MATH 416H HW 6

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1. a) Rank would be 3 and Nullity would be 2 as the matrix is already in reduced row-echlon form and the number of pivots is the rank and the number of none pivot column is Nullity.

   b)

$$\left[\begin{array}{ccccc|c} 1 & 2 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{array}\right] \rightarrow \begin{cases} x_1 = -2x_2 - x_4 \\ x_2 = x_2 \\ x_3 = -x_4 \\ x_4 = x_4 \\ x_5 = 0 \end{cases}$$

   Take $x_2 = 1, x_4 = 1$, seperatly, we have a basis consisting 2 element:
   $\left\{ \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ -1 \\ 1 \\ 0 \end{pmatrix} \right\}$

2. a) Scaler Multiplication: multiplying a scaler does not change the symetric of the matrix.

$$\begin{aligned} A &= A^T \\ a_{ij} &= A_{ji} \quad 0 \leq ij \leq n \\ ka_{ij} &= ka_{ji} \quad k \in F \end{aligned}$$

   Vector Addition: Adding two such matrix also does not change such symetry:

$$\begin{aligned} A &= A^T & B &= B^T \\ a_{ij} &= a_{ji} & b_{ij} &= b_{ji} \\ a_{ij} + b_{ij} &= a_{ji} + b_{ji} \end{aligned}$$

   Consider $a_{ij} = 0, \ \forall i, j$, such matrix will be the additive identity. And these operations do fullfill the 8 properties as in the question,

$M_{n,n}(F)$ is already a vector space. And S_n is close under the 2 operations, thus it is a subspace.

b) Notice that one possible set of basis would be:

$$\left\{ \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right\}$$

3. T is injective, then$\forall w \in W$, $\exists v \in V$ such that $w = T(v)$. By definition, $T^*(\psi) = \psi \circ T(v)$, $\psi \in W^*$. Consider $T^*(\psi)(v) = 0$

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$$\psi(T(v)) = 0 \ \forall v \in V$$

As $T(v)$ is surjective on W, which means that $\psi(w) = 0, \ \forall w \in W$. Thus, ψ is a zero map, thus, $N(T^*) = \{\overrightarrow{0}\}$. Thus, by rank/nullity, the T^* is injective.

4. $\forall \ell \in U^0$, $\ell_1(x) + \ell_2(x) = 0 + 0$. Thus, $\exists \ell_3$ such that $\ell_1 + \ell_2 = \ell_3$. Thus U^0 is closed under addition. $\forall \lambda \in F$, $\lambda \ell(x) = \lambda \times 0 = 0$. Thus, it is also closed under scaler multiplication. Also, $\ell + \ell = \ell$ as $0 + 0 = 0$, thus, there also exists a zero element. Thus, U^0 is a subspace.

5. Consider a map: $\pi : V \to V/U$, $\pi(v) = v + U$. Thus, $\forall u \in U$, $\exists v$ that $u = v + U$ by definition. Thus, π is surjective. Thus, according to 3., the dual map $\pi^* : (V/U)^* \to V^*$ is injective. Thus, $N(\pi^*) = \overrightarrow{0}$. In this case, profed by 4., $\overrightarrow{0} = U^0 = \{\ell\}$. Thus, accroding to the 1st isomorphism law, $(V/U)^*/N(\pi^*) \to R(\pi^*)$ is isomorphic.
Claim: $R(\pi^*) = U^0$
For any $\psi \in R(\pi^*)$, $\exists w^* \in (V/U)^*$ that
$\psi = \pi^*(w^*) = w^*(\pi(v)) = w^*(v + U)$. For any $u \in U$, $\psi = w^*(u + U) = w^*(\overrightarrow{0} + U) = 0$. Thus, all ψ sends u to 0. Thus $R(\pi^*) \subseteq U^0$. Also, for any $u^* \in U^0$, $\exists w^*$ that $\pi^*(w^*) = u^*$, for example consider such map: $\gamma(w) = 0$. Thus, $U^0 \subseteq R(\pi^*)$. Thus $U^0 = R(\pi^*)$. Also, as $\dim(N(\pi^*)) = 0$ due to injectivity, $(V/U)^*/N(\pi^*) = (V/U)^*$. Thus, due to first law of isomorphism, the map $(V/U)^* \to U^0$ is isomorphic.

6.

Subspace: $\forall w \in W$, $w_1(0) + w_2(0) = 0 + 0 = 0$, Thus $(w_1 + w_2) \in W$.
$\forall \lambda \in F$, $\lambda w(0) = \lambda \times 0 = 0$. Thus $\lambda w \in W$
consider $f(x) = 0$, $w + f = w$. Thus there exists a $\overrightarrow{0}$.
Thus it is a subspace.

Isomorphism: Consider the map $T : V \to \mathbb{R}, T(f) = f(0)$.
Claim that T is linear:
prof: $T(\lambda f) = \lambda f(0) = \lambda(T(f)), T(g+f) = g(0)+f(0) = T(g)+T(f)$
Thus it is linear.
Claim that $N(T) = W$:
prof: $\forall f \in N(T), T(f) = 0$ meaning that $f(0) = 0$. Thus $N(T) = W$.
Claim that $R(T) = \mathbb{R}$.
prof: Suppose it is not surjective, then $\exists \lambda \in \mathbb{R}$, does not exist such f which $T(f) = \lambda$. However, consider the map that $f(x) = \lambda$, then $f(0) = \lambda, T(f) = \lambda$ which raises a contradiction. Thus it is surjective.
Thus, through first Isomorphism law, $V/W \to \mathbb{R}$ is a isomorphic map, and they are isomorphic.

7. a) $u_1, u_2 \in U$, $T(u_1), T(u_2) \in W$, $T(u_1) + T(u_2) = T(u_1 + u_2) \in W$ as T is linear and U is it self a subspace.
$\lambda \in F$, $\lambda T(u) \in W$ as $T(u) \in W$ and W is a vector space that itself shall be close under scaler multiplication.

b) According to the 2nd isomorphic law, $U/(U \cap N(T))$ will be isomorphic with $(U + N(T))/N(T)$. Consider a new subspace of V, $U + N(T)$. Thus according to the 1st isomorphic law, $(U + N(T))/N(T)$ will be isomorphic with $R(T(U + N(T))) = R(T(U) + T(N(T))) = R(T(U) + \overrightarrow{0})$ as profed in a), $T(U)$ is a subspace, then $\overrightarrow{0} \in T(U)$, Thus $R(T(U + N(T))) = T(U)$. Thus, $T(U)$ is isomorphic with $(U+N(T))/N(T)$. Thus, $T(U)$ is also isomorphic with $U/(U\cap N(T))$.

8.

well define: If $v_1 = v_2$, then $T(v_1) = T(v_2)$ as T is well defined, then $\bar{T}(v_1) = \bar{T}(v_2)$. Thus, $\bar{T}$ is well defined.

linear: $T((v_1+U)+(v_2+U)) = T(v_1+v_2+U) = T(v_1+v_2) = T(v_1)+T(v_2) = \bar{T}(v_1 + U) + \bar{T}(v_2 + U)$.
$\bar{T}(\lambda(v + U)) = \bar{T}(\lambda v + U) = T(\lambda v) = \lambda T(v) = \lambda\bar{T}(v)$.

Thus it is well defined and linear.