Physics 225, Fall 2024: Homework 12
Due 12/5/24

All solutions must clearly show the steps or reasoning you used to arrive at your result. You will lose points for poorly written solutions or incorrect reasoning/explanations; answers given without explanation will not be graded.

Name:
Section:

1. **Index notation practice.** Here you will use the Levi-Civita symbol to quickly derive several useful derivative identities: this is a good example of the power of index notation!

   (a) Derive the identity $\nabla \cdot (\nabla \times \mathbf{A}) = 0$ for any vector field $\mathbf{A}$. The index heights work out most easily if you treat $\mathbf{A}$ to have lower indices A_k, which is equivalent to thinking of $\mathbf{A}$ as a row vector. This should be a *very* quick computation, no more than a few lines!

   (b) There is a lowered-index version of the Levi-Civita symbol, ϵ_{ijk}, defined identically to ϵ^{ijk}. Show that the product of the two Levi-Civita symbols can be written in terms of the Kronecker delta:
   $$\epsilon_{abc}\epsilon^{ijk} = \delta^i{}_a\delta^j{}_b\delta^k{}_c + \delta^i{}_b\delta^j{}_c\delta^k{}_a + \delta^i{}_c\delta^j{}_a\delta^k{}_b - \delta^i{}_b\delta^j{}_a\delta^k{}_c - \delta^i{}_a\delta^j{}_c\delta^k{}_b - \delta^i{}_c\delta^j{}_b\delta^k{}_a \qquad (1)$$
   Hint: the only values the left-hand side can take are -1, 1, or 0. Every term on the right-hand side corresponds to a unique permutation of (a,b,c), so you just have to check the signs and verify that the right-hand side vanishes whenever the left-hand side does.

   (c) By contracting the first index of Eq. (1), derive the contraction identity
   $$\epsilon_{ibc}\epsilon^{ijk} = \delta^j{}_b\delta^k{}_c - \delta^j{}_c\delta^k{}_b. \qquad (2)$$
   Hint: you will have some traces in your expression. Remember your result from problem 2b of HW 3!

   (d) The *Laplacian operator* ∇^2 is defined to act on vectors as
   $$\nabla^2\mathbf{F} \equiv \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)\mathbf{F} \Longleftrightarrow \partial^i\partial_i F_a. \qquad (3)$$
   Use your identity in Eq. (2) to derive the "curl-of-a-curl" identity
   $$\nabla \times (\nabla \times \mathbf{F}) = \nabla(\nabla \cdot \mathbf{F}) - \nabla^2\mathbf{F}. \qquad (4)$$
   We will use this equation in an important way over the next 2 weeks!

2. **Line integral practice.** (30 points) Compute the line integrals $\int_C \mathbf{F} \cdot d\mathbf{r}$ of the following vector fields $\mathbf{F}$ over the corresponding paths C:

   (a) $\mathbf{F} = \mathbf{r}$, along the straight line C defined by $x(t) = t$, $y(t) = 3t - 1$, $z(t) = 2t$, from the points $(1, 2, 2)$ to $(3, 8, 6)$.

   (b) $\mathbf{F} = (2xy, z^2, 3)$, along the path C which is the intersection of the paraboloid $z = x^2 + y^2$ with the plane $y = 2x$, from $(0, 0, 0)$ to $(2, 4, 20)$.

   (c) $\mathbf{F} = (-y, x, 0)$ counterclockwise along the circle C defined by $x^2 + y^2 = 4$ from $(2, 0)$ to $(-2, 0)$. *Hint: choose a good coordinate system!*

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