MATH 416H HW 9

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1. Note the basis as $\{b_1, b_2\}$, as $b_1, b_2 \in V$, $b_{11} + b_{12} + b_{13} = 0$ similar to b_2. consider such vector $m = \begin{pmatrix} 1+0i \\ 0+0i \\ -1+0i \end{pmatrix}$, then, the orthogonal basis would have a property like: $\langle m \cdot n \rangle = 0$ which is:

$$\begin{aligned} 1 \times \bar{n}_1 + 0 \times \bar{n}_2 - 1\bar{n}_3 = 0 \\ \bar{n}_1 - \bar{n_3} = 0 \end{aligned}$$

   Therefore, $m = \begin{pmatrix} 1+0i \\ 0+0i \\ -1+0i \end{pmatrix}, n = \begin{pmatrix} 1+0i \\ -2+0i \\ 1+0i \end{pmatrix}$ is a set of orthogonal vectors. Claim that it is a basis.
   Notice that $\forall v \in V, \exists v = (v_1 + \frac{1}{2}v_2)m - \frac{1}{2}v_2 n$, therefore, it is a basis.

2. $T(x_1, x_2, \cdots, x_n) = \sum_{i,j=1}^n \delta_i^j x_{ij}$, .

$$\begin{aligned} T(a+b, x_2, \cdots, x_n) &= (a_{11} + b_{11}) + \sum_{i,j=2}^n \delta_i^j x_{ij} \\ &\neq a_{11} + \sum_{i,j=2}^n \delta_i^j x_{ij} + b_{11} + \sum_{i,j=2}^n \delta_i^j x_{ij} \\ &= (a_{11} + b_{11}) + 2\sum_{i,j=2}^n \delta_i^j x_{ij} \\ &= T(a, x_2, \cdots, x_n) + T(b, x_2, \cdots, x_n) \\ &\neq T(a+b, x_2, \cdots, x_n) \end{aligned}$$

   Therefore, it is not n-linear.

3. Consider $v, w \in W$

$$\begin{aligned}
T(v+w) &= T(v) + T(w) \\
&= 0 + 0 \\
&\text{Therefore it is close under addition} \\
T(\lambda v) &= \lambda T(v) \\
&= 0 \\
&\text{Therefore it is close under scaler multiplication} \\
v + \overrightarrow{0} &= v \\
&\text{Thus ther exists a zero vector}
\end{aligned}$$

4. a)

$$\begin{aligned}
\langle x, y \rangle &= \langle \sum \alpha_i v_i, \sum \beta_j v_j \rangle \\
&= \sum \alpha_i \langle v_i, \sum \beta_j v_j \rangle \\
&= \sum_{i=1}^{n} \alpha_i \overline{\langle \sum_{j=1}^{n} \beta_j v_j, v_i \rangle} \\
&= \sum_{i=1}^{n} \left(\alpha_i \sum_{j=1}^{n} \overline{\beta_j} \overline{\langle v_j, v_i \rangle} \right) \\
&= \sum_{i=1}^{n} \left(\alpha_i \sum_{j=1}^{n} \overline{\beta_j} \langle v_i, v_j \rangle \right) \\
&= \sum_{i=1}^{n} \left(\alpha_i \sum_{j=1}^{n} \overline{\beta_j} \delta_i^j \right) \\
&= \sum \alpha_k \beta_k
\end{aligned}$$

b)

$$\begin{aligned}
\langle x, v_k \rangle &= \sum_{i=1}^{n} \delta_i^k \alpha_k \times 1 \\
&= \alpha_k \\
\overline{\langle y, v_k \rangle} &= \sum_{i=1}^{n} \delta_i^k \overline{\beta}_k \times 1 \\
&= \overline{\beta}_k \\
\sum \langle x, v_k \rangle \overline{\langle y, v_k \rangle} &= \sum \alpha_k \overline{\beta_k} \\
&= \langle x, y \rangle
\end{aligned}$$

5. It is not a real inner product. If it is a inner product, then $\langle A, A\rangle = \overrightarrow{0}$ then $A = \overrightarrow{0}$. However, consider $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $AA = \overrightarrow{0}$, $\text{Tr}(A, A) = 0 + 0$ which is 0 while A is not $\overrightarrow{0}$. Thus, it is not a real inner product

6.

$||u + v||^2$

$$\begin{aligned} ||u+v||^2 &= \langle u+v, u+v\rangle \\ &= \langle u, u+v\rangle + \langle v, u+v\rangle \\ &= \overline{\langle u+v, u\rangle} + \overline{\langle u+v, v\rangle} \\ &= \overline{\langle u, u\rangle} + \overline{\langle v, u\rangle} + \overline{\langle u, v\rangle} + \overline{\langle v, v\rangle} \\ &= ||u||^2 + \langle u, v\rangle + \overline{\langle u, v\rangle} + ||v||^2 \\ &= 2^2 + (2+i) + (2-i) + 3^2 \\ &= 17 \end{aligned}$$

$||u - v||^2$

$$\begin{aligned} ||u-v||^2 &= \langle u-v, u-v\rangle \\ &= \langle u, u-v\rangle - \langle v, u-v\rangle \\ &= \overline{\langle u-v, u\rangle} - \overline{\langle u-v, v\rangle} \\ &= \overline{\langle u, u\rangle} - \overline{\langle v, u\rangle} - \left(\overline{\langle u, v\rangle} - \overline{\langle v, v\rangle}\right) \\ &= ||u||^2 - \langle u, v\rangle - \overline{\langle u, v\rangle} + ||v||^2 \\ &= 4 - (2+i) - (2-i) + 3^2 \\ &= 9 \end{aligned}$$

$\langle u + iv, v + iu\rangle$

$$\begin{aligned} \langle u+iv, v+iu\rangle &= \langle u, v\rangle + \langle u, iu\rangle + \langle iv, v\rangle + \langle iv, iu\rangle \\ &= \langle u, v\rangle - i\langle u, u\rangle + i\langle v, v\rangle + \langle v, u\rangle \\ &= (2+i) - i \cdot 4 + i \cdot 9 + (2-i) \\ &= 4 + 5i \end{aligned}$$

7.

$$
\begin{aligned}
\overline{\left\langle \sum \beta_j b_j, \sum \alpha_i b_i \right\rangle} &= \overline{\sum \beta_k \overline{\alpha_k}} \\
&= \sum \alpha_k \overline{\beta_k} \\
&= \left\langle \sum \alpha_i b_i, \sum \beta_j b_j \right\rangle \\
\left\langle \sum \lambda \alpha_i b_i + \sum \mu \gamma_l b_l, \sum \beta_j b_j \right\rangle &= \left\langle \sum \lambda \alpha_i b_i + \mu \gamma_i b_i, \sum \beta_j b_j \right\rangle \\
&= \sum (\lambda \alpha_i + \mu \gamma_i) \overline{\beta_i} \\
&= \lambda \sum \alpha_i \overline{\beta_i} + \mu \sum \gamma_i \overline{\beta_i} \\
&= \lambda \left\langle \sum \alpha_i b_i, \sum \beta_j b_j \right\rangle + \mu \left\langle \sum \gamma_i b_i, \sum \beta_j b_j \right\rangle \\
\left\langle \sum \alpha_i b_i, \sum \alpha_j b_j \right\rangle &= \sum \alpha_k \overline{\alpha}_k \\
&= \sum ||\alpha_k||^2 \geq 0 \\
\text{if } \left\langle \sum \alpha_i b_i, \sum \alpha_j b_j \right\rangle &= 0 \\
\text{then } \sum ||\alpha_k||^2 &= 0 \\
\text{since} ||a_k||^2 \geq 0, ||\alpha_k||^2 &= 0 \\
\text{if } v &= 0 \\
\text{then } \alpha_i &= 0 \\
\text{Thus } \langle v, v \rangle &= 0
\end{aligned}
$$

Therefore it is a Hermitian inner product

8.

$$
\begin{aligned}
&\frac{1}{4} \left(||x+y||^2 - ||x-y||^2 - i||x-iy||^2 + i||x+iy||^2 \right) \\
&= \frac{1}{4} \left(\langle x+y, x+y \rangle - \langle x-y, x-y \rangle - i \langle x-iy, x-iy \rangle + i \langle x+iy, x+iy \rangle \right)
\end{aligned}
$$

Calculate each part seperatly:

$$
\begin{aligned}
\langle x+y, x+y\rangle &= \langle x,x\rangle + \langle x,y\rangle + \langle y,x\rangle + \langle y,y\rangle \\
\langle x-y, x-y\rangle &= \langle x,x\rangle + \langle x,-y\rangle + \langle -y,x\rangle + \langle -y,-y\rangle \\
-\langle x-y, x-y\rangle &= -\langle x,x\rangle + \langle x,y\rangle + \langle y,x\rangle - \langle y,y\rangle \\
\langle x+y, x+y\rangle - \langle x-y, x-y\rangle &= 2\langle x,y\rangle + 2\langle y,x\rangle \\
\langle x-iy, x-iy\rangle &= \langle x,x\rangle + \langle x,-iy\rangle + \langle -iy,x\rangle + \langle -iy,-iy\rangle \\
&= \langle x,x\rangle + i\langle x,y\rangle - i\langle y,x\rangle + (-i\times i)\langle y,y\rangle \\
&= \langle x,x\rangle + i\langle x,y\rangle - i\langle y,x\rangle + \langle y,y\rangle \\
-i\langle x-iy, x-iy\rangle &= -i\langle x,x\rangle + \langle x,y\rangle - \langle y,x\rangle - i\langle y,y\rangle \\
\langle x+iy, x+iy\rangle &= \langle x,x\rangle + \langle x,iy\rangle + \langle iy,x\rangle + \langle iy,iy\rangle \\
&= \langle x,x\rangle - i\langle x,y\rangle + i\langle y,x\rangle + (i\times -i)\langle y,y\rangle \\
&= \langle x,x\rangle - i\langle x,y\rangle + i\langle y,x\rangle + \langle y,y\rangle \\
+i\langle x+iy, x+iy\rangle &= i\langle x,x\rangle + \langle x,y\rangle - \langle y,x\rangle + i\langle y,y\rangle \\
-i\langle x-iy, x-iy\rangle + i\langle x+iy, x+iy\rangle &= 2\langle x,y\rangle - 2\langle y,x\rangle \\
\frac{1}{4}\left(||x+y||^2 - ||x-y||^2 - i||x-iy||^2 + i||x+iy||^2\right) &= \frac{1}{4}(2\langle x,y\rangle + 2\langle y,x\rangle + 2\langle x,y\rangle - 2\langle y,x\rangle) \\
&= \langle x,y\rangle
\end{aligned}
$$