Exam 1 Notes

Format:

- ≥ 1 Sep. of Variables
- ≥ 1 Integrating Factor
- From Equation to Interpretation "Math Storytelling"
- NO interest in long multi-part where missing A means you miss B, C, D...
- $\frac{1}{2}$, $\frac{1}{2}$ math and written answer.
- ≥ 1 Diagram $\rightarrow$ Eq.
- ≥ 1 Text $\rightarrow$ Diagram

How to take Exam

- Show up 8 AM Th.
- Get File (Word, .tex)
- Turn in, w/ photos or tablet notes at 9:15-9:20 window on canvas.
- Questions $\rightarrow$ DM on Zoom.

Q1 Vector Fields

How to Draw a Vector Field

From ODE → Graph.

$$\frac{dy}{dx} = 0.2xy$$

Goal: Plot slope $\left(\frac{dy}{dx}\right)$ of our solution $(y(x))$ in x,y coordinates

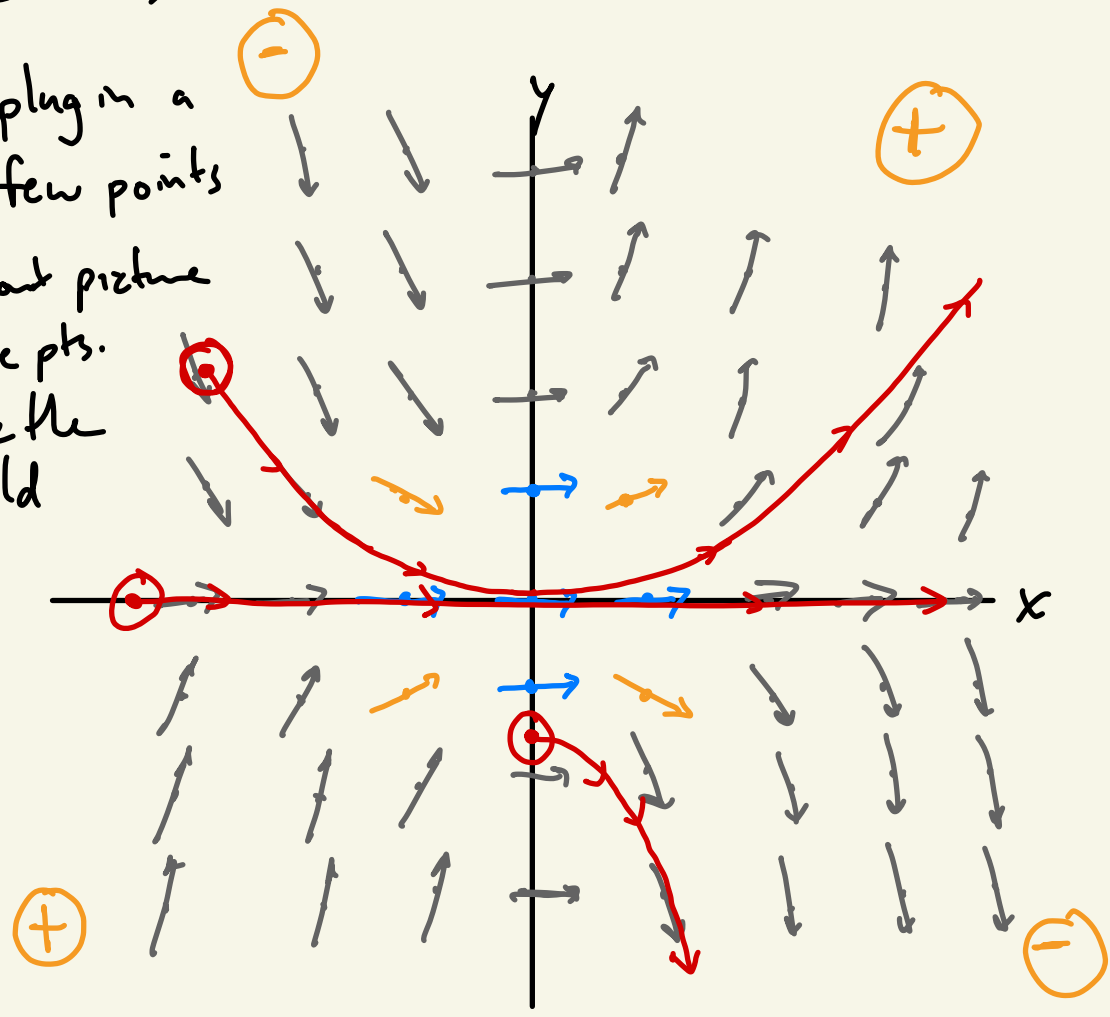
(x,y)	dy/dx	
$(0,0)$	0	→ flat arrows
$(1,0)$	0	
$(-1,0)$	0	
$(0,1)$	0	
$(0,-1)$	0	
$(1,1)$	0.2	
$(-1,1)$	-0.2	

① Axes

② plug in a few points

③ flesh out picture w/ more pts.

④ complete the field



⑤ a solution is like a piece of cotton in the wind.

Q2

- Recursion $n(t+1)$
- Difference Δn
- Differential $\frac{dn}{dt}$

Which are discrete?

Which one is continuous?

↓

Diff. Eq. have differentials dx, dy

$\frac{dy}{dx} = \dots$ $\frac{dn}{dt} = \dots$

Differential Equation.

$\frac{dn}{dt} = \lim_{\Delta t \to 0} \frac{f(n(t)) - n(t)}{\Delta t}$ (D.E.)

lim of $\frac{\Delta n}{\Delta t}$ as $\Delta t \to 0$

Recursion is a recipe for the next value.

Recursions start w/ $n(t+1) = \dots$ (function of $n(t)$ on RHS)

"Update rule"

$n(t+1) = f(n(t))$

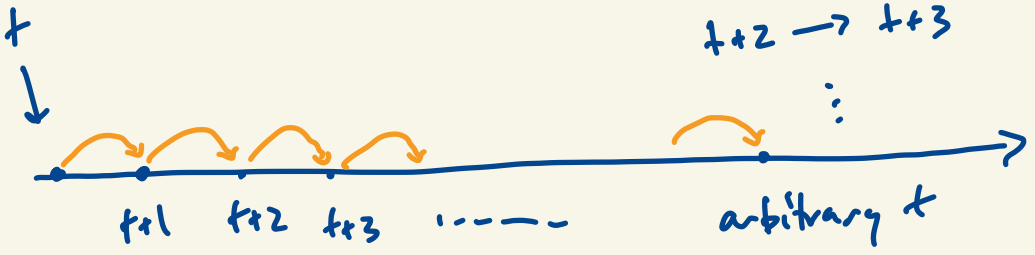
"Bootstrap"

$t \to t+1$

$t+1 \to t+2$

$t+2 \to t+3$

⋮



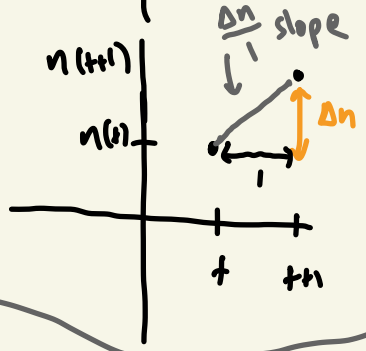
diff between t and t+1 is... $\Delta n(t)$ = difference equation.

Recursion

If you have $n(t+1) = f(n(t))$

you can easily subtract $n(t)$ from both sides

$n(t+1) - n(t) = f(n(t)) - n(t)$



$\Delta n = f(n(t)) - n(t)$

Difference Eqn.

Q3 Integrating Factor.

$$\frac{dy}{dx} = y + e^{-x}, \quad y(2) = e^{\pi}$$

If a 1st order linear ODE isn't separable → I.F.

① I.F. standard form.

$$\frac{dy}{dx} - 1y = e^{-x}$$

- $\frac{dy}{dx}$: no coeff on $\frac{dy}{dx}$
- $-1y$: $\frac{dy}{dx}$ term
- y term
- e^{-x}: all the other extra x stuff on RHS.

② My I.F. $\mu(x) = e^{\int \square\, dx}$

$$\mu(x) = e^{\int -1\, dx}$$

$$\mu(x) = e^{-x}$$

③ Multiply both sides by $\mu(x)$.

$$e^{-x}\left(\frac{dy}{dx} - y\right) = e^{-x} e^{-x}$$

always $\frac{d}{dx}(y \cdot \mu)$

④ Magic! RHS collapses

$$\int \frac{d}{dx}\left(y\, e^{-x}\right) dx = \int e^{-2x}\, dx$$

$$y e^{-x} = \frac{e^{-2x}}{-2} + c$$

⑤ solve for y...

$$y = \frac{e^{-x}}{-2} + c e^{x}$$

⑥ Initial Condition (Plug in → get const.)

$$e^{\pi} = \frac{e^{-2}}{-2} + c e^{2} \rightsquigarrow \boxed{\frac{e^{\pi} + \frac{1}{2}e^{-2}}{e^{2}} = c}$$

⑦ Plug in

$$\boxed{y = \frac{e^{\pi} + \frac{1}{2}e^{-2}}{e^{2}} e^{x} - \frac{1}{2}e^{-x}}$$

$$y' = y + e^{-x} y$$

$$y' = y\left(1 + e^{-x}\right)$$

$$\frac{dy}{y} = \left(1 + e^{-x}\right) dx$$

$$\frac{dy}{dx} - x^{1997} y = e^{x}$$

linear

linear

$$\frac{dy}{dx} = y + e^{-xy}$$

nonlinear.

$$\frac{dy}{dx} - xy = e^{-xy}$$

Order: 1st

ODE: ✓

Linear or Nonlinear?

Non!

Can use I.F. on

- 1st order ✓
- Linear ✗
- ODEs ✓

Q4 Haploid Models

$$s_c = (b_A - d_A) - (b_a - d_a)$$

s_c: positive when A winning
negative when a winning

- "Competition" like a race.
- Which allele (a, A) outruns the other one?

typical birth/death

$$W_A = (1-d_A)(1+b_A)$$

Exponential Growth: $n_A(t+1) = W_A\, n_A(t)$

$n_a(t+1) = W_a\, n_a(t)$

positive growth.

$$\frac{dn}{dt} = c\, n(t)\left(1 - \frac{n(t)}{K}\right)$$

The thing you really want to know:

What fraction of total pop $n_A + n_a$ is A vs a?

$$p = \frac{n_A}{n_A + n_a} \quad \text{(fraction A)} \qquad q = 1 - p = \frac{n_a}{n_A + n_a} \quad \text{(fraction a)}$$

$$\frac{dp}{dt} = s_c\, p(t)\left(1 - p(t)\right)$$

positive or negative

like a carrying capacity of 1

$\lim_{\Delta t \to 0} \frac{\Delta p}{\Delta t}$ (Lec. 6)

$$\rightarrow p(t+1) = \frac{W_A\, p(t)}{W_A p(t) + W_a(1-p(t))} = \frac{V_A\, p(t)}{V_A p(t) + (1-p(t))}$$

Let $V_A = \frac{W_A}{W_a}$ "relative fitness"

Q5 Explicit vs Implicit Solutions.

Say you're solving...

and you can solve to get $y(x) = \ldots f(x)$ (and no y on RHS)

Explicit

Say you're solving...

and you <u>cannot</u> get y by itself.

$y(x) \neq f(x)$

Implicit

Ex: $y(x) = 1 - e^{y(x)}$