Calculating Biological Quantities

CSCI 2897

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Last time on CSCI 2987: Consumer-Resource Models

$$\frac{dn_1}{dt} = f(n_1) - g(n_1, n_2)$$

$$\frac{dn_2}{dt} = \epsilon g(n_1, n_2) - h(n_2)$$

$f(n_1)$: rate of change of the resource via means other than consumption $(n_2 = 0)$.

$g(n_1, n_2)$: rate of consumption of the resource by the consumer.

ϵ : the conversion factor by which resource units $\rightarrow$ consumer units.

$h(n_2)$: rate at which the number of consumers changes without resources $(n_1 = 0)$.

TABLE 3.3
Consumer-resource models. Examples of functions that can be used in the consumer-resource model (3.16), where n_1 refers to the level of resources (e.g., number of prey) and n_2 refers to the level of consumers (e.g., number of predators).

Function	Description
$f(n_1) = \theta$	Inflow of resources at a constant rate
$f(n_1) = -\psi$	Outflow of resources at a constant rate
$f(n_1) = r\, n_1$	Constant per capita growth of resource species
$f(n_1) = r n_1\left(1 - \frac{n_1}{K}\right)$	Per capita growth of resource species declines linearly with resource level (logistic)
$f(n_1) = r\, n_1\, e^{-\alpha n_1}$	Per capita growth of resource species declines exponentially with resource level
$g(n_1, n_2) = a\, c\, n_1\, n_2$	Linear (type I) rate of resource consumption
$g(n_1,n_2) = \frac{a\, c\, n_1}{b + n_1} n_2$	Saturating (type II) rate of resource consumption
$g(n_1,n_2) = \frac{a\, c\, n_1^k}{b + n_1^k} n_2$	Generalized (type III) rate of resource consumption
$h(n_2) = \delta\, n_2$	Constant per capita death rate of consumer
$h(n_2) = (\delta\, n_2)\, n_2$	Per capita death rate of consumer increases linearly with consumer population size

Otto & Day, Table 3.3

Lecture 9 Plan

1. **Let's reverse engineer an equation:**

   what does the equation tell us about the biology?

2. **New math: the "integrating factor" method.**

Working Backward: Equations to Interpretation

By now, we're pretty good at taking a system and working forward, from behavior into a diagram and then into equations.

Sometimes, it's useful to be able to work in reverse, by taking an equation, and then interpreting the equation in terms of biology.

For example, here's the logistic equation with constant hunting or harvesting.

$$\frac{dn}{dt} = rn(t)\left(1 - \frac{n(t)}{K}\right) - \theta$$

Working Backward: Equations to Interpretation

However, all 3 of these equations could be described as "logistic growth with constant hunting or harvesting." So what's the difference in *meaning*?

1. $$\frac{dn}{dt} = rn(t)\left(1 - \frac{n(t)}{K}\right) - \theta$$

2. $$\frac{dn}{dt} = rn(t)\left(1 - \frac{n(t)}{K}\right) - Hn(t)$$

3. $$\frac{dn}{dt} = rn(t)\left(1 - \frac{n(t)}{K}\right) - H(n(t) - P)$$

- What is the story being told by each of these?
- What can we say about the "mechanism" of constant harvesting implied by each equation?

Working Backward: Equations to Interpretation

However, all 3 of these equations could be described as "logistic growth with constant hunting or harvesting." So what's the difference in *meaning*?

1. $$\frac{dn}{dt} = rn(t)\left(1 - \frac{n(t)}{K}\right) - \theta$$

$$\frac{dn}{dt} = -\theta$$

- The rate of consumption isn't changing. (because θ is a constant.)
- Rate of consumption is independent of population size.
- ex: each year, if CO gives away the same # of elk tags (regardless of elk pop'n)

- could think of $-\theta$ as emigration out of the area, instead of hunting.

Working Backward: Equations to Interpretation

Nota Bene
"note well"
NB:

However, all 3 of these equations could be described as "logistic growth with constant hunting or harvesting." So what's the difference in *meaning*?

$$2.\ \frac{dn}{dt} = rn(t)\left(1 - \frac{n(t)}{K}\right) - Hn(t)$$

NB: we can also think about individual experience in an equation
Here: each indiv. experience same risk of being hunted.
Last slide: risk ↑ as pop'n ↓

- removing algae from a pond.

↓

- more algae when n(t) large. less when n(t) is small.

- Hunting occurs at a constant per-capita rate.

Working Backward: Equations to Interpretation

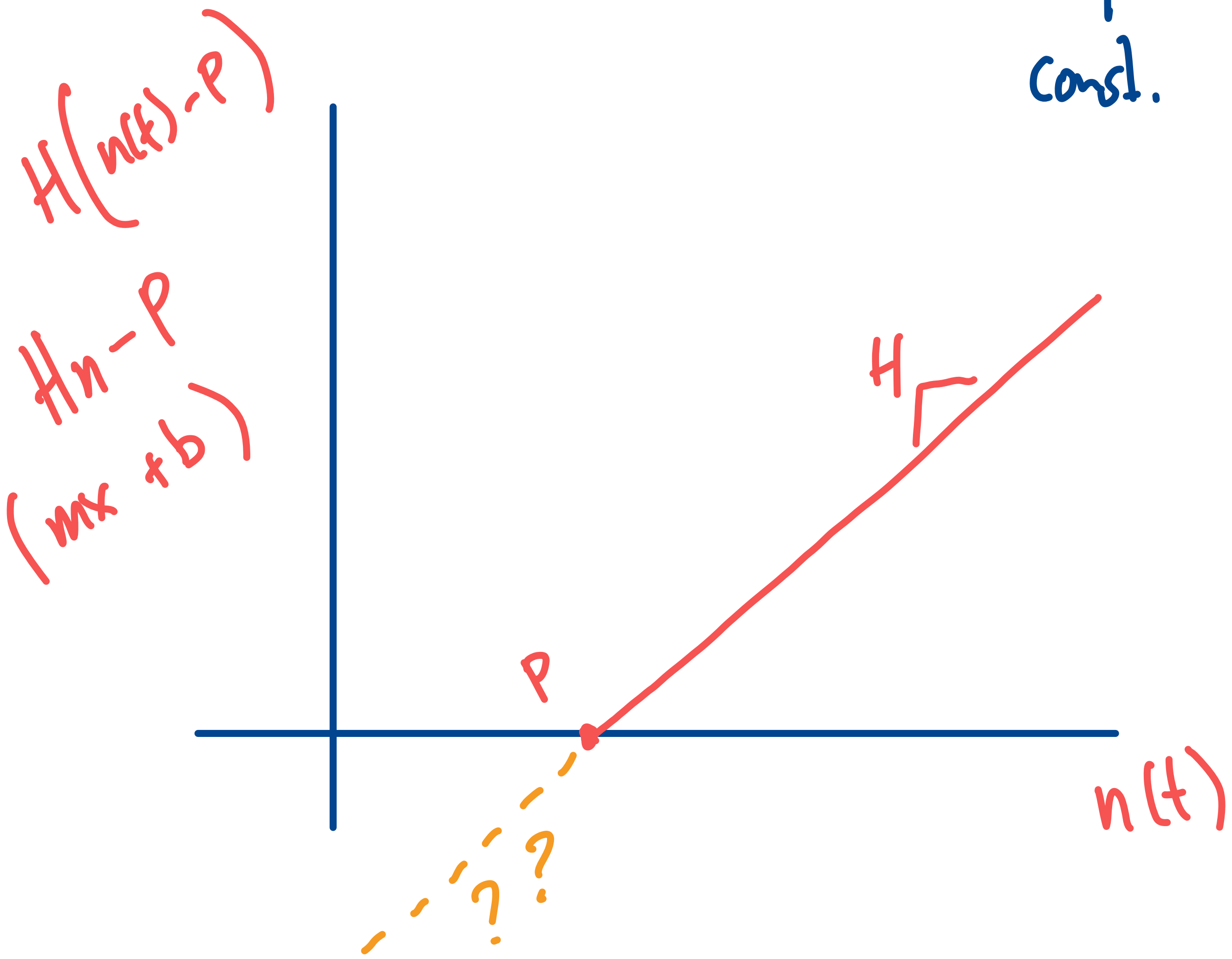
However, all 3 of these equations could be described as "logistic growth with constant hunting or harvesting." So what's the difference in *meaning*?

3. $$\frac{dn}{dt} = rn(t)\left(1 - \frac{n(t)}{K}\right) - \overbrace{H(n(t) - P)}$$

const. pop'n const P

→ becomes = 0 when n(t) = P
⇒ Hunting ceases at pop'n size P.

- When n(t) < P → restocking of population. "Reverse hunting".
- When n(t) > P → Hunting.



$$\frac{dn}{dt} = [\text{logistic}] - Hn(t) + HP$$

Const per-capita harvesting

constant total rate of restocking.

Working Backward: Equations to Interpretation

However, all 3 of these equations could be described as "logistic growth with constant hunting or harvesting." So what's the difference in *meaning*?

$$3.\ \frac{dn}{dt} = rn(t)\left(1 - \frac{n(t)}{K}\right) - H(n(t) - P)$$

Equilibrium?

10 E.C. HW 2.

tell me about equilibria of this equation.

① math

② interpretation in words. Algae in the pond.

Let's level up our ODE game

Warmup:

$$\frac{dy}{dx} - 3y = 0$$

$$\frac{dy}{dx} = 3y$$

$$\int \frac{dy}{y} = \int 3\,dx$$

$$e^{\ln y} = e^{3x + c}$$

$$y = e^{3x+c}$$

$$e^{a+b} = e^a e^b$$

$$y = e^c e^{3x}$$

$$\boxed{y = k e^{3x}}$$

What if we make it *just* a little different?

$$\frac{dy}{dx} - 3y = 1$$

deriv of y

just y

To make the green terms equal, ... set them equal!! (solve)

$$\frac{d}{dx}\left[\mu(x)\, y(x)\right] = \mu \frac{dy}{dx} + \frac{d\mu}{dx} y$$

multiply original equation by μ

$$\mu \frac{dy}{dx} - 3\mu y = \mu$$

\mu

$$\frac{d\mu}{dx} = -3\mu$$

$$\frac{d\mu}{\mu} = -3\,dx$$

$$\ln \mu = -3x + c$$

$$\boxed{\mu = ke^{-3x}}$$

sub back in.

After subbing in μ...

$$\cancel{k}e^{-3x}\frac{dy}{dx} - 3\cancel{k}e^{-3x} y = 1\cancel{k}e^{-3x}$$

$$e^{-3x}\frac{dy}{dx} - 3e^{-3x} y = e^{-3x}$$

$$\cancel{\int \frac{d}{dx}}\left[e^{-3x} y\right]\cancel{dx} = \int e^{-3x}\,dx$$

$$e^{-3x} y = \frac{e^{-3x}}{-3} + c$$

$$\boxed{y = \frac{1}{-3} + ce^{3x}}$$

Recap:

$$\frac{dy}{dx} - 3y = 1$$

- We observed that if there were a function called $\mu(x)$, then (product rule):

$$\frac{d}{dx}\left[\mu(x)y(x)\right] = \mu\frac{dy}{dx} + y\frac{d\mu}{dx}$$

- We compared this to our ODE: one $\dfrac{dy}{dx}$ term and one y term!
- Then, we matched up terms to figure out what $\mu(x)$ should be.
- This required us to solve another ODE (sep. of vars.) which we did.
- Then we integrated both sides and solved for $y(x)$.

Example:

$$\frac{dy}{dx} + y = x, \text{ with } y(0) = 4$$

$$\frac{d}{dx}(\mu y) = \mu \frac{dy}{dx} + \frac{d\mu}{dx} y$$

$$\mu \frac{dy}{dx} + \mu y = \mu x$$

$\Rightarrow \mu = \frac{d\mu}{dx}$ to get left hand sides to match.

(sep. of vars)

$$\mu = e^x$$

Plug in μ ...

$$e^x \frac{dy}{dx} + e^x y = x e^x$$

$$\frac{d}{dx}\left[e^x y\right] = e^x \frac{dy}{dx} + e^x y$$

MAGIC

$$\int \frac{d}{dx}\left[e^x y\right] dx = \int x e^x \, dx$$

$$\boxed{\cancel{e^{-x}}}\ \cancel{e^x}\, y = \left[\int x e^x \, dx\right] e^{-x}$$

$$y = e^{-x}\left[\int x e^x \, dx\right]$$

Example (continued): $\frac{dy}{dx} + y = x$, with $y(0) = 4$

side quest

$$\int x e^x \, dx$$

$u = x$
$dv = e^x dx$

want to be able to do this.

$$\int u \, dv = uv - \int v \, du$$

ultraviolet voodoo

what I have

where I'm going

$du = 1$

$v = \int dv = \int e^x dx = e^x$

$$\int x e^x = x e^x - \int e^x \cdot 1 \, dx$$

$$= x e^x - e^x + c$$

sub in

$$y = e^{-x}\left(x e^x - e^x + c\right)$$

$$y = x - 1 + c e^{-x}$$

$y = 4, \; x = 0$

$4 = 0 - 1 + c e^0$

$4 = -1 + c$

$5 = c$

$$\boxed{y = x - 1 + 5e^{-x}}$$