

Problem 1.

- (i) Let $x, y \in \text{Tor}(M)$ and $r \in R$. We need to show that $rx + y \in \text{Tor}(M)$ in order to show that $\text{Tor}(M)$ is a submodule of M . Since both $x, y \in \text{Tor}(M)$, then there exists nonzero $r_x, r_y \in R$ such that $r_x x = 0$ and $r_y y = 0$. Now we show that $r_x r_y (rx + y) = 0$:

$$\begin{aligned} r_x r_y (rx + y) &= r_x r_y r x + r_x r_y y \\ &= r_x r_y r x + r_x (r_y y) \\ &= r_x r_y r x \\ &= r_y r (r_x x) = 0 \end{aligned}$$

And since both r_x and r_y are nonzero therefore $r_x r_y$ is nonzero as well. This proves that $rx + y \in \text{Tor}(M)$ and thus $\text{Tor}(M)$ is a submodule of M .

- (ii) Consider \mathbb{Z}_6 as a \mathbb{Z}_6 module. By definition we have:

$$\text{Tor}(\mathbb{Z}_6) = \{2, 3, 4\}$$

Which doesn't even form a group, let alone a submodule of M .

- (iii) Let $m \neq 0 \in M$, where M is a R -module. And let $r_1 r_2 = 0$ where $r_1, r_2 \in R$ and $r_1, r_2 \neq 0$. If $r_2 m = 0$ then we are done since $m \in \text{Tor}(M)$. Otherwise consider $(r_1 r_2)m = r_1(r_2 m) = 0$ Since $r_2 m \neq 0$ then $r_2 m \in \text{Tor}(M)$. This completes the proof.

Problem 2.

Let $x, y \in \bigcup_{k=1}^{\infty} N_k$ and $r \in R$. Therefore there exists $i, j \in \mathbb{N}$ such that $x \in N_i$ and $y \in N_j$. WLOG we can assume $i \leq j$. Since $N_i \subseteq N_j$ therefore we have $x, y \in N_j$. Since N_j is a submodule of M then we have:

$$\begin{aligned} rx + y &\in N_j \subseteq \bigcup_{k=1}^{\infty} N_k \\ \implies rx + y &\in \bigcup_{k=1}^{\infty} N_k \end{aligned}$$

Which shows that it is a submodule of M .

Problem 3.

Let M be an R -module, where scalar product is non-trivial. Now consider the $M \times M$ as a R -module. With componentwise addition and multiplication.

$$\begin{aligned} M_1 &= \{(m, 0) | m \in M\} \\ M_2 &= \{(0, m) | m \in M\} \end{aligned}$$

M_1 and M_2 are both sub-modules of $M \times M$ since:

$$\begin{aligned}\forall x, y \in M_1, \forall r \in R : rx + y &= r(m_x, 0) + (m_y, 0) \\ &= (rm_x, 0) + (m_y, 0) \\ &= (rm_x + m_y, 0) \in M_1\end{aligned}$$

Last lines is a direct result of M being a R -module. Now consider $M_1 \cup M_2$. And $x \neq 0 \in M_1$, $y \neq 0 \in M_2$ and $r \in R$ such that $rx \neq 0$:

$$rx + y = r(m_x, 0) + (0, m_y) = (rm_x, m_y)$$

Since both $rm_x \neq 0$ and $m_y \neq 0$, then $(rm_x, m_y) \notin M_1 \cup M_2$. This shows that $M_1 \cup M_2$ is not a sub-module of $M \times M$.

Problem 4.

Again consider $M = \mathbb{Z}_6$ as a \mathbb{Z}_6 -module. And let $I = N = M$ We have $Tor_I(N) = \{2, 3, 4\}$ and $Ann_I(N) = \{0\}$ Thus $Tor_I(N) \neq Ann_I(N)$.

Problem 5.

(a) Let $r \in Ann_{\mathbb{Z}}(M)$. Then for any $(a, b, c) \in M$ we have $r(a, b, c) = (0, 0, 0)$. Let $a = 1_{\mathbb{Z}_{24}}, b = 1_{\mathbb{Z}_{15}}, c = 1_{\mathbb{Z}_{50}}$. Therefore we have:

$$r(a, b, c) = (r, r, r) = (0, 0, 0)$$

This shows that:

$$\left. \begin{array}{l} 24 \mid r \\ 15 \mid r \\ 50 \mid r \end{array} \right\} \implies 2^3 \times 3 \times 5^2 = 600 \mid r \implies r \in 600\mathbb{Z}$$

And since for any $r \in 600\mathbb{Z}$ we have:

$$600(a, b, c) = (600a, 600b, 600c) = (0, 0, 0)$$

Which implies $r \in Ann_{\mathbb{Z}}(M)$, then we have $Ann_{\mathbb{Z}}(M) = 600\mathbb{Z}$.

(b) If $(a, b, c) \in Ann_M(2\mathbb{Z})$ then we have:

$$\begin{aligned}2 \in 2\mathbb{Z} : 2(a, b, c) &= (2a, 2b, 2c) = (0, 0, 0) \\ \implies a &\in \{0, 12\} \cong \mathbb{Z}_2 \\ b &\in \{0\} \cong \mathbb{Z}_1 \\ c &\in \{0, 25\} \cong \mathbb{Z}_2\end{aligned}$$

This shows that $Ann_M(2\mathbb{Z}) \cong \mathbb{Z}_2 \times \mathbb{Z}_2$. It is easy to see that for any $2i \in 2\mathbb{Z}$ it works.

Problem 6.

Let $\varphi \in \text{Hom}_{\mathbb{Z}}(\mathbb{Z}_{30}, \mathbb{Z}_{21})$. Then φ is a group homomorphism from \mathbb{Z}_{30} to \mathbb{Z}_{21} . Let r be the generator of \mathbb{Z}_{30} having order 30. Then it must mapped to an element with order s where $s \mid 30$. And since $s \mid 21$ then we can only have $s = 1, 3$. If $s = 3$, there are only two elements with order 3 in \mathbb{Z}_{21} , namely 7 and 14. Also since $R = \mathbb{Z}$ then if φ is a homomorphism we have:

$$\varphi(rm) = \varphi(m + \cdots + m) = r\varphi(m)$$

Thus φ is also a module homomorphism. Therefore any module homomorphism from \mathbb{Z}_{30} to \mathbb{Z}_{21} is uniquely identified with $\varphi(1)$. And we have only 3 such homomorphism:

$$\begin{aligned}\varphi(1) = 7 &\implies \varphi(m) = 7m \pmod{21} \\ \varphi(1) = 14 &\implies \varphi(m) = 14m \pmod{21} \\ \varphi(1) = 0 &\implies \varphi(m) = m \pmod{21}\end{aligned}$$

Which form a group of order 3, \mathbb{Z}_3 .

Now since any \mathbb{Z} -module homomorphism is uniquely identified with $\varphi(1)$, and we have:

$$\left. \begin{aligned} \text{Ord}_{\mathbb{Z}_n}(\varphi(1)) \mid n \\ \text{Ord}_{\mathbb{Z}_n}(\varphi(1)) \mid \text{Ord}_{\mathbb{Z}_m}(1) = m \end{aligned} \right\} \implies \text{Ord}_{\mathbb{Z}_n}(\varphi(1)) \mid (n, m)$$

We only need to show that homomorphisms that map 1 to a members with order r in \mathbb{Z}_n where $r \mid (n, m)$ form the group $\mathbb{Z}_{(m, n)}$. Let $\sigma \in \text{Hom}_{\mathbb{Z}}(\mathbb{Z}_m, \mathbb{Z}_n)$, where $\sigma(1) = s$ and $\text{Ord}(s) = (m, n)$. Now it is easy to see that $\sigma^r(1) = \sigma^{r-1}(s) = \cdots = s^r$. This shows that $\text{Ord}(\sigma) = \text{Ord}(s) = (m, n)$. On the otherhand the number of elements in \mathbb{Z}_n such as s where $\text{Ord}_{\mathbb{Z}_n}(s) \mid (n, m)$ is exactly (m, n) . This proves that σ generates the whole group and thus we have:

$$\text{Hom}_{\mathbb{Z}}(\mathbb{Z}_m, \mathbb{Z}_n) \cong \mathbb{Z}_{(m, n)}$$

Problem 7.

First we show that eM and $(1 - e)M$ is submodule of M :

$$\begin{aligned}x, y \in M &\implies ex, ey \in eM, r \in R \\ r(ex) + ey &= rex + ey = erx + ey = e(rx + y) \in eM\end{aligned}$$

Note that we used $re = er$ and also that since M is module then $rx + y \in M$. Similarly for $(1 - e)M$ we have:

$$\begin{aligned}r((1 - e)x) + (1 - e)y &= rx - rex + y - ey \\ &= rx + y - (erx - ey) \\ &= rx + y - e(rx - y) \in (1 - e)M\end{aligned}$$

It is obvious that since both eM and $(1-e)M$ are R module, so is $eM \oplus (1-e)M$. It remains to show an isomorphism:

$$\begin{aligned}
\varphi : M &\rightarrow eM \oplus (1-e)M \\
m &\mapsto (em, (1-e)m) \\
\varphi(m+m') &= (e(m+m'), (1-e)(m+m')) \\
&= (em, (1-e)m) + (em', (1-e)m') \\
&= \varphi(m) + \varphi(m'). \\
\varphi(rm) &= (erm, (1-e)rm) \\
&= (rem, (r-re)m) \\
&= (rem, r(1-e)m) \\
&= r(em, (1-e)m) = r\varphi(m).
\end{aligned}$$

This shows that φ is a homomorphism. Now to show that it is an isomorphism we have to show that it is both surjective and injective. Suppose $em + (1-e)m' \in eM \oplus (1-e)M$. For any $m, m' \in M$ we know $em + (1-e)m' \in M$. Thus we have:

$$\begin{aligned}
\varphi(em + (1-e)m') &= (e(em + (1-e)m'), (1-e)(em + (1-e)m')) \\
&= ((e^2m + (e-e^2)m'), (e-e^2)m + (1-2e+e^2)m')
\end{aligned}$$

Since $e^2 = e$:

$$= (em, (1-e)m')$$

This gives us that φ is surjective. For injective suppose $m, m' \in M$ where $m \neq m'$:

$$\begin{aligned}
\varphi(m) &= \varphi(m') \\
\implies (em, (1-e)m) &= (em', (1-e)m') \\
\implies em &= em', (1-e)m = (1-e)m' \\
\implies e(m-m') &= 0, (1-e)(m-m') = 0 \\
\implies e(m-m') + (1-e)(m-m') &= 0 \\
\implies m-m' &= 0 \implies m = m'
\end{aligned}$$

Which is a contradiction. This gives us that φ is bijective as well. Therefore φ is an isomorphism and thus we have:

$$M \cong eM \oplus (1-e)M$$

Problem 8.

- (i) Suppose that R/I is free module with basis $E \subseteq R/I$ and I is non-trivial. Let $r+I \in E$. And let $s_1, s_2 \in I$ such that $s_1 \neq s_2$ and $s_1r + I = s_2r + I$. This always exists since I is non-trivial. Since $r+I \in E$ then we must have: $s_1(r+I) \neq s_2(r+I)$. But we have:

$$s_1(r+I) = s_1r + I = s_2r + I = s_2(r+I)$$

Which is a contradiction. Thus R/I is not a free module.

- (ii) (\Leftarrow) If I is a principal ideal of R then we have $I = \langle a \rangle$ for some $a \in R$. Since I is an ideal of R then it's also a submodule of R . We show that $\{a\}$ is a basis for I over R . Any element in I is of the form ra for some $r \in R$. We have to show that this is unique. Since $\{a\}$ is linearly independent. Suppose there exists some $r_1 \neq r_2 \in R$ such that $r_1a = r_2a$:

$$r_1a = r_2a \implies ar_1 = ar_2 \implies a(r_1 - r_2) = 0$$

And since R is a integral domain then either $a = 0$ or $r_1 = r_2$. And since $a = 0$ makes a trivial ideal I then we have $r_1 = r_2$. Therefore I is a free module over R .

(\Rightarrow) If I is a free module over R with basis E . If E has more than 1 element such a and b , then we have:

$$ba + (-a)b \in R$$

Since R is integral domain then we have: $ba - ab = 0$. And since $a, b \in E$ then they are linearly independent. Which means that $b = -a = 0$. Which is a contradiction supposing a and b are different elements in E . This shows that E has exactly 1 element a . And each element in I can be expressed as ra for some $r \in R$. This shows that $I = \langle a \rangle$. And thus I is a principal ideal of R .

Problem 9.

We prove that it is a module homomorphism. Consider φ :

$$\begin{aligned} \varphi : Hom_R(M_1 \times M_2, N) &\rightarrow Hom_R(M_1, N) \times Hom_R(M_2, N) \\ \sigma(-, -) &\mapsto (\sigma(-, 0), \sigma(0, -)) \end{aligned}$$

It is obvious that both $\sigma(-, 0)$ and $\sigma(0, -)$ are homomorphism. It remains to show that φ is a homomorphism:

$$\begin{aligned} \varphi(\sigma + \delta)(a, b) &= ((\sigma + \delta)(a, 0), (\sigma + \delta)(0, b)) \\ &= (\sigma(a, 0) + \delta(a, 0), \sigma(0, b) + \delta(0, b)) \\ &= (\sigma(a, 0), \sigma(0, b)) + (\delta(a, 0), \delta(0, b)) \\ &= \varphi(\sigma)(a, b) + \varphi(\delta)(a, b) \end{aligned}$$

Also

$$\begin{aligned} \varphi(r\sigma)(a, b) &= (r\sigma(a, 0), r\sigma(0, b)) \\ &= r(\sigma(a, 0), \sigma(0, b)) \\ &= r\varphi(\sigma)(a, b) \end{aligned}$$

Thus φ is homomorphism. To show that it is an isomorphism we show that φ has an inverse.

$$\begin{aligned} \psi : Hom_R(M_1, N) \times Hom_R(M_2, N) &\rightarrow Hom_R(M_1 \times M_2, N) \\ (\sigma, \delta) &\mapsto \pi \end{aligned}$$

Where $\pi(a, b) = (\sigma(a), \delta(b))$. It is obvious that $\varphi^{-1} = \psi$. First we have to show that π is a homomorphism.

$$\begin{aligned}
\pi((a_1, b_1) + (a_2, b_2)) &= \pi(a_1 + a_2, b_1 + b_2) \\
&= (\sigma(a_1 + a_2), \delta(b_1 + b_2)) \\
&= (\sigma(a_1) + \sigma(a_2), \delta(b_1) + \delta(b_2)) \\
&= (\sigma(a_1), \delta(b_1)) + (\sigma(a_2), \delta(b_2)) \\
&= \pi(a_1, b_1) + \pi(a_2, b_2). \\
\pi(r(a_1, b_1)) &= \pi((ra_1, rb_1)) \\
&= (\sigma(ra_1), \delta(rb_1)) \\
&= (r\sigma(a_1), r\delta(b_1)) \\
&= r(\sigma(a_1), \delta(b_1)) \\
&= r\pi(a_1, b_1).
\end{aligned}$$

This shows that π is a homomorphism. Now to show that ψ is homomorphism.

$$\begin{aligned}
\psi((\sigma_1, \delta_1) + (\sigma_2, \delta_2))(a, b) &= \psi(\sigma_1 + \sigma_2, \delta_1 + \delta_2)(a, b) \\
&= ((\sigma_1 + \sigma_2)(a), (\delta_1 + \delta_2)(b)) \\
&= (\sigma_1(a) + \sigma_2(a), \delta_1(b) + \delta_2(b)) \\
&= (\sigma_1(a), \delta_1(b)) + (\sigma_2(a), \delta_2(b)) \\
&= \psi(\sigma_1, \delta_1)(a, b) + \psi(\sigma_2, \delta_2)(a, b)
\end{aligned}$$

Also:

$$\begin{aligned}
\psi(r(\sigma, \delta))(a, b) &= \psi(r\sigma, r\delta)(a, b) \\
&= (r\sigma(a), r\delta(b)) \\
&= r(\sigma(a), \delta(b)) \\
&= r\psi(\sigma, \delta)(a, b)
\end{aligned}$$

This shows that ψ is a homomorphism. And since it is the inverse of φ , therefore φ is an module isomorphism. Thus we have:

$$Hom_R(M_1 \times M_2, N) \cong_R Hom_R(M_1, N) \times Hom_R(M_2, N)$$

Problem 10.