MATH 416H HW 7

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Due: Oct 23 Edit: October 21, 2024

1. First, as U is a vector space, a set of basis can be write out. $\{u_1, \cdots, u_n\}$, also, since U is also a subset of V. We can extend the basis into a basis of V: $\{u_1, \cdots, u_n, z_1, \cdots, z_m\}$. Now consider the dual space V^* by theromes profed in class, a basis of V^* is $\{u_1^*, \cdots, u_n^*, z_1^*, \cdots, z_m^*\}$ and, by definition, $\{z_1^*, \cdots, z_m^*\} \subseteq U^0$ as $\forall u \in U$, $u = \sum_{i=1}^n \lambda_i u_i + \sum_{j=1}^m 0 \times z_i$. Thus, $\forall u \in U$, $z_j^*(u) = 0$. Since U^0 us a subset of V, then a basis of U^0 must also be a subset of V's.

   Suppose $\{z_1, \cdots, z_m\}$ is not a basis for U^0, then $\exists k$ that for some $u^{*0} \in U^0$, $u^*0 = \lambda u_k^* + \mu_1 z_1^* + \cdots + \mu_m z_m^*$ However, consider $u_k \in U$ (the one in basis). $u^{*0}(u_k) = \lambda u_k^*(u_k) + \mu_1 z_1^*(u_k) + \cdots + \mu_m z_m^*(u_k) = \lambda + 0 + \cdots + 0 \neq 0$. Thus, there is a contradiction, therefore, $\{z_1^*, \cdots, z_m^*\}$ is a basis of U^0.

   $$\dim(V) = m + n \qquad \dim(U) = n \qquad \dim(U^0) = m$$

   Thus, $\dim(U^0) + \dim(U) = \dim(V)$.

2. a) $\forall w^* \in W$, $T^*(w^*) = w^* \circ T(v)$. $\forall w^* \in N(T^*)$, $T^*(w^*) = \vec{0} = v^{*0}$, $v^{*0} \in V^0$, Thus, $N(T^*) = \{w^* | \forall v \in V, w^*(T(v)) = 0\}$ or $\{w^* | w^* \in W^*, \forall w \in R(T), w^*(w) = 0\}$.

   $(R(T))^0 = \{w^* | w^*(w) = 0, w^* \in W^*, \forall w \in R(T)\}$. Therefore, $N(T^*) = (R(T))^0$

   b)

   $$N(T^*) = (R(T))^0 \qquad \text{(by part a)}$$
   $$\dim(N(T^*)) = \dim(W) - \dim(R(T)) \qquad \text{(by problem 1)}$$
   $$\dim(W^*) - \dim(R(T^*)) = \dim(W) - \dim(R(T)) \qquad \text{(by rank nullity)}$$
   $$\because \dim(W) = \dim(W^*)$$
   $$\therefore \dim(R(T^*)) = \dim(R(T))$$

   c) Any $m \times n$ matrix A would be equavelent to $T : V \to W$ where $\dim(V) = n$, $\dim(W) = m$, and A^T would be equavelent with $T^* : W^* \to V^*$. As profed by b), $R(T^*) = R(T)$. Thus, they do have same rank.

3. By 2.a) if T^* is injective then $N(T^*) = \overrightarrow{0} = (R(T))^0$. Then by 1., we have $\dim(R(T)) = \dim(W^*) - \dim((R(T))^0) = \dim(W^*) = \dim(W)$. as $R(T) \subseteq W$, therefore, $R(T) = W$ Thus, the transformation is surjective.

4.

$$
\begin{aligned}
(\lambda b)(v_1 + v_2, \gamma w) &= \lambda b(v_1 + v_2, \gamma w) \\
&= \lambda \gamma b(v_1, w) + \lambda \gamma b(v_2, w) \\
&= (\lambda \gamma b)(v_1, w) + (\lambda \gamma b)(v_2, w) && \text{(Close under scaler multi.)} \\
(b_1 + b_2)(v_1 + v_2, \gamma w) &= b_1(v_1 + v_2, \gamma w) + b_2(v_1 + v_2, \gamma w) \\
&= b_1(v_1, w) + b_1(v_2, w) + b_2(v_1, w) + b_2(v_2, w) \\
&= b_1(v_1, w) + b_2(v_1, w) + b_1(v_2, w) + b_2(v_2, w) \\
&= (b_1 + b_2)(v_1, w) + (b_1 + b_2)(v_2, w) && \text{(Close under vector add)}
\end{aligned}
$$

a)

$$
\begin{aligned}
(b_1 + b_2)(v, w) &= b_1(v, w) + b_2(v + w) \\
&= b_2(v, w) + b_1(v + w) \\
&= (b_2 + b_1)(v, w) && \text{(vector ddition is communitive)}
\end{aligned}
$$

b)

$$
\begin{aligned}
((b_1 + b_2) + b_3)(v, w) &= (b_1 + b_2)(v, w) + b_3(v, w) \\
&= b_1(v, w) + b_2(v + w) + b_3(v, w) \\
&= b_1(v, w) + (b_2 + b_3)(v, w) \\
&= (b_1 + (b_2 + b_3))(v, w) && \text{(vector addition is associative)}
\end{aligned}
$$

c) Consider the bilinear map $b_0(v, w) = 0$

$$
\begin{aligned}
(b + b_0)(v, w) &= b(v, w) + b_0(v, w) \\
&= b(v, w) + 0 \\
&= b(v, w) && (\exists \overrightarrow{0} \text{ such that } b + \overrightarrow{0} = b)
\end{aligned}
$$

d)

$$
\begin{aligned}
(b + (-1 \cdot b))(v, w) &= b(v, w) + (-1) \times b(v, w) \\
&= 0 \\
&= b_0 && (\forall b, \ \exists - b, \ b + (-b) = 0)
\end{aligned}
$$

e)

$$
\begin{aligned}
(1 \cdot b)(v, w) &= 1 \times b(v, w) \\
&= b(v, w)
\end{aligned}
$$

f)

$$\lambda((\mu \cdot b)(v, w)) = \lambda(\mu b(v, w))$$
$$= \lambda \mu b(v, w)$$
$$= \mu \lambda b(v, w)$$
$$= \mu((\lambda \cdot b)(v, w))$$

g)

$$\lambda(b_1 + b_2)(v, w) = \lambda(b_1(v, w) + b_2(v, w))$$
$$= \lambda b_1(v, w) + \lambda b_2(v, w)$$
$$= (\lambda b_1)(v, w) + (\lambda b_2)(v, w)$$

h)

$$((\lambda + \mu)b)(v, w) = (\lambda + \mu)b(v, w)$$
$$= \lambda b(v, w) + \mu b(v, w)$$
$$= (\lambda b)(v, w) + (\mu b)(v, w)$$

5.   a)

$$w_0^{\#}(v_1 + v_2) = b(v_1 + v_2, w_0)$$
$$= b(v_1, w_0) + b(v_2, w_0)$$
$$= w_0^{\#}(v_1) + w_0^{\#}(v_2)$$
$$w_0^{\#}(\lambda v) = b(\lambda v, w_0)$$
$$= \lambda b(v, w_0)$$
$$= \lambda w_0^{\#}(v)$$

Thus it is linear.

b) Mark the transformation by T instead of $\#$.

$$T(\lambda w_0) = b(v, \lambda w_0)$$
$$= \lambda b(v, w_0)$$
$$= \lambda T(w_0)$$
$$T(w_1 + w_2) = b(v, w_1 + w_2)$$
$$= b(v, w_1) + b(v, w_2)$$
$$= T(w_1) + T(w_2)$$

Thus, it is linear.

c) $\forall w \in N(\#)$, $\#(w) = 0$ which is $\forall v \in V$, $w^{\#}(v) = 0$ which is $\forall v \in V$, $b(v, w) = 0$. Therefore, it is correct.

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d) As proved in c) $N(\#) = \{w_0 \in W | b(w^*, w_0) = 0 \text{ for all } w^* \in W^*\}$. For any w_0 we can write a bsis of W containing w_0. Thus, the dual vector space would also have a basis containing w_0^*. Suppose it exists some $w_0 \neq 0$ that $\forall w^* \in W^*$, $b(w^*, w_0) = 0$, consider $w_0^* \in W^*$, $b(w_0^*, w_0) = w_0^*(w_0) = 1 \neq 0$. Thus there is a contradiction. Therefore, $N(\#) = \{0\}$. Thus, the map is injective.

6. a)
$$\sigma = (1,2)(3,1)(4,5)$$

b) Since there are 3 swaps, the sign is -1

7. It shall equal to $1 + 1 + 1 = 3$

8.