Calculating Biological Quantities

CSCI 2897

- HW 2 due today
- O.H. after class
- O.H. → #pin post #general

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Last time on CSCI 2987: The integrating factor

$$\frac{dy}{dx} - 3y = 1$$

$$\left[\frac{dy}{dx} - 3y\right] e^{-3x} = [1]\, e^{-3x}$$

$$\frac{d}{dx}\left[e^{-3x}y\right] = e^{-3x}$$

$$\int \frac{d}{dx}\left[e^{-3x}y\right]\, dx = \int e^{-3x}\, dx$$

$$e^{-3x}y = \frac{e^{-3x}}{-3} + c$$

$$y = \frac{1}{-3} + ce^{3x}$$

Note: you could solve this using S.o.V.

Today

1. **Wrapping up the Integrating Factor method.**
2. **Measles and the SIR model.**

want $\frac{d}{dx}\left[\mu(x)\cdot y(x)\right] = \mu(x)\frac{dy}{dx} + \frac{d\mu}{dx}y(x)$ (Prod. Rule)

① ② ③ ?

My ODE $\mu(x)\frac{dy}{dx} + \mu(x)y(x) = f(x)\mu(x)$

I need $\mu(x) = \frac{d\mu}{dx}$

$\int dx = \int \frac{d\mu}{\mu} \Rightarrow x + c = \ln\mu \Rightarrow \mu = ke^{x}$

One key point: only the *left side* affects $\mu(x)$

(once we get the equation into "standard form")

Integrating factor

$$\left(\frac{dy}{dx} + y = x\right)e^x \qquad \int \frac{d}{dx}\left(y \cdot e^x\right)dx = \int x e^x \, dx \qquad y e^x = \int x e^x \, dx$$

$$\left(\frac{dy}{dx} + y = 1998x^2\right)e^x \qquad \int \frac{d}{dx}\left(y \cdot e^x\right)dx = \int 1998 x^2 e^x \, dx \qquad y e^x = 1998\int x^2 e^x \, dx$$

$$\left(\frac{dy}{dx} + y = f(x)\right)e^x \qquad \int \frac{d}{dx}\left(y \cdot e^x\right)dx = \int f(x) e^x \, dx \qquad y e^x = \int f(x) e^x \, dx$$

$$\frac{d}{dx}\left(y \cdot e^x\right) = \frac{dy}{dx}e^x + e^x y$$

$$\frac{dy}{dx} + P(x)\,y(x) = f(x)$$

$P(x) = 1$

Integrating factors — general version

$$\frac{dy}{dx} + P(x)y = f(x)$$

$$\frac{d}{dx}\left[\mu(x)\, y(x)\right] = \mu(x)\frac{dy}{dx} + \boxed{\frac{d\mu}{dx}}\, y(x)$$

match

$$\mu(x)(\text{ODE}): \quad \mu(x)\frac{dy}{dx} + \boxed{P(x)\,\mu(x)}\, y(x) = f(x)\,\mu(x)$$

need: $\frac{d\mu}{dx} = \mu(x)\,P(x)$ S.O.V. $\Rightarrow$ $\frac{d\mu}{\mu} = P(x)\,dx$

$$\ln \mu = \int P(x)\,dx$$

$$\mu = e^{\int P(x)\,dx}$$

Formula

mult this to the ODE

$$e^{\int P(x)dx}\left(\frac{dy}{dx} + P(x)\,y(x)\right) = f(x)\,e^{\int P(x)dx}$$

$$\frac{d}{dx}\left[y(x)\,e^{\int P(x)dx}\right] = f(x)\,e^{\int P(x)dx}$$

integrate...

$$y(x)\,e^{\int P(x)dx} = \int f(x)\,e^{\int P(x)dx}\,dx$$

② solve for y ① solve

Integrating factors — general version

$$\frac{dy}{dx} + P(x)y = f(x)$$

1. Get the equation into this standard form.
2. The integrating factor is $\mu(x) = e^{\int P(x)dx}$. Multiply both sides by $\mu(x)$.
3. Write the LHS as $\dfrac{d}{dx}\left[\mu(x)y(x)\right]$ and integrate both sides with respect to dx
4. Solve for $y(x)$.
5. Plug in the initial condition $y(x_0) = y_0$.
6. 🎉🎉🎉

Practice makes the master

$y(1) = e + 1$

Solve $x\dfrac{dy}{dx} - 4y = x^5 e^x$, with ~~$y(0)=1$~~

$$\frac{dy}{dx} + P(x)y = f(x)$$

1. Get the equation into this standard form.
2. The integrating factor is $\mu(x) = e^{\int P(x)dx}$. Multiply both sides by $\mu(x)$.
3. Write the LHS as $\dfrac{d}{dx}\left[\mu(x)y(x)\right]$ and integrate both sides with respect to dx
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6. 🎉🎉🎉

① $\dfrac{dy}{dx} - \dfrac{4}{x}y = x^4 e^x$

② $\mu(x) = e^{\int P(x)dx}$ $\quad P(x) = -\dfrac{4}{x}$ so $\int P(x)dx = -4\int \dfrac{1}{x}dx = -4\ln x$

$\Rightarrow \mu(x) = e^{-4\ln x} = e^{\ln x^{-4}} = x^{-4}$

rules of logs

e and ln cancel

$x^{-4}\dfrac{dy}{dx} - \dfrac{4x^{-4}}{x}y = \cancel{x^{-4}x^4}^{\,1} e^x$

③ $\dfrac{d}{dx}\left[x^{-4}y(x)\right] = e^x$

$\int \dfrac{d}{dx}\left[x^{-4}y(x)\right]dx = \int e^x dx$

$x^{-4}y(x) = e^x + c$

④ so... $\boxed{y(x) = x^4 e^x + cx^4}$

⑤ Let $x=1$ then $y = 1\cdot e + c$

$y = e + c$

Let's say we want the answer to have $c=1$... then $y = e+1$

$\boxed{y(x) = x^4 e^x + x^4}$

Final thoughts 1 — linear first order ODEs

Remember that a linear first-order ODE can be written in this form:

$$a_1(x)\frac{dy}{dx}+a_0(x)y = g(x)$$

Today we worked with equations that were in the form

$$\frac{dy}{dx}+P(x)y = f(x)$$

What is the relationship between P, f, and a_1, a_0?

$$P(x) = \frac{a_0(x)}{a_1(x)} \qquad f(x) = \frac{g(x)}{a_1(x)}$$

what if $a_1(x) = 0$ at some point?

Final thoughts 2 — linear first order ODEs

Remember that a linear first-order ODE can be written in this form:

$$a_1(x)\frac{dy}{dx}+a_0(x)y = g(x)$$

S.o.V.

When $g(x) = 0$, we call the linear first order ODE **homogeneous**.

$a_1(x)\,y' + a_0(x)\,y = 0$ $\quad y' = \dfrac{-a_0(x)}{a_1(x)}y$ $\quad \dfrac{dy}{y} = \dfrac{-a_0(x)}{a_1(x)}dx \xrightarrow{\text{S.o.V.}}$ solve. $y = ke^{\int \frac{-a_0}{a_1}dx}$

When $g(x)$ is anything other than 0, we call the ODE **nonhomogeneous**.

Integrating factors.

Final thoughts 2 — linear first order ODEs

Remember that a linear first-order ODE can be written in this form:

$$a_1(x)\frac{dy}{dx}+a_0(x)y = g(x)$$

When $g(x) = 0$, we call the linear first order ODE **homogeneous**.

When $g(x)$ is anything other than 0, we call the ODE **nonhomogeneous**.

Remember: people name ideas and create vocabulary/jargon when they think **those things are important.**

- It's a nice thing to **learn another field's vocabulary,** because **you learn what they think is important.** Try to see this as an opportunity, rather than a barrier—don't let vocab push you out or exclude you.
- **Be gentle with outsiders** from other fields when they try to learn your vocabulary too.

Measles and the SIR model

- **Measles** is an infectious disease caused by *Measles morbillivirus* (MeV).
- MeV is a single-stranded RNA virus, that infects only humans.
- Measles causes fever, cough, runny nose, inflamed eyes, and a rash.
- It is spread by contact—coughing, sneezing, or secretions.

Measles and the SIR model

AIDS
COVID-19

HIV
SARS-CoV-2

- **Measles** is an infectious disease caused by *Measles morbillivirus* (MeV).

~~Malaria~~

P. falciparum

- MeV is a single-stranded RNA virus, that infects only humans.
- Measles causes fever, cough, runny nose, inflamed eyes, and a rash.
- It is spread by contact—coughing, sneezing, or secretions.

- Case fatality rate is dependent on care (0.3% in the U.S, but up to 20% in some places.)
- Immunosuppressive — leads to complications.
- Around 0.1% of cases lead to encephalitis, with potential brain damage.

Measles and the SIR model

- To model an infectious disease like measles, we use a **compartmental model** call the "SIR" model:

S: Susceptible **I:** Infected (Infectious) **R:** Recovered.

- Rules:

1. Each person is a member of only one compartment in each time step.
2. People can move between compartments according to these rules:

$S \times S \longrightarrow$ Remain S

$I \times I \longrightarrow$ Remain I

$R \times R \longrightarrow$ Remain R.

if $S \times I \longrightarrow$ S becomes I w.p. c

Rate at which S and I contact each other is given by a.

I by itself will recover at rate γ

$$S \xrightarrow{acSI} I \xrightarrow{\gamma I} R$$

$a \cdot c = \beta$

The canonical SIR model:

$$S \xrightarrow{\beta S I} I \xrightarrow{\gamma I} R$$
$$I \xrightarrow{\delta I} \Omega$$

① Three Pop's : S, I, R

$N = S + I + R$ total # of people.

②
$$\frac{dS}{dt} = \underbrace{-\beta S I}$$
$$\frac{dI}{dt} = \underbrace{\beta S I} \underbrace{- \gamma I} \underbrace{- \delta I}$$
$$\frac{dR}{dt} = \underbrace{\gamma I}$$
$$\frac{d\Omega}{dt} = \underbrace{\delta I}$$

Typical:

$$\frac{dS}{dt} = -\beta S I$$
$$\frac{dI}{dt} = \beta S I - \gamma I$$
$$\frac{dR}{dt} = \gamma I$$

What if $I = 0$?
(no infections)

$$\frac{dS}{dt} = 0$$
$$\frac{dI}{dt} = 0$$
$$\frac{dR}{dt} = 0$$

nothing happens.

What if $S = 0$?
(no susceptible)

$$\frac{dS}{dt} = 0$$
$$\frac{dI}{dt} = -\gamma I$$
$$\frac{dR}{dt} = \gamma I$$

$I(t) = ke^{-\gamma t}$