MATH 416H HW 3

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1. $\forall W \subseteq V$ that W is a subspace containing S, (span S) $\subseteq W$ as W is itself a vector space. Thus, (span S) $\subseteq \bigcap_i W_i$. Suppose it exisist such $v \in V$, that $v \notin$ span S, while $v \in \bigcap_i W_i$. Consider a subspace span $S \subseteq V$ while S$\subseteq$span S. Thus, span $S = W_i$ for some i. Thus, by definition of intersections, such vector does not exist, thus, $\bigcap_i W_i \subseteq$ span S Thus, the span of S equals the intersection of all the subspaces of V containing S.

2. For the 2 vectors to be linear independent, the only solution to $\begin{cases} \lambda_1 + \lambda_2 = 0 \\ x\lambda_1 + y\lambda_2 = 0 \end{cases}$ should be $\begin{cases} \lambda_1 = 0 \\ \lambda_2 = 0 \end{cases}$. for $\lambda_1 + \lambda_2 = 0$, there could be $\lambda_1 = \lambda_2 = 0$ or $\lambda_1 = -\lambda_2$ As the vectors should be linear independent, then, $-\lambda_1 x + \lambda_2 y \neq 0$, thus $x \neq y$.

   For vectors with $x \neq y$, there is $\begin{cases} \lambda_1 + \lambda_2 = 0 \\ x\lambda_1 + y\lambda_2 = 0 \end{cases}$ with only one set of solution of $\lambda_1 = \lambda_2 = 0$, Thus they are linearly independent with condition $x \neq y$.

3. i: $\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$ is one of the vectors that can form a basis. suppose they are linear dependent, then exisist $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{R}$, that $\lambda_1 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + \lambda_2 \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + \lambda_3 \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ while $\exists \lambda_i \neq 0$ however $\begin{cases} \lambda_2 + 2 \times \lambda_3 = 0 \\ \lambda_2 + \lambda_3 = 0 \end{cases}$ Showing that $\lambda_2 = \lambda_3 = 0$ and thus, as $\lambda_1 + 0 + 0 = 0$ $\lambda_1 = 0$, Thus they are linear independent, as there are 3 elements while the dimension of $\mathbb{R}^3$ is also 3, it is a basis.

   ii: $\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$ is one of the vectors that can form a basis. suppose they are linear dependent, then exisist $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{R}$, that $\lambda_1 \begin{pmatrix}0\\0\\1\end{pmatrix} + \lambda_2 \begin{pmatrix}1\\1\\0\end{pmatrix} + \lambda_3 \begin{pmatrix}1\\0\\1\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}$

while $\exists \lambda_i \neq 0$ however $\begin{cases} \lambda_2 + \lambda_3 = 0 \\ \lambda_2 = 0 \end{cases}$ Showing that $\lambda_2 = \lambda_3 = 0$ and thus, as $\lambda_1 + 0 + 0 = 0$ $\lambda_1 = 0$, Thus they are linear independent, as there are 3 elements while the dimension of $\mathbb{R}^3$ is also 3, it is a basis.

4. As $x + y + z = \overrightarrow{0}$, thus, $z = -x - y$. In $\forall v \in$span $\{y, z\}$ $\exists \lambda_1, \lambda_2 \in F$ that $v = \lambda_1 y + \lambda_2 z = \lambda_1 y + \lambda_2(-x - y) = (\lambda_1 - \lambda_2)y - \lambda_2 x$, $(\lambda_1 - \lambda_2 \in F)$, Thus v is also in span $\{x, y\}$. Thus, span $\{x, y\}$ =span $\{y, z\}$

5.

6. $\forall w \in R(T)$, $\exists \lambda_i \in F$ that $w = T(\sum_i^n \lambda_i v_i)$. As the map is linear, there is $T(\sum_i^n \lambda_i v_i) = \sum_i^n \lambda_i (T(v_i))$, Thus, $w \in$span $T(v_i)$, Also, as profed in class, $T(\overrightarrow{0}) = \overrightarrow{0}$ for linear transformations, and as $T(v)$ is , then $T(\sum_i^n \lambda_i v_i) = \overrightarrow{0}$ only when $\sum_i^n \lambda_i v_i = \overrightarrow{0}$ which is that all $\lambda_i = 0$. Therefore, $\sum_i^n \lambda_i (T(v_i)) = \overrightarrow{0}$ only when all $\lambda_i = 0$. Thus, $T(v_i)$ is linear independent. Thus it is one of the basis of $R(T)$. If the mapping not injective, meaning that $T(v_i)$ is not nessesarily linear independent as there might be some other vectors in the null space of T, leading to a set of λ_i not all equal to 0 that gives $\sum_i^n \lambda_i (T(v_i)) = \overrightarrow{0}$.

7. a) these two operations are closed under the set of all maps between X and W.

   i. $(f + g)(x) = f(x) + g(x) = g(x) + f(x) = (g + f)(x)$ addition is communitivitive.

   ii. $(f + (g + h))(x) = f(x) + (g(x) + h(x)) = (f(x) + g(x)) + h(x) = ((f + g) + h)(x)$ addition is associative.

   iii. Consider $f(x) = 0$, $(f + g)(x) = f(x) + g(x) = 0 + g(x) = g(x)$ zero vector do exisist.

   iv. $\forall f \in$MAP(X, W), $-f(x) + f(x) = \overrightarrow{0}$ for all vector do exist a negative vecotr.

   v. Consider $\lambda = 1$ $1f(x) = f(x)$, multiplicative identity exists.

   vi. $\lambda, \mu \in \mathbb{R}$, $\lambda(\mu f(x)) = (\mu\lambda) f(x)$

   vii. $\lambda \in \mathbb{R}$, $\lambda(f + g)(x) = \lambda(f(x) + g(x)) = \lambda f(x) + \lambda g(x) = (\lambda f + \lambda g)(x)$

   viii. $\lambda, \mu \in \mathbb{R}$, $(\lambda + \mu) f(x) = \lambda f(x) + \mu f(x) = (\lambda f + \mu f)(x)$

   Thus it is a vector space.

   b) $\forall$ linear maps $f(x), g(x)$, $(f + g)(\lambda x) = f(\lambda x) + g(\lambda x) = \lambda f(x) + \lambda g(x) = \lambda(f + g)(x)$, thus, addition is closed. Consider $\lambda, \mu \in \mathbb{R}$, $(\lambda f)(\mu x) = \lambda f(\mu x) = \mu\lambda f(x)$. Thus scaler multiplication is also closed. The other eight properties were checked in part a), thus, it is subspace.

8. $\forall f \in$Hom $(\mathbb{R}^3, \mathbb{R})$,