MATH 461 Homework 8

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6.27 a)

$$\begin{aligned}
f_Z(z) &= \int_0^\infty f_{Z,W}(z,w)dw \\
&= \int_0^\infty \lambda_1\lambda_2 e^{-\lambda_1 zw} e^{-\lambda_2 w}\frac{1}{w}dw \\
&= \lambda_1\lambda_2 \int_0^\infty \frac{1}{w} e^{-(\lambda_1 z+\lambda_2)w} dw \\
&= \frac{\lambda_1\lambda_2}{\lambda_1 z+\lambda_2}; z \geq 0
\end{aligned}$$

b)

$$\begin{aligned}
P(X_1 < X_2) &= \int_0^\infty P(X_1 < x)\times f_{X_2}(x)dx \\
&= \int_0^\infty (1-e^{-\lambda_1 x})\lambda_2 e^{-\lambda_2 x}dx \\
&= \lambda_2 \int_0^\infty e^{-\lambda_2 x}dx - \lambda_2 \int_0^\infty e^{-(\lambda_1+\lambda_2)x}dx \\
&= \lambda_2 \frac{1}{\lambda_2} - \lambda_2 \frac{1}{\lambda_1+\lambda_2} \\
&= \frac{\lambda_1}{\lambda_1+\lambda_2}
\end{aligned}$$

6.29 a) $\mu = 2200 \times 2 = 4400$ and $\sigma = 230 \times \sqrt{2} = 325.27.z = (5000 - 4400)/325.27 = 1.84$ $P(x > 5000) = P(z > 1.845) = 1 - P(z < 1.845) = 1 - 0.9671 = 0.0329$

b) $z = (2000 - 2200)/230 = -\frac{200}{230} = -0.87$ $P(X > 2000) = 1 - P(z < -0.87) = 1 - 0.1922 = 0.8078$

$$\begin{aligned}
P(X \geq 2) &= P(X=2) + P(X=3) \\
&= \binom{3}{2}(0.8078)^2(1-0.8078) + \binom{1}{1}(0.8078)^3 \\
&= 0.9034
\end{aligned}$$

6.31 a) $\mu_x = n_1 p_1 = 50.4$ $\sigma_X^2 = n_1 p_1(1-p_1) = 37.6992.$
$\mu_y = n_2 p_2 = 47.2$ $\sigma_Y^2 = n_2 p_2(1-p_2) = 36.0608.$
$\mu_z = \mu_X + \mu_y = 97.6$ $\sigma_Z^2 = \sigma_X^2 + \sigma_Y^2 = 73.76$
$\sigma_Z = 8.5884$

$$\begin{aligned} P(Z \geq 110) &= 1 - P(z < \frac{110 - 0.5 - 97.6}{8.5884}) \\ &= 1 - 0.9177 \\ &= 0.0823 \end{aligned}$$

b) $E(Y - X) = E(Y) - E(X) = -3.2$

$$\begin{aligned} P(D \geq 0) &= P(z \geq \frac{0 - 3.2 - 0.5}{8.85884}) \\ &= 0.3783 \end{aligned}$$

6.34 a)

$$\begin{aligned} P(N > 2) &= 1 - P(N = 0, 1, 2) \\ &= 1 - e^{-2.2} - 2.2e^{-2.2} - \frac{2.2^2}{2!}e^{-2.2} \\ &= 0.3773 \end{aligned}$$

b)

$$\begin{aligned} P(N_1 > 4) &= 1 - \sum_{k=0}^{4} \frac{4.4^k}{k!} e^{-4.4} \\ &= 0.44882 \end{aligned}$$

c)

$$\begin{aligned} P(N_2 > 5) &= 1 - P(N_2 = 0, 1, 2, 3, 4, 5) \\ &= 1 - \sum_{k=0}^{4} \frac{6.6^k}{k!} e^{-6.6} \\ &= 0.64533 \end{aligned}$$

6.38 a)

$$\begin{aligned} P(Y = y, X = x) &= P(Y = y | X = x) \times P(X = x) \\ &= \frac{1}{x} \times \frac{1}{5} \\ &= \frac{1}{5x} \end{aligned}$$

b)

$$\begin{aligned} P(Y=i) &= \sum_{x=i}^{5} P(Y=i, X=x) \\ &= \sum_{x=i}^{5} (\frac{1}{5x}) P(X|Y=i) \qquad\qquad = \frac{\frac{1}{5x}}{\sum_{x=i}^{5}(\frac{1}{5x})} \end{aligned}$$

c) No they are not as $P(Y=y) \times P(X=x) \neq P(Y=y, X=x)$

6.40 a)

x	1	2
$P(x\|y=1)$	1/2	1/2
$P(x\|y=2)$	1/3	2/3

b) They are not independent.

$$\begin{aligned} p_{x,y}(1,1) &= 1/8 \\ p_x(1) &= 3/8 \\ p_y(1) &= 1/4 \\ p_y(1)p_x(1) &= 3/32 \neq p_{x,y}(1) \end{aligned}$$

c) i.

$$\begin{aligned} P(XY \leq 3) &= P(1,1) + P(1,2) + P(2,1) \\ &= 1/8 + 1/4 + 1/8 \\ &= 1/2 \end{aligned}$$

ii.

$$\begin{aligned} P(X+Y > 2) &= 1 - P(1,1) \\ &= 7/8 \end{aligned}$$

iii.

$$\begin{aligned} P(X/y > 1) &= P(2,1) \\ &= 1/8 \end{aligned}$$

6.41 a)

$$\begin{aligned} f(x) &= \int_0^\infty xe^{-x(y+1)}dy \\ &= e^{-x} \\ f(y) &= \int_0^\infty xe^{-x(y+1)}dx = \frac{1}{(y+1)^2} \\ Y = y, f(x,y) = \frac{f(x,y)}{f(y)} &= \frac{xe^{-x(y+1)}}{\frac{1}{(y+1)^2}} \\ &= x(y+1)^2e^{-x(y+1)} \\ X = x, f(x,y) = \frac{f(x,y)}{f(x)} &= \frac{xe^{-x(y+1)}}{e^{-x}} \\ &= xe^{-xy} \end{aligned}$$

b)

$$\begin{aligned} P(XY < z) &= \int_0^\infty \int_0^{z/x} f(x,y)dydx \\ &= \int_0^\infty \int_0^{z/x} xe^{-x(1+y)}dydx \\ &= \int_0^\infty (1 - e^{-z}x^{-x})dx \\ &= (1 + e^{-x}) \end{aligned}$$

6.42

$t \leq x$

$$P(Y \leq t|X = x) = \frac{3}{2x^3}(x^2(t+x) - t^3/3 + x^3/3)$$

$t < -x$

$$P(Y \leq t|X = x) = 0$$

$t > x$

$$P(Y \leq t|X = x) = 1$$