MATH 416H HW 4

James Liu

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1. It is $\sum_{i=1}^{m} a_i x_i$

2. It is $\begin{pmatrix} yb_1 \\ yb_2 \\ \vdots \\ yb_n \end{pmatrix}$

3.

$$E_1 A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{bmatrix} = \begin{bmatrix} a_{21} & a_{22} & a_{23} & a_{24} \\ a_{11} & a_{12} & a_{13} & a_{14} \\ 3 \times a_{31} & 3 \times a_{32} & 3 \times a_{33} & 3 \times a_{34} \end{bmatrix}$$

$$E_2 A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{bmatrix} = \begin{bmatrix} a_{31} & a_{32} & a_{33} & a_{34} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{11} & a_{12} & a_{13} & a_{14} \end{bmatrix}$$

$$E_3 A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -5 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} - 5a_{31} & a_{22} - 5a_{32} & a_{23} - 5a_{33} & a_{24} - 5a_{34} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{bmatrix}$$

4. a) $\iota(w) = w,\ \iota(v) = v,\ \iota(w) + \iota(v) = w + v = \iota(w + v)$
   $\lambda\iota(v) = \lambda \cdot v = \iota(\lambda v)$. Thus, the inclusion map is linear.

   b) i: $\forall u \in U,\ T|_U(u) = w = T(u),\ \iota_U(u) = u$. Thus $T(\iota_U(u)) = T(u) = w = T|_U(u)$. Thus $T|_U = T \circ \iota_U$
   As the composition of 2 linear maps is still linear, $T|_U$ is linear.

5. a)

forward: Suppose $\exists w_1, w_1' \in W_1,\ w_2, w_2', \in W_2$, that $v = w_1 + w_2 = w_1' + w_2'$. Thus:

$$w_1 - w_1' = w_2' - w_2$$
$$w_1 - w_1' \in W_1 \quad w_2' - w_2 \in W_2$$
$$\text{as } W_1 \cap W_2 = \{\overrightarrow{0}\}$$
$$w_1 - w_1' = w_2' - w_2 = \overrightarrow{0}$$
$$w_1 = w_1' \quad w_2 = w_2'$$

Thus, if $V = W_1 \oplus W_2$, $\forall v \in V$, exists a unique w_1, w_2 that $v = w_1 + w_2$

backward: If $\forall v \in V$, $\exists w_1 \in W_1,\ w_2 \in W_2$ that $v = w_1 + w_2$, $V = W_1 + W_2$ by definition. Suppose that $W_1 \cap W_2 \neq \{\overrightarrow{0}\}$, then $\exists w \in W_1, W_2$. Thus, forsome $v \in W_1$, or $v = w_1 + \overrightarrow{0}$, Thus define $k = w_1 - w$, Therefore $\exists v = w_1 - w + w$, where $w \neq \overrightarrow{0}$. However, there are only one set of w_1, w_2 that $v = w_1 + w_2$, Therefroe, $W_1 \cap W_2 = \overrightarrow{0}$

b) **Existence**:

$\forall v \in V$, exists a unique set $w_1 \in W_1,\ w_2 \in W_2$, that $v = w_1 + w_2$ as profed in 5.a). Define:

$$\forall v = w_1 + w_2$$
$$\iota_1(v) = w_1,\ \iota_2(v) = w_2$$
$$\iota_1' : W_1 \to V\ \iota_1'(w_1) = w_1 + \overrightarrow{0},\ \iota_2' : W_2 \to V\ \iota_2'(w_2) = w_2 + \overrightarrow{0}$$

$\forall T_1, T_2$ define: $T(v) = T_1 \circ \iota_1(v) + T_2 \circ \iota_2(v)$.
$\forall w_1 \in W_1$: $T|_{W_1} = T_1(w_1) + T_2(\overrightarrow{0})$, as $T_1,\ T_2$ are linear, there is $T_2(\overrightarrow{0}) = \overrightarrow{0}$. Thus, $T|_{W_1} = T_1$ similarly, $T|_{W_2} = T_2$. Thus, such map exists.
Linearity:

$$\begin{aligned}
T(v_1 + v_2) &= T(w_{11} + w_{12} + w_{21} + w_{22}) \\
&= T_1(w_{11} + w_{21}) + T_2(w_{12} + w_{22}) \\
&= T_1(w_{11}) + T_1(w_{21}) + T_2(w_{12}) + T_1(w_{22}) \\
&= T_1(w_{11}) + T_2(w_{12}) + T_1(w_{21}) + T_1(w_{22}) \\
&= T(v_1) + T(v_2) \\
T(\lambda v_1) &= T(\lambda(w_1 + w_2)) \\
&= T(\lambda w_1 + \lambda w_2) \\
&= T_1(\lambda w_1) + T_2(\lambda w_2) \\
&= \lambda(T_1(w_1) + T_2(w_2)) \\
&= \lambda T(v_1) \qquad \text{Thus, } T \text{ is linear.}
\end{aligned}$$

Uniqueness:

Suppose it exists such T' that $T'|_{W_1} = T_1,\ T'|_{W_2} = T_2,\ T' \neq T$. Then $T'(v) \neq T(v)$. However:

$$\begin{aligned} T'(v) &= T'(w_1 + w_2) \\ &= T'(w_1) + T'(w_2) \\ &= T'|_{W_1}(w_1) + T'|_{W_2}(w_2) \\ &= T_1(w_1) + T_2(w_1) \\ &= T(v) \end{aligned}$$

Thus, it is unique.

6. $\forall v \in V, T(S(v)) = v$, Thus, T is Surjective, Thus, $R(T) = dim(V)$, Thus $N(T) = \{\overrightarrow{0}\}$ Suppose T is not injective, then $\exists v, w \in V$ that $v \neq w$ that $T(v) = T(w)$, then $T(v) - T(w) = \overrightarrow{0} = T(v - w)$, which raises a contradiction. Thus, T is a bijection. Thus T is invertable. As T is invertable, $\exists T^{-1}$ that $T \circ T^{-1} = \mathrm{id}_V$. Therefore, $T^{-1} = S$. Thus, $S \circ T = T^{-1} \circ T = \mathrm{id}_V$

7. Take a random set of $a_1, \cdots, a_k, \cdots \in \mathbb{N}, S(a_1, a_2, \cdots, a_k, \cdots) = (0, a_1, a_2, \cdots, a_k, \cdots)$, $T(0, a_1, a_2, \cdots, a_k, \cdots) = (a_1, a_2, \cdots, a_k, \cdots)$. Thus, $T \circ S = \mathrm{id}_V$

8. $\forall x_i \in \mathbb{R},\ v = (x_1, \cdots, x_n)P(v) = (x_1, 0, \cdots, 0)$, and $P(P(v)) = P(x_1, 0, \cdots, 0) = (x_1, 0, \cdots, 0) = P(v)$ Thus, $P \circ P = P$
   $N(P) = (0, x_2, \cdots, x_n),\ R(P) = (x_1, 0, \cdots, 0),\ \mathbb{R}^n = N(P) + R(P) = (x_1, x_2, \cdots, x_n)$. Also, as $R(P) \cap N(P) = (0, 0, \cdots, 0) = \overrightarrow{0}$. Thus, $\mathbb{R}^n = N(P) \oplus R(P)$