MATH 416H HW 7

James Liu

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1. First, as U is a vector space, a set of basis can be write out. $\{u_1, \cdots, u_n\}$, also, since U is also a subset of V. We can extend the basis into a basis of V: $\{u_1, \cdots, u_n, z_1, \cdots, z_m\}$. Now consider the dual space V^* by theromes profed in class, a basis of V^* is $\{u_1^*, \cdots, u_n^*, z_1^*, \cdots, z_m^*\}$ and, by definition, $\{z_1^*, \cdots, z_m^*\} \subseteq U^0$ as $\forall u \in U$, $u = \sum_{i=1}^n \lambda_i u_i + \sum_{j=1}^m 0 \times z_i$. Thus, $\forall u \in U$, $z_j^*(u) = 0$. Since U^0 us a subset of V, then a basis of U^0 must also be a subset of V's.

   Suppose $\{z_1, \cdots, z_m\}$ is not a basis for U^0, then $\exists k$ that for some $u^{*0} \in U^0$, $u^*0 = \lambda u_k^* + \mu_1 z_1^* + \cdots + \mu_m z_m^*$ However, consider $u_k \in U$ (the one in basis). $u^{*0}(u_k) = \lambda u_k^*(u_k) + \mu_1 z_1^*(u_k) + \cdots + \mu_m z_m^*(u_k) = \lambda + 0 + \cdots + 0 \neq 0$. Thus, there is a contradiction, therefore, $\{z_1^*, \cdots, z_m^*\}$ is a basis of U^0.

   $$\dim(V) = m + n \qquad \dim(U) = n \qquad \dim(U^0) = m$$

   Thus, $\dim(U^0) + \dim(U) = \dim(V)$.

2. a) $\forall w^* \in W$, $T^*(w^*) = w^* \circ T(v)$. $\forall w^* \in N(T^*)$, $T^*(w^*) = \overrightarrow{0} = v^{*0}$, $v^{*0} \in V^0$, Thus, $N(T^*) = \{w^* | \forall v \in V, w^*(T(v)) = 0\}$ or $\{w^* | w^* \in W^*,\ \forall w \in R(T), w^*(w) = 0\}$.

   $(R(T))^0 = \{w^* | w^*(w) = 0,\ w^* \in W^*,\ \forall w \in R(T)\}$. Therefore, $N(T^*) = (R(T))^0$

   b)

   $$\begin{aligned} N(T^*) &= (R(T))^0 && \text{(by part a)} \\ \dim(N(T^*)) &= \dim(W) - \dim(R(T)) && \text{(by problem 1)} \\ \dim(W^*) - \dim(R(T^*)) &= \dim(W) - \dim(R(T)) && \text{(by rank nullity)} \\ \because \dim(W) &= \dim(W^*) \\ \therefore \dim(R(T^*)) &= \dim(R(T)) \end{aligned}$$

   c) Any $m \times n$ matrix A would be equavelent to $T : V \to W$ where $\dim(V) = n$, $\dim(W) = m$, and A^T would be equavelent with $T^* : W^* \to V^*$. As profed by b), $R(T^*) = R(T)$. Thus, they do have same rank.

3. By 2.a) if T^* is injective then $N(T^*) = \overrightarrow{0} = (R(T))^0$. Then by 1., we have $\dim(R(T)) = \dim(W^*) - \dim((R(T))^0) = \dim(W^*) = \dim(W)$. as $R(T) \subseteq W$, therefore, $R(T) = W$ Thus, the transformation is surjective.

4.

$$\begin{aligned}
(\lambda b)(v_1 + v_2, \gamma w) &= \lambda b(v_1 + v_2, \gamma w) \\
&= \lambda\gamma b(v_1, w) + \lambda\gamma b(v_2, w) \\
&= (\lambda\gamma b)(v_1, w) + (\lambda\gamma b)(v_2, w) && \text{(Close under scaler multi.)} \\
(b_1 + b_2)(v_1 + v_2, \gamma w) &= b_1(v_1 + v_2, \gamma w) + b_2(v_1 + v_2, \gamma w) \\
&= b_1(v_1, w) + b_1(v_2, w) + b_2(v_1, w) + b_2(v_2, w) \\
&= b_1(v_1, w) + b_2(v_1, w) + b_1(v_2, w) + b_2(v_2, w) \\
&= (b_1 + b_2)(v_1, w) + (b_1 + b_2)(v_2, w) && \text{(Close under vector add)}
\end{aligned}$$

a)

$$\begin{aligned}
(b_1 + b_2)(v, w) &= b_1(v, w) + b_2(v + w) \\
&= b_2(v, w) + b_1(v + w) \\
&= (b_2 + b_1)(v, w) && \text{(vector ddition is communitive)}
\end{aligned}$$

b)

$$\begin{aligned}
((b_1 + b_2) + b_3)(v, w) &= (b_1 + b_2)(v, w) + b_3(v, w) \\
&= b_1(v, w) + b_2(v + w) + b_3(v, w) \\
&= b_1(v, w) + (b_2 + b_3)(v, w) \\
&= (b_1 + (b_2 + b_3))(v, w) && \text{(vector addition is associative)}
\end{aligned}$$

c) Consider the bilinear map $b_0(v, w) = 0$

$$\begin{aligned}
(b + b_0)(v, w) &= b(v, w) + b_0(v, w) \\
&= b(v, w) + 0 \\
&= b(v, w) && (\exists \overrightarrow{0} \text{ such that } b + \overrightarrow{0} = b)
\end{aligned}$$

d)

$$\begin{aligned}
(b + (-1 \cdot b))(v, w) &= b(v, w) + (-1) \times b(v, w) \\
&= 0 \\
&= b_0 && (\forall b,\ \exists -b,\ b + (-b) = 0)
\end{aligned}$$

e)

$$\begin{aligned}
(1 \cdot b)(v, w) &= 1 \times b(v, w) \\
&= b(v, w)
\end{aligned}$$

f)

$$\begin{aligned}\lambda((\mu \cdot b)(v,w)) &= \lambda(\mu b(v,w)) \\ &= \lambda\mu b(v,w) \\ &= \mu\lambda b(v,w) \\ &= \mu((\lambda \cdot b)(v,w))\end{aligned}$$

g)

$$\begin{aligned}\lambda(b_1+b_2)(v,w) &= \lambda(b_1(v,w)+b_2(v,w)) \\ &= \lambda b_1(v,w)+\lambda b_2(v,w) \\ &= (\lambda b_1)(v,w)+(\lambda b_2)(v,w)\end{aligned}$$

h)

$$\begin{aligned}((\lambda+\mu)b)(v,w) &= (\lambda+\mu)b(v,w) \\ &= \lambda b(v,w)+\mu b(v,w) \\ &= (\lambda b)(v,w)+(\mu b)(v,w)\end{aligned}$$

5. a)

$$\begin{aligned}w_0^{\#}(v_1+v_2) &= b(v_1+v_2,w_0) \\ &= b(v_1,w_0)+b(v_2,w_0) \\ &= w_0^{\#}(v_1)+w_0^{\#}(v_2) \\ w_0^{\#}(\lambda v) &= b(\lambda v,w_0) \\ &= \lambda b(v,w_0) \\ &= \lambda w_0^{\#}(v)\end{aligned}$$

Thus it is linear.

b) Mark the transformation by T instead of $\#$.

$$\begin{aligned}T(\lambda w_0) &= b(v,\lambda w_0) \\ &= \lambda b(v,w_0) \\ &= \lambda T(w_0) \\ T(w_1+w_2) &= b(v,w_1+w_2) \\ &= b(v,w_1)+b(v,w_2) \\ &= T(w_1)+T(w_2)\end{aligned}$$

Thus, it is linear.

c) $\forall w \in N(\#)$, $\#(w)=0$ which is $\forall v \in V$, $w^{\#}(v)=0$ which is $\forall v \in V$, $b(v,w)=0$. Therefore, it is correct.

d) As proved in c) $N(\#) = \{w_0 \in W | b(w^*, w_0) = 0 \text{ for all } w^* \in W^*\}$. For any w_0 we can write a bsis of W containing w_0. Thus, the dual vector space would also have a basis containing w_0^*. Suppose it exists some $w_0 \neq 0$ that $\forall w^* \in W^*$, $b(w^*, w_0) = 0$, consider $w_0^* \in W^*$, $b(w_0^*, w_0) = w_0^*(w_0) = 1 \neq 0$. Thus there is a contradiction. Therefore, $N(\#) = \{0\}$. Thus, the map is injective.

6. a)

$$\sigma = (1,2)(3,1)(4,5)$$

b) Since there are 3 swaps, the sign is -1

7. It shall equal to $1+1+1 = 3$

8.

k-linear:

$$\begin{aligned}
(T^*\alpha)(v_1, \cdots, v_{i1} + v_{i2}, \cdots, v_k) &= \alpha(T(v_1), \cdots, T(v_{i1} + v_{i2}), \cdots, T(v_k)) \\
&= \alpha(T(v_1), \cdots, T(v_{i1}) + T(v_{i2}), \cdots, T(v_k)) \\
&= \alpha(T(v_1), \cdots, T(v_{i1}), \cdots, T(v_k)) + \alpha(T(v_1), \cdots, T(v_{i1}), \cdots, T(v_k \\
&= (T^*\alpha)(v_1, \cdots, v_{i1}, \cdots, v_k) + (T^*\alpha)(v_1, \cdots, v_{i2}, \cdots, v_k) \\
(T^*\alpha)(v_1, \cdots, \lambda v_i, \cdots, v_k) &= \alpha(T(v_1), \cdots, T(\lambda v_i), \cdots, T(v_k)) \\
&= \alpha(T(v_1), \cdots, \lambda T(v_i), \cdots, T(v_k)) \\
&= \lambda\alpha(T(v_1), \cdots, T(v_i), \cdots, T(v_k)) \\
&= \lambda (T^*\alpha)(v_1, \cdots, v_i, \cdots, v_k)
\end{aligned}$$

alternating:

$$\begin{aligned}
(T^*\alpha)(v_k, v_1, \cdots, v_{k-1}) &= \alpha(T(v_k), T(v_1), \cdots, T(v_{k-1})) \\
&= -1 \times \alpha(T(v_1), \cdots, T(v_{k-1}), T(v_k)) \\
&= -1 \times (T^*\alpha)(v_1, \cdots, v_k)
\end{aligned}$$