MATH 461 Homework 4

James Liu

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4.21 a) I believe is $E[X]$ as the chance of selecting a child on a bigger bus is larger while the probability of selecting a child on a smaller bus is also smaller. However, they are just same for $E[Y]$

b) $E(X) = 40 \cdot \frac{40}{148} + 33 \cdot \frac{33}{148} + 25 \cdot \frac{25}{148} + 50 \cdot \frac{50}{148} = \frac{2907}{74} \approx 39.2838$
$E(Y) = 40 \cdot \frac{1}{4} + 33 \cdot \frac{1}{4} + 25 \cdot \frac{1}{4} + 50 \cdot \frac{1}{4} = \frac{40+33+25+50}{4} = 37$
Thus $E[X] > E[Y]$

4.23 a) Suppose I bought x lb of the commodity at the first week, then at second week, the expactation of the money i hold will be:

$$1000 - 2x + 2x \times \frac{1}{2} + 4x \times \frac{1}{2} = 1000 + x$$

Thus, The best stratagy would be apending all the money on first week.

b) Suppose I bought x lb of the commodity at the first week, then at second week, the expactation of the money i hold will be:

$$x + \frac{1000 - 2x}{1}\frac{1}{2} + \frac{1000 - 2x}{4}\frac{1}{2} = 650 - \frac{x}{4}$$

Thus, The best stratagy would be saving all the money for the second week.

4.32 If a pool is negative, then it costs only one test which is:
$E = 0.9^{10} \times 1 = 0.3486$
The rest pools will need 11 tests which is:
$E = (1 - 0.9^{10}) \times 11 = 7.1645.$
Thus the expacted total of tests per group is:
$E_{tot} = 0.3486 + 7.1645 = 7.5132$

4.35

a) The probability of getting 2 same color balls is $\frac{\binom{2}{1}\binom{5}{2}}{\binom{10}{2}} = \frac{4}{9}$

The probability of getting 2 different balls is $1 - \frac{4}{9} = \frac{5}{9}$.
Thus, the expactation would be:$\frac{4}{9} \times 1.1 - \frac{5}{9} = \frac{-1}{15} \approx -0.067$

b) Var$= E(X^2) - (E(X))^2 = (1.1)^2 \times \frac{4}{9} + 1 \times \frac{5}{9} - (\frac{-1}{15})^2 = \frac{49}{45} \approx 1.0889$

4.37

$$\begin{aligned}
\text{Var}(X) &= E[X^2] - (E(X))^2 \\
&= 40^2 \cdot \frac{40}{148} + 33^2 \cdot \frac{33}{148} + 25^2 \cdot \frac{25}{148} + 50^2 \cdot \frac{50}{148} - \left(\frac{2907}{5476}\right)^2 \\
&= 82.2033 \\
\text{Var}(Y) &= E[Y^2] - (E(Y))^2 \\
&= 40^2 \cdot \frac{1}{4} + 33^2 \cdot \frac{1}{4} + 25^2 \cdot \frac{1}{4} + 50^2 \cdot \frac{1}{4} - 37^2 \\
&= 84.5
\end{aligned}$$

4.38

$$Var(X) = 5 = E(X^2) - (E(X))^2 = E(X^2) - 1, \text{ thus, } E(X^2) = 6$$

a)

$$\begin{aligned}
E[(2+X)^2] &= \sum_x (2+x)^2 p(x) \\
&= \sum_x (4 + 4x + x^2) p(x) \\
&= \sum_x 4p(x) + 4 \cdot xp(x) + x^2 p(x) \\
&= 4 + 4E(X) + E(X^2) \\
&= 4 + 4 + 6 = 14
\end{aligned}$$

b)

$$\begin{aligned}
\text{Var}(4+3X) &= 3^2 \times \text{Var}(X) \\
&= 9 \times 5 = 45
\end{aligned}$$

4.40 $\binom{5}{4}\left(\frac{1}{3}\right)^2 \frac{2}{3} + \left(\frac{1}{3}\right)^5 = \frac{11}{243} = 0.045267$

4.45 $P(\text{on}) = \frac{1}{3},\ P(\text{off}) = \frac{2}{3}$

$$\begin{aligned}
P_{\text{on},3}(\text{pass}) &= \binom{3}{2}\left(\frac{4}{5}\right)^2 \frac{1}{5} + \left(\frac{4}{5}\right)^2 \\
&= \frac{112}{125} = 0.896 \\
P_{\text{off},3}(\text{pass}) &= \binom{3}{2}\left(\frac{2}{5}\right)^2 \frac{3}{5} + \left(\frac{2}{5}\right)^2 \\
&= \frac{44}{125} = 0.352 \\
E(\text{pass,3}) &= P_{\text{on},3}(\text{pass})p(\text{on}) + P_{\text{off},3}(\text{pass})p(\text{off}) \\
&= \frac{112}{125} \times \frac{1}{3} + \frac{44}{125} \times \frac{2}{3} \\
&= \frac{8}{15} \approx 0.53333 \\
P_{\text{on},5}(\text{pass}) &= \binom{5}{3}\left(\frac{4}{5}\right)^3\left(\frac{1}{5}\right)^2 + \binom{5}{4}\left(\frac{4}{5}\right)^4 \frac{1}{5} + \left(\frac{4}{5}\right)^5 \\
&= 0.94208 \\
P_{\text{off},5}(\text{pass}) &= \binom{5}{3}\left(\frac{2}{5}\right)^3\left(\frac{3}{5}\right)^2 + \binom{5}{4}\left(\frac{2}{5}\right)^4 \frac{3}{5} + \left(\frac{2}{5}\right)^5 \\
&= 0.31744 \\
E(\text{pass,5}) &= P_{\text{on},5}(\text{pass})p(\text{on}) + P_{\text{off},5}(\text{pass})p(\text{off}) \\
&= 0.94208 \times \frac{1}{3} + 0.31744 \times \frac{2}{3} \\
&= 0.525653
\end{aligned}$$

Thus, the student should choose the 3 examer's group.

4.48

$$\begin{aligned}
P(\text{return}) &= 1 - \binom{10}{1}(0.99)^9 \times 0.01 - (0.99)^{10} \\
&= 0.004266 \\
P(\text{return only one}) &= \binom{3}{1} 0.004266 \times (1 - 0.004266)^2 \\
&= 0.1269
\end{aligned}$$

4.50 a)

$$
\begin{aligned}
p(6h) &= \binom{10}{6} p^6 (1-p)^4 \\
P(h,t,t) \cap P(6h) &= p \times (1-p)^2 \binom{7}{5} p^5 (1-p)^2 \\
P(h,t,t|6h) &= \frac{P(h,t,t) \cap P(6h)}{p(6h)} \\
&= \frac{p \times (1-p)^2 \binom{7}{5} p^5 (1-p)^2}{\binom{10}{6} p^6 (1-p)^4} \\
&= \frac{1}{10}
\end{aligned}
$$

b)

$$
\begin{aligned}
p(6h) &= \binom{10}{6} p^6 (1-p)^4 \\
P(t,h,t) \cap P(6h) &= (1-p)p \times (1-p) \binom{7}{5} p^5 (1-p)^2 \\
P(t,h,t|6h) &= \frac{P(h,t,t) \cap P(6h)}{p(6h)} \\
&= \frac{p \times (1-p)^2 \binom{7}{5} p^5 (1-p)^2}{\binom{10}{6} p^6 (1-p)^4} \\
&= \frac{1}{10}
\end{aligned}
$$

4.55 It is a typical Poisson distribution senario. $\lambda_1 = 3, \lambda_2 = 4.2$.

$$
\begin{aligned}
P(\text{no err typist 1}) &= e^{-3} \frac{3^0}{0!} \\
&= 0.049787 \\
P(\text{no err typist 2}) &= e^{-4.2} \frac{4.2^0}{0!} \\
&= 0.014996 \\
P(\text{no err}) &= 0.5 \times 0.049787 + 0.5 \times 0.014996 \\
&= 0.032391
\end{aligned}
$$

4.57 a)

$$
\begin{aligned}
P(X \geqslant 3) &= 1 - P(X = 0, 1, 2) \\
&= 1 - \left(e^{-3} \frac{3^0}{0!} + e^{-3} \frac{3^1}{1!} + e^{-3} \frac{3^2}{2!} \right) \\
&= 0.57681
\end{aligned}
$$

b)

$$
\begin{aligned}
P(X \geqslant 1) &= 1 - e^{-3} \\
&= 0.950213 \\
P(X \geqslant 3 | X \geqslant 1) &= \frac{P(X \geqslant 3)}{P(X \geqslant 1)} \\
&= \frac{0.57681}{0.950213} \\
&= 0.607032
\end{aligned}
$$

4.59 Since the question asks for a approximation, we can approximate it with Poisson distribution with $\lambda = np = 0.5$

a)

$$
\begin{aligned}
P(X \geq 1) &= 1 - P(X = 0) \\
&= 1 - e^{-0.5} \\
&= 0.393469
\end{aligned}
$$

b)

$$
\begin{aligned}
P(X = 1) &= e^{-0.5} \frac{0.5^1}{1!} \\
&= 0.303265
\end{aligned}
$$

c)

$$
\begin{aligned}
P(X \geq 2) &= 1 - P(X = 0) - P(X = 1) \\
&= 1 - e^{-0.5} - 0.303265 \\
&= 0.090204
\end{aligned}
$$

4.61 Use a Poisson distribution with $\lambda = np = 1.4$.

$$
\begin{aligned}
P(X \geq 2) &= 1 - P(X = 0) - P(X = 1) \\
&= 1 - e^{-1.4} - e^{-1.4} \times \frac{1.4^1}{1!} \\
&= 0.408167
\end{aligned}
$$

4.63 We can use Poisson distribution to estimate this.

a) use $\lambda = np = 5 \times \frac{1}{2} = 2.5$, then $P(X = 0) = e^{-2.5} = 0.082085$

b) $P(X \geq 4) = 1 - P(X = 0, 1, 2, 3) = 0.242424$