MATH 416H HW 1

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1. **i.** True.

Functions folowing natural addition and multiplication is closed under the two operations while is also naturally follows the following 8 properties.

a. + is communitive, $\overrightarrow{x} + \overrightarrow{y} = \overrightarrow{y} + \overrightarrow{x}$

b. + is associative, $\overrightarrow{x} + (\overrightarrow{y} + \overrightarrow{z}) = (\overrightarrow{x} + \overrightarrow{y}) + \overrightarrow{z}$

c. $\exists\, \overrightarrow{0} \in V$ such that $\forall \overrightarrow{v} \in V,\ \overrightarrow{0} + \overrightarrow{v} = \overrightarrow{v}$

d. $\forall \overrightarrow{v} \in V, \exists - \overrightarrow{v}$

e. $\forall v \in V,\ 1 \times \overrightarrow{v} = \overrightarrow{v}$

f. $\forall \lambda, \mu \in \mathbb{R},\ \lambda(\mu \cdot \overrightarrow{v}) = (\lambda\mu) \cdot \overrightarrow{v}$

g. $\forall \lambda \in \mathbb{R},\ \lambda(\overrightarrow{v} + \overrightarrow{w}) = \lambda\overrightarrow{v} + \lambda\overrightarrow{w}$

h. $\forall \lambda, \mu \in \mathbb{R},\ (\lambda + \mu)\overrightarrow{v} = \lambda\overrightarrow{v} + \mu\overrightarrow{v}$

ii. False.

Consider $F(x) = 1$, $-1 \cdot F(x) = -1$, it is not close under scaler multiplication.

iii. False.

Consider $F_1(x) = x^2 + x^3$ and $F_2(x) = x^2 - x^3$ as both of them are at degree 3. However, $F_1 + F_2 = 2x^2$ is nolonger degree 3 which means it is not close under addition and thus not a vector space.

2. Apply the Gaussian illimination:

$$\left[\begin{array}{ccc|c} 1 & 3 & 2 & 2 \\ 1 & 6 & 1 & 3 \\ 2 & 3 & 5 & 5 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 3 & 2 & 2 \\ 0 & 3 & -1 & 1 \\ 0 & -3 & 1 & 1 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 3 & 2 & 2 \\ 0 & 3 & -1 & 1 \\ 0 & 0 & 0 & 2 \end{array}\right]$$

Therefore, there is no solution

3. Yes, it is linear.
Assume two different maps $f(x)$ and $g(x)$,

$$\begin{aligned}T(\lambda f + \mu g) &= \int_1^2 \lambda f(x) + \mu g(x)\ dx \\ &= \int_1^2 \lambda f(x)\ dx + \int_1^2 \mu g(x)\ dx \\ &= \lambda \int_1^2 f(x)\ dx + \mu \int_1^2 g(x)\ dx \\ &= \lambda T(f) + \mu T(g)\end{aligned}$$

Thus, it is linear.

4.

a. $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 13 \\ 31 \end{bmatrix}$

b. $\begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 7 \\ 3 \\ 2 \end{bmatrix}$

c. $\begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 5 \\ 8 \\ 11 \\ 4 \end{bmatrix}$

5.

a. Apply Gaussian illimination

$$\left[\begin{array}{cccc|c} 1 & 2 & -1 & 2 & 3 \\ 3 & 7 & 0 & 5 & 8 \\ -1 & 0 & 7 & -2 & -1 \end{array}\right] \to \left[\begin{array}{cccc|c} 1 & 2 & -1 & 2 & 3 \\ 0 & 1 & 3 & -1 & -1 \\ 0 & 2 & 6 & 0 & 2 \end{array}\right] \to$$

$$\left[\begin{array}{cccc|c} 1 & 0 & -7 & 4 & 5 \\ 0 & 1 & 3 & -1 & -1 \\ 0 & 0 & 0 & 2 & 4 \end{array}\right] \to \left[\begin{array}{cccc|c} 1 & 0 & -7 & 0 & -3 \\ 0 & 1 & 3 & 0 & 1 \\ 0 & 0 & 0 & 1 & 2 \end{array}\right]$$

Therefore:

$$\begin{cases} x_1 \quad - 7x_3 = -3 \\ x_2 + 3x_3 = 1 \\ x_4 = 2 \end{cases}$$

Thus: $\begin{cases} x_1 = -3 + 7s \\ x_2 = 1 - 3s \\ x_3 = s \\ x_4 = 2 \end{cases}$

b. Apply Gaussian illimination

$$\left[\begin{array}{cccc|c} 1 & 2 & -1 & 2 & a \\ 3 & 7 & 0 & 5 & b \\ -1 & 0 & 7 & -2 & c \end{array}\right] \to \left[\begin{array}{cccc|c} 1 & 2 & -1 & 2 & a \\ 0 & 1 & 3 & -1 & b-3a \\ 0 & 2 & 6 & 0 & c+a \end{array}\right] \to$$

$$\left[\begin{array}{cccc|c} 1 & 0 & -7 & 4 & 7a-2b \\ 0 & 1 & 3 & -1 & b-3a \\ 0 & 0 & 0 & 2 & c-2b+7a \end{array}\right] \to \left[\begin{array}{cccc|c} 1 & 0 & -7 & 0 & -7a+2b-2c \\ 0 & 1 & 3 & 0 & \frac{1}{2}c+\frac{1}{2}a \\ 0 & 0 & 0 & 1 & \frac{1}{2}c-1b+\frac{7}{2}a \end{array}\right]$$

Thus:

$$\begin{cases} x_1 = -7a + 2b - 2c + 7s \\ x_2 = \frac{1}{2}c + \frac{1}{2}a - 3s \\ x_3 = s \\ x_4 = \frac{1}{2}c - 1b + \frac{7}{2}a \end{cases}$$

Therefore, changing the right hand side will still give a solution.

6. quiz question

Let V be a vector space containing all C^0(contiuous) functions. Try prof that $F(\overrightarrow{x}) = 3\overrightarrow{x} + 1$ is not linear for $x \in V$