NPRE200 HW 6

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A. 1)

$$\frac{\mathrm{d}n}{\mathrm{d}t} = S + (\eta - 1)\Sigma_a\phi + D\nabla^2\phi$$

As we are using a non-multiplying material, we have $\eta = 0$, then:

$$\frac{\mathrm{d}n}{\mathrm{d}t} = S - \Sigma_a\phi + D\nabla^2\phi$$

Solving for a steady state, we have: $\frac{\mathrm{d}n}{\mathrm{d}t} = 0$

$$\begin{aligned}
0 &= S - \Sigma_a\phi + D\nabla^2\phi \\
\Sigma_a\phi - D\nabla^2\phi &= S \\
D\nabla^2\phi - \Sigma_a\phi &= -S \\
\nabla^2\phi - \frac{\Sigma_a}{D}\phi &= \frac{S}{D} = \frac{J}{D} \\
\nabla^2\phi - \frac{1}{L^2}\phi &= \frac{J}{D} \\
\phi(x) &= C_1 e^{\frac{x}{L}} + C_2 e^{-\frac{x}{L}} - \frac{JL^2}{D} \\
&= A\sinh(\frac{x}{L}) + B\cosh(\frac{x}{L}) - \frac{JL^2}{D}
\end{aligned}$$

Apply initial conditions that $\phi(a) = 0$ and $\phi(0) = J$

$$
\begin{aligned}
J &= -D\nabla\phi(0) \\
J &= -D\left(\frac{A}{L}\cosh\left(\frac{0}{L}\right) + \frac{B}{L}\sinh\left(\frac{0}{L}\right)\right) \\
J &= \frac{-DA}{L} + 0 \\
A &= -\frac{JL}{D} \\
\phi(a) &= 0 \\
0 &= A\sinh\left(\frac{a}{L}\right) + B\cosh\left(\frac{a}{L}\right) - \frac{JL^2}{D} \\
\frac{JL^2}{D} &= -\frac{JL}{D}\sinh\left(\frac{a}{L}\right) + B\cosh\left(\frac{a}{L}\right) \\
B\cosh\left(\frac{a}{L}\right) &= \frac{JL^2}{D} + \frac{JL}{D}\sinh\left(\frac{a}{L}\right) \\
B &= \frac{JL}{D}\left(\frac{L}{\cosh\left(\frac{a}{L}\right)} + \frac{\sinh\left(\frac{a}{L}\right)}{\cosh\left(\frac{a}{L}\right)}\right) \\
\phi(x) &= A\sinh(\frac{x}{L}) + B\cosh(\frac{x}{L}) - \frac{JL^2}{D} \\
&= -\frac{JL}{D}\sinh(\frac{x}{L}) + \frac{JL}{D}\left(\frac{L}{\cosh\left(\frac{a}{L}\right)} + \frac{\sinh\left(\frac{a}{L}\right)}{\cosh\left(\frac{a}{L}\right)}\right)\cosh\left(\frac{x}{L}\right) - \frac{JL^2}{D} \\
&= \frac{4J\sinh\left(\frac{a-x}{L}\right)}{\sinh\left(\frac{a}{L}\right) + 2\frac{D}{L}\cosh\left(\frac{a}{L}\right)}
\end{aligned}
$$

2)

$$
\begin{aligned}
J_x &= -D\phi'(a) \\
&= \frac{4DJ\cosh\left(\frac{a-x}{L}\right)}{\frac{1}{L}\sinh\left(\frac{a}{L}\right) + 2\frac{D}{L^2}\cosh\left(\frac{a}{L}\right)} \\
\frac{J_x(a)}{J} &= \frac{4D\cosh\left(\frac{a-a}{L}\right)}{\frac{1}{L}\sinh\left(\frac{a}{L}\right) + 2\frac{D}{L^2}\cosh\left(\frac{a}{L}\right)} \\
&= \frac{4D}{\frac{1}{L}\sinh\left(\frac{a}{L}\right) + 2\frac{D}{L^2}\cosh\left(\frac{a}{L}\right)}
\end{aligned}
$$

B. 3)

$$
\begin{aligned}
\nabla^2 &= \frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}r^2\frac{\mathrm{d}}{\mathrm{d}r} \\
\phi(r) &= \frac{1}{r}\Psi(r) \\
\nabla^2\phi - \frac{1}{L^2}\phi &= 0 \\
\frac{d\Psi}{dr^2} - \frac{1}{L^2}\Psi(r) &= 0 \\
\Psi(r) &= Ae^{r/L} + Ce^{-r/L} \\
\phi(r) &= \frac{A}{r}e^{r/L} + \frac{C}{r}e^{-r/L} \\
\phi(\infty) &= 0 \\
A &= 0 \\
P &= \lim_{r\to 0}\left[-4\pi r^2 D\frac{d}{dr}\phi(r)\right] \\
\phi'(r) &= \frac{d}{dr}\left[\frac{C}{r}e^{-r/L}\right] \\
-4\pi r^2 D\phi'(r) &= -4\pi r^2 D(-e^{r/L} \times C(\frac{1}{r^2} + \frac{1}{rL})) \\
P &= 4\pi DCe^{-0/L} + 4\pi\frac{0}{L}DCe^{0/L} \\
&= 4\pi DC \\
C &= \frac{P}{4\pi D} \\
\phi(r) &= \frac{P}{4\pi Dr}e^{-r/L}
\end{aligned}
$$