{y,ball} &= \frac{\left(\sqrt{15}c\right)\sin(\theta)}{\sqrt{1+\left(\sqrt{15}\right)\sin(\theta)}}
\end{aligned}
$$

for some t, the snow ball hits $(0,0)$, thus:

$$
\begin{aligned}
v_x t &= 4 \\
v_y t &= 1 \\
\frac{v_x}{v_y} &= 4 \\
\frac{\frac{\gamma\left(4\beta c+\left(\sqrt{15}c\right)\cos(\theta)\right)}{\sqrt{1+\gamma\left(4\beta+\left(\sqrt{15}\right)\cos(\theta)\right)}}}{\frac{\left(\sqrt{15}c\right)\sin(\theta)}{\sqrt{1+\left(\sqrt{15}\right)\sin(\theta)}}} &= 4 \\
\frac{\gamma\left(4\beta c+\left(\sqrt{15}c\right)\cos(\theta)\right)}{\sqrt{1+\gamma\left(4\beta+\left(\sqrt{15}\right)\cos(\theta)\right)}} \times \frac{\sqrt{1+\left(\sqrt{15}\right)\sin(\theta)}}{\left(\sqrt{15}c\right)\sin(\theta)} &= 4 \\
\gamma = \frac{1}{\sqrt{1-\beta^2}} &= \frac{1}{\sqrt{0.51}} \\
\beta &= 0.7
\end{aligned}
$$

plug back into the equation, we get: $\cos(\theta) = 0.97998$ or -0.047908

b)

$$
\begin{aligned}
p &= \sqrt{p_x^2 + p_y^2} \\
p^2 &= \left(\gamma\beta 4mc + \left(\sqrt{15}mc\right)\cos(\theta)\right)^2 + \left(15m^2c^2\right)\sin^2(\theta) \\
p &= \sqrt{m^2c^2(30 + 8\beta\gamma(\sqrt{15}\cos(\theta) + 2))} \\
v &= \frac{pc^2}{E} \\
&= \frac{pc^2}{4mc^2} \\
&= \frac{\sqrt{30 + 8\beta\gamma(\sqrt{15}\cos(\theta) + 2)}}{4c} \\
t &= L/v \\
&= \frac{\sqrt{16+1}c}{\frac{\sqrt{30+8\beta\gamma(\sqrt{15}\cos(\theta)+2)}}{4c}} \\
&= \frac{4\sqrt{17}c^2}{\sqrt{30 + 8\beta\gamma(\sqrt{15}\cos(\theta) + 2)}} \\
&= 1.7 \times 10^{17} \text{ or } 2.2319 \times 10^{17}\ s
\end{aligned}
$$

c)

$$
\begin{aligned}
\tau = \frac{t}{\gamma_{ball}} &= \frac{tp}{mv} \\
&= 6.12 \times 10^{-34} s
\end{aligned}
$$

d)

$$
\begin{aligned}
p' &= \sqrt{15}mc \\
p &= \sqrt{m^2c^2(30 + 8\beta\gamma(\sqrt{15}\cos(\theta) + 2))}
\end{aligned}
$$

3. a)

$$
\begin{aligned}
\frac{f_{\text{obs}}}{f_{\text{emit}}} &= \sqrt{\frac{1+\beta}{1-\beta}} \\
\frac{550}{700} &= \sqrt{\frac{1+\beta}{1-\beta}} \\
\beta &= -\frac{12}{37} \\
v &= \frac{12}{37}c = 9.72 \times 10^7 \text{ m/s}
\end{aligned}
$$

b)

$$
\begin{aligned}
v &= 4 \times 10^4 \text{km/hr} \\
&= 4 \times 10^4 \times 10^3 \div 60^2 \text{ m/s} \\
&= 1.11 \times 10^4 \text{m/s} \\
&= 3.707 \times 10^{-5} c \\
\beta &= 3.707 \times 10^{-5} \\
\frac{f_{obs}}{f_{emit}} &= \sqrt{\frac{1+\beta}{1-\beta}} \\
f_{obs} &= 700.026 \text{ nm}
\end{aligned}
$$

A taylor expansion would be appropriate due to the significantly small value of β.