MATH 416H HW 2

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1. No it is no longer a vector space, as:

$$\begin{aligned}\exists \lambda, \mu \in \mathbb{R},\ (\lambda+\mu)\overrightarrow{v} &= ((\lambda+\mu)^2 \times v_1, \cdots, (\lambda+\mu)^2 \times v_n)^T \\ &= ((\lambda^2+\mu^2+2\lambda\mu) \times v_1, \cdots, (\lambda^2+\mu^2+2\lambda\mu) \times v_n)^T \\ &\neq ((\lambda^2+\mu^2) \times v_1, \cdots, (\lambda^2+\mu^2) \times v_n)^T \\ &= \lambda\overrightarrow{v} + \mu\overrightarrow{v}\end{aligned}$$

   Thus, as it is not a vector space.

2. Yes it is a vecotr space, as:

   a) addition is communitive as $ab = ba$
   b) addition is associative as $a(bc) = (ab)c$
   c) For $a = 1$, $\forall b \in V$, $ab = b$, thus $1 = \overrightarrow{0}$
   d) $\forall a \in V$, $-a + a = a^{-1} \times a^1 = a^0 = 1 = \overrightarrow{0}$
   e) $\forall a \in V$, $a^1 = a$
   f) $\lambda(\mu a) = (a^\mu)^\lambda = a^{\mu\lambda} = (\lambda\mu)a$
   g) $\lambda(a+b) = (ab)^\lambda = a^\lambda b^\lambda = \lambda a + \lambda b$
   h) $(\lambda+\mu)a = a^{\lambda+\mu} = a^\lambda \cdot a^\mu = \lambda a + \mu a$
   i) It is also closed onder vector multiplication and addition as any power of a positive realnumber will stay positive and thus any multiple is also still positive thus remains in the $(0, \infty)$ range.

   For example: $\lambda = 2, \mu = 1, a = 2$, as $2^{(1+2)} = 8 \neq 6 = 2^1 + 2^2$ Thus, it is not a vector space.

3. No it is not in the span. Suppose it is then $\exists a_1, a_2 \in \mathbb{R}$ such that $a_1(3,4,2)^T + a_2(1,3,3)^T = (-1,2,3)^T$ which is:

$$\begin{cases} 3a_1 + a_2 &= -1 \\ 4a_1 + 3a_2 &= 2 \\ 2a_1 + 3a_2 &= 3 \end{cases}$$

Deriving it will result in:

$$\begin{cases} 3a_1 + a_2 & = & -1 \\ -5a_1 + 0a_2 & = & 5 \\ -7a_1 + 0a_2 & = & 6 \end{cases} \to \begin{cases} a_1 = -1 \\ a_1 = -\frac{6}{7} \end{cases} \tag{1}$$

Which raises a contradiction and thus, such a_1, a_2 does not exist thus, it is not in the span.

4. as v is in span of the subspace, then:$\exists a_1, \cdot, a_n, a_i \in \mathbb{R}$ that $v = a_1 \overrightarrow{v}_1 + \cdots + a_n \overrightarrow{v}_n$ Thus, $-1v + a_1 \overrightarrow{v}_1 + \cdots + a_n \overrightarrow{v}_n = 0$ thus, the set is linear dependent.

5. As U, W are subspace of V, then $\overrightarrow{0} \in U, W$ thus $\overrightarrow{0} \in U \cap W$ Also, consider that $x, y \in U \cap W$, then $x, y \in W$ and $x, y \in U$, Thus, $x + y \in U$ as U is a subspace, similarly, $x + y \in W$ therefore. $x + y \in U \cap W$ similarly, $\forall \lambda \in \mathbb{R}$, $\lambda x \in U$, and $\lambda x \in V$, therefore, the addition and multiplication are closed in $U \cap W$ following V while the intersection contains the zero vector making it not empty, thus, the intersection is still a subspace.

6. a) As profed in class, any set of vector S that spans V will have a element number n that is larger or equal to any set that is linear independent. and for any vector space of dimension m, it has a basis consisting of m elements. And any additional vector in the vector space will be lienar dependent with this base. Thus, basis is the largest linear independent subsets in a vector space. Therefore, a subset of a vector space that spans the vector space must have elements that is larger or equal to the number of elements in a base. Thus

   b) As V have a dimension of n, then exists a base $\{v_1, \cdots, v_n\} \subseteq V$ and spans V with n linearly independent vectors $v_1, \cdots, v_n$. Thus, for any other vector, $\forall v \in V, \exists \lambda_1, \cdots, \lambda_n \in F$ that $v = \lambda_1 v_1 + \cdots + \lambda_n v_n$ and $\exists \mu_1, \cdots, \mu_n \in F$ that $-v = \mu_1 v_1 + \cdots + \mu_n v_n$.
   Thus, $v + \mu_1 v_1 + \cdots + \mu_n v_n = 0$ and it is linear dependent with the set of basis. Thus, the maximum numebr of elements is n.

7. $s' \in$ span S, thus, $\exists \lambda_i \in F$ that $s' = \sum_i \lambda_i s'_i$ where $s'_i \in S$. Also, as $S' \subset S'$, there is $\forall s'_i \in S'$, $s'_i \in S$, thus, as s' can be expressed in $s' = \sum_i \lambda_i s'_i$, $s' \in$ span S, thus span $S' \subset$ span S

8. As $v \in$ span S, $\exists \lambda_i \in F$, $\forall s_i \in S$ that $v = \sum_i \lambda_i s_i$ $\forall w \in$ span $(S \cup \{v\})$, $\exists \lambda_i, \lambda'_i, \mu \in F$,
   that $w = \sum_i \lambda_i s_i + \mu = \sum_i \lambda_i s_i + \mu \sum_i \lambda'_i s_i = \sum_i (\lambda_i + \mu \lambda'_i) s_i$, $(\lambda_i + \mu \lambda'_i) \in F$ thus, set $\lambda_i + \mu \lambda'_i = \alpha_i$, then $w = \sum_i \alpha_i s_i$ Thus, span S =span $(S \cup \{v\})$

9. $\forall \lambda_i, \mu_i, a, b \in F$:

$$\begin{aligned} F(a\lambda_1 + b\mu_1, \cdots, a\lambda_n + b\mu_n) &= a\lambda_1 v_1 + b\mu_1 v_1 + \cdots + a\lambda_n v_n + b\mu_n v_n \\ &= a\lambda_1 v_1 + \cdots + a\lambda_n v_n + b\mu_1 v_1 + b\mu_n v_n \\ &= a(\lambda_1 v_1 + \cdots + \lambda_n v_n) + b(\mu_1 v_1 + \mu_n v_n) \\ &= aF(\lambda_1, \cdots, \lambda_n) + bF(\mu_1, \cdots, \mu_n) \end{aligned}$$

Therefore it is a linear map.