Calculating Biological Quantities

CSCI 2897

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2021, Lecture ~~6~~ 7

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Last time on CSCI 2897..

not yet

1. **Haploid & $\boxed{\text{diploid}}$ ~~models~~ models of natural selection**

const.

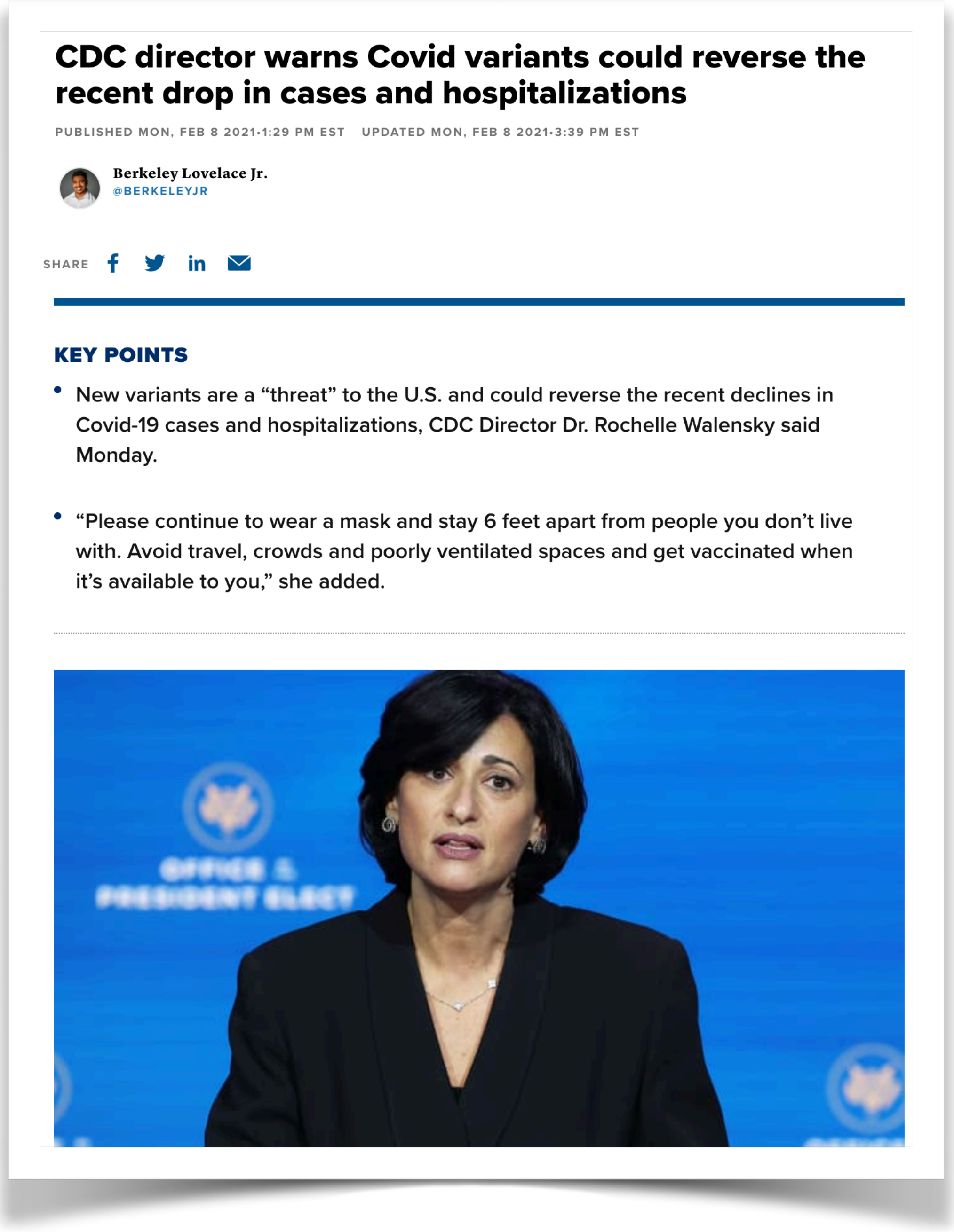
$$\frac{dp}{dt} = s_c p(t)\big(1 - p(t)\big)$$

p(t) = fraction of population type/variant A (vs a)

$$s_c = (b_A - d_A) - (b_a - d_a)$$

birth death
A

a



CDC director warns Covid variants could reverse the recent drop in cases and hospitalizations

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KEY POINTS

- New variants are a "threat" to the U.S. and could reverse the recent declines in Covid-19 cases and hospitalizations, CDC Director Dr. Rochelle Walensky said Monday.
- "Please continue to wear a mask and stay 6 feet apart from people you don't live with. Avoid travel, crowds and poorly ventilated spaces and get vaccinated when it's available to you," she added.

Lecture 7 Plan

1. **Equilibrium solutions**
2. **Lotka-Volterra Model of Competition**

Equilibrium

A system at **equilibrium** does not change over time. (Plural: **equilibria**.)

For a discrete time model, at equilibrium, it must be true that:

$n(t+1) = n(t)$ (no change) $\Delta n = 0$

For a continuous time model, at equilibrium, it must be true that:

$$\frac{dn}{dt} = 0$$ (no change)

rate of change

Equilibrium

$c \cdot x(1-x) = 0$ what is x?

$c=0$ $x=0$ $x=1$

A system at **equilibrium** does not change over time. (Plural: **equilibria**.)

What is the equilibrium / what are the equilibria for our haploid frequency equation?

$$\frac{dp}{dt} = s_c p(t)\big(1 - p(t)\big)$$

$s_c\, \underbrace{p(t)}_{p=0}\, \underbrace{\big(1-p(t)\big)}_{p=1} = 0$

① set $\frac{dp}{dt} = 0$ (no change)

② solve the equation for the variables, but not for the parameters

not interested

"steady state solutions"

- If the system is at $p=0$, it will always be at $p=0$.
- If the system is at equilibrium, it will always be at that eq!.

Note: we're always solving for equilibrium values of the *variables*, not the *parameters*.

Stability

equil.

An equilibrium is **locally stable** if a system near that equilibrium approaches it. This property is called **locally attracting**.

If I jiggle/bump the system a little, does it come back to equil.? YES

An equilibrium is **globally stable** if a system approaches that equilibrium *regardless* of its initial position.

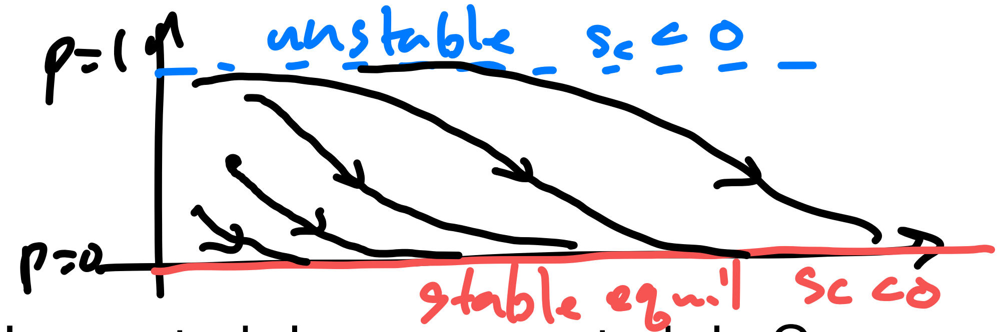
An equilibrium is **unstable** if a system near the equilibrium moves away from it. This property is called **repelling**.

If I "bump" the system a little does it come back to equil? NO

Liz.
Bradley.

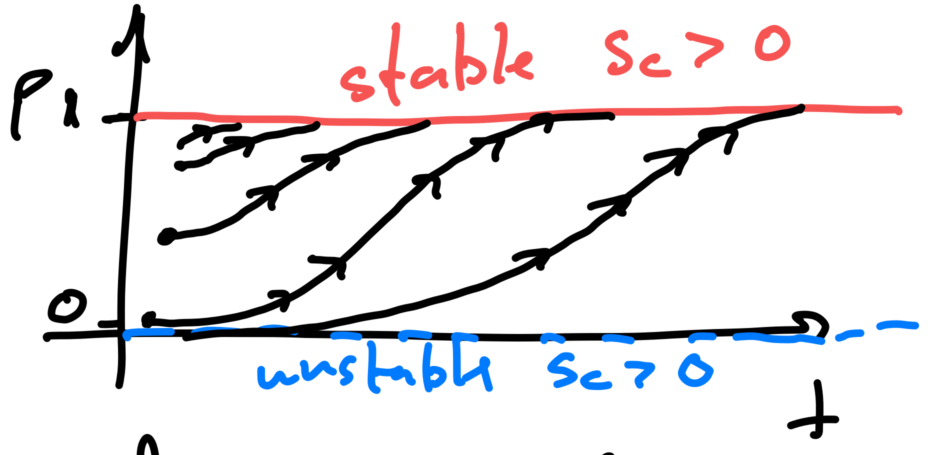
bump = some perturbation $\varepsilon > 0$ e.g. 0.000000000001

Stability



Are the equilibria for our haploid allele frequency equation stable or unstable?

$$\frac{dp}{dt} = s_c p(t)\big(1 - p(t)\big)$$



$p = 0$

Let $p = p_{equil} + \varepsilon$ $\quad \underline{\varepsilon > 0}$, but v. small

$$\boxed{s_c = (b_A - d_A) - (b_a - d_a)}$$

+ or −

$$\frac{dp}{dt} = s_c \, \underset{}{\overset{+}{(\varepsilon)}}\overset{+}{(1-\varepsilon)}$$

positive or negative?

when $s_c > 0$ $\quad$ $s_c < 0$

- If $s_c > 0$, $p = 0$ is unstable.
- If $s_c < 0$, $p = 0$ is stable

$p = 1$

Let $p = p_{equil} - \varepsilon$, $\varepsilon > 0$ but v. small

$$\frac{dp}{dt} = s_c (1-\varepsilon)\big(1-(1-\varepsilon)\big)$$

$$= s_c (1-\varepsilon)(\varepsilon)$$

positive when $s_c > 0$, negative when $s_c < 0$

- If $s_c > 0$ $\Rightarrow$ $p = 1$ stable
- If $s_c < 0$ $\Rightarrow$ $p = 1$ unstable

Bonus

Identify the equilibrium/a of the logistic growth equation, and characterize stability.

$$\dot{n} = r\underbrace{n}_{=0}\underbrace{\left(1-\frac{n}{K}\right)}_{=0} = 0$$

$r > 0$

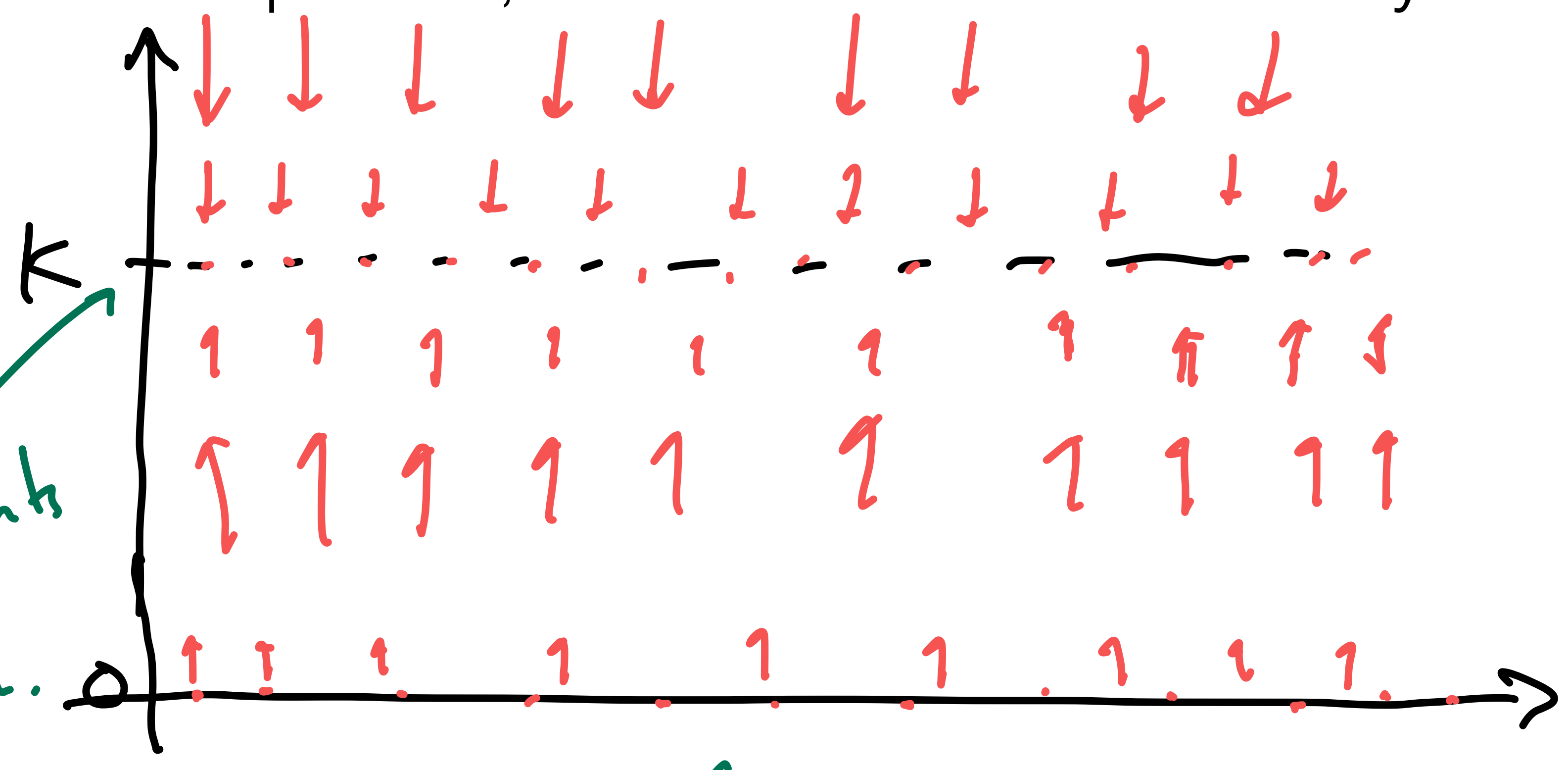
$\boxed{n=0}$ pop = 0 $n = \varepsilon$

$\left(1-\frac{n}{K}\right) = 0$

$\Rightarrow \boxed{n = K}$ pop = carrying capacity K

$n = K + \varepsilon$ $n = K - \varepsilon$

dir. field points to stable equilibrium.



dir field points away from unstable equilibrium.

Lotka-Volterra Competition

Imagine that there are two species, with population sizes $n_1(t)$ and $n_2(t)$.

Let's imagine that each one has the property from Logistic Growth where its growth rate R depends on its population size n, so we have $R_1(n_1)$ and $R_2(n_2)$.

What if one species' growth rate depended on the size of the other population?

Specifically, suppose that species i experiences competition *as if its own species had population* $n_i(t) + \alpha_{ij}\ n_j(t)$. (Here, i could be 1 or 2).

my pop. feels like its size

$$\left(n_1(t) + \alpha_{12}\, n_2(t) \right)$$

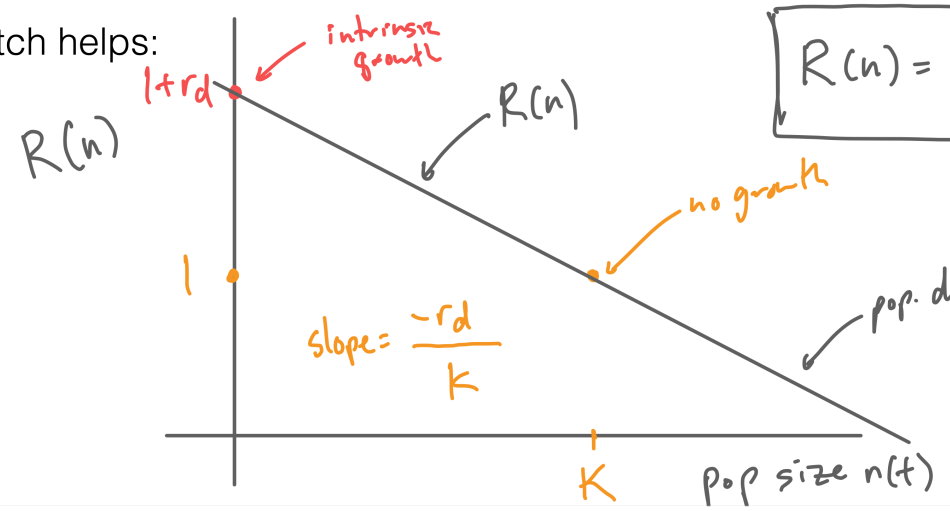
Lotka-Volterra Competition

Remember when we derived the Logistic Growth equation?

Logistic growth in discrete time

$y = mx + b$
$y = b + xm$

- Let's say that when the population size is zero, $R(0) = 1 + r_d$. intercept.
  - This is called the **intrinsic rate of growth.**
  - It's what happens when there aren't resource limitations (= prev. model).
- Let's say that $R(n)$ decreases until it becomes 1, at *some* value of n.
- A sketch helps:



$$\boxed{R(n) = (1+r_d) + n(t)\left(\frac{-r_d}{K}\right)}$$

$n_i(t) + \alpha_{ij} n_2(t)$

Logistic growth in discrete time

- If we write $n(t+1) = R(n)\ n(t)$, we now get
- $n(t+1) = \left[(1+r_d) - \frac{r_d}{K} n(t)\right] n(t)$

$$n(t+1) = n(t) + r_d\left(1 - \frac{n(t)}{K}\right) n(t)$$

We're now going to modify that equation for $R(n)$.

Lotka-Volterra Competition

used to be $n_i(t)$ for log. growth.

Let $R_i = (1 + r_i) + \left(\frac{-r_i}{K_i}\right)\left(n_i(t) + \alpha_{ij} n_j(t)\right)$

"index" species by i, j.

$i=1, j=2$

$i=2, j=1$

$n(t+1) = R\, n(t)$

Let's plug in this reproductive factor into *each* of our two update equations:

$$n_1(t+1) = \left[(1+r_1) + \left(\frac{-r_1}{K_1}\right)\left(n_1(t) + \alpha_{12} n_2(t)\right)\right] \cdot n_1(t)$$

$$n_2(t+1) = \left[(1+r_2) + \left(\frac{-r_2}{K_2}\right)\left(n_2(t) + \alpha_{21} n_1(t)\right)\right] \cdot n_2(t)$$

Lotka-Volterra Competition

We can write similar equations in continuous time:

$$\frac{dn_1}{dt} = r_1 \, n_1(t) \left[1 - \frac{n_1(t) + \alpha_{12} n_2(t)}{K_1} \right]$$

$$\frac{dn_2}{dt} = r_2 \, n_2(t) \left[1 - \frac{n_2(t) + \alpha_{21} n_1(t)}{K_2} \right]$$

cf: $\frac{dn}{dt} = r n(t) \left(1 - \frac{n(t)}{K}\right)$

α_{12} and α_{21} need not be same.

- α_{12} = impact that species 1 feels from species 2.
- α_{21} = impact that 2 feels from 1.

Lotka-Volterra Competition

Quick check: if the species don't interact, then: $\alpha_{12} = 0$, $\alpha_{21} = 0$

which implies that…

$$\frac{dn_1}{dt} = r_1 n_1(t)\left(1 - \frac{n_1(t) + \cancel{\alpha_{12}\, n_2(t)}^{\,0}}{K_1}\right) = r_1\, n_1(t)\left(1 - \frac{n_1(t)}{K_1}\right)$$ Log. growth.

$$\frac{dn_2}{dt} = r_2 n_2(t)\left(1 - \frac{n_2(t) + \cancel{\alpha_{21}\, n_1(t)}^{\,0}}{K_2}\right) = r_2\, n_2(t)\left(1 - \frac{n_2(t)}{K_2}\right)$$ Log. growth.

Interpretation: If species don't interact $\longrightarrow$ Log. growth.

Also note: this model is *symmetric* in that relabeling $1 \leftrightarrow 2$ produces the same equations.

Lotka-Volterra…Competition?

$r_1 = 1, \quad n_1(t) = 1, \quad K_1 = 10$
$n_2(t) = 2$

$$\frac{dn_1}{dt} = r_1 n_1(t)\left(1 - \frac{n_1(t) + \alpha_{12}\, n_2(t)}{K_1}\right)$$

$$\frac{dn_2}{dt} = r_2 n_2(t)\left(1 - \frac{n_2(t) + \alpha_{21}\, n_1(t)}{K_2}\right)$$

$\alpha_{12} = 0$

$$\Rightarrow \frac{dn_1}{dt} = 1 \cdot 1\left(1 - \frac{1 + 0 \cdot 2}{10}\right)$$

$$= 1\left(1 - \frac{1}{10}\right) = \frac{9}{10}$$

What if α_{12} is negative? How does an increase in n_2 affect $\dfrac{dn_1}{dt}$?

$\alpha_{12} = -1$

$$\frac{dn_1}{dt} = 1 \cdot 1\left(1 - \frac{1 + (-1) \cdot 2}{10}\right)$$

$$= 1\left(1 - \frac{1-2}{10}\right) = \left(1 - \frac{-1}{10}\right) = \frac{11}{10}$$

pop n_2 helps grow n_1!

Lotka-Volterra…Competition?

α_{12}	α_{21}	Relationship
0	0	none
+	+	competitive
+	−	parasitic
−	+	parasitic
−	−	mutualistic
−	0	commensal
0	−	commensal

$$\frac{dn_1}{dt} = r_1 n_1(t)\left(1 - \frac{n_1(t)+\alpha_{12}\, n_2(t)}{K_1}\right)$$

$$\frac{dn_2}{dt} = r_2 n_2(t)\left(1 - \frac{n_2(t)+\alpha_{21}\, n_1(t)}{K_2}\right)$$

Let's code up the Lotka-Volterra model to explore!

$$\frac{dn_1}{dt} = r_1 n_1(t)\left(1 - \frac{n_1(t)+\alpha_{12}\, n_2(t)}{K_1}\right)$$

$$\frac{dn_2}{dt} = r_2 n_2(t)\left(1 - \frac{n_2(t)+\alpha_{21}\, n_1(t)}{K_2}\right)$$

with initial conditions

$$n_1(0) = a$$

$$n_2(0) = b$$