PHYS 225 HW 6

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1. a) Suppose $\dim(A) = n \times m,\ \dim(B) = m \times n$:

$$
\begin{aligned}
(AB)^i_j &= \sum_k^m A^i_k B^k_j \\
\mathrm{Tr}(AB) &= \sum_{i,j}^n \delta^i_j (AB)^i_j = \sum_{i,j}^n \sum_k^m \delta^i_j A^i_k B^k_j = \sum_i^n (AB)^i_i = \sum_i^n \sum_k^m A^i_k B^k_i \\
(BA)^i_j &= \sum_k^n B^i_k A^k_j \\
\mathrm{Tr}(BA) &= \sum_{i,j}^m \delta^i_j (BA)^i_j = \sum_{i,j}^m \sum_k^n \delta^i_j B^i_k A^k_j = \sum_i^n (BA)^i_i = \sum_i^m \sum_k^n B^i_k A^k_i
\end{aligned}
$$

As $\sum_i^n \sum_k^m A^i_k B^k_i = \sum_k^n \sum_i^m A^k_i B^i_k$, thus $\mathrm{Tr}(AB) = \mathrm{Tr}(BA)$

b)

$$
\begin{aligned}
\mathrm{Tr}(ABC) &= \mathrm{Tr}((AB)C) = \mathrm{Tr}(C(AB)) = \mathrm{Tr}(CAB) \\
\mathrm{Tr}(ABC) &= \mathrm{Tr}(A(BC)) = \mathrm{Tr}((BC)A) = \mathrm{Tr}(BCA) \\
\text{Thus, } \mathrm{Tr}(ABC) &= \mathrm{Tr}(CAB) = \mathrm{Tr}(BCA)
\end{aligned}
$$

c)

$$
\begin{aligned}
\mathrm{Tr}(A_1 A_2 \cdots A_{n-1} A_n) &= \mathrm{Tr}((A_1 A_2 \cdots A_{n-1}) A_n) = \mathrm{Tr}(A_n (A_1 A_2 \cdots A_{n-1})) \\
&= \mathrm{Tr}(A_n A_1 A_2 \cdots A_{n-1})
\end{aligned}
$$

2.

Identity: Claim: $\exists M, N$ that $M^T M = I$, $NM = N$, For square matrixes, consider the Identity matrix I, first, $I^T I = I$, Thus $I \in O(n)$.

Inverse: $\forall M \in O(n)$, $M^T M = I$, thus $M^T \in O(n)$ and $M^T = M^{-1}$

Closure: For any $M, N \in O(n)$

$$\begin{aligned} MN &= A \\ N^T M^T MN &= N^T M^T A \\ I &= N^T M^T A \\ I &= (MN)^T A \\ (MN)^T (MN) &= I \end{aligned}$$

Thus, it is closed.

3.

Identity: $\exists (R(\theta), \overrightarrow{a})$ that, for any $(R(\phi), \overrightarrow{b})$, $(R(\theta), \overrightarrow{a})(R(\phi), \overrightarrow{b}) = (R(\phi), \overrightarrow{b})$, Take $\theta = 0, \overrightarrow{a} = \overrightarrow{0}$, then by not rotating and adding nothing, the transformation is not change.

Inverse: $\forall (R(\phi), \overrightarrow{b})$, $(R(-\phi), -\overrightarrow{b})(R(\phi), \overrightarrow{b}) = I$

Closure: $(R(\theta), \overrightarrow{a})(R(\phi), \overrightarrow{b}) = (R(\phi + \theta), \overrightarrow{a} + \overrightarrow{b})$. Thus it is closed.