MATH 416H HW 11

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Note that A^i_j, A_{ij} all means that the ith row and jth collumn of the matrix A

1. Consider the base: $\{\frac{1}{0!}x^0, \frac{1}{1!}x^1, \frac{1}{2!}x^2 \cdots, \frac{1}{n!}x^n\}$,

   Then, $A = [T]_{\mathcal{B}\mathcal{B}} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & 0 & \cdots & 0 & 0 \end{bmatrix}$

   Which is in jordan normal form

2. Notice that $\begin{pmatrix} 1 & 3 \\ 3 & 1 \end{pmatrix}$ is diagonalizable. Therefore, $USDS^{-1}U^{-1}$ is a diagonal matrix. Then, diagonalize $\begin{pmatrix} 1 & 3 \\ 3 & 1 \end{pmatrix}$. There is $\lambda_1 = 4, \lambda_2 = -2$ and $v_1 = (s, -s)^T, v_2 = (s, s)^T$ Then the equation becomes:

   $$U \begin{bmatrix} s & s \\ s & -s \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{2s} & \frac{1}{2s} \\ \frac{1}{2s} & -\frac{1}{2s} \end{bmatrix} U^*$$

   Consider the specific case where $US = I, U^*S^{-1} = I$ while setting the imaginary part to zero, we have $UU = I, US = I, US^{-1} = I$, then $S = S^{-1}$, we have $\begin{bmatrix} \frac{1}{2s} & \frac{1}{2s} \\ \frac{1}{2s} & -\frac{1}{2s} \end{bmatrix} = \begin{bmatrix} s & s \\ s & -s \end{bmatrix}$, then $s = \frac{\sqrt{2}}{2}$, then $U = S^{-1} = \begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \end{bmatrix} = U^*$.

3. a) Since the standard basis of $\mathbb{C}^n$ is orthornomal, $\langle e_i, e_j \rangle = \delta^i_j$ and $\varphi e_i = b_i$, then We have $\langle \varphi(e_i), \varphi(e_j) \rangle = \langle b_i, b_j \rangle = \delta^i_j = \langle e_i, e_j \rangle$. Thus the map is unitary.

   b) The transformation that sents $\{e_1, e_2, \cdots, e_n\}$ into $\mathcal{B}'$ is Unitary with similar prof done in a). And for a Unitary map we can write out a matrix, note $\varphi = [\varphi] = \mathcal{T}$ that $[\varphi] = \mathcal{T}$ is the matrix representation.

Then its inverse would be $\mathcal{T}^*$ which is also unitary as: $\mathcal{T}^*(\mathcal{T}^*)^* = \mathcal{T}^*\mathcal{T} = I$. Then consider $\varphi' \circ \varphi$ ($\varphi' = \mathcal{T}'$):

$$\begin{aligned}
\varphi' \circ \varphi &= \mathcal{T}'\mathcal{T} = T \\
TT^* &= \mathcal{T}'\mathcal{T}(\mathcal{T}'\mathcal{T})^* \\
&= \mathcal{T}'\mathcal{T}\mathcal{T}^*\mathcal{T}'^* \\
&= I
\end{aligned}$$

Therefore, $\varphi' \circ \varphi$ is unitary. And such transformation transforms from $\mathcal{B}$ to $\mathcal{B}'$. Which is $[\mathrm{id}_V]_{\mathcal{B},\mathcal{B}'}$

4.

Forward: For any Hermitian matrix A that is positive definite, forall $v \in V$ $\langle v, Av\rangle > 0$ then for all eigenvectors v_i There is:

$$\begin{aligned}
Av_i &= \lambda_i v_i \\
\langle v_i, Av_i\rangle &= \langle v_i, \lambda_i v_i\rangle \\
&= \overline{\lambda_i}\langle v_i, v_i\rangle \\
&= \overline{\lambda_i}||v_i||^2
\end{aligned}$$

For any non-zero vector v_i, $||v_i||^2 > 0$, then for $\langle v_i, Av_i\rangle > 0$, $\overline{\lambda_i}||v_i||^2 > 0$ Thus $\lambda_i > 0$ also as it is a hermitian matrix, all of its eigenvalues are real.

Backward: Assume tha $\forall \lambda_i > 0$. According to the spectral theorem, $\forall A$, $\exists U$ that is unitary that $UAU^* = D$ which is the diagonal matrix with all of A's eigenvalues on its main diagonal. Then we have:

$$\begin{aligned}
\langle v, Av\rangle &= \langle Uv, UAv\rangle \\
&= \langle UU^*v, UAU^*v\rangle \\
&= \langle Iv, Dv\rangle
\end{aligned}$$

Consider a set of orthornomal basis of V, $\{b_1, \cdots, b_n\}$. $v = \sum \mu_i b_i$, $\mu_i \in \mathbb{C}$, then $Dv = \displaystyle\sum_{i=1}^{n} \lambda_i \mu_i b_i$, Thus:

$$\begin{aligned}
\langle v, Dv\rangle &= \sum_{i=1}^{n} \mu_i \overline{\lambda_i \mu_i}||b_i||^2 \\
&= \sum_{i=1}^{n} \lambda_i ||\mu_i||^2||b_i||^2
\end{aligned}$$

since $||\mu_i||^2 > 0$, $||b_i||^2 > 0$ and $\forall \lambda_i > 0$(and real as it is from a hermitian matrix), $\sum_{i=1}^{n} \lambda_i||\mu_i||^2||b_i||^2 > 0$ Therefore the matrix is positive definite.

5. Since A is hermitian, then $\exists U, D$ that $A = UDU^*$ where U is unitary and D is diagonal with entries equaling to eigenvalues. Therefore, as A is positve definite, all entries of D would be positve real numbers.
   then write the matrix as $D_{ij} = \delta_{ij}\lambda_i$, then $\exists D'$ that $D'_{ij} = \delta_{ij}\sqrt{\lambda_i}$, then $D'D' = \Gamma$, and $\Gamma_{ij} = \sum_{k=1}^{n} \delta_{ik}\sqrt{\lambda_k}\delta_{kj}\sqrt{\lambda_k} = \delta_{ij}\lambda_i = D$, Therefore: $A = UD'D'U^*$ As D' only have real positve entries in its main diagonal, $D'^* = D'$, then take $S = D'U^*$, then $S^* = (D'U^*)^* = UD'^* = UD'$, Thus, $S^*S = UD'D'U^* = A$

6. a)

Forward:

$$\begin{aligned}
A^* &= -A \\
-A^* &= A \\
-\sqrt{-1}A^* &= \sqrt{-1}A \\
-iA^* &= iA \\
(iA)^* &= -iA^*
\end{aligned}$$

Therefore, if A is skew-Hermitian then $A^* = -A$

Backward: If $\sqrt{-1}A$ is Hermitian, then:

$$\begin{aligned}
(iA)^* &= iA \\
iA &= -iA^* \\
A &= -A^*
\end{aligned}$$

   b) For any Hermitian matrix H, forall eigenvalues λ_i of H, $\lambda_i \in \mathbb{R}$. As iA is Hermitian, $\exists U,\ iA = U\Lambda U^*$ where Λ only have real diagonal entries. Then $A = U(\frac{1}{i}\Lambda)U^*$. $\forall r \in \mathbb{R}, \frac{r}{i} = -ri$ which is purely imaginary (if include r=0) therefore $\left(\frac{1}{i}\Lambda\right)$ have only pure imaginary (including zero) diagonal entries. Also, as U is unitary and $UU^* = I$, $A = U\left(\frac{1}{i}\Lambda\right)U^{-1}$ which means that $\left(\frac{1}{i}\Lambda\right)$ is the matrix containing all of its eigenvalues which means that they are all purely imaginary.

   c) Using lemma 31.4 profed in class, as iA is herimitan, then $iA = U\Lambda U^*$ and $A = U\left(\frac{1}{i}\Lambda\right)U^*$ which is unitary equavalent to the diagonal matrix $-i\Lambda$ meaning that it includes all eigenvectors of A.

7. a)

$$\begin{aligned}
[B, A] &= AB - BA \\
&= -(BA - AB) \\
&= -[A, B]
\end{aligned}$$

b)

$$
\begin{aligned}
-[A,B] &= [B,A] \\
&= BA - AB \\
&= (-B)(-A) - (-A)(-B) \\
&= B^*A^* - A^*B^* \\
[A,B]^* &= (AB - BA)^* \\
&= (AB)^* - (BA)^* \\
&= B^*A^* - A^*B^* \\
&= -[A,B]
\end{aligned}
$$

Therefore it is also skew-hermitian.

c)

$$
\begin{aligned}
[A,B] &= AB - BA \\
&= A^*B^* - B^*A^* \\
&= A^*B^* - (AB)^* \\
[A,B]^* &= B^*A^* - A^*B^* \\
&= (AB)^* - A^*B^* \\
&= -[A,B]
\end{aligned}
$$

Therefore, in general $[A,B]$ is not hermitian. And is when $(AB)^* = A^*B^*$

8. a) Since A is Hermitian, note $[T]_{\mathcal{E}\mathcal{E}} = A$, then $A = U\Lambda U^*$ where $U \in U(n)$ and thus $U^* = U^{-1}$, since $Av_i = \lambda_i$, As Λ is the diagonal matrix with eigenvalues entries. Then on eigenvector basis $\mathcal{B}$, $\Lambda v_i = \lambda_i v_i$. Thus, $[T]_{\mathcal{B}\mathcal{B}} = \Lambda$. $A = [\mathrm{id}]_{\mathcal{E}\mathcal{B}}[T]_{\mathcal{B}\mathcal{B}}[\mathrm{id}]_{\mathcal{B}\mathcal{E}} = U\Lambda U^{-1}$ Therefore, $U = [v_1, v_2, \cdots, v_n]$ (collumns would be the eigenvectors of A). Claim that $\sum v_j v_j^* = UU^* = I$. Note the k's entry of v_i as $v_i(k)$

$$
\begin{aligned}
(UU^*)_{ij} &= \sum_{k=1}^{n} U_k^i U_j^{*k} \\
&= \sum_{k=1}^{n} v_k(i) v_k^*(j) \\
&= \sum_{j=1}^{n} v_j v_j^*
\end{aligned}
$$

b) As stated above, we have $A = U\Lambda U^*$

$$\begin{aligned}
(\Lambda U^*)_{ij} &= \sum_{k=1}^{n} \Lambda^i_k U^{*k}_j \\
&= \sum_{k=1}^{n} \delta^i_k \lambda_i U^{*k}_j \\
(U(\Lambda U^*))_{ij} &= \sum_{\ell=1}^{n} U^i_\ell (\Lambda U^*)^\ell_j \\
&= \sum_{\ell=1}^{n} \left(U^i_\ell \sum_{k=1}^{n} \delta^\ell_k \lambda_\ell U^{*k}_j \right) \\
&= \sum_{\ell=1}^{n} \left(U^i_\ell \lambda_\ell U^{*\ell}_j \right) \\
&= A
\end{aligned}$$

Therefore, $A = \sum_{j=1}^{n} \lambda_j v_j v_j^*$