MATH 416H HW 3

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1. $\forall W \subseteq V$ that W is a subspace containing S, $S \subseteq \bigcap_i W_i$ as Ws are vector spaces, thus they are closed on the 2 operations, thus, linear combinations of S are also in any W containing S. Thus, (span S) $\subseteq \bigcap_i W_i$.

   Suppose that $\bigcap_i W_i \not\subseteq$ span S, then it exisist such $v \in V$, that $v \notin$ span S, while $v \in \bigcap_i W_i$. Consider a special subspace (span S) $\subseteq V$, $S \subseteq$ span$S$, thus $spanS \in \{W_i\}$ while span $S \subseteq$span S. Therefore, any $v \notin span\ S, v \notin \bigcap_i W_i$. Thus, by definition of intersections, such vector does not exist, thus, $\bigcap_i W_i \subseteq$ span S

   Thus, the span of S equals the intersection of all the subspaces of V containing S.

2. For the 2 vectors to be linear independent, the only solution to $\begin{cases} \lambda_1 + \lambda_2 = 0 \\ x\lambda_1 + y\lambda_2 = 0 \end{cases}$ should be $\begin{cases} \lambda_1 = 0 \\ \lambda_2 = 0 \end{cases}$. for $\lambda_1 + \lambda_2 = 0$, there could be $\lambda_1 = \lambda_2 = 0$ or $\lambda_1 = -\lambda_2$ As the vectors should be linear independent, then, $-\lambda_1 x + \lambda_2 y \neq 0$, thus $x \neq y$.

   For vectors with $x \neq y$, there is $\begin{cases} \lambda_1 + \lambda_2 = 0 \\ x\lambda_1 + y\lambda_2 = 0 \end{cases}$ with only one set of solution of $\lambda_1 = \lambda_2 = 0$, Thus they are linearly independent with condition $x \neq y$.

3. i: $\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$ is one of the vectors that can form a basis. suppose they are linear dependent, then exisist $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{R}$, that $\lambda_1 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + \lambda_2 \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + \lambda_3 \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ while $\exists \lambda_i \neq 0$ however $\begin{cases} \lambda_2 + 2 \times \lambda_3 = 0 \\ \lambda_2 + \lambda_3 = 0 \end{cases}$ Showing that $\lambda_2 = \lambda_3 = 0$ and thus, as $\lambda_1 + 0 + 0 = 0$ $\lambda_1 = 0$, Thus they are linear independent,

as there are 3 elements while the dimension of $\mathbb{R}^3$ is also 3, it is a basis.

ii: $\begin{pmatrix}1\\0\\1\end{pmatrix}$ is one of the vectors that can form a basis. suppose they are linear dependent, then exisist $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{R}$,

that $\lambda_1 \begin{pmatrix}0\\0\\1\end{pmatrix} + \lambda_2 \begin{pmatrix}1\\1\\0\end{pmatrix} + \lambda_3 \begin{pmatrix}1\\0\\1\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}$

while $\exists \lambda_i \neq 0$ however $\begin{cases} \lambda_2 + \lambda_3 = 0 \\ \lambda_2 = 0 \end{cases}$ Showing that $\lambda_2 = \lambda_3 = 0$ and thus, as $\lambda_1 + 0 + 0 = 0$ $\lambda_1 = 0$, Thus they are linear independent, as there are 3 elements while the dimension of $\mathbb{R}^3$ is also 3, it is a basis.

4. As $x + y + z = \overrightarrow{0}$, thus, $z = -x - y$. In $\forall v \in$span $\{y, z\}$ $\exists \lambda_1, \lambda_2 \in F$ that $v = \lambda_1 y + \lambda_2 z = \lambda_1 y + \lambda_2(-x - y) = (\lambda_1 - \lambda_2)y - \lambda_2 x$, $(\lambda_1 - \lambda_2 \in F)$, Thus v is also in span $\{x, y\}$. Thus, span $\{x, y\}$ =span $\{y, z\}$

5. $N(T) = \{x | T(x) = \overrightarrow{0}\}$, for $\mathcal{P}_n$, $N(T) = \{x | \frac{d}{dx}x = 0\} = k + 0x + 0x^2 + \cdots + 0x^n$, for some $k \in \mathbb{R}$
$R(T) = \{x | T(v) = x, v \in V\}$, for $\mathcal{P}_n$, $R(T) = \{x | \frac{d}{dx}v = x,\ v \in V\} = k_0 + k_1 x + k_2 x^2 + \cdots + k_{n-1}x^{n-1} + 0x^n$ for some $k_i \in \mathbb{R}$
For example, take $v_1 = 1$ and $v_2 = x$, $v_1 \in R(T)$ and $T(v_2) = v_1 \in N(T)$. Thus, the intersection is not empty.

6. $\forall w \in R(T)$, $\exists \lambda_i \in F$ that $w = T(\sum_i^n \lambda_i v_i)$. As the map is linear, there is $T(\sum_i^n \lambda_i v_i) = \sum_i^n \lambda_i (T(v_i))$, Thus, $w \in$span $T(v_i)$, thus $T(v_i)$ spans $R(T)$, Also, as profed in class, $T(\overrightarrow{0}) = \overrightarrow{0}$ for linear transformations, and as $T(v)$ is , then $T(\sum_i^n \lambda_i v_i) = \overrightarrow{0}$ only when $\sum_i^n \lambda_i v_i = \overrightarrow{0}$ which is that all $\lambda_i = 0$. Therefore, $\sum_i^n \lambda_i (T(v_i)) = \overrightarrow{0}$ only when all $\lambda_i = 0$. Thus, $T(v_i)$ is linear independent. Thus it is one of the basis of $R(T)$.

   If the mapping not injective, meaning that $T(v_i)$ is not nessesarily linear independent as there might be some other vectors in the null space of T, leading to a set of λ_i not all equal to 0 that gives $\sum_i^n \lambda_i (T(v_i)) = \overrightarrow{0}$.

7. a) these two operations are closed under the set of all maps between X and W.
      i. $(f + g)(x) = f(x) + g(x) = g(x) + f(x) = (g + f)(x)$ addition is communitivitive.
      ii. $(f + (g + h))(x) = f(x) + (g(x) + h(x)) = (f(x) + g(x)) + h(x) = ((f + g) + h)(x)$ addition is associative.
      iii. Consider $f(x) = 0$, $(f + g)(x) = f(x) + g(x) = 0 + g(x) = g(x)$ zero vector do exisist.
      iv. $\forall f \in$MAP(X, W), $-f(x) + f(x) = \overrightarrow{0}$ for all vector do exist a negative vecotr.

v. Consider $\lambda = 1$ $1f(x) = f(x)$, multiplicative identity exists.

vi. $\lambda, \mu \in \mathbb{R}$, $\lambda(\mu f(x)) = (\mu\lambda)f(x)$

vii. $\lambda \in \mathbb{R}$, $\lambda(f+g)(x) = \lambda(f(x)+g(x)) = \lambda f(x) + \lambda g(x) = (\lambda f + \lambda g)(x)$

viii. $\lambda, \mu \in \mathbb{R}$, $(\lambda+\mu)f(x) = \lambda f(x) + \mu f(x) = (\lambda f + \mu f)(x)$

Thus it is a vector space.

b) $\forall$ linear maps $f(x), g(x)$, $(f+g)(\lambda x) = f(\lambda x) + g(\lambda x) = \lambda f(x) + \lambda g(x) = \lambda(f+g)(x)$, thus, addition is closed. Consider $\lambda, \mu \in \mathbb{R}$, $(\lambda f)(\mu x) = \lambda f(\mu x) = \mu\lambda f(x)$. Thus scaler multiplication is also closed. The other eight properties were checked in part a), thus, it is subspace.

8. $\forall w \in \mathbb{R}^3$, $w = \lambda_1 e_1 + \lambda_2 e_2 + \lambda_3 e_3$ where e_i are basis in $\mathbb{R}^3$. Thus $\forall v \in$Hom$(\mathbb{R}^3, \mathbb{R})$, $v = f(w) = f(\lambda_1 e_1 + \lambda_2 e_2 + \lambda_3 e_3)$ Since $f(x)$ is linear, then $f(\lambda_1 e_1 + \lambda_2 e_2 + \lambda_3 e_3) = \lambda_1 f(e_1) + \lambda_2 f(e_2) + \lambda_3 f(e_3)$. Consider 3 linear maps l_1, l_2, l_3 in the prompt. For standard basis, e_1, e_2, e_3, $w = \lambda_1 e_1 + \lambda_2 e_2 + \lambda_3 e_3$. $l_1(w) = \lambda_1$, $l_2(w) = \lambda_2$, $l_3(w) = \lambda_3$, Thus, $f(w) = l_1(w)f(e_1) + l_2(w)f(e_2) + l_3(w)f(e_3)$. Thus, as $f(e_i) \in \mathbb{R}$, l_1, l_2, l_3 do span Hom$(\mathbb{R}^3, \mathbb{R})$.
Next, prof that the set is linearly independent. Suppose they are not, then exists $\lambda_1, \lambda_2, \lambda_3 \in F$ not all zero, while $\lambda_1 l_1 + \lambda_2 l_2 + \lambda_3 l_3 = \overrightarrow{0}$ as $\overrightarrow{0} = (f(w) = 0)$, $\forall w \in \mathbb{R}^3$. As w is a random vector is $\mathbb{R}^3$, $\lambda_1 l_1 + \lambda_2 l_2 + \lambda_3 l_3 = \lambda_1 x_1 + \lambda_2 x_2 + \lambda_3 x_3 = 0$. However, does not exist such combinations of set of λ_i Therefore, they are linear independent.
Thus, the set of maps is a set of basis for Hom$(\mathbb{R}^3, \mathbb{R})$