MATH 416H HW 6

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1. a) Rank would be 3 and Nullity would be 2 as the matrix is already in reduced row-echlon form and the number of pivots is the rank and the number of none pivot column is Nullity.

   b)

$$\left[\begin{array}{ccccc|c} 1 & 2 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{array}\right] \to \begin{cases} x_1 = -2x_2 - x_4 \\ x_2 = x_2 \\ x_3 = -x_4 \\ x_4 = x_4 \\ x_5 = 0 \end{cases}$$

   Take $x_2 = 1, x_4 = 1$, seperatly, we have a basis consisting 2 element: $\left\{ \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ -1 \\ 1 \\ 0 \end{pmatrix} \right\}$

2. a) Scaler Multiplication: multiplying a scaler does not change the symetric of the matrix.

$$\begin{aligned} A &= A^T \\ a_{ij} &= A_{ji} \quad 0 \le ij \le n \\ ka_{ij} &= ka_{ji} \quad k \in F \end{aligned}$$

   Vector Addition: Adding two such matrix also does not change such symetry:

$$\begin{aligned} A &= A^T & B &= B^T \\ a_{ij} &= a_{ji} & b_{ij} &= b_{ji} \\ a_{ij} + b_{ij} &= a_{ji} + b_{ji} \end{aligned}$$

   And these operations do fullfill the 8 properties as in the question, $M_{n,n}(F)$ is already a vector space. And S_n is close under the 2 operations, thus it is a subspace.

b) Notice that one possible set of basis would be:

$$\left\{ \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right\}$$

3. T is injective, then$\forall w \in W$, $\exists v \in V$ such that $w = T(v)$. By definition, $T^*(\psi) = \psi \circ T(v)$, $\psi \in W^*$. Consider $T^*(\psi)(v) = 0$

$$\begin{aligned} T^*(\psi)(v) &= 0 \\ \psi(T(v)) &= 0 \; \forall v \in V \end{aligned}$$

As $T(v)$ is surjective on W, which means that $\psi(w) = 0$, $\forall w \in W$. Thus, ψ is a zero map, thus, $N(T^*) = \{\overrightarrow{0}\}$. Thus, by rank/nullity, the T^* is injective.

4. Vector addition: