Calculating Biological Quantities

CSCI 2897

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Last time on CSCI 2987: The SIR Model

E (exposed)

$S \longrightarrow I \longrightarrow \cancel{R}$

$$\dot{S} = -\beta SI$$

$$\dot{I} = \beta SI - \gamma I$$

$$\dot{R} = \gamma I$$

in literature, often work with "normalized" version without explicitly saying so!

- SEIR : COVID
- SI : HSV
- SIRS : Influenza... COVID
- include SEI for mosquitos. let them interact w/ SIR model for humans → malaria dengue zika
- spatially embedded.
  - SIR for Boulder
  - SIR for Denver
  - Hwys, 36, 93 mixing 2 pops.

Rescaling the SIR model?

Instead of "How many people are infected?"

→ "What fraction of pop'n is infected?"

Before: $S + I + R = N$, total pop. size.

let $s = \frac{S}{N}$, $i = \frac{I}{N}$, $r = \frac{R}{N}$ $\Rightarrow$ $s + i + r = 1$

$r = 1 - s - i$

- I don't need to track r...
- Just tell me s, i, we're able to determine r.

$\Rightarrow \dot{s} = \frac{1}{N}\dot{S}$, $\dot{i} = \frac{1}{N}\dot{I}$, $\dot{r} = \frac{1}{N}\dot{R}$

$$\frac{1}{N}\dot{R} = \gamma I \frac{1}{N}$$

$$\dot{r} = \gamma i$$

$$\boxed{\dot{r} = \gamma i}$$

$$\frac{1}{N}\dot{S} = -\beta S I \frac{1}{N}$$

$$\dot{s} = -\beta S i \frac{N}{N}$$

$$\dot{s} = -\beta N s i$$

$$\boxed{\dot{s} = \bar{\beta} s i}$$

$$\frac{1}{N}\dot{I} = (\beta S I - \gamma I)\frac{1}{N}$$

$$\dot{i} = \beta S i - \gamma i$$

$$\dot{i} = \beta N s i - \gamma i$$

$$\boxed{\dot{i} = \bar{\beta} s i - \gamma i}$$

Let $\underset{\text{new}}{\bar{\beta}} = \underset{\text{old}}{\beta} N$

These 3 eqns represent rates of change in population, as either fractions or absolutes.

What's one thing we can do with a set of ODEs?

What are the equilibria?

$$\dot{S} = -\beta S I = 0$$
$$\dot{I} = \beta S I - \gamma I = 0$$
$$\dot{R} = \gamma I = 0$$

$-\beta S \cdot 0 = 0$ ✓ no restrictions on S.

$\uparrow$

$\beta S \cdot 0 - \gamma \cdot 0 = 0$ ✓ no restrictions on S.

$\uparrow$

$\Rightarrow I = 0$

$$\boxed{\Rightarrow I = 0, \quad S + R = 1 \quad (\text{no restrictions on } S, R).}$$

If nobody is infected, S and R don't change from whatever they are.

Breakout:

I claim that: $$\frac{dS}{dt} + \frac{dI}{dt} + \frac{dR}{dt} = 0$$

Can you think of at least one way to show why this is true using math?

Can you explain why this is true in words?

Breakout:

I claim that: $\frac{dS}{dt} + \frac{dI}{dt} + \frac{dR}{dt} = 0$

Can you think of at least one way to show why this is true using math?

Can you explain why this is true in words?

① $\dot{S} = -\beta S I$

$\dot{I} = \beta S I - \gamma I$

$\dot{R} = \gamma I$

$\dot{S} + \dot{I} + \dot{R} = \cancel{-\beta S I + \beta S I} - \cancel{\gamma I + \gamma I} = 0$ ✓

② $S + I + R = 1$

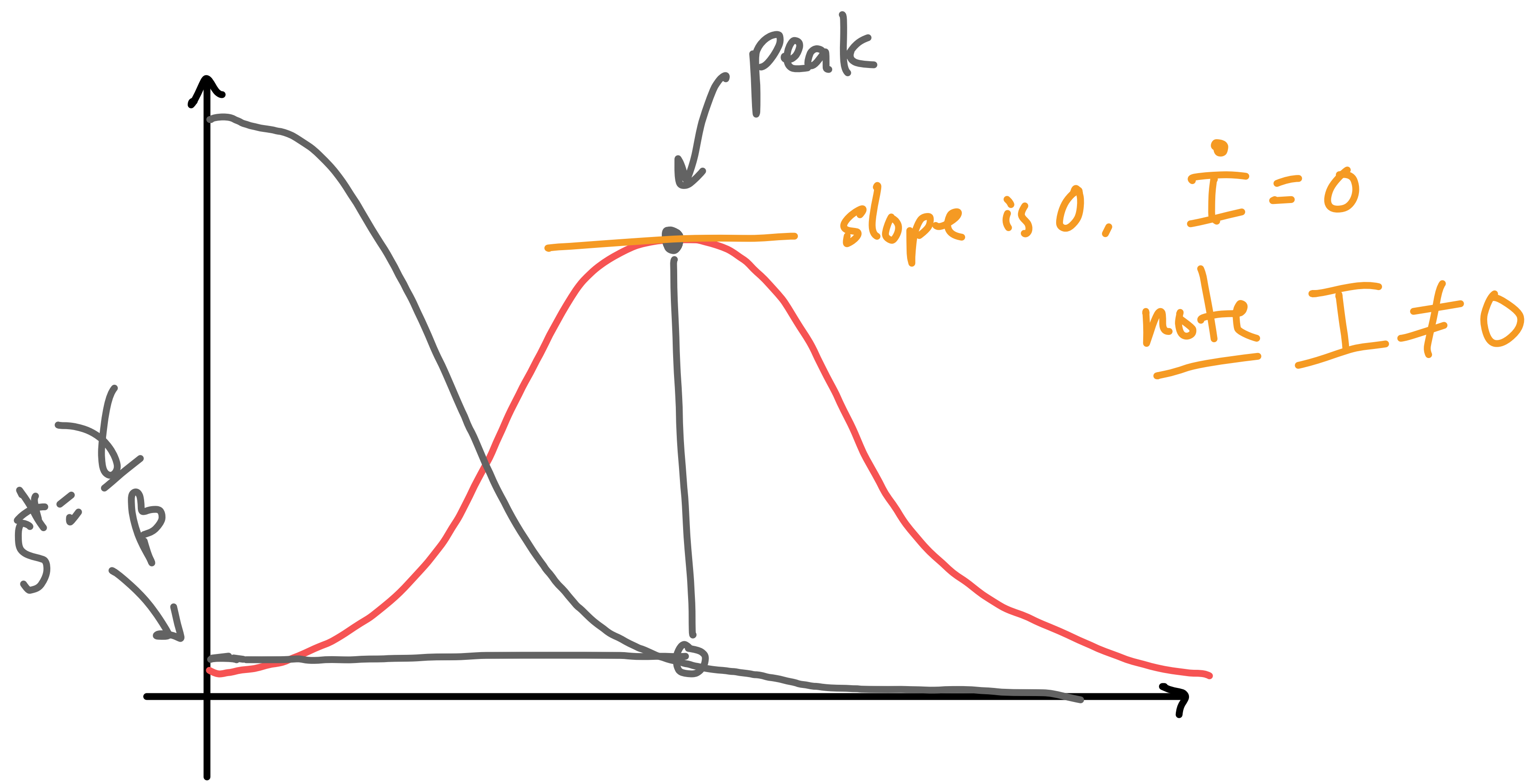
$\dot{S} + \dot{I} + \dot{R} = 0$ ✓

↓

Pop'n is fixed, so $\dot{S} + \dot{I} + \dot{R} = 0$

Conservation of Population

Analysis: when does the epidemic peak?



$$\frac{dI}{dt} = 0 = \beta S I - \gamma I$$

$$\Rightarrow \beta S - \gamma = 0$$

$$\boxed{S^* = \frac{\gamma}{\beta}}$$

If you tell me S, I can tell you whether Infections ↑ or ↓!

$$\dot{I} = \beta S I - \gamma I$$

$$\dot{I} = I(\beta S - \gamma)$$

<u>note</u>: $I \geq 0$

$\Rightarrow$ sign of $\dot{I}$ depends on sign of $\beta S - \gamma$!

<u>If</u> $S > \frac{\gamma}{\beta} \longrightarrow$ growth

<u>If</u> $S < \frac{\gamma}{\beta} \longrightarrow$ decline

Analysis: vaccination & herd immunity?

- Imagine that we vaccinate a fraction v of the population!
- How many do we need to vax, to make $\dot{I} < 0$?

Key: If vax fraction v,

then from a starting S

we get to a new $(1-v)S$

$\uparrow$

$1-v$ fraction

not vaccinated

Prev slide: want: $S < \frac{\gamma}{\beta}$

after vax

vax effect

before vax

$$(1-v)S < \frac{\gamma}{\beta}$$

$$1-v < \frac{\gamma}{S\beta}$$

$$-v < \frac{\gamma}{S\beta} - 1$$

$$v > 1 - \frac{\gamma}{\beta} \cdot \frac{1}{S}$$

$$\boxed{v > 1 - \frac{S^*}{S}}$$

For a totally susceptible pop'n

$$\boxed{v > 1 - \gamma/\beta}$$

Analysis: vaccination & herd immunity?

Interpret $v > 1 - \frac{\gamma}{\beta}$ to have $\dot{I} < 0$ (epidemic dies out)

News: COVID-19 variants may be more transmissible.
Goal: get to herd immunity via vaccination...
=> need more vax,

we get to control γ, β

γ: rate at which ppl. recover

β: transmissibility

Increase recovery → Increase γ → Increase $\frac{\gamma}{\beta}$ → $1-\frac{\gamma}{\beta}$ close to 0 → lowered need for vaccine

Increase transmiss. → Increase β → Decrease $\frac{\gamma}{\beta}$ → $1-\frac{\gamma}{\beta}$ close to 1 → increased need for vaccine.

Analysis: the basic reproductive number

R_0 : # of new infections typically caused by a single infection in a susceptible population.

this R is not "Recovered" R... think of this as a rate. "R" for "Reproductive"

$$\dot{S} = -\beta S I$$
$$\dot{I} = \beta S I - \gamma I$$

new infections (per time): $\beta S I$ (per time)

new infections (per infected): $\frac{\beta S I}{I} = \beta S$

to get the total new infections, need to multiply by the duration of an infection... $\frac{1}{\gamma}$

$$R = \frac{\beta S}{\gamma}$$

let $S = 1$ $\boxed{R_0 = \frac{\beta}{\gamma}}$