MATH 416H HW 8

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1. a)

$$\det\left(\begin{bmatrix} 1 & 2 & 0 \\ 1 & 1 & 5 \\ 1 & -3 & 0 \end{bmatrix}\right) = -5 \times \begin{vmatrix} 1 & 2 \\ 1 & -3 \end{vmatrix}$$
$$= 25$$

b)

$$\begin{aligned}
\det\left(\begin{bmatrix} 4 & -6 & -4 & 4 \\ 2 & 1 & 0 & 0 \\ 0 & -3 & 1 & 3 \\ -2 & 2 & -3 & -5 \end{bmatrix}\right) &= 2 \times \begin{vmatrix} 2 & -3 & -2 & 2 \\ 2 & 1 & 0 & 0 \\ 0 & -3 & 1 & 3 \\ -2 & 2 & -3 & -5 \end{vmatrix} \\
&= 2 \times \begin{vmatrix} 2 & -3 & -2 & 2 \\ 0 & 4 & 2 & -2 \\ 0 & -3 & 1 & 3 \\ 0 & -1 & -5 & -3 \end{vmatrix} \\
&= -2 \times \begin{vmatrix} 2 & -3 & -2 & 2 \\ 0 & -1 & -5 & -3 \\ 0 & -3 & 1 & 3 \\ 0 & 4 & 2 & -2 \end{vmatrix} \\
&= -2 \times \begin{vmatrix} 2 & -3 & -2 & 2 \\ 0 & -1 & -5 & -3 \\ 0 & 0 & 16 & 12 \\ 0 & 0 & -18 & -14 \end{vmatrix} \\
&= -2 \times 4 \times 2 \times \begin{vmatrix} 2 & -3 & -2 & 2 \\ 0 & -1 & -5 & -3 \\ 0 & 0 & 4 & 3 \\ 0 & 0 & -9 & -7 \end{vmatrix} \\
&= -16 \times 2 \times (-1) \begin{vmatrix} 4 & 3 \\ -9 & -7 \end{vmatrix} \\
&= 32 \times (-28 - (-27)) \\
&= -32
\end{aligned}$$

2.

$$
\begin{aligned}
\det\left(\begin{bmatrix}1 & x & x^2\\ 1 & y & y^2\\ 1 & z & z^2\end{bmatrix}\right) &= \begin{vmatrix}1 & x & x^2\\ 0 & (y-x) & (y^2-x^2)\\ 0 & (z-x) & (z^2-x^2)\end{vmatrix}\\
&= 1\times((y-x)(z-x)(z+x)-(z-x)(y-x)(y+x))\\
&= (y-x)(z-x)(z+x-(y+x))\\
&= (y-x)(z-x)(z-y)
\end{aligned}
$$

3.

forward: If column of B forms a basis in F^n, then all the column vectors are linearly independent with each other. Thus, noteing the columns in matrix A as $v_1, \cdots, v_n$. Then, for some $x_i \in F$, $x_1v_1+\cdots+x_nv_n = 0$ means $x_1 = \cdots = x_n = 0$. Thus, $Ax = 0$ give only solution of $x = \overrightarrow{0}$. Therefore $N(T) = \{\overrightarrow{0}\}$. Thus the map is invertable. Thus $\det(A) \neq 0$

backward: If $\det(A) \neq 0$,then the function is invertable meaning that $N(T) = \{\overrightarrow{0}\}$. Thus, the only solution for $Ax = 0$ is $x = \overrightarrow{0}$, or the only solution for $x_1v_1 + \cdots x_nv_n = 0$ is $x_1 = \cdots = x_n = 0$. Therefore, the column vectors are linearly independent to each other. As there are in total n linearly independent vectors in F, it forms a basis in F^n

4. As $\dim(V) = 1$, then we can write out a basis for V, $\{b\}$, $\forall v \in V$, $v = \lambda b$ for some $\lambda \in F$. Thus: Suppose that $T(b) = v$, $v = \lambda b$. For any other vector $w \in V$, $w = \mu b$, then $T(w) = T(\mu b) = \mu T(b) = \mu\lambda b = \lambda(\mu b) = \lambda w$. Thus, there exsist a unique λ for such map.

5. a)

$$\det(N^k) = 0 = (\det(N))^k = \det(N)$$

b)

$$\begin{bmatrix}0 & 1 & 2\\ 0 & 3 & 4\\ 0 & 5 & 6\end{bmatrix}$$

6.

$$
\begin{aligned}
\det(A) &= \det(A^T)\\
\det(AA^T) = \det(A)\det(A^T) &= \det(I) = 1\\
\det(A)^2 &= 1\\
\det(A) = \det(A^T) &= \pm 1
\end{aligned}
$$

7. a)

$$\begin{aligned}
b(v_1+v_2,w) &= \ell_1(v_1+v_2)\ell_2(w)\\
&= \ell_1(v_1)\ell_2(w)+\ell_1(v_2)\ell_2(w)\\
&= b(v_1,w)+b(v_2,w)\\
b(\lambda v,w) &= \ell_1(\lambda v)\ell_2(w)\\
&= \lambda\ell_1(v)\ell_2(w)\\
&= \lambda(b(v,w))
\end{aligned}$$

Similar the two properties can be profed for ℓ_2 and thus it is bilinear.

b)

$$\begin{aligned}
(\ell_1\wedge\ell_2)(v,w) &= b(v,w)-b(w,v)\\
(\ell_1\wedge\ell_2)(v_1+v_2,w) &= b(v_1+v_2,w)-b(w,v_1+v_2)\\
&= b(v_1,w)+b(v_2,w)+b(w,v_1)+b(w,v_2)\\
&= (b(v_1,w)+b(w,v_1))+(b(v_2,w)+b(w,v_2))\\
&= (\ell_1\wedge\ell_2)(v_1,w)+(\ell_1\wedge\ell_2)(v_2,w)\\
(\ell_1\wedge\ell_2)(\lambda v,w) &= b(\lambda v,w)-b(w,\lambda v)\\
&= \lambda b(v,w)-\lambda b(w,v)\\
&= \lambda(b(v,w)-b(w,v))\\
&= \lambda(\ell_1\wedge\ell_2)(v,w)
\end{aligned}$$

Similar the two properties can be profed for w, and thus it is bilinear.

$$\begin{aligned}
(\ell_1\wedge\ell_2)(v,w) &= b(v,w)-b(w,v)\\
(\ell_1\wedge\ell_2)(w,v) &= b(w,v)-b(v,w)\\
&= -(b(v,w)-b(w,v))\\
&= -(\ell_1\wedge\ell_2)(v,w)
\end{aligned}$$

Thus, it is alternating.

8. a) $\alpha,\beta\in \mathrm{Alt}^k(V)$, Define vector addition and scaler multiplication as following:

$$\begin{aligned}
(\alpha+\beta)(v_1,\cdots,v_k) &= \alpha(v_1,\cdots,v_k)+\beta(v_1,\cdots,v_k)\\
(\lambda\alpha)(v_1,\cdots,v_k) &= \lambda\alpha(v_1,\cdots,v_k)
\end{aligned}$$

And zero vector as $T \in \mathrm{Alt}^k(V), T(x) = 0$.

$$\begin{aligned}
(\alpha+\beta)(v_1+v_1',\cdots,v_k) &= \alpha(v_1+v_1',\cdots,v_k)+\beta(v_1+v_1',\cdots,v_k)\\
&= \alpha(v_1,\cdots,v_k)+\beta(v_1,\cdots,v_k)+\alpha(v_1',\cdots,v_k)+\beta(v_1',\cdots,v_k)\\
&= (\alpha+\beta)(v_1,\cdots,v_k)+(\alpha+\beta)(v_1',\cdots,v_k)\\
(\alpha+\beta)(\lambda v_1,\cdots,v_k) &= \alpha(\lambda v_1,\cdots,v_k)+\beta(\lambda v_1,\cdots,v_k)\\
&= \lambda(\alpha+\beta)(v_1,\cdots,v_k)\\
(\alpha+\beta)(v_2,\cdots,v_k,v_1) &= -\alpha(v_1,\cdots,v_k)-\beta(v_1,\cdots,v_k)\\
&= -(\alpha+\beta)(v_1,\cdots,v_k)\\
(\lambda\alpha)(v_1+v_1',\cdots,v_k) &= \lambda\alpha(v_1+v_1',\cdots,v_k)\\
&= (\lambda\alpha)(v_1,\cdots,v_k)+(\lambda\alpha)(v_1',\cdots,v_k)\\
(\lambda\alpha)(\mu v_1,\cdots,v_k) &= \lambda\alpha(\mu v_1,\cdots,v_k)\\
&= \mu\lambda\alpha(v_1,\cdots,v_k)\\
&= \mu(\lambda\alpha)(v_1,\cdots,v_k)\\
(\lambda\alpha)(v_2,\cdots,v_k,v_1) &= -\lambda\alpha(v_1,\cdots,v_k)\\
&= -(\lambda\alpha)(v_1,\cdots,v_k)\\
(\alpha+T)(v_1,\cdots,v_k) &= \alpha(v_1,\cdots,v_k)+T(v_1,\cdots,v_k)\\
&= \alpha(v_1,\cdots,v_k)
\end{aligned}$$

And the other 8 properties also holds, Thus, it is a vector space.

b) Choose a basis for V, $\{b_1, b_2\}$ and $\{b_1^*, b_2^*\}$ be a basis for the dual basis. Then $\forall v_1, v_2 \in V, (b_1^* \wedge b_2^*)(v_1, v_2) = b_1^*(v_1)b_2^*(v_2) - b_1^*(v_2)b_2^*(v_1)$. Take $v_1 = b_1, v_2 = b_2$, $(b_1^* \wedge b_2^*)(b_1, b_2) = 1 \times 1 - 0 \times 0 = 1 \neq 0$ Thus the wedge is not zero map. $\forall v_1, v_2 \in V, \alpha \in \mathrm{Alt}^2(V)$

$$\begin{aligned}
\alpha(v_1,v_2) &= \alpha(\lambda_1 b_1+\lambda_2 b_2, \mu_1 b_1+\mu_2 b_2)\\
&= \alpha(\lambda_1 b_1, \mu_1 b_1+\mu_2 b_2)+\alpha(\lambda_2 b_2, \mu_1 b_1+\mu_2 b_2)\\
&= \alpha(\lambda_1 b_1, \mu_1 b_1)+\alpha(\lambda_1 b_1, \mu_2 b_2)+\alpha(\lambda_2 b_2, \mu_1 b_1)+\alpha(\lambda_2 b_2, \mu_2 b_2)\\
&= \lambda_1\mu_1\alpha(b_1,b_1)+\lambda_1\mu_2\alpha(b_1,b_2)+\lambda_2\mu_1\alpha(b_2,b_1)+\lambda_2\mu_2\alpha(b_2,b_2)\\
&= (\lambda_1\mu_2-\lambda_2\mu_1)\alpha(b_1,b_2)\\
(b_1^*\wedge b_2^*)(v_1,v_2) &= b_1^*(\lambda_1 b_1+\lambda_2 b_2)b_2^*(\mu_1 b_1+\mu_2 b_2)-b_2^*(\lambda_1 b_1+\lambda_2 b_2)b_1^*(\mu_1 b_1+\mu_2 b_2)\\
&= \lambda_1\mu_2-\lambda_2\mu_1
\end{aligned}$$

Thus $\alpha(v_1)(v_2) = \alpha(b_1, b_2)((b_1^* \wedge b_2^*)(v_1, v_2))$, therfore it spans and the dimension is 1.

c) Using same notation with b).

span:

$$\begin{aligned}
\alpha(v_1, v_2) &= \alpha(\lambda_1 b_1 + \lambda_2 b_2 + \lambda_3 b_3, \mu_1 b_1 + \mu_2 b_2 + \mu_3 b_3) \\
&= (\lambda_1\mu_2 - \lambda_2\mu_1)\alpha(b_1, b_2) + (\lambda_1\mu_3 - \lambda_3\mu_1)\alpha(b_1, b_3) + (\lambda_2\mu_3 - \lambda_3\mu_2)\alpha(b_2, b_3) \\
(b_1^* \wedge b_2^*)(v_1, v_2) &= \lambda_1\mu_2 - \lambda_2\mu_1 \\
(b_1^* \wedge b_3^*)(v_1, v_2) &= \lambda_1\mu_3 - \lambda_3\mu_1 \\
(b_2^* \wedge b_3^*)(v_1, v_2) &= \lambda_2\mu_3 - \lambda_3\mu_2 \\
\alpha(v_1, v_2) &= \alpha(b_1, b_2) \cdot (b_1^* \wedge b_2^*)(v_1, v_2) + \alpha(b_1, b_3) \cdot (b_1^* \wedge b_3^*)(v_1, v_2) \\
&+ \alpha(b_2, b_3) \cdot (b_2^* \wedge b_3^*)(v_1, v_2)
\end{aligned}$$

linear independent: $\forall v_1, v_2 \in V$, if:

$$\chi_1(b_1^* \wedge b_2^*)(v_1, v_2) + \chi_2(b_1^* \wedge b_3^*)(v_1, v_2) + \chi_3(b_2^* \wedge b_3^*)(v_1, v_2) = 0$$

Take $v_1 = b_1 + b_2, v_2 = b_2 + b_3$, then we have:

$$\begin{aligned}
(b_1^* \wedge b_2^*)(v_1, v_2) &= \lambda_1\mu_2 - \lambda_2\mu_1 = 1 \\
(b_1^* \wedge b_3^*)(v_1, v_2) &= \lambda_1\mu_3 - \lambda_3\mu_1 = 1 \\
(b_2^* \wedge b_3^*)(v_1, v_2) &= \lambda_2\mu_3 - \lambda_3\mu_2 = 1
\end{aligned}$$

Thus, for the equality to hold, $\chi_1 = \chi_2 = \chi_3$ Thus they are linearly independent.

Thus, it is a set of basis.