MATH 416H HW 10

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1.

$$\begin{aligned}
\det(\bar{A})_{ij} &= \sum_{\sigma \in S_n} (\text{sign}(\sigma))\bar{a}_{\sigma(1)1} \cdots \bar{a}_{\sigma(n)n} \\
&= \sum_{\sigma \in S_n} (\text{sign}(\sigma))\overline{a_{\sigma(1)1} \cdots a_{\sigma(n)n}} \\
&= \sum_{\sigma \in S_n} \overline{(\text{sign}(\sigma))a_{\sigma(1)1} \cdots a_{\sigma(n)n}} \\
&= \overline{\sum_{\sigma \in S_n} (\text{sign}(\sigma))a_{\sigma(1)1} \cdots a_{\sigma(n)n}} \\
&= \overline{\det(A)}
\end{aligned}$$

2. a)

$$\begin{aligned}
b^{\#}((1,0)^T) &= b((1,0)^T, (x,y)^T) \\
&= x - 0y = e_1^*
\end{aligned}$$

Which is $\forall v \in \mathbb{R}^2, v = \mu_1 e_1 + \mu_2 e_2$

$$\begin{aligned}
b^{\#}((0,1)^T)(v) &= \mu_1 b^{\#}(e_1)(e_1) + \mu_2 b^{\#}(e_1)(e_2) \\
&= \mu_1 b^{\#}(e_1)(e_1) \\
&= \mu_1(1-0) \\
&= \mu_1
\end{aligned}$$

There fore, by definition, it is e_1^*. similarly:

$$\begin{aligned}
b^{\#}((0,1)^T) &= b((0,1)^T, (x,y)^T) \\
&= 0x - y = -e_2^*
\end{aligned}$$

$$\begin{aligned}
b^{\#}((0,1)^T)(v) &= \mu_1 b^{\#}(e_2)(e_1) + \mu_2 b^{\#}(e_2)(e_2) \\
&= -\mu_2 b^{\#}(e_2)(e_2) \\
&= -\mu_2(1) \\
&= -\mu_2
\end{aligned}$$

Thus, as $\{e_1, e_2\}$ is a set of standard basis, the $\{e_1^*, e_2^*\}$ is the dual basis and there is a isomorphism.

b)

$$
\begin{aligned}
v &= (1,1)^T \\
b((1,1)^T, (1,1)^T) &= 1 - 1 = 0 \\
v' &= (1,0) \\
b(v', v') &= 1 - 0 = 1 > 0 \\
v'' &= (0,1) \\
b(v'', v'') &= 0 - 1 = -1 < 0
\end{aligned}
$$

3. a) As it is spaned by 3 vectors and they are linear independent, $\dim(E) = 3$, consider $f_1(x) = \frac{1}{\sqrt{2}}$, then we have:

$$
\begin{aligned}
\langle f_1, f_1 \rangle &= \int_{-1}^{1} \frac{1^2}{\sqrt{2}^2} \mathrm{d}t \\
&= \frac{1}{2} - \left(-\frac{1}{2}\right) \\
&= 1
\end{aligned}
$$

to find orthongonal vectors to f_1, $\langle f_1, f_2 \rangle = 0$ should hold. Which is:

$$
\begin{aligned}
\int_{-1}^{1} \frac{1}{\sqrt{2}} f_2(t) \mathrm{d}t &= 0 \\
\text{Consider } f_2(t) &= \frac{\sqrt{3}}{\sqrt{2}} t \\
\text{Then } \int_{-1}^{1} \frac{1}{\sqrt{2}} f_2(t) \mathrm{d}t &= \int_{-1}^{1} \frac{1}{\sqrt{2}} \frac{\sqrt{3}}{\sqrt{2}} t \mathrm{d}t \\
&= 0 \\
\langle f_2(t), f_2(t) \rangle &= \int_{-1}^{1} \frac{3}{2} t^2 \mathrm{d}t \\
&= \frac{1}{2} t^3 \Big|_{-1}^{1} \\
&= \frac{1}{2} - \left(-\frac{1}{2}\right) \\
&= 1
\end{aligned}
$$

Set $f_3 = a + bt + ct^2$, for some $a, b, c \in \mathbb{R}$ then, for f_1:

$$
\begin{aligned}
&\int_{-1}^{1} \frac{1}{\sqrt{2}}a + \frac{1}{\sqrt{2}}bt + \frac{1}{\sqrt{2}}ct^2 \mathrm{d}t = 0 \\
&\Rightarrow \int_{-1}^{1} \frac{1}{\sqrt{2}}a\mathrm{d}t + \int_{-1}^{1} \frac{1}{\sqrt{2}}bt\mathrm{d}t + \int_{-1}^{1} \frac{1}{\sqrt{2}}ct^2\mathrm{d}t = 0 \\
&\Rightarrow \sqrt{2}a + 0 + \frac{\sqrt{2}}{3}c = 0 \\
&\Rightarrow c = -3a
\end{aligned}
$$

For f_2:

$$
\begin{aligned}
&\int_{-1}^{1} \sqrt{\frac{3}{2}}at + \sqrt{\frac{3}{2}}bt^2 + \sqrt{\frac{3}{2}}ct^3\mathrm{d}t \\
&\Rightarrow 0 + \frac{\sqrt{6}}{3}b + 0 = 0 \\
&\Rightarrow b = 0
\end{aligned}
$$

For itself:

$$
\begin{aligned}
&\int_{-1}^{1} (a - 3at^2)^2\mathrm{d}t = 1 \\
&\Rightarrow \int_{-1}^{1} a^2 - 6a^2t^2 + 9a^2t^4\mathrm{d}t = 1 \\
&\Rightarrow 2a^2 - 4a^2 + \frac{18}{5}a^2 = 1 \\
&\Rightarrow \frac{8}{5}a^2 = 1 \\
&\Rightarrow a = \sqrt{\frac{5}{8}}
\end{aligned}
$$

Therefore, $f_3(t) = \sqrt{\frac{5}{8}} - 3 \cdot \sqrt{\frac{5}{8}}t^2$
Therefore, $\{f_1, f_2, f_3\}$ where:

$$
\begin{aligned}
f_1(t) &= \frac{1}{\sqrt{2}} \\
f_2(t) &= \sqrt{\frac{3}{2}}t \\
f_3(t) &= \sqrt{\frac{5}{8}} - 3 \cdot \sqrt{\frac{5}{8}}t^2
\end{aligned}
$$

is a set of orthonormal basis.

b) define $f'(t)$ as the projection, then:

$$
\begin{aligned}
f' &= f_1\langle f_1, f'\rangle + f_2\langle f_2, f'\rangle + f_3\langle f_3, f'\rangle \\
f_1\langle f_1, f'\rangle &= \int_{-1}^{1} \frac{1}{\sqrt{2}}|t|\mathrm{d}t \\
&= \frac{1}{\sqrt{2}}\left(\int_0^1 \frac{1}{\sqrt{2}}t\mathrm{d}t - \int_{-1}^{0} \frac{1}{\sqrt{2}}t\mathrm{d}t\right) \\
&= \frac{1}{\sqrt{2}}\left(\frac{1}{2\sqrt{2}} + \frac{1}{2\sqrt{2}}\right) \\
&= \frac{1}{2} \\
f_2\langle f_2, f'\rangle &= \sqrt{\frac{3}{2}}t\left(\int_{-1}^{1} \sqrt{\frac{3}{2}}t|t|\mathrm{d}t\right) \\
&= \sqrt{\frac{3}{2}}t\left(\int_0^1 \sqrt{\frac{3}{2}}t^2\mathrm{d}t - \int_{-1}^{0} \sqrt{\frac{3}{2}}t^2\mathrm{d}t\right) \\
&= 0 \\
f_3\langle f_3, f'\rangle &= \left(\sqrt{\frac{5}{8}} - 3\cdot\sqrt{\frac{5}{8}}t^2\right)\left(\int_{-1}^{1}\left(\sqrt{\frac{5}{8}} - 3\cdot\sqrt{\frac{5}{8}}t^2\right)|t|\mathrm{d}t\right) \\
&= \left(\sqrt{\frac{5}{8}} - 3\cdot\sqrt{\frac{5}{8}}t^2\right)\left(\int_{0}^{1}\left(\sqrt{\frac{5}{8}} - 3\cdot\sqrt{\frac{5}{8}}t^2\right)t\mathrm{d}t - \int_{-1}^{0}\left(\sqrt{\frac{5}{8}} - 3\cdot\sqrt{\frac{5}{8}}t^2\right)t\mathrm{d}t\right) \\
&= \left(\sqrt{\frac{5}{8}} - 3\cdot\sqrt{\frac{5}{8}}t^2\right)\left(-\frac{\sqrt{10}}{8}\right) \\
&= -\frac{5}{16} + \frac{15}{16}t^2 \\
f' &= \frac{3}{16} + \frac{15}{16}t^2
\end{aligned}
$$

4. a) Note that $\dim(V) = j, \dim(E) = i, 0 < i < j$. Say that $\{b_1, \cdots, b_j\}$ is a set of orthonormal basis for V and $b_1, \cdots, b_i$ is a set of orthonormal basis for E, and P_E as $f_E(v)$. Then, $\forall v \in V$, $v = \mu_1 b_1 + \cdots + \mu_j b_j$. and $f_E(v) = \mu_1 b_1 + \cdots + \mu_i b_i$. Therefore, by definition of eigenvalue and eigenvector, there are 2 eigenvalues and j eigenvectors.

$$
\begin{gathered}
\lambda_1 = 1 \text{ with } \{b_1, \cdots, b_i\} \text{ as eigenvectors} \\
\lambda_2 = 0 \text{ with } \{b_{j-i}, \cdots, b_j\} \text{ as eigenvectors}
\end{gathered}
$$

b) in the coordinate system discribed above, the matrix would be $j \times j$ with only values on its main diagonal and first i rows of its main diagonal will be 1 and rest will be zeros. Therefore, $\mathrm{Tr}(P_E) = i = \dim(E)$

5. Decompose that $v_\parallel, w_\parallel \in R(P)$, $v_\perp, w_\perp \in N(P)$ and $v = v_\parallel + v_\perp, w = w_\parallel + w_\perp$. Then:

$$\begin{aligned}
\langle P(v), w\rangle &= \langle v_\parallel, w_\parallel + w_\perp\rangle \\
&= \langle v_\parallel, w_\parallel\rangle + \langle v_\parallel, w_\perp\rangle \\
&= \langle v_\parallel, w_\parallel\rangle + 0 \\
\langle w, P(v)\rangle &= \langle v_\parallel + v_\perp, w_\parallel\rangle \\
&= \langle v_\parallel, w_\parallel\rangle
\end{aligned}$$

Therefore, the claim is true.

6. $\forall A, B \in U(n)$

$$\begin{aligned}
(AB)(AB)^* &= ABB^*A^* \\
&= AIA^* \\
&= I
\end{aligned}$$

Therefore, $\forall A, B \in U(n), AB \in U(n)$
$I \in U(n)$ as $II^* = I$
For $A \in U(n)$,$A^{-1} = A^*$ and $(A^*)^*A^* = AA^* = I$ Thus $A^* = A^{-1} \in U(n)$
Thus, it is a group.

7. $\forall A, B \in O(n)$, $(AB)(AB)^T = ABB^TA^T = AIA^T = I$ Thus, $\forall A, B \in O(n), AB \in O(n)$, Also, $II^T = I$, $I \in O(n)$. $\forall A \in O(n)$, $A^TA = I$ and $A^T(A^T)^T = A^TA = I$, therefore $A^{-1} \in O(n)$ Thus, $O(n)$ is a group.

8.

$$\begin{aligned}
\rho(\sigma)_{ij} &= \delta^j_{\sigma(i)} \\
(\rho(\sigma)^T)_{ij} &= \delta^i_{\sigma(j)} \\
(\rho(\sigma)\rho(\sigma)^T)_{ij} &= \sum_{k=1} \delta^k_{\sigma(i)}\delta^i_{\sigma(k)} \\
&= \delta^k_k \\
&= I
\end{aligned}$$

Therefore, they are orthongonal.