# Calculating Biological Quantities

## CSCI 2897

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**2021, Lecture 8**

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# Last time on CSCI 2897..

## 1. Equilibrium solutions

no change

$\Delta n = 0$

$n(t+1) = n(t)$

$\frac{dn}{dt} = 0$

Predator-Prey is different → later today!

Recipe for finding equilibria:

① set $n(t+1) = n(t) = n$ (constant) not changing

① set $\frac{dn}{dt} = 0$

→ ② solve for $n$.

③ interpret.

## 2. Lotka-Volterra Model of Competition

$$\frac{dn_1}{dt} = r_1 n_1(t)\left(1 - \frac{n_1(t) + \alpha_{12}\, n_2(t)}{K_1}\right)$$

$$\frac{dn_2}{dt} = r_2 n_2(t)\left(1 - \frac{n_2(t) + \alpha_{21}\, n_1(t)}{K_2}\right)$$

Observations:

- there are two equations. $n_1$, $n_2$ ⇒ tracking two versions of same thing.
- $n_1 = f(n_1, n_2)$ $n_2 = f(n_1, n_2)$ coupled via $\alpha_{12}$, $\alpha_{21}$.
- very similar look to Logistic Growth.
- very similar to each other.

# Lecture 8 Plan

1. **HW Comments**
2. **Consumer-Resource Models**

- HW 2 posted
- Due in a week.
- HW1 regrade/grade questions?

  email or slack!

# Homework 1

First, you record your observations every 10 minutes. Each time you record your observations, you address 20% of the unread notifications. Second, you notice that new notifications are arriving at a growing rate: 10 in the first 10 minutes, 20 in the second 10 minutes, 30 in the next 10 minutes, 40, 50, . . . .

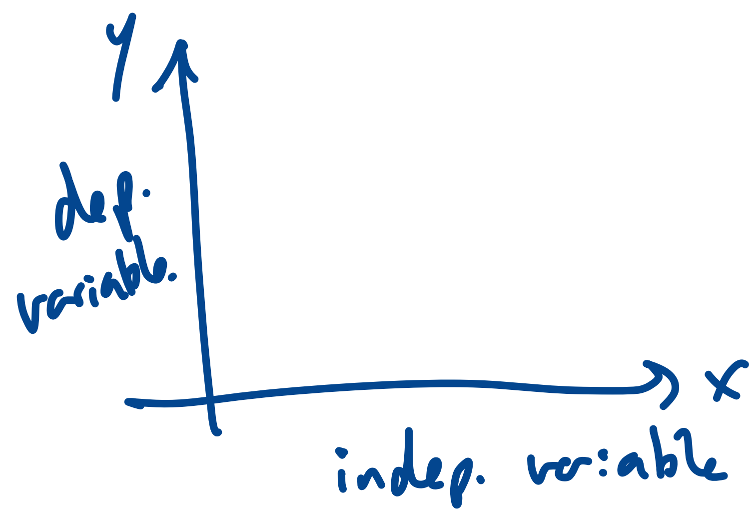
1) Discrete or Continuous

2) Variable?

dependent

$n(t)$

independent variable.




if timescale of one minute

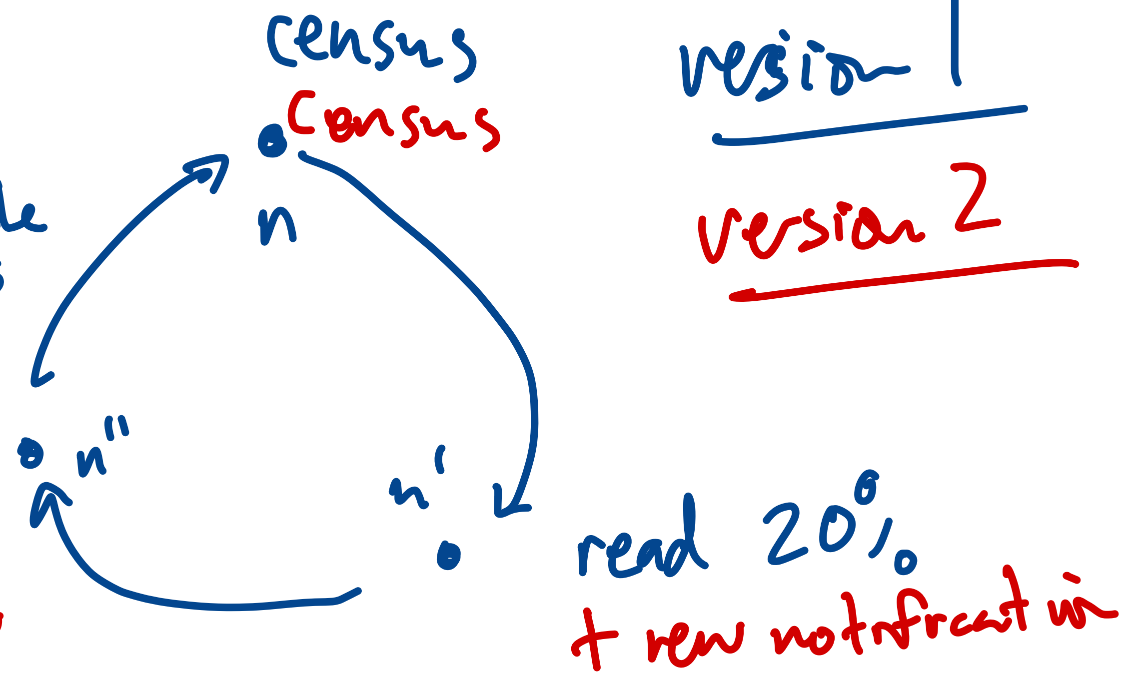
10t

3)

if timescale = 10 mins

t new notifications

read 20%




version 1

version 2

$$n(t+1) = n(t)(1-0.2) + t$$

$$\boxed{n(t+1) = (n(t)+t)(1-0.2)}$$

different numbers!

# Homework 1

Trig functions: $\frac{1}{\cos x} = \sec x$ $\quad \frac{1}{\sin x} = \csc x$

$$\boxed{\frac{\sec x}{\cos x} = 1}$$

helpful!!

Front cover of Zill:

**Trigonometric:**

**9.** $\frac{d}{dx}\sin x = \cos x$ **10.** $\frac{d}{dx}\cos x = -\sin x$ **11.** $\frac{d}{dx}\tan x = \sec^2 x$

**12.** $\frac{d}{dx}\cot x = -\csc^2 x$ **13.** $\frac{d}{dx}\sec x = \sec x \tan x$ **14.** $\frac{d}{dx}\csc x = -\csc x \cot x$

# Consumer-Resource Models

So far we've been thinking about resources as constant.

- light striking a patch of land
- nutrients in a river flowing past a location

resources aren't being depleted.

But in many situations, the resources *get depleted* as they are consumed.

- Bears eat salmon—and decrease the salmon population as a consequence!

We can account for these phenomena using a *consumer-resource* model.

one variable another variable.

⇒ two equations

# Consumer-Resource Models: General Structure

\omega → $\omega$

\Omega → $\Omega$

Here is the general form of a consumer-resource model. What do you see?

only on $n_1$ → growth  depends on $n_1$, and $n_2$ → decline

Res

$$\frac{dn_1}{dt} = f(n_1) - g(n_1, n_2)$$

$n_2$ is responsible for $n_1$'s decline. ⇒ $n_2$ kills $n_1$!

$f(n_1)$ just a function of $n_1$

$h(n_2)$ " " " " $n_2$

but minus in front of $h$? ⇒ $f$ is about ↑ $n_1$?

$h$ is about ↓ $n_2$?

how many $n_2$s do you make per $n_1$ dead?

growth  decline

Cons

$$\frac{dn_2}{dt} = \epsilon g(n_1, n_2) - h(n_2)$$

$n_1$'s death at the hands of $n_2$ leads to more $n_2$! ⇒ $n_2$ eats $n_1$, and grows.

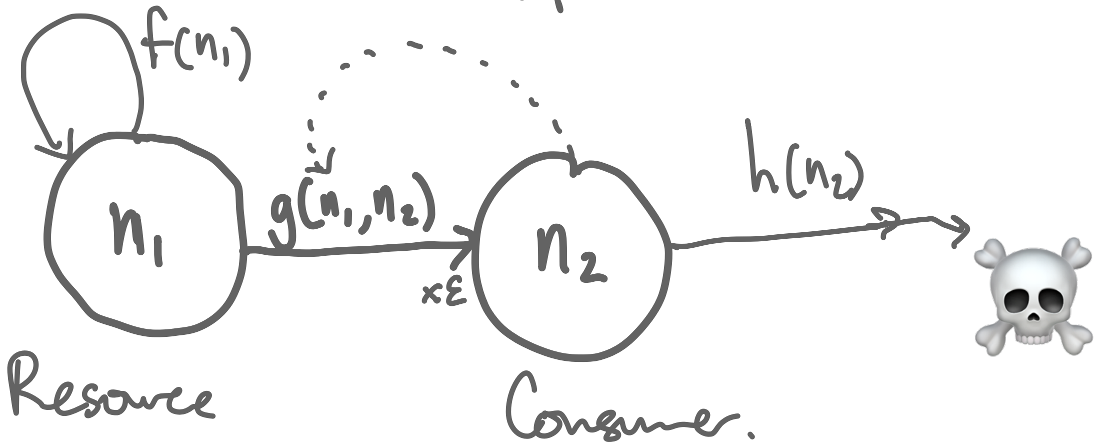
only on $n_2$. Apex predator? Its death is not dependent on $n_1$.

$g(n_1, n_2)$ shows up in both

have $\varepsilon$ in front of it in $n_2$ equation

+ in $n_2$ eqn, but − in $n_1$ equation!

Construct a flow diagram:



# Consumer-Resource Models: General Structure

$$\frac{dn_1}{dt} = f(n_1) - g(n_1, n_2)$$

$$\frac{dn_2}{dt} = \epsilon g(n_1, n_2) - h(n_2)$$

$f(n_1)$ : rate of change of the resource via means other than consumption $(n_2 = 0)$.

$g(n_1, n_2)$ : rate of consumption of the resource by the consumer.

$\epsilon$ : the conversion factor by which resource units $\rightarrow$ consumer units.

$h(n_2)$ : rate at which the number of consumers changes without resources $(n_1 = 0)$.

# Consumer-Resource Models: General Structure

$$\frac{dn_1}{dt} = f(n_1) - g(n_1, n_2)$$

$$\frac{dn_2}{dt} = \epsilon g(n_1, n_2) - h(n_2) + \varepsilon_{candy}\, g(n_2, candy)$$

$$\frac{dcandy}{dt} = \sin(t) - \delta\, candy - g(n_2, candy)$$

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**TABLE 3.3**
Consumer-resource models. Examples of functions that can be used in the consumer-resource model (3.16), where $n_1$ refers to the level of resources (e.g., number of prey) and $n_2$ refers to the level of consumers (e.g., number of predators).

| Function | Description |
|---|---|
| $f(n_1) = \theta$ | Inflow of resources at a constant rate |
| $f(n_1) = -\psi$ | Outflow of resources at a constant rate |
| $f(n_1) = r\, n_1$ | Constant per capita growth of resource species |
| $f(n_1) = r n_1\left(1 - \frac{n_1}{K}\right)$ | Per capita growth of resource species declines linearly with resource level (logistic) |
| $f(n_1) = r\, n_1\, e^{-\alpha n_1}$ | Per capita growth of resource species declines exponentially with resource level |
| $g(n_1, n_2) = a\, c\, n_1\, n_2$ | Linear (type I) rate of resource consumption |
| $g(n_1,n_2) = \frac{a\, c\, n_1}{b + n_1} n_2$ | Saturating (type II) rate of resource consumption |
| $g(n_1,n_2) = \frac{a\, c\, n_1^k}{b + n_1^k} n_2$ | Generalized (type III) rate of resource consumption |
| $h(n_2) = \delta\, n_2$ | Constant per capita death rate of consumer |
| $h(n_2) = (\delta\, n_2)\, n_2$ | Per capita death rate of consumer increases linearly with consumer population size |

Otto & Day, Table 3.3

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# Consumer-Resource Models: Const. Immigration

$$\frac{dn_1}{dt} = \theta - a\,c\,n_1\,n_2$$

const.

two coeffs. a, c

rate of depletion / consumption = product of $n_1$ $n_2$

"law of mass action"

↳ from chemistry.

$$\frac{dn_2}{dt} = \epsilon\,a\,c\,n_1\,n_2 - \delta\,n_2$$

a : prob. that a contact results in consumption.

c : rate of contact between
$n_1$s and $n_2$s per $n_1$, $n_2$

# Consumer-Resource Models: Const. Immigration

$$\frac{dn_1}{dt} = \theta - a\ c\ n_1\ n_2$$ nutrient

$$\frac{dn_2}{dt} = \epsilon\ a\ c\ n_1\ n_2 - \delta\ n_2$$ algae

**Example**: a nutrient flows into a lake at a constant rate.
A population of algae uses that nutrient to grow.

# Consumer-Resource Models: Const. Immigration

test Q:
beavers...
take log?
something
something.

$$\frac{dn_1}{dt} = \cancel{\theta}^{\,0} - a\,\cancel{c}^{\,0}\,n_1\,n_2$$

beavers dam the inflow stream $\theta = 0$

$$\frac{dn_2}{dt} = \epsilon\,a\,\cancel{c}^{\,0}\,n_1\,n_2 - \delta\,n_2$$

bears can't get to salmon due to fence. $c = 0$

$$\frac{dn_1}{dt} = \theta$$

$$n_1(t) = \theta t + c_1$$

$$\frac{dn_2}{dt} = -\delta n_2$$

$$n_2(t) = c_2 e^{-\delta n_2}$$

**Example**: a nutrient flows into a lake at a constant rate. A population of algae uses that nutrient to grow.

**Extension**: How could we explore a situation where the nutrient no longer flows into the lake? What other scenarios might we explore?

# Consumer-Resource Models: Const. Immigration

$$\frac{dn_1}{dt} = \theta - a\ c\ n_1\ n_2 \quad = 0$$

$$\frac{dn_2}{dt} = \epsilon\ a\ c\ n_1\ n_2 - \delta\ n_2 = 0$$

What is an equilibrium solution to these equations?

set $\frac{dn_1}{dt} = 0$ AND $\frac{dn_2}{dt} = 0$

$a c n_1 n_2 = \theta$ (*)

$a c n_1 n_2 = \frac{\delta n_2}{\varepsilon}$

$\theta = \frac{\delta n_2}{\varepsilon} \rightarrow \boxed{n_2 = \frac{\theta \varepsilon}{\delta}}$

into (*)

$a c n_1 \left( \frac{\cancel{\theta} \varepsilon}{\delta} \right) = \cancel{\theta}$

$\rightarrow \boxed{n_1 = \frac{\delta}{\varepsilon a c}}$

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Otto & Day, Table 3.3

# Consumer-Resource Models: Predator-Prey

only term that changed.

$$\frac{dn_1}{dt} = r\, n_1 - a\, c\, n_1\, n_2$$

$$\frac{dn_2}{dt} = \epsilon\, a\, c\, n_1\, n_2 - \delta\, n_2$$

What happens to $n_1$ if there is no $n_2$?

$$\frac{dn_1}{dt} = r\, n_1 - a\, c\, n_1\, \cancel{n_2}^{\,0}$$

set $n_2 = 0$

$$\frac{dn_1}{dt} = r\, n_1$$

prey experiences exponential growth in the absence of $n_2$ (predator).

# Consumer-Resource Models: Predator-Prey

$$\frac{dn_1}{dt} = r\, n_1 - a\, c\, n_1\, n_2$$

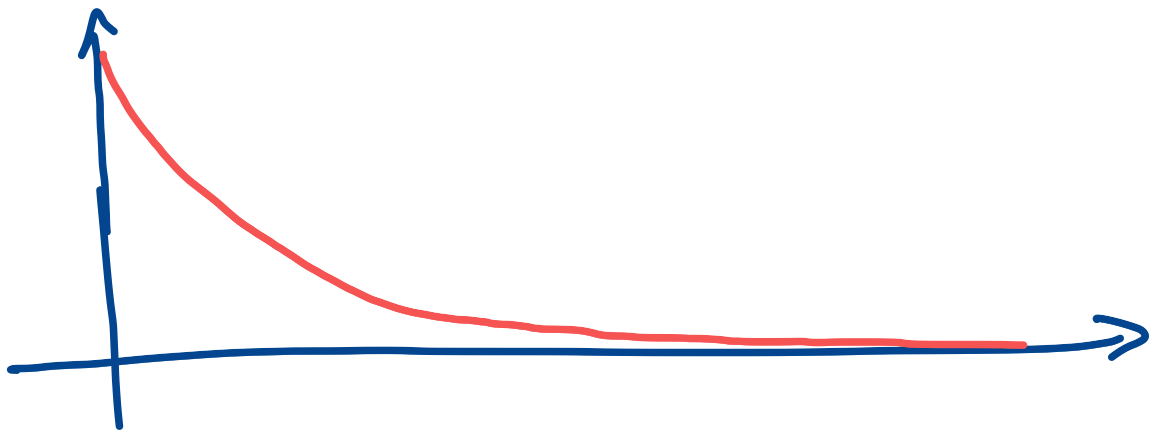
What happens to $n_2$ if there is no $n_1$?

$$\frac{dn_2}{dt} = \epsilon\, a\, c\, \cancelto{0}{n_1}\, n_2 - \delta\, n_2$$

$$\frac{dn_2}{dt} = -\delta\, n_2$$

"exponential" decrease in predator without the prey.

exponential decrease can be very slow.



# Consumer-Resource Models: Predator-Prey

$$\frac{dn_1}{dt} = r\ n_1 - a\ c\ n_1\ n_2 \quad = 0$$

$$\frac{dn_2}{dt} = \epsilon\ a\ c\ n_1\ n_2 - \delta\ n_2 \quad = 0$$

$0 = 0$ ✓

$$\frac{1}{n_1} r n_1 = a c n_1 n_2 \frac{1}{n_1}$$

$$\delta n_2 = \varepsilon a c n_1 n_2$$

assme that $n_2 = 0$

$r n_1 = 0 \Rightarrow n_1 = 0$ $\boxed{(0,0)}$

What is an equilibrium solution to these equations?

assume $n_1 = 0$ $\delta n_2 = 0$ $\Rightarrow n_2 = 0$

$\boxed{\text{no predators, no prey : } (0,0)}$

assume $n_1 \neq 0$ $r = a c n_2$

$\underset{\text{growth rate}}{r} = \underset{\text{rate of predation}}{a c n_2} \rightarrow \boxed{n_2 = \frac{r}{ac}}$

$$\frac{\delta \cancel{r}}{\cancel{ac}} = \varepsilon a c n_1 \frac{\cancel{r}}{\cancel{ac}} \quad \boxed{n_1 = \frac{\delta}{\varepsilon a c}}$$

# Working Backward: Equations to Interpretation

By now, we're pretty good at taking a system and working forward, from behavior into a diagram and then into equations.

Sometimes, it's useful to be able to work in reverse, by taking an equation, and then interpreting the equation in terms of biology.

For example, here's the logistic equation with constant hunting or harvesting.

$$\frac{dn}{dt} = rn(t)\left(1 - \frac{n(t)}{K}\right) - \theta$$

# Working Backward: Equations to Interpretation

However, all 3 of these equations could be described as "logistic growth with constant hunting or harvesting." So what's the difference in *meaning*?

1. $$\frac{dn}{dt} = rn(t)\left(1 - \frac{n(t)}{K}\right) - \theta$$

2. $$\frac{dn}{dt} = rn(t)\left(1 - \frac{n(t)}{K}\right) - Hn(t)$$

3. $$\frac{dn}{dt} = rn(t)\left(1 - \frac{n(t)}{K}\right) - H(n(t) - P)$$

- What is the story being told by each of these?
- What can we say about the "mechanism" of constant harvesting implied by each equation?