MATH 461 Homework 6

James Liu

Due: Oct 18 Edit: October 18, 2024

5.6 a)

$$\int_0^\infty \frac{1}{4}x^2 e^{-\frac{x}{2}}\mathrm{d}x = 4$$

b)

$$\int_{-1}^1 cx(1-x^2)\mathrm{d}x = 0$$

As it is an odd function.

c)

$$\int_5^\infty x \times \frac{5}{x^2}\mathrm{d}x = 5(\ln(\infty) - \ln(5)) = \infty$$

5.10 a)

$$\int_5^{15} \frac{1}{60}\mathrm{d}x + \int_{20}^{30} \frac{1}{60}\mathrm{d}x + \int_{35}^{40} \frac{1}{60}\mathrm{d}x + \int_{50}^{60} \frac{1}{60}\mathrm{d}x = 4 \times \frac{10}{60} = \frac{2}{3}$$

b)

$$\int_{10}^{15} \frac{1}{60}\mathrm{d}x + \int_{20}^{30} \frac{1}{60}\mathrm{d}x + \int_{35}^{40} \frac{1}{60}\mathrm{d}x + \int_{50}^{60} \frac{1}{60}\mathrm{d}x + \int_{65}^{70} \frac{1}{60}\mathrm{d}x = 3 \times \frac{10}{60} + 2 \times \frac{5}{60} = \frac{2}{3}$$

5.12

Case 1:

$$\begin{aligned} E(Y) &= \left(\left(\int_0^{25} |x-0|\mathrm{d}x\right) + \left(\int_{25}^{75} |x-50|\mathrm{d}x\right) + \left(\int_{75}^{100} |x-100|\mathrm{d}x\right)\right) \times \frac{1}{100} \\ &= 12.5 \end{aligned}$$

Case 2:

$$\begin{aligned} E(Y) &= \left(\left(\int_0^{37.5} |x-25|\mathrm{d}x\right) + \left(\int_{37.5}^{62.5} |x-50|\mathrm{d}x\right) + \left(\int_{62.5}^{100} |x-75|\mathrm{d}x\right)\right) \times \frac{1}{100} \\ &= 9.375 < 12.5 \end{aligned}$$

Thus, it is right.

5.13 a)

$$\int_{10}^{30} \frac{1}{30} \mathrm{d}x = \frac{2}{3}$$

b)

$$\frac{\int_{25}^{30} \frac{1}{30} \mathrm{d}x}{\int_{0}^{15} \frac{1}{30} \mathrm{d}x} = \frac{5}{15} = \frac{1}{3}$$

5.15 a)

$$\begin{aligned} P(X > 5) = P\left(\frac{X-\mu}{\sigma} > \frac{5-10}{6}\right) &= P\left(z > \frac{5-10}{6}\right) \\ &= 1 - P\left(z \leq -0.8333\right) \\ &= 1 - 0.2023 = 0.7977 \end{aligned}$$

b)

$$\begin{aligned} P(4 < X < 16) &= P\left(\frac{4-10}{6} < \frac{X-\mu}{\sigma} < \frac{16-10}{6}\right) \\ &= P(-1 < z < 1) = P(z < 1) - P(z < -1) \\ &= 0.8413 - 0.1587 = 0.6826 \end{aligned}$$

c)

$$P(X < 8) = P\left(\frac{X-\mu}{\sigma} < \frac{8-10}{6}\right) = P\left(z < -0.3333\right) = 0.3695$$

d)

$$P(X < 20) = P\left(\frac{X-\mu}{\sigma} < \frac{20-10}{6}\right) = P(z < 1.6667) = 0.9522$$

e)

$$\begin{aligned} P(X > 16) &= P\left(\frac{X-\mu}{\sigma} > \frac{16-10}{6}\right) = P(z > 1) \\ &= 1 - P(z \leq 1) = 1 - 0.8413 = 0.1587 \end{aligned}$$

5.18

$$\begin{aligned} P(\frac{x-\mu}{\sigma} > \frac{9-5}{\sigma}) &= 0.2 \\ 1 - P(z \leq \frac{4}{\sigma}) &= 0.2 \\ P(z \leq \frac{4}{\sigma}) &= 0.84 \\ \sigma = 4/0.84 &= 4.761 \\ \sigma^2 &= 22.675 \end{aligned}$$

5.21 a)

$$P(X > 74) = P(\frac{X-\mu}{\sigma}) > \frac{74-\mu}{\sigma} = P(z > \frac{74-71}{2.5}) = 1 - P(z \leq 1.2) = 1 - 0.8849 = 0.115$$

b)

$$P(X > 77) = P(\frac{X-\mu}{\sigma}) > \frac{77-\mu}{\sigma} = P(z > \frac{77-71}{2.5}) = 1 - P(z \leq 2.4) = 1 - 0.9918 = 0.0082$$

$$P(X > 72) = P(\frac{X-\mu}{\sigma}) > \frac{72-\mu}{\sigma} = P(z > \frac{72-71}{2.5}) = 1 - P(z \leq 0.4) = 1 - 0.6554 = 0.3466$$

5.23 a)

$$\begin{aligned}
\mu &= np = 166.6667 \\
\sigma &= \sqrt{np(1-p)} = 11.7851 \\
P\left(150 \leq X \leq 200\right) &= P\left(\frac{150-\mu}{\sigma} \leq \frac{X-\mu}{\sigma} \leq \frac{200-\mu}{\sigma}\right) \\
&= P\left(\frac{150-0.5}{11.785} \leq z \leq \frac{200-0.5}{11.7851}\right) \\
&= P\left(\frac{150-0.5-166.6667}{11.785} \leq z \leq \frac{200+0.5-166.6667}{11.7851}\right) \\
&= P\left(\frac{149.5-166.6667}{11.785} \leq z \leq \frac{200.5-166.6667}{11.7851}\right) \\
&= P\left(-1.46 \leq z \leq 2.87\right) \\
&= \varphi\left(2.87\right) - \varphi\left(-1.46\right) = 0.998 - 0.072 = 0.9258
\end{aligned}$$

b)

$$\begin{aligned}
\mu &= np = 800 \times \frac{1}{5} = 160 \\
\sigma &= \sqrt{np(1-p)} = 11.314 \\
P(X < 150) &= p(z < \frac{150-0.5-160}{11.314}) \\
&= P(z < -0.92) = 0.17
\end{aligned}$$

5.25

$$\begin{aligned}
\mu &= np = 150 \times 0.05 = 7.5 \\
\sigma &= \sqrt{np(p-1)} = 2.67 \\
z &= \frac{10-7.5}{2.67} = 0.938 \\
P(z \leq 0.938) &= 0.826
\end{aligned}$$

5.28

$$
\begin{aligned}
\mu = np &= 200 \times 0.12 = 24 \\
\sigma &= \sqrt{np(p-1)} = 4.595 \\
z = \frac{20-24}{4.595} &= -0.8703 \\
P(z \geq 0.8703) &= 1 - P(z \leq -0.8703) \\
&= 1 - 0.273173 = 0.726827
\end{aligned}
$$

5.32 a)

$$P(X > 2) = \exp(-2\lambda) = e^{-1} = 0.3678$$

b)

$$\frac{P(X \geq 10 \text{ and } X > 9)}{P(X > 9)} = \frac{\exp(-5)}{\exp(-4.5)} = 0.6065$$

5.33

$$e^{-1} = 0.36787$$

5.40

$$
\begin{aligned}
F_Y(y) = P(Y \leq y) = P(e^X \leq y) = P(X \leq \ln(y)) = F_X(\ln(y)) \\
F_Y(y)' = F_X(\ln(y))\frac{1}{y}
\end{aligned}
$$

As X is uniform on $(0,1)$, then $f_Y = \frac{1}{y}$ when $1 < y < e$ and 0 other wise.