Calculating Biological Quantities

CSCI 2897

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Last time on CSCI 2897..

1. **How to verify that a function is a solution of an ODE.** ✓
2. **Solving an ODE initial value problem *numerically* by stepping along the solution.** ✓
3. **Logistic & Exponential Growth** ✓

- vector fields
- long term behavior

Lecture 5 Plan

1. **Finding the analytical solution to exponential growth.**
2. **Separation of variables (general)** ✓
3. **Separability** (idea)
4. **Finding the analytical solution to logistic growth**

Exponential Growth in Continuous Time

- Let $n(t)$ be the population at time t
- Let r be the growth rate of the population
- Then our ODE is $\dfrac{dn}{dt} = r\,n$

rate of change of pop = (const × current pop.) "is proportional to current population."

- **Separation of Variables** (SoV) is a mathematical technique we can use to solve this ODE.

$\dfrac{dn}{dt}$ = ratio of a little change in n dn

and a little change in t dt

Separation of Variables — Exponential Growth

Goal: get all the n terms on the LHS and all the t terms on the RHS.

$$\frac{dn}{dt} = r\,n$$

$$\int \frac{dn}{n} = \int r\,dt$$

$$= \int n^{-1}\,dn \qquad = r\int dt = r(t + c_1)$$

$$= rt + rc_1$$

$$= \ln n + c_3 \qquad = rt + c_2$$

$$\ln n + c_3 = rt + c_2$$

$$\ln n = rt + c_4$$

$$e^{(\cancel{\ln} n)} = e^{(rt + c)} \quad \leftarrow \text{rel'ship btwn } n, t$$

$$n = e^{rt + c} \qquad e^{a+b} = e^a e^b$$

$\downarrow$ rules of exp.

$$n = e^{rt} e^{c} \qquad k = e^c \text{ (also a constant)}$$

$$\boxed{n = k e^{rt}}$$

Separation of Variables — Exponential Growth

Goal: get all the n terms on the LHS and all the t terms on the RHS.

$$\frac{dn}{dt} = r\ n \qquad \to \qquad n(t) = ke^{rt}$$

Followup: Verify that what we found is indeed a solution to the ODE.

① $$\frac{dn}{dt} = \frac{d}{dt} n(t) = \frac{d}{dt} ke^{rt}$$
$$= k \frac{d}{dt} e^{rt}$$
$$= k e^{rt} r$$

② $$\frac{dn}{dt} = r\ n$$

$$ke^{rt} r = r ke^{rt} \quad \checkmark$$

Separation of Variables — General I

General Goal: get all the n terms on the LHS and all the t terms on the RHS.

$$\frac{dy}{dx} = g(x)$$

derivative

RHS only a function of x.

$$dy = g(x)\,dx$$

$$\int dy = \int g(x)\,dx$$

$G(x)$ = antideriv. of $g(x)$

$$y + c_1 = G(x) + c_2$$

$$\boxed{y = G(x) + c}$$

Antiderivative:

$$\frac{d}{dx}G(x) = g(x)$$

Separation of Variables — General I

General Goal: get all the n terms on the LHS and all the t terms on the RHS.

$$\frac{dy}{dx} = g(x) \qquad \rightarrow \qquad y(x) = \int g(x)\ dx = G(x) + c$$

where $G(x)$ is the antiderivative of $g(x)$.

Followup: Verify that what we found is indeed a solution to the ODE.

$$y(x) = G(x) + c$$

$$\frac{dy}{dx} = \frac{d}{dx}\left(G(x) + c\right)$$

$$= g(x) + 0$$

$$\frac{dy}{dx} = g(x)$$

$$g(x) = g(x) \checkmark$$

Separation of Variables — General II

General Goal: get all the $\overset{y}{h}$ terms on the LHS and all the $\overset{x}{t}$ terms on the RHS.

$$\frac{dy}{dx} = \underbrace{g(x)}_{\sim}\ \underbrace{h(y)}_{\text{new}}$$

$$\frac{dy}{h(y)} = g(x)\,dx$$

$$\int \frac{1}{h(y)}\,dy = \int g(x)\,dx$$

let $p(y) = \frac{1}{h(y)}$

$$\int p(y)\,dy = G(x) + c$$

$$P(y) = G(x) + c$$

if possible, solve for y.
→ explicit solution $y = \text{RHS}$

else, implicit solution
function of y = RHS

Separation of Variables — Recipe

1. Get your equation into this form: $\dfrac{dy}{dx} = g(x)\ h(y)$
2. Identify $g(x)$ and $h(y)$.
3. Divide both sides by $h(y)$, bring dx to RHS.
4. Integrate both sides—don't forget your constant!
5. Solve for $y(x)$ if possible.

Separation of Variables — Example I

$$\frac{dy}{dx} = xy$$

① $\frac{dy}{dx} = g(x)\,h(y)$

② $g(x) = x$

$h(y) = y$

③ $\xleftarrow{\text{LHS}} y = x \xrightarrow{\text{RHS}}$ $\quad \int \frac{dy}{y} = \int x\,dx$

④ $\int \quad = \int$ $\quad \ln y = \frac{x^2}{2} + c$

$e \qquad e$

⑤ solve for y

$$y = e^{x^2/2 + c}$$

$$= e^{x^2/2}\, e^{c}$$

$$\boxed{y = k e^{x^2/2}}$$

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} + c$$

unless $n = -1$

If $n = -1$ $\quad \int x^{-1}\,dx = \ln x + c$

Separation of Variables — Example II

$$\frac{dy}{dx} = -\frac{x}{y}$$

$$\int y\,dy = -\int x\,dx$$

$$\frac{y^2}{2} = -\frac{x^2}{2} + c$$

$$y^2 = -x^2 + 2c$$

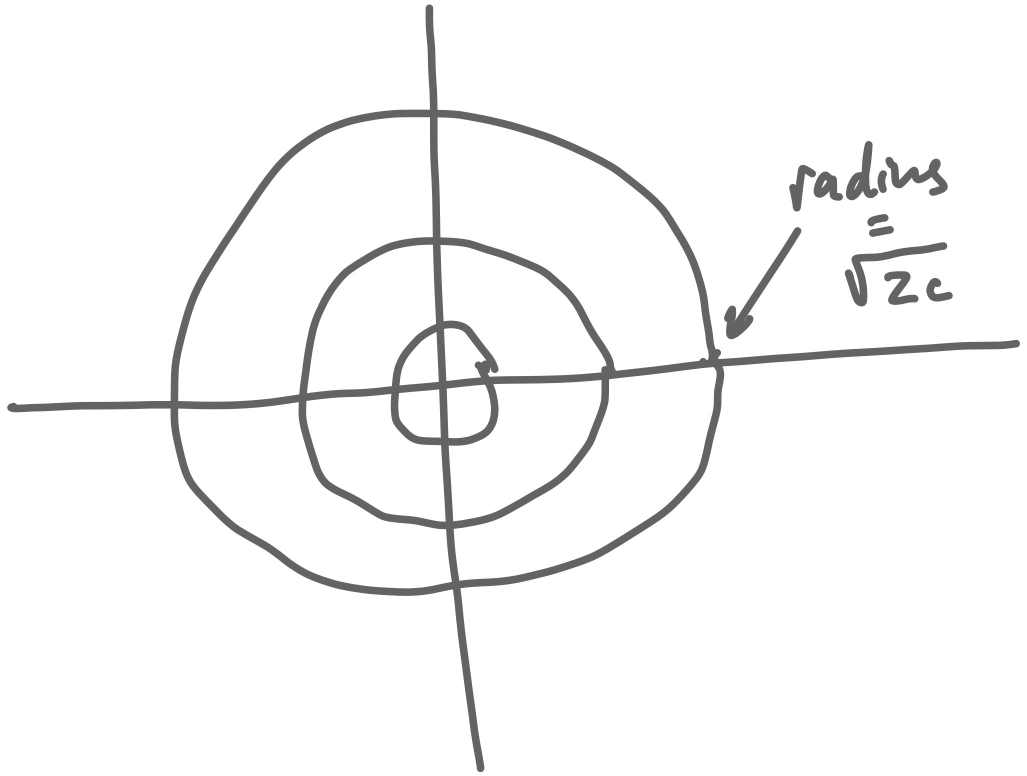
$$y = \pm\sqrt{2c - x^2}$$ explicit sol'n.

new branch

$y^2 + x^2 = 2c$ ← circle!

$x^2 + y^2 = \text{radius}^2$

centered at the origin.



$r^2 = 2c$

$r = \sqrt{2c}$

let $k^2 = 2c$ $\quad k = \sqrt{2c}$

$x^2 + y^2 = k^2$

Separation of Variables — Example III

$$\frac{dy}{dx} = \sin 5x$$

should we get $\dfrac{\cos 5x}{5}$ or $-\dfrac{\cos 5x}{5}$?

$$\frac{d}{dx}\cos 5x = -\sin 5x \cdot 5$$

$$\int dy = \int \sin 5x \, dx$$

$$y = \frac{1}{5}\left(-\cos 5x\right) + c$$

$$y = -\frac{\cos 5x}{5} + c$$

(General Form 1)

Separation of Variables — Example IV

$$\frac{dy}{dx} = x + y$$

I cannot algebra this!

$$dy = (x+y)\,dx$$

$$\frac{dy}{y} = \frac{(x+y)\,dx}{y}$$

$$\frac{dy}{y} = \left(1 + \frac{x}{y}\right)dx$$

damn!

$$\frac{dy}{dx} - y = x$$

$$dy - y\,dx = x\,dx$$

damn!

Cannot separate!

Separability

$\frac{dy}{dx} = x + y$ nope!

When we can write a first-order ODE in the form $\dfrac{dy}{dx} = g(x)\ h(y)$,

we call that equation **separable**, or say that it has **separable variables**.

Q: Why do we care?

A: We can solve separable equations using SoV. But if we cannot separate the variables, well… we can't use separation of variables to solve!

Real World Examples: Separability

Suppose that you are collecting data on avian malaria among the local Chickadee population, here in nests around Boulder.

Real World Examples: Separability

Suppose that you are collecting data on avian malaria among the local Chickadee population, here in nests around Boulder. **Suddenly**, a *crazy comp bio professor* leaps out from behind a tree and *shouts at you*:

Real World Examples: Separability

$\frac{dy}{dx} = g(x)\, h(y)$

Suppose that you are collecting data on avian malaria among the local Chickadee population, here in nests around Boulder. **Suddenly**, a *crazy comp bio professor* leaps out from behind a tree and *shouts at you*: Which ODEs are separable?!???

1. $\frac{dy}{dx} = (x+1)^2$

separable: no y on RHS. $dy = (x+1)^2\, dx$ $\quad h(y) = 1$

2. $\frac{dy}{dx} = (x+y)^2$

not separable: x and y added. $\frac{dy}{dx} = x^2 + xy + y^2 \ldots$ can't write as $g(x)\, h(y)$

3. $\frac{dy}{dx} = y^2 e^x \ln x^y$

$\frac{dy}{dx} = y^2 \cdot e^x \cdot \ln x^y = y^2 \cdot e^x \cdot y \ln x$

$= \underbrace{y^3}\cdot\underbrace{e^x \ln x}$ ✓

log rule!

$a \log b = \log b^a$

$\log c + \log d = \log cd$

$\log c - \log d = \log \frac{c}{d}$

4. $\frac{dy}{dx} = e^{3x+2y}$

$= \underbrace{e^{3x}}_{g(x)}\, \underbrace{e^{2y}}_{h(y)}$

Revisiting Logistic Growth

Recall our Logistic Growth Equation:

$$\frac{dn}{dt} = rn\left(1 - \frac{n}{K}\right)$$

Is this equation separable? yep!

no t on RHS!

Revisiting Logistic Growth

Partial fractions $\frac{1}{n(1-\frac{n}{K})} = \frac{1}{n} + \frac{1/K}{1-\frac{n}{K}}$

$$\frac{dn}{dt} = rn\left(1 - \frac{n}{K}\right)$$

$$\frac{dn}{n\left(1-\frac{n}{K}\right)} = r\,dt$$

$$\left(\frac{1}{n} + \frac{\frac{1}{K}}{1-\frac{n}{K}}\right)dn = r\,dt$$

$$\underbrace{\int \frac{1}{n}\,dn}_{①} + \underbrace{\int \frac{1/K}{1-\frac{n}{K}}\,dn}_{②} = \underbrace{\int r\,dt}_{③}$$

① $\int \frac{1}{n}\,dn = \ln n + c$

③ $\int r\,dt = r\int dt = rt + c$

② $\frac{1}{K}\int \frac{1}{1-\frac{n}{K}}\,dn = \int \frac{1}{K-n}\,dn$

$u = K - n$

$du = -dn$

$-du = dn$

$= -\int \frac{1}{u}\,du \rightarrow -\ln u + c$

$= -\ln(K-n) + c$

Revisiting Logistic Growth

$\alpha = e^c$ const. $\quad \beta = \frac{1}{\alpha}$ const.

$$\frac{dn}{dt} = rn\left(1 - \frac{n}{K}\right)$$

① ② ③

$$\ln n - \ln(K-n) = rt + c$$

$$\ln \frac{n}{K-n} = rt + c$$

$$\frac{n}{K-n} = e^{rt+c}$$

$$\frac{n}{K-n} = \alpha e^{rt}$$

$$n = (K-n)\alpha e^{rt}$$

$$n = K\alpha e^{rt} - n\alpha e^{rt}$$

$$n + n\alpha e^{rt} = K\alpha e^{rt}$$

$$n(1+\alpha e^{rt}) = K\alpha e^{rt}$$

$$n = \frac{K\alpha e^{rt}}{1+\alpha e^{rt}} \cdot \frac{\frac{1}{\alpha e^{rt}}}{\frac{1}{\alpha e^{rt}}}$$

$$n = \frac{K}{\frac{1}{\alpha e^{rt}} + 1}$$

$$\boxed{n(t) = \frac{K}{1 + \beta e^{-rt}}}$$

Revisiting Logistic Growth

$$\frac{dn}{dt} = rn\left(1 - \frac{n}{K}\right)$$

leads to a solution

$$n(t) = \frac{K}{1 + CKe^{-rt}}$$

Revisiting Logistic Growth

$$\frac{dn}{dt} = rn\left(1 - \frac{n}{K}\right) \qquad \rightarrow \qquad n(t) = \frac{K}{1 + CKe^{-rt}}$$

What happens when $t = 0$?

What happens when $t \rightarrow \infty$?

Examples of logistic growth

- Mable & Otto (2001) — cultivated both haploid & diploid *S. cerevisiae* (yeast) in two separate flasks.
- Diploid yeast cells are *bigger* and thus take up more resources.