MATH 416H HW 6

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1. a) Rank would be 3 and Nullity would be 2 as the matrix is already in reduced row-echlon form and the number of pivots is the rank and the number of none pivot column is Nullity.

   b)

$$\left[\begin{array}{ccccc|c} 1 & 2 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{array}\right] \rightarrow \begin{cases} x_1 = -2x_2 - x_4 \\ x_2 = x_2 \\ x_3 = -x_4 \\ x_4 = x_4 \\ x_5 = 0 \end{cases}$$

   Take $x_2 = 1, x_4 = 1$, seperatly, we have a basis consisting 2 element:
$$\left\{ \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ -1 \\ 1 \\ 0 \end{pmatrix} \right\}$$

2. a) Scaler Multiplication: multiplying a scaler does not change the symetric of the matrix.

$$\begin{aligned} A &= A^T \\ a_{ij} &= A_{ji} \quad 0 \leq ij \leq n \\ ka_{ij} &= ka_{ji} \quad k \in F \end{aligned}$$

   Vector Addition: Adding two such matrix also does not change such symetry:

$$\begin{aligned} A &= A^T & B &= B^T \\ a_{ij} &= a_{ji} & b_{ij} &= b_{ji} \\ a_{ij} + b_{ij} &= a_{ji} + b_{ji} \end{aligned}$$

   Consider $a_{ij} = 0,\ \forall i, j$, such matrix will be the additive identity. And these operations do fullfill the 8 properties as in the question,

$M_{n,n}(F)$ is already a vector space. And S_n is close under the 2 operations, thus it is a subspace.

b) Notice that one possible set of basis would be:

$$\left\{ \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right\}$$

3. T is injective, then$\forall w \in W$, $\exists v \in V$ such that $w = T(v)$. By definition, $T^*(\psi) = \psi \circ T(v)$, $\psi \in W^*$. Consider $T^*(\psi)(v) = 0$

$$\begin{aligned} T^*(\psi)(v) &= 0 \\ \psi(T(v)) &= 0 \ \forall v \in V \end{aligned}$$

   As $T(v)$ is surjective on W, which means that $\psi(w) = 0$, $\forall w \in W$. Thus, ψ is a zero map, thus, $N(T^*) = \{\overrightarrow{0}\}$. Thus, by rank/nullity, the T^* is injective.

4. $\forall \ell \in U^0$, $\ell_1(x) + \ell_2(x) = 0 + 0$. Thus, $\exists \ell_3$ such that $\ell_1 + \ell_2 = \ell_3$. Thus U^0 is closed under addition. $\forall \lambda \in F$, $\lambda \ell(x) = \lambda \times 0 = 0$. Thus, it is also closed under scaler multiplication. Also, $\ell + \ell = \ell$ as $0 + 0 = 0$, thus, there also exists a zero element. Thus, U^0 is a subspace.

5. Consider a map: $\pi : V \to V/U$, $\pi(v) = v + U$. Thus, $\forall u \in U$, $\exists v$ that $u = v + U$ by definition. Thus, π is surjective. Thus, according to 3., the dual map $\pi^* : (V/U)^* \to V^*$ is injective. Thus, $N(\pi^*) = \overrightarrow{0}$. In this case, profed by 4., $\overrightarrow{0} = U^0 = \{\ell\}$. Thus, accroding to the 1st isomorphism law, $(V/U)^*/N(\pi^*) \to R(\pi^*)$ is isomorphic.