MATH 416H HW 5

James Liu

Due: Oct 2 Edit: September 30, 2024

1.

$$\left(\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 3 & 7 & 3 & 0 & 1 & 0 \\ 2 & 3 & 4 & 0 & 0 & 1 \end{array}\right) \to \left(\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -3 & 1 & 0 \\ 0 & -1 & 2 & -2 & 0 & 1 \end{array}\right) \to$$

$$\left(\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -3 & 1 & 0 \\ 0 & 0 & 2 & -5 & 1 & 1 \end{array}\right) \to \left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 7 & -2 & 0 \\ 0 & 1 & 0 & -3 & 1 & 0 \\ 0 & 0 & 1 & -2.5 & 1 & 1 \end{array}\right) \to$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 9.5 & -2.5 & -0.5 \\ 0 & 1 & 0 & -3 & 1 & 0 \\ 0 & 0 & 1 & -2.5 & 0.5 & 0.5 \end{array}\right)$$

Thus $A^{-1} = \begin{pmatrix} \frac{19}{2} & -\frac{5}{2} & -\frac{1}{2} \\ -3 & 1 & 0 \\ -\frac{5}{2} & \frac{1}{2} & \frac{1}{2} \end{pmatrix}$

2. A) As profed in class, $([id]_{\mathcal{SB}})^{-1} = [id]_{\mathcal{BS}}$. Thus:

$$([id]_{\mathcal{SB}})^{-1}[T]_{\mathcal{SS}}[id]_{\mathcal{SB}} = [id]_{\mathcal{BS}}[T]_{\mathcal{SS}}[id]_{\mathcal{SB}}$$

Say that $\forall v \in \mathbb{R}^n,\ T(v) = v'$, Then $[T]_{\mathcal{BB}}[v]_{\mathcal{BB}} = [v']_{\mathcal{BB}}$. We have:

$$\begin{aligned} [id]_{\mathcal{BS}}[T]_{\mathcal{SS}}[id]_{\mathcal{SB}}[v]_{\mathcal{B}} &= [id]_{\mathcal{BS}}[T]_{\mathcal{SS}}[v]_{\mathcal{S}} \\ &= [id]_{\mathcal{BS}}[v']_{\mathcal{S}} \\ &= [v']_{\mathcal{B}} \end{aligned}$$

Thus, $[T]_{\mathcal{BB}} = ([id]_{\mathcal{SB}})^{-1}[T]_{\mathcal{SS}}[id]_{\mathcal{SB}}$

b) let $\mathcal{S}$ be the standard basis. Then:

$$\begin{aligned}
[T]_{\mathcal{BB}} &= \left([id]_{\mathcal{SB}}\right)^{-1}[T]_{\mathcal{SS}}[id]_{\mathcal{SB}} \\
&= \left(\begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}\right)^{-1}\begin{pmatrix} 3 & 1 \\ 1 & -2 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} \\
&= \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 3 & 1 \\ 1 & -2 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} \\
&= \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 4 & 5 \\ -1 & -3 \end{pmatrix} \\
&= \begin{pmatrix} 9 & 13 \\ -5 & -8 \end{pmatrix}
\end{aligned}$$

3. $\forall u \in U,\ u = \lambda_1 b_1 + \cdots + \lambda_n b_n$. Then $T(u) = T(\lambda_1 b_1 + \cdots + \lambda_n b_n)$, then due to linearity, $T(u) = \lambda_1 T(b_1) + \cdots + \lambda_n T(b_n)$. Thus, the set $\{T(b_1), \cdots, T(b_n)\}$ spans $\{T(u) |\ \forall u \in U\}$.

   i: The set $\{T(b_1), \cdots, T(b_n)\}$ is linearly independent, then the claim is right.

   ii: The set is not linear independent. Meaning that $\exists \mu_1, \cdots, \mu_n$ that $\mu_1 T(b_1) + \cdots + \mu_n T(b_n) = 0$, then do a operation that subsitutes $T(b_n)$ with $-(\mu_1 T(b_1) + \cdots + \mu_n T(b_{n-1}))$ in $T(u) = \lambda_1 T(b_1) + \cdots + \lambda_n T(b_n)$. We can do this operation till the remaining set is linear independent which shows that the original claim was right.

   Thus $\{T(b_1), \cdots, T(b_n)\}$ do have a subset that can serve as a base for $T(U)$.

4. Suppose that there exists a inverse T^{-1}, then:

$$\begin{aligned}
T^{-1} \circ T \circ T &= 0 \\
T &= 0
\end{aligned}$$

   $T = 0$ is not invertable as it is obviously not injection, which raises a contradiction, meaning T is not invertable.

5. i: Suppose $\exists v \in V$ that $v \in N(P),\ v \in R(P)$, then $P(v) = \overrightarrow{0}$ as $v \in N(T)$, also as $P(P(v)) = P(v)$ as stated. Due to P is linear, $P(\overrightarrow{0}) = \overrightarrow{0}$. Thus, $\forall w \in N(P)$. $P(w) = \overrightarrow{0}$.
   Thus, $N(P) \cap R(P) = \{\overrightarrow{0}\}$

   ii:

$$\begin{aligned}
\dim(N(P) + R(P)) &= \dim(N(P)) + \dim(R(P)) - \dim(R(P) \cap N(P)) \\
&= \dim(N(P)) + \dim(R(P)) - \dim(\{\overrightarrow{0}\}) \\
&= \dim(N(P)) + \dim(R(P))
\end{aligned}$$

As $R(P) = \dim(V) - N(P)$ from rank/nullity therom,
$\dim(V) = \dim(R(P)) + \dim(N(T))$. As null space and range are all vector spaces, then there exists sets of basis: $\{n_i\}$ and $\{r_i\}$ spanning $N(P)$ and $R(T)$ expanding such basis into $\dim(V)$, then they must be linear independent, as the intercetion only contains $\overrightarrow{0}$. Suppose it does not, then $\exists\lambda_1, \cdots, \lambda_n$ that $\sum_i \lambda_i r_i = n_i$ then exists $n_i \in R(T)$ and vice versa. Therefore, as the set of vectors $\{r_1, \cdots, r_n, n_1, \cdots, n_i\}$ are linear independent with
$\dim(\text{span}(r_1, \cdots, r_n, n_1, \cdots, n_i)) = \dim(V)$ they are a sets of basis of V, Thus $V = R(P) + N(P)$, thus, it is a direct sum.

6. As $\{b_i\}$ is a set of basis for V, then $\forall b \in V, \exists\lambda_1, \cdots, \lambda_n$ that $b = \sum_{i=1}^n \lambda_i b_i$. Thus, for all $T \in \text{Hom}(V, \mathbb{R})$,
$T(b) = T(\sum_{i=1}^n \lambda_i b_i) = \sum_{i=1}^n \lambda_i T(b_i) \times 1 = \sum_{i=1}^n \lambda_i T(b_i) \times \ell_i(b_i)$. As $T(b_i) \in \mathbb{R}$, set $\mu_i = \lambda_i \times T(b_i)$, then $T(b) = \sum_{i=1}^n \mu_i \ell_i(b_i)$. Thus, the set of vectors spans $\text{Hom}(V, \mathbb{R})$ Also, for $1 \leq j \leq n$ $\sum_{i=1}^n \lambda_i \ell_i(b_j) = \lambda_j$. for $\sum \lambda_i \ell_i = 0$, when running this sum through all the basis vectors, all the coefficients (λ_i) must all equal to zero meaning that the set of ℓ_i are linearly independent, thus, it is a set of basis for $\text{Hom}(V, \mathbb{R})$

7. $T_{\mathcal{B}\mathcal{B}} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $[id]_{\mathcal{B}\mathcal{B}'} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$ $[id]_{\mathcal{B}'\mathcal{B}} = ([id]_{\mathcal{B}'\mathcal{B}})^{-1} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{pmatrix}$
Thus:

$$T_{B'B} = [id]_{\mathcal{B}'\mathcal{B}} T_{\mathcal{B}\mathcal{B}} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & \frac{1}{2} \\ 0 & -\frac{1}{2} \end{pmatrix}$$

8. a) $\forall n_{k+1} \in N(T^{k+1}), T(T^k(n_{k+1})) = N(T^{k+1}) = \overrightarrow{0}$. As T is a linear map, $T(v) = \overrightarrow{0}$ then $v = \overrightarrow{0}$. Thus, $T^k(n_{k+1}) = \overrightarrow{0}$, Thus, $N(T^k) \subsetneqq N(T^{k+1})$.

for any $w = T^{k+1}(v)$, $w = T \circ T^k = \underbrace{T \circ \cdots \circ T}_{k} = T^k \circ T = T^k(T(v))$

Thus, $R^{k+1} \subsetneqq R^k$

b) Suppose it does not exists such k, then as range itself is a subspace, if they do not equal, then the $\dim(R(T^{k+1})) \neq \dim(R(T^k))$, as profed in a), $R^{k+1} \subsetneqq R^k$. Thus, $\forall k \in \mathbb{N}$, $\dim(R(T^{k+1})) < \dim(R(T^k))$. According to rank/nullity, $\dim(N(T)) = \dim(V) - \dim(R(T))$, there is no such thing as negative dimension, therefore the largest value for $\dim(R(T))$ is $\dim(V)$. However, According to the asumption, $\dim(R(T^2)) < \dim(R(T)) \leq \dim(V)$, $\dim(R(T)) \neq 0$ as if it is, there does not exist such $\dim(R(T^2)) < 0$ meaning the assumption is wrong. Then is needs to be at least 1.
Then $\dim(R(T^2)) < \dim(V) - 1$. then through mathematical induction, setting $\dim(R(T^k)) \leq \dim(V) - (k-1)$, for same reason stated above, $\dim(R(T^{k+1})) \leq \dim(V) - (k-1) - 1$. Then for

$k = \dim(V) + 1$, $\dim(R(T^k)) \leq \dim(V) - (\dim(V) - 1) = 0$, then $\dim(R(T^{k+1}))$ does not exist then there is a contradiction, and thus, $\exists k$ that $\dim(R(T^k)) = \dim(R(T^{k+1}))$, as in a), $R^{k+1} \subsetneqq R^k$, then $\exists k$ that $R(T^k) = R(T^{k+1})$.

c) If $\exists R(T^k) = R(T^{k+1})$, then $\exists k$, $R(T^k) = R(T \circ T^k)$, Note $r_k = R(T^k)$, then $r_k = R(T(r^k))$.
Induction:
$\exists k$ that $R(T^k) = R(T^{k+1})$, suppose $R(T^{k+s}) = R(T^{k+s+1})$, claim that $R(T^{k+s+1}) = R(T^{k+s+2})$.
prof:
$R(T^{k+s}) = R(T^{k+s+1})$ means that $r_{k+s} = R(T(r_{k+s})) = r_{k+s+1}$, then $R(T^{k+s+2}) = R(T(r_{k+s+1})) = R(T(r_{k+s})) = r_{k+s+1} = R(T^{k+s+1})$.
Thus, by mathematical induction, the original statement was right