Elliptic Curves - Spring 2025 Problem Set 3

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Problem 1.

Note that the curve described in the problem is $F(x,y,z) = x^3 + axz^2 + bz^3 - y^2z$. Now first we show that F is smooth iff we have $\Delta = 4a^3 + 27b^2 \neq 0$. First suppose that F is smooth, this means that:

$$\frac{\partial F}{\partial x} = 3x^2 + az^2 \qquad \frac{\partial F}{\partial y} = -2yz \qquad \frac{\partial F}{\partial z} = 3z^2 + 2axz - y^2$$

at least one of the above is nonzero for any point in the curve.

Problem 2.

Problem 3.

Problem 4.

Problem 5.

Let $F(X,Y,Z) = X^3+pY^3+p^2Z^3$. For the sake of contradiction suppose we have some projective point (a,b,c) such that $F(a,b,c) = 0$, where $a,b,c \in \mathbb{Q}$. There exists some $t \in \mathbb{Z}$ such that $ta, tb, tc \in \mathbb{Z}$. Since we have (a,b,c) (ta,tb,tc), then $F(ta,tb,tc) = 0$. This means that F has some integer root (ta,tb,tc). For simplicity put $(a,b,c) = (ta,tb,tc)$. Let us define $\nu_p(a)$ to be the biggest power of p dividing a. In other words, $\nu_p(a) = \alpha$ means that $p^\alpha \mid a$ and $p^{\alpha+1} \nmid a$. Now suppose $\nu_p(a) = \alpha$, $\nu_p(b) = \beta$ and $\nu_p(c) = \gamma$. Then we have $\nu_p(a^3) = 3\alpha$, $\nu_p(pb^3) = 3\beta + 1$ and $\nu_p(p^2c^3) = 3\gamma + 2$. This shows that

$$\nu_p(a^3) \neq \nu_p(pb^3) \neq \nu_p(p^2c^3)$$

WLOG suppose that $\nu_p(a^3) < \nu_p(pb^3) < \nu_p(p^2c^3)$. This means that $\nu_p(a^3+pb^3+p^2c^3) = 3\alpha$. Now:

$$\left.\begin{array}{l} a^3 + pb^3 + p^2c^3 = 0 \\ p^{3\beta+1} \mid 0 \\ p^{3\beta+1} \mid pb^3 + p^2c^3 \end{array}\right\} \implies p^{3\beta+1} \mid a^3$$

But since $\nu_p(a^3) = 3\alpha < 3\beta + 1$, we arrive at a contradiction. This shows that we had not rational root in the first place.