MATH 416H HW 12

James Liu

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1. Consider a set of basis $\mathcal{B}$ for V, as it is a linear map,let $[T]_{\mathcal{B}\mathcal{B}} = A$ be the matrix representation of the map, then $p_T = \det(x\mathrm{id}_V - [T]_{\mathcal{B}\mathcal{B}})$. If expand the determinant by the first row of every sub matrix, we can see that there will be a term like: $(x-A_{11})(x-A_{22})(x-A_{33})\cdots[(x - A_{n-1,n-1})(x - A_{nn}) - (A_{n-1,n}A_{n,n-1})]$ the highest power will be the power of multiplying every term on the main diagonal. And as A is a $n \times n$ matrix, there will be n terms on the main diagonal which means that the degree is n.

2. a) By definition, $S(w) = T_W(w)$, then $S \circ S(w) = T_W \circ T_W(w)$ which is equivalent to $[S]^2 w = [T]_W^2 w$ and similarly $[S]^k w = [T]_W^k w$ therefore:

$$\begin{aligned} p(S) &= a_1\mathrm{id}_W + a_2 S + a_3 S^2 + \cdots \\ &= a_1\mathrm{id}_W + a_2 T|_W + a_3 T|_W^2 + \cdots \\ &= p(T)|_W \end{aligned}$$

   b) Note the minimal polynomial of T as m_T.

$$\begin{aligned} m_T &= a_0x^0 + a_1x^1 + \cdots + a_nx^n \\ m_T(S) &= a_0\mathrm{id}_W + a_1S^1 + \cdots + a_nS^n \\ &= a_0\mathrm{id}_W + a_1T|_W^1 + \cdots + a_nT|_W^n = m_T(T|_W) \end{aligned}$$

$$\begin{aligned} &\text{as } m_T(T) = 0 \\ &\quad m_T(T|_W)w = m_T(S)w = 0,\ \forall w \in W \end{aligned}$$

   Since $m_S(x)$ is the minimal polynomial of S, it divides any polynomial that annihilates S.

3. a)

$$
\begin{aligned}
p_T(x) &= \det(x \text{ id} - A) \\
&= \begin{vmatrix} x-2 & -1 & 0 & 1 \\ 0 & x-3 & -1 & 0 \\ 0 & 1 & x-1 & 0 \\ 0 & -1 & 0 & x-3 \end{vmatrix} \\
&= (x-2)(x-3)[(x-3)(x-1) - (-1) \times 1] \\
&= (x-2)(x-3)[x^2 - 4x + 4] \\
&= (x-2)^3(x-3)
\end{aligned}
$$

b)

$$
\begin{aligned}
\det(A - x \text{ id}) &= 0 \\
(x-2)^3(x-3) &= 0 \\
\lambda_1 = 2 \quad \lambda_2 &= 3 \\
(A-2)v &= \begin{pmatrix} 0 & -1 & 0 & 1 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 0 & 1 \end{pmatrix} v = 0 \\
v &= \begin{bmatrix} s \\ k \\ k \\ k \end{bmatrix} \\
(A-3)v &= \begin{pmatrix} -1 & -1 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} v = 0 \\
v &= \begin{bmatrix} s \\ 0 \\ 0 \\ s \end{bmatrix}
\end{aligned}
$$

Therefore , there are 3 distinct eigenvectors:

$$\lambda = 2$$

$$v_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad v_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

$$\lambda = 3$$

$$v_3 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

$\forall v \in E_\lambda$, v is either linear combination of v_1 and v_2 or a complex multiply of v_3

c) A is not diagonalizable as the number of distinct eigenvectors does not equal to 4.

4. a)

$$\det(A - x \text{ id}) = 0$$

$$\begin{vmatrix} -1-x & 3 & 0 \\ 0 & 2-x & 0 \\ 2 & 1 & -1-x \end{vmatrix} = 0$$

$$v = (-1-x)(-1-x)(2-x)$$

$$\lambda_1 = -1, \quad \lambda_2 = 2$$

$$(A + 1 \text{ id})v = 0$$

$$\begin{pmatrix} 0 & 3 & 0 \\ 0 & 3 & 0 \\ 2 & 1 & 0 \end{pmatrix} v = 0$$

$$(A - 2 \text{ id})v = 0$$

$$v = \begin{pmatrix} 0 \\ 0 \\ s \end{pmatrix}$$

$$\begin{pmatrix} -3 & 3 & 0 \\ 0 & 0 & 0 \\ 2 & 1 & -3 \end{pmatrix} v = 0$$

$$v = \begin{pmatrix} s \\ s \\ s \end{pmatrix}$$

$$J_A = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

b)

$$\lambda = -1$$
$$v_1 = (0,0,1)^T$$
$$\lambda = 2$$
$$v_3 = (1,1,1)^T$$

As for $\lambda = -1$, the geometrix multiplicity is 1 and algebratic is 2:

$$Av_2 = \lambda_1 v_2 + v_1$$
$$(A+1)v_2 = v_1$$
$$\begin{bmatrix} 0 & 3 & 0 \\ 0 & 3 & 0 \\ 2 & 1 & 0 \end{bmatrix} v_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$
$$v_2 = \begin{bmatrix} \frac{1}{2} \\ 0 \\ 0 \end{bmatrix}$$
$$K = \begin{bmatrix} 0 & 0.5 & 1 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix}$$
$$K^{-1} = \begin{bmatrix} 0 & -1 & 1 \\ 2 & -2 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$
$$A = KJK^{-1}$$
$$J = K^{-1}AK$$
$$K = S^{-1}$$
$$K^{-1} = S$$

5.

$$\begin{bmatrix} 7 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 7 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 7 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 7 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 3 \end{bmatrix}$$

As the minimal polynomial indicates that the for 7, the algebratic multiplicity is 4 and the largest jordan block size is 2 similar for 3, the algebratic multiplicity is 3 whith largest jordan block size being 2.

6.

$$\begin{aligned} S(A) &= \lambda A \\ A^T &= \lambda A \\ S(A^T) &= S(\lambda A) \\ A &= \lambda \times A^T \\ &= \lambda^2 A \\ \lambda &= \pm 1 \end{aligned}$$

Consider when $\lambda = 1$

$$A^T = A$$
$$\text{note such matrix as:} X$$

When $\lambda = -1$

$$A^T = -A$$
$$\text{note such matrix as:} Y$$

$$\begin{aligned} A &= \frac{1}{2}(A + A^T) + \frac{1}{2}(A - A^T) \\ \frac{1}{2}(A + A^T)^T &= \frac{1}{2}(A^T + A) = X \\ \frac{1}{2}(A - A^T)^T &= -\frac{1}{2}(A - A^T) = Y \\ A &= \frac{1}{2}X + \frac{1}{2}Y \end{aligned}$$

Which means that $S(A)$ do have a eigen basis which means that it is diagonalizable.

7. a)

exists a zero vector: For $M = [0]_n$(n×n zero matrix), $\forall A \in \mathcal{H}_n,\ A + M = A$

addition: $\forall A \in \mathcal{H}_n, \mu \in \mathbb{R}$

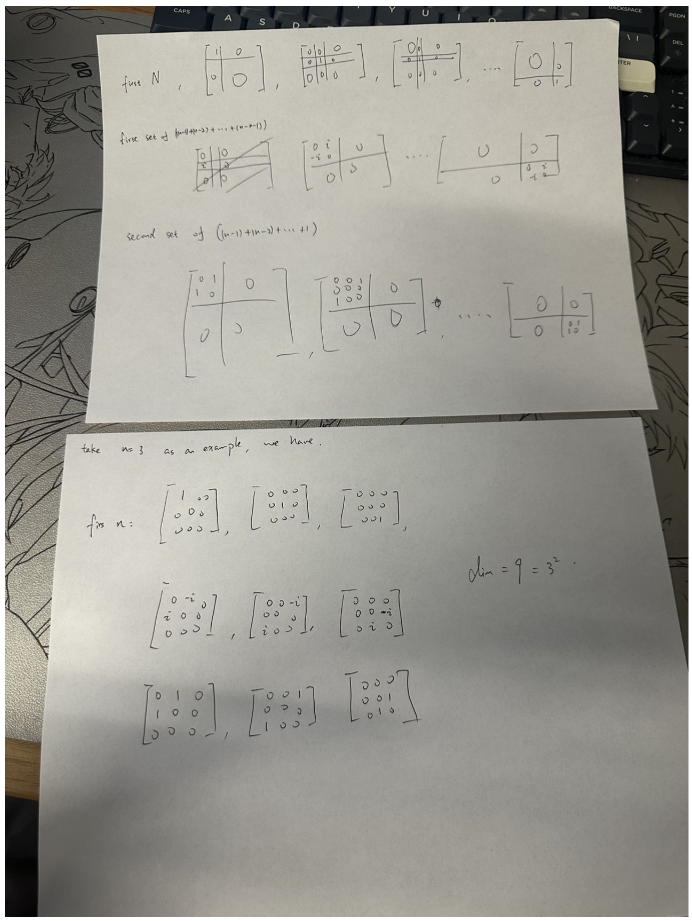
$$\begin{aligned} (A + B)_{ij} &= A_{ij} + B_{ij} \\ &= \overline{A}_{ji} + \overline{B}_{ji} \\ &= (A + B)^*_{ij} \end{aligned}$$

multiplication:

$$\begin{aligned} \lambda A_{ij} &= \lambda A_{ij} \\ (\lambda A_{ij})^* &= \lambda (A_{ij})^* \end{aligned}$$

And it also follows the rest axioms about algebra which makes it a vector space over $\mathbb{R}$.
However, scaler multiplication is no longer close under $\mathbb{C}$ thus it is not a vector space over$\mathbb{C}$

b) the dimention will be $\dim(\mathcal{H}_n) = n+2\times((n-1)+(n-2)+\cdots+1) = n^2$, $\mathcal{B} = \{b_1, b_2, \cdots, b_n^2\}$. The basis would be:



8. $\mathcal{B} = \{E_{11}, E_{12}, E_{21}, E_{22}\}$. Then, we have:

$$T = \begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix}$$

$$\det(T - x \ \mathrm{id}_4) = 0$$

$$\begin{vmatrix} 3-x & 0 & 0 & 0 \\ 0 & 2-x & 1 & 0 \\ 0 & 1 & 2-x & 0 \\ 0 & 0 & 0 & 3-x \end{vmatrix} = 0$$

$$(3-x)^2[(2-x)^2 - 1] = 0$$

$$(3-x)^3(1-x) = 0$$

$$\lambda_1 = 3 \ \lambda_2 = 1$$

find eigenvectors:

$$\lambda_1 = 3$$

$$(T - 3 \ \mathrm{id}_4)v = 0$$

$$v = (s, k, k, t)$$

$$\lambda_2 = 1$$

$$(T - 1 \ \mathrm{id}_4)v = 0$$

$$v = (0, s, -s, 0)$$

Therefore it does have a eigenbasis, which gives:

$$J = \begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$b_1 = (1, 0, 0, 0) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

$$b_2 = (0, 1, 1, 0) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$b_3 = (0, 0, 0, 1) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

$$b_4 = (0, 1, -1, 0) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$