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Problem 1.

(*i*) Let $H_n(e, x, t)$ be the predicate for halting problem for program e with input x after t steps. We know that H_n is computable. Now if W_x is finite, then it has a maximum, thus we have:

$$\begin{aligned} e \in Fin &\iff \exists_y \forall_z (l(z) > y \to \overline{H_n}(e, l(z), r(z))) \\ &\implies Fin \in \Sigma_2 \end{aligned}$$

In other words there exists some y such that for any z such that $l(z) > y$, then $\phi_e(l(z)) \uparrow$ or $l(z) \notin W_e$.

(*ii*) We use the same H_n from the previous part:

$$\begin{aligned} e \in Inf &\iff \forall_y \exists_z (l(z) > y \land H_n(e, l(z), r(z))) \\ &\implies Inf \in \Pi_2 \end{aligned}$$

In other words for all y there exists some z such that $l(z) > y$ and $\phi_e(l(z)) \downarrow$ or $l(z) \in W_e$.

(*iii*) If $e \in Cof$ it means that there exists some y such that for any $z > y$, $\phi_e(z) \downarrow$. Here we use the S_n predicate with $S_n(e, x, y, t)$ which is the halting problem for program e with input x and output y after t steps.

$$\begin{aligned} e \in Cof &\iff \exists_y \forall_z \exists_v (z > y \to S_n(e, z, l(v), l(t))) \\ &\implies Cof \in \Sigma_3 \end{aligned}$$

(*iv*) We use the same S_n from the last part:

$$\begin{aligned} e \in Total &\iff \forall_y \exists_z (S_n(e, y, l(z), r(y))) \\ &\implies Total \in \Pi_2 \end{aligned}$$

Which means that if ϕ_e is total, then for any y there exists some z such that $\phi_e(y)$ halts after $r(z)$ steps with output $l(z)$.

(*v*) We know that $\Pi_2 \subset \Pi_3$. Know we can also wirte Total from the last part like this:

$$\begin{aligned} e \in Total &\iff \exists_a \forall_y \exists_z (S_n(e, y, l(z), r(z))) \\ &\implies Total \in \Sigma_3 \end{aligned}$$

Therefore we have $Total \in \Delta_3$.

Problem 2.

We rewrite the right hand side:

$$\begin{aligned}
&(\exists y)[\forall z(x < y < z) \wedge (\forall u)(\exists v)(u + v = x + y)] \\
&\equiv (\exists y)[\forall z((x < y < l(z)) \wedge \exists v(r(z) + v = x + y))] \\
&\equiv (\exists y)(\forall z)[(x < y < l(z)) \wedge (\exists v)(r(z) + v = x + y)] \\
&\equiv (\exists y)(\forall z)(\exists v)[(x < y < l(z)) \wedge (r(z) + v = x + y)] \\
&\equiv (\exists y)(\forall z)(\exists v)T(x, y, z, v)
\end{aligned}$$

And since T is easily seen to be decidable, thus $A \in \Sigma_3$.

Problem 3.

(i) Since $A \in \Sigma_n$ then we have:

$$x \in A \iff (\exists y_1)(\forall y_2)\dots[T(x, y_1, y_2, \dots, y_n)]$$

Where T is a decidable predicate. And since $B \leq_m A$ then there exists some total computable function $\delta : \mathbb{N} \to \mathbb{N}$ such that $x \in B \iff \delta(x) \in A$. Thus we have:

$$x \in B \iff \delta(b) \in A \iff (\exists y_1)(\forall y_2)\dots[T(\delta(x), y_1, y_2, \dots, y_n)]$$

Therfore $B \in \Sigma_n$.

(ii) Suppose we have decidable predicates T and S such that:

$$\begin{aligned}
x \in A &\iff (\exists y_1)(\forall y_2)\dots[T(x, y_1, y_2, \dots, y_n)] \\
x \in B &\iff (\exists y_1)(\forall y_2)\dots[S(x, y_1, y_2, \dots, y_n)] \\
\implies x \in A \cap B &\iff (\exists y_1)(\forall y_2)\dots[T(x, y_1, \dots, y_n) \wedge S(x, y_1, \dots, y_n)] \\
\implies x \in A \cup B &\iff (\exists y_1)(\forall y_2)\dots[T(x, y_1, \dots, y_n) \vee S(x, y_1, \dots, y_n)] \\
&\implies A \cap B, A \cup B \in \Sigma_n
\end{aligned}$$

Problem 4.

Suppose we know there exists some $S \in \Sigma_n$ such that $S \notin \Pi_n$, And we have $P \in \Pi_n$ such that $P \notin \Sigma_n$. Therefore since $\Delta_n = \Sigma_n \cap \Pi_n$ we have $\Delta_n \subsetneq \Sigma_n, \Pi_n$. We also know that $\Sigma_{n-1} \subsetneq \Sigma_n$, And since $\Sigma_{n-1} \subset \Pi_n$, Thus we have $\Sigma_{n-1} \subsetneq \Delta_n$. This proves the problem.

Problem 5.

For this we show that for any $n \in \mathbb{N}$, Σ_n is countable. Which is true since any set in Σ_n can be mapped to a predicate T, (Supposing we fix $(\exists y_1)(\forall y_2)\dots,(\exists y_n)$ and their position in the predicate), And we have only countable number of predicates since their characteristic function is computable. Now for any $n \in \mathbb{N}$ we have that $\Sigma_n, \Pi_n \subset \Sigma_{n+1}$. Which is countable. Now we know that all of the subsets of $\mathbb{N}$ are an uncountable set, And if the statement in the problem is true, then union of some (countable) number of countable sets, is uncountable, which is not possible, This shows that the statement in the problem is wrong.