MATH 416H HW 4

James Liu

Due: Sep 25 Edit: September 22, 2024

1. It is $\sum_{i=1}^{m} a_i x_i$

2. It is $\begin{pmatrix} yb_1 \\ yb_2 \\ \vdots \\ yb_n \end{pmatrix}$

3.

$$E_1A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{bmatrix} = \begin{bmatrix} a_{21} & a_{22} & a_{23} & a_{24} \\ a_{11} & a_{12} & a_{13} & a_{14} \\ 3 \times a_{31} & 3 \times a_{32} & 3 \times a_{33} & 3 \times a_{34} \end{bmatrix}$$

$$E_2A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{bmatrix} = \begin{bmatrix} a_{31} & a_{32} & a_{33} & a_{34} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{11} & a_{12} & a_{13} & a_{14} \end{bmatrix}$$

$$E_3A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -5 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} - 5a_{31} & a_{22} - 5a_{32} & a_{23} - 5a_{33} & a_{24} - 5a_{34} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{bmatrix}$$

4. a) $\iota(w) = w,\ \iota(v) = v,\ \iota(w) + \iota(v) = w + v = \iota(w + v)$
   $\lambda\iota(v) = \lambda \cdot v = \iota(\lambda v)$. Thus, the inclusion map is linear.

   b) i:

5. a)

forward: Suppose $\exists w_1, w_1' \in W_1,\ w_2, w_2', \in W_2$, that $v = w_1 + w_2 = w_1' + w_2'$. Thus:

$$w_1 - w_1' = w_2' - w_2$$
$$w_1 - w_1' \in W_1 \quad w_2' - w_2 \in W_2$$
$$\text{as } W_1 \cap W_2 = \{\overrightarrow{0}\}$$
$$w_1 - w_1' = w_2' - w_2 = \overrightarrow{0}$$
$$w_1 = w_1' \quad w_2 = w_2'$$

Thus, if $V = W_1 \oplus W_2$, $\forall v \in V$, exists a unique w_1, w_2 that $v = w_1 + w_2$

backward: If $\forall v \in V$, $\exists w_1 \in W_1,\ w_2 \in W_2$ that $v = w_1 + w_2$, $V = W_1 + W_2$ by definition. Suppose that $W_1 \cap W_2 \neq \{\overrightarrow{0}\}$, then $\exists w \in W_1, W_2$. Thus, forsome $v \in W_1$, or $v = w_1 + \overrightarrow{0}$, Thus define $k = w_1 - w$, Therefore $\exists v = w_1 - w + w$, where $w \neq \overrightarrow{0}$. However, there are only one set of w_1, w_2 that $v = w_1 + w_2$, Therefroe, $W_1 \cap W_2 = \overrightarrow{0}$

b)

6. $\forall v \in V, T(S(v)) = v$, Thus, T is Surjective, Thus, $R(T) = dim(V)$, Thus $N(T) = \{\overrightarrow{0}\}$ Suppose T is not injective, then $\exists v, w \in V$ that $v \neq w$ that $T(v) = T(w)$, then $T(v) - T(w) = \overrightarrow{0} = T(v - w)$, which raises a contradiction. Thus, T is a bijection. Thus T is invertable. As T is invertable, $\exists T^{-1}$ that $T \circ T^{-1} = \mathrm{id}_V$. Therefore, $T^{-1} = S$. Thus, $S \circ T = T^{-1} \circ T = \mathrm{id}_V$

7. Take a random set of $a_1, \cdots, a_k, \cdots \in \mathbb{N}, S(a_1, a_2, \cdots, a_k, \cdots) = (0, a_1, a_2, \cdots, a_k, \cdots)$, $T(0, a_1, a_2, \cdots, a_k, \cdots) = (a_1, a_2, \cdots, a_k, \cdots)$. Thus, $T \circ S = \mathrm{id}_V$

8. $\forall x_i \in \mathbb{R},\ v = (x_1, \cdots, x_n) P(v) = (x_1, 0, \cdots, 0)$, and $P(P(v)) = P(x_1, 0, \cdots, 0) = (x_1, 0, \cdots, 0) = P(v)$ Thus, $P \circ P = P$
   $N(P) = (0, x_2, \cdots, x_n)$, $R(P) = (x_1, 0, \cdots, 0)$, $\mathbb{R}^n = N(P) + R(P) = (x_1, x_2, \cdots, x_n)$. Also, as $R(P) \cap N(P) = (0, 0, \cdots, 0) = \overrightarrow{0}$. Thus, $\mathbb{R}^n = N(P) \oplus R(P)$