MATH 461 Homework 4

James Liu

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4.72

$$P(i=4) = \binom{4}{4} \times (0.6)^4 = 0.1296$$
$$P(i=5) = (\binom{5}{4} - \binom{4}{4}) \times (0.6)^4(0.4) = 0.2592$$
$$P(i=5) = (\binom{6}{4} - \binom{5}{4}) \times (0.6)^4(0.4)^2 = 0.2903$$
$$P(i=5) = (\binom{7}{4} - \binom{5}{4}) \times (0.6)^4(0.4)^3 = 0.28201$$
$$P(2\text{in}2) = 0.6^2 = 0.36$$
$$P(2\text{in}3) = (\binom{3}{2} - \binom{2}{2})0.6^2 0.4 = 0.288$$
$$P(4win) = \sum P(i) = 0.9092 > 0.648 = P(2win)$$

The stronger team has a high rate of wining when it needs to win 4 times compared with 2. Subtraction is due to that it counts the combinations that the games ends before i.

4.73

$$P(4) = \binom{2}{1}\binom{4}{4}0.5^4 0.5^0 = 0.125$$
$$P(5) = \binom{2}{1}(\binom{5}{4} - \binom{4}{4})0.5^4 05^1 = 0.25$$
$$P(6) = \binom{2}{1}(\binom{6}{4} - \binom{5}{4})0.5^4 05^2 = 0.3125$$
$$P(6) = \binom{2}{1}(\binom{7}{4} - \binom{6}{4})0.5^4 05^3 = 0.3125$$
$$E(X) = 4P(4) + 5P(5) + 6P(6) + 7P(7) = 5.8125$$

4.77 For combinations of first $N - (N - K)$ experiment, the last trial must be from the box needed to be emptied, thus, finding combination like

$\binom{2n-k-1}{n-k}$ give all the possible ways of arranging such experiments. And there are 2 boxes, the answer should be:

$$2 \cdot \binom{2n-k-1}{n-k}(0.5^{2n-k})$$

4.78

$$P(win) = \binom{4}{2}\frac{4 \cdot 3 \cdot 4 \cdot 3}{8 \cdot 7 \cdot 6 \cdot 5} = \frac{18}{35} = 0.514286$$
$$P(n) = (1 - P(win))^{n-1}P(win) = \left(\frac{17}{35}\right)^{n-1}\frac{18}{35}$$

4.79 a)

$$P(X = 0) = \frac{\binom{94}{10}}{\binom{100}{10}} = 0.5223$$

b)

$$P(X = 1) = \frac{\binom{94}{9}\binom{6}{1}}{\binom{100}{10}} = 0.3686$$
$$P(X = 2) = \frac{\binom{94}{8}\binom{6}{2}}{\binom{100}{10}} = 0.096458$$
$$P(X > 2) = 1 - P(X = 0, 1, 2) = 1 - 0.5223 - 0.3686 - 0.0964 = 0.012551$$

4.84 a) For each box, the probability of it being empty after 10 balls is $P_i(empty) = (1 - P_i)^{10}$
Thus $E(X) = \sum_{i=1}^{5} 1 \times (1 - p_i)^{10}$

b) Similarly: $E(X) = \sum_{i=1}^{5} \binom{10}{1}(1 - p_i)^9 p_i$

4.85 $E(X) = \sum_{i=1}^{k} 1 - (1 - p_i)^n$

5.1 $f(x)$ is a probability distribution function then $\int f \mathrm{d}x = 1$

a)

$$\int_{-1}^{1} c(1 - x^2)\mathrm{d}x = 1$$
$$x - \frac{1}{3}x^3\Big|_{-1}^{1} = \frac{1}{c}$$
$$c = \frac{3}{4}$$

b)

$$\int_{-1}^{x} \frac{3}{4}(1-s^2)\mathrm{d}s = \frac{3}{4}\left.(s-\frac{1}{3}s^3)\right|_{-1}^{x}$$
$$= \frac{3}{4}(x-\frac{1}{3}x^3) - \frac{3}{4}(-1+\frac{1}{3})$$
$$= \frac{3}{4}(-\frac{1}{3}x^3 + x + \frac{2}{3})\ (-1 < x < 1)\text{otherwise } 0$$

5.2

$$\int_{0}^{\infty} Cxe^{-x/2}\mathrm{d}x = 1$$
$$\left.\Big((-2x-4)e^{-x/2}\Big)\right|_{0}^{\infty} = \frac{1}{c}$$
$$0-(0-4) = \frac{1}{c}$$
$$c = \frac{1}{4}$$
$$\int_{0}^{5} \frac{xe^{-x/2}}{4} = \frac{1}{4}\left.\Big((-2x-4)e^{-x/2}\Big)\right|_{0}^{5}$$
$$= 0.7127$$

5.4 a)

$$\int_{20}^{\infty} \frac{10}{x^2}\mathrm{d}x = -\left.\frac{10}{x}\right|_{20}^{\infty}$$
$$P(X > 20) = 0.5$$

b)

$$\int_{10}^{x} \frac{10}{s^2}\mathrm{d}s = -\left.\frac{10}{s}\right|_{10}^{x}$$
$$= 1 - \frac{10}{x}(10 < x)\text{otherwise } 0$$

c) If assume that the time of functioning is independent, then we have:

$$P(X > 15) = 0 - \frac{10}{15} = \frac{2}{3}$$

$$P(X \geq 3) = \binom{6}{3}(\frac{1}{3})^3(\frac{2}{3})^3 + \binom{6}{4}(\frac{1}{3})^2(\frac{2}{3})^4 + \binom{6}{5}(\frac{1}{3})^1(\frac{2}{3})^5 + \binom{6}{6}(\frac{1}{3})^0(\frac{2}{3})^6 = 0.89986$$

5.5

$$
\begin{aligned}
\int_0^x 5(1-s)^4 \mathrm{d}s &= 0.99 \\
-(1-s)^5|_0^x &= 0.99 \\
1-(1-x)^5 &= 0.99 \\
x &= 0.6018
\end{aligned}
$$

Thus, 601.8 gallons.