

MATH 416H HW 4

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1. It is $\sum_{i=1}^m a_i x_i$

2. It is $\begin{pmatrix} yb_1 \\ yb_2 \\ \vdots \\ yb_n \end{pmatrix}$

3.

$$E_1 A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{bmatrix} = \begin{bmatrix} a_{21} & a_{22} & a_{23} & a_{24} \\ a_{11} & a_{12} & a_{13} & a_{14} \\ 3 \times a_{31} & 3 \times a_{32} & 3 \times a_{33} & 3 \times a_{34} \end{bmatrix}$$

$$E_2 A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{bmatrix} = \begin{bmatrix} a_{31} & a_{32} & a_{33} & a_{34} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{11} & a_{12} & a_{13} & a_{14} \end{bmatrix}$$

$$E_3 A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -5 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} - 5a_{31} & a_{22} - 5a_{32} & a_{23} - 5a_{33} & a_{24} - 5a_{34} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{bmatrix}$$

4. a) $\iota(w) = w$, $\iota(v) = v$, $\iota(w) + \iota(v) = w + v = \iota(w + v)$
 $\lambda \iota(v) = \lambda \cdot v = \iota(\lambda v)$. Thus, the inclusion map is linear.

b) i:

5. a)

forward: Suppose $\exists w_1, w'_1 \in W_1$, $w_2, w'_2 \in W_2$, that $v = w_1 + w_2 = w'_1 + w'_2$. Thus:

$$\begin{aligned} w_1 - w'_1 &= w'_2 - w_2 \\ w_1 - w'_1 &\in W_1 \quad w'_2 - w_2 \in W_2 \\ \text{as } W_1 \cap W_2 &= \{\vec{0}\} \\ w_1 - w'_1 &= w'_2 - w_2 = \vec{0} \\ w_1 &= w'_1 \quad w_2 = w'_2 \end{aligned}$$

Thus, if $V = W_1 \oplus W_2$, $\forall v \in V$, exists a unique w_1, w_2 that $v = w_1 + w_2$

backward: If $\forall v \in V, \exists w_1 \in W_1, w_2 \in W_2$ that $v = w_1 + w_2$, $V = W_1 + W_2$ by definition. Suppose that $W_1 \cap W_2 \neq \{\vec{0}\}$, then $\exists w \in W_1, W_2$. Thus, for some $v \in W_1$, or $v = w_1 + \vec{0}$, Thus define $k = w_1 - w$, Therefore $\exists v = w_1 - w + w$, where $w \neq \vec{0}$. However, there are only one set of w_1, w_2 that $v = w_1 + w_2$, Therefore, $W_1 \cap W_2 = \vec{0}$

b)

6. $\forall v \in V, T(S(v)) = v$, Thus, T is Surjective, Thus, $R(T) = \dim(V)$, Thus $N(T) = \{\vec{0}\}$ Suppose T is not injective, then $\exists v, w \in V$ that $v \neq w$ that $T(v) = T(w)$, then $T(v) - T(w) = \vec{0} = T(v - w)$, which raises a contradiction. Thus, T is a bijection. Thus T is invertable. As T is invertable, $\exists T^{-1}$ that $T \circ T^{-1} = \text{id}_V$. Therefore, $T^{-1} = S$. Thus, $S \circ T = T^{-1} \circ T = \text{id}_V$
7. Take a random set of $a_1, \dots, a_k, \dots \in \mathbb{N}$, $S(a_1, a_2, \dots, a_k, \dots) = (0, a_1, a_2, \dots, a_k, \dots)$, $T(0, a_1, a_2, \dots, a_k, \dots) = (a_1, a_2, \dots, a_k, \dots)$. Thus, $T \circ S = \text{id}_V$
8. $\forall x_i \in \mathbb{R}, v = (x_1, \dots, x_n) P(v) = (x_1, 0, \dots, 0)$, and $P(P(v)) = P(x_1, 0, \dots, 0) = (x_1, 0, \dots, 0) = P(v)$ Thus, $P \circ P = P$
 $N(P) = (0, x_2, \dots, x_n)$, $R(P) = (x_1, 0, \dots, 0)$, $\mathbb{R}^n = N(P) \oplus R(P) = (x_1, x_2, \dots, x_n)$. Also, as $R(P) \cap N(P) = (0, 0, \dots, 0) = \vec{0}$. Thus, $\mathbb{R}^n = N(P) \oplus$