

**Problem 1.** Let  $\mathcal{C}$  be a category. Prove that for any finite diagram  $\mathcal{D} : \mathcal{I} \rightarrow \mathcal{C}$  there exists  $\lim_{\leftarrow \mathcal{I}} \mathcal{D}$  iff category  $\mathcal{C}$  has a Terminal object and for any two morphisms with the same domain and codomain, there exists an Equalizer and also for any  $A, B \in \mathcal{C}_0$ ,  $A \times B \in \mathcal{C}_0$ .

( $\Rightarrow$ ) Suppose for any finite diagram limit exists.

- (i) let  $\mathcal{I}_0 = \{\}$ . since  $X = \lim_{\leftarrow \mathcal{I}_0} \mathcal{D}_0$  exists, and the image of  $\mathcal{I}$  in  $\mathcal{C}$  is empty, therefore any object of  $\mathcal{C}$  like  $Y$  with any morphisms, commutes with this empty Image of  $\mathcal{I}$ . Which means that there exists a unique morphism  $Y \xrightarrow{\varphi} X$ . Therefore we can say that there exists a unique morphism from any object of  $\mathcal{C}$  to  $X$ . This shows that  $X$  is terminal object of the category  $\mathcal{C}$ .
- (ii) Let  $A, B \in \mathcal{C}_0$ . Let  $\mathcal{I}$  be the category of 2 with only identity morphisms.

$$\begin{array}{ccc} \text{id} & & \text{id} \\ \Downarrow & & \Downarrow \\ \cdot & & * \end{array}$$

Where  $\mathcal{D}$  maps  $\cdot$  to  $A$  and  $*$  to  $B$ . Let  $X = \lim_{\leftarrow \mathcal{I}} \mathcal{D}$  with morphisms  $P_A$  and  $P_B$  be the limit of this diagram. Now for any  $f$  and  $g$  we know that there exists a unique morphism  $\varphi$  from  $Y$  to  $X$  such that the diagram commutes.

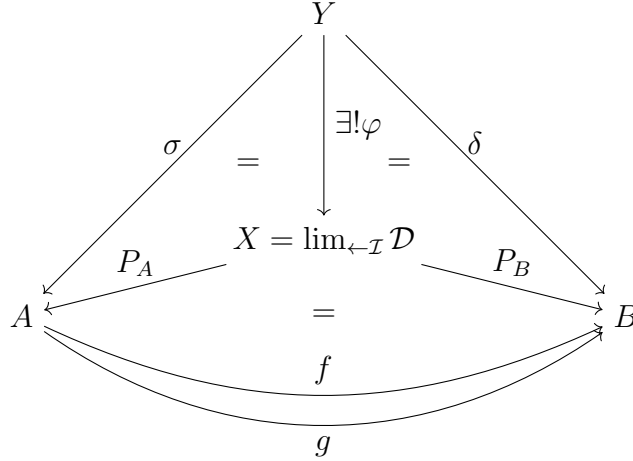
$$\begin{array}{ccccc} & & Y & & \\ & \swarrow f & \downarrow \exists! \varphi & \searrow g & \\ & & X = \lim_{\leftarrow \mathcal{I}} \mathcal{D} & & \\ & \swarrow P_A & & \searrow P_B & \\ A & & & & B \\ \uparrow \text{id} & & & & \uparrow \text{id} \end{array}$$

Since for any  $f$  and  $g$ ,  $Y$  commutes with the image of  $\mathcal{I}$ , therefore we can say that for any  $f$  and  $g$  there exists a unique morphism from  $Y$  to  $X$  such that the diagram above commutes. With this description  $X$  is the product of  $A$  and  $B$ .

- (iii) As for equalizer of two morphisms  $f$  and  $g$  from  $A$  to  $B$ , let  $\mathcal{I}$  be the category of 2 with 2 non-trivial morphisms.

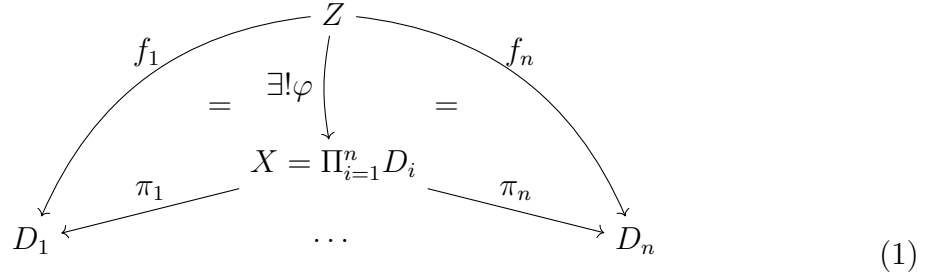
$$\begin{array}{ccc} \cdot & \xrightarrow{\quad} & * \\ & \curvearrowright & \\ & \xrightarrow{\quad} & \end{array}$$

And let functor  $\mathcal{D}$  be in a way that maps  $\cdot$  to  $A$  and  $*$  to  $B$  and two non-trivial morphisms to  $f$  and  $g$ . Let  $X = \lim_{\leftarrow \mathcal{I}} \mathcal{D}$  and morphisms  $P_A$  and  $P_B$  be the limit of this diagram  $\mathcal{D}$ .



Since  $X$  commutes with image of  $\mathcal{I}$  we have  $fP_A = gP_A = P_B$ . Now let  $Y$  be in a way such that it commutes with Image of  $\mathcal{I}$ . Then we have:  $f\sigma = g\sigma = \delta$ . Now since  $X$  is the limit of this diagram we know that there exists a unique  $\varphi$  from  $Y$  to  $X$  such that it commutes. This shows that  $(X, P_A)$  is exactly the equalizer of  $f$  and  $g$ . Since we have  $fP_A = gP_A$ . And also for any other  $Y$  and  $\sigma$  such that  $f\sigma = g\sigma$  there exists a unique morphism from  $Y$  to  $X$  such that  $\sigma = P_A\varphi$ .

( $\Leftarrow$ ) Now suppose we have all 3 properties in  $\mathcal{C}$ . We want to prove that for any arbitrary finite index category  $\mathcal{I}$  and diagram  $\mathcal{D}$  there exists a limit. Let all of the objects in image of  $\mathcal{I}$  be  $D_1, D_2, \dots, D_n$ . Since product of any number of objects is available in  $\mathcal{C}$  consider  $X = \prod_{i=1}^n D_i$ . With morphisms  $\pi_i$ . But this diagram doesn't necessarily commute. For any  $(Z, f_1, \dots, f_n)$  where  $Z \xrightarrow{f_i} D_i$ , that commutes with image of  $\mathcal{I}$  there exists a unique morphism from  $Z$  to  $X$  such that the diagram commutes.



Now suppose object  $Y$  with morphisms  $\pi_1, \dots, \pi_n$  to  $D_1, \dots, D_n$  where for the set  $U$  of morphisms in image of  $\mathcal{I}$  diagram commutes. in other words:

$$\begin{aligned} \forall k \in U \quad s.t \quad D_i \xrightarrow{k} D_j \\ \implies \pi_j = k\pi_i \end{aligned}$$

Also for any  $(Z, f_1, \dots, f_n)$  that commutes with image of  $\mathcal{I}$  then there exists a unique morphism ( $\varphi$ ) from  $Z$  to  $Y$  such that the diagram commutes or:  $\forall i : f_i = \pi_i\varphi$ . Let  $k$  be a morphism in image of  $\mathcal{I}$  where  $k : D_i \rightarrow D_j \notin U$ . This means that  $\pi_j \neq k\pi_i$ . Now let  $(Y', \sigma)$  be the equalizer of  $\pi_j$  and  $k\pi_i$ . Therefore  $\pi_j\sigma = k\pi_i\sigma$ . We will show that  $Y'$

has all the properties of  $Y$  with a bigger  $U$ .

Let  $(Z, f_1, \dots, f_n)$  be in such a way that it commutes with image of  $\mathcal{I}$ . Then we know there exists a unique morphism from  $Z$  to  $Y$  such that the diagram commutes.

$$\begin{array}{c}
 \begin{array}{ccccc}
 & & Z & & \\
 & \swarrow & \downarrow \exists! \delta & \searrow & \\
 & & Y' & & \\
 & \swarrow \exists! \varphi & \downarrow \sigma & \searrow & \\
 & & Y & & \\
 \begin{array}{ccc}
 \swarrow \pi_i & & \searrow \pi_j \\
 D_i & & D_j
 \end{array}
 \end{array}
 \\
 \begin{array}{ccc}
 \text{Curved arrow } f_i: Z \rightarrow D_i & \text{Curved arrow } f_j: Z \rightarrow D_j & \text{Curved arrow } k: D_i \rightarrow D_j
 \end{array}
 \end{array}
 \quad (2)$$

$$\left. \begin{array}{l} f_j = k f_i \\ f_i = \pi_i \varphi \\ f_j = \pi_j \varphi \end{array} \right\} \implies \pi_j \varphi = k \pi_i \varphi$$

And since  $Y'$  is equalizer of  $\pi_j$  and  $k\pi_i$  then there exists a unique morphism ( $\delta$ ) from  $Z$  to  $Y'$  such that  $\varphi = \sigma\delta$ . If we name  $\pi'_i = \pi_i\sigma$  it is easy to see that  $\pi'_j = k\pi'_i$ . And also for any  $r \in U$  that  $D_x \xrightarrow{r} D_y$  then we know  $\pi_y = r\pi_x$ . therefore  $\pi'_y = \pi_y\sigma = r\pi_x\sigma = r\pi'_x$ . Therefore  $(Y', \pi'_1, \dots, \pi'_n)$  commutes with a bigger set of morphisms in image of  $\mathcal{I}$ . and also  $\delta$  commutes with the diagram since:

$$\pi'_i \delta = \pi_i \sigma \delta = \pi_i \varphi = f_i$$

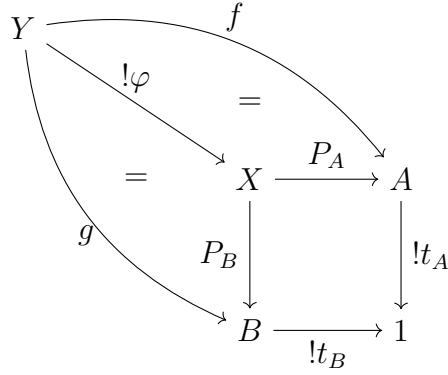
Also uniqueness of  $\delta$  follows from uniqueness of  $\varphi$ . This shows that we can obtain the object with a  $U$  consisting of all of morphisms in image of  $\mathcal{I}$ . Which shows that  $\lim_{\leftarrow \mathcal{I}} \mathcal{D}$  exists.

**Problem 2.** Let  $\mathcal{C}$  be a category. Prove that for any finite diagram  $\mathcal{D} : \mathcal{I} \rightarrow \mathcal{C}$  there exists  $\lim_{\leftarrow \mathcal{I}} \mathcal{D}$  iff category  $\mathcal{C}$  has terminal object and for any two morphisms with the same co-domain, there exists pull-back.

We just have to show that the set of pull-back and terminal is the same as the set of terminal, product and equalizer.

(i) Pull-back, Terminal  $\implies$  Product

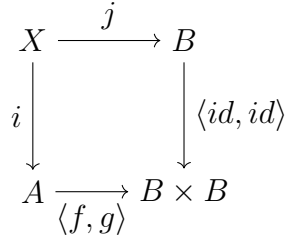
If  $A, B \in \mathcal{C}_0$  we can obtain  $A \times B$  with this pull-back:



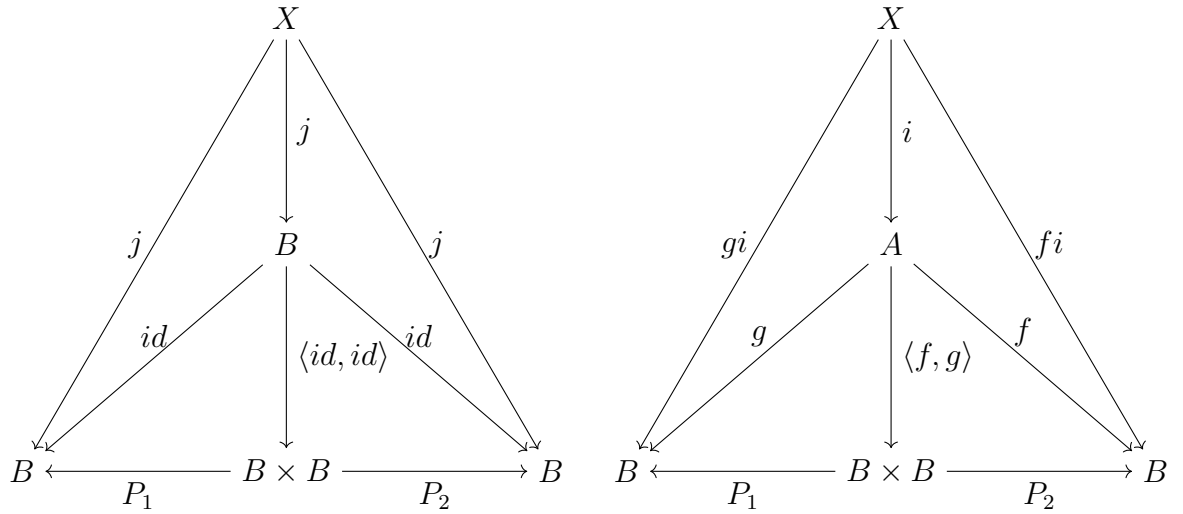
Since for any  $f$  and  $g$  the diagram commutes since  $1$  is terminal object and there is only one morphism from  $Y$  to  $1$ . And since  $X$  is the pull-back then there exists a unique morphism from  $Y$  to  $X$  such that it commutes with  $f$  and  $g$ . This shows that  $X$  is the product of  $A$  and  $B$ .

(ii) Pull-back  $\Rightarrow$  Equalizer

Consider the product  $B \times B$ . We want to find the equalizer of  $f, g : A \rightarrow B$ . Let  $X$  be the pull-back of  $\langle f, g \rangle$  and  $\langle id, id \rangle$ .



With  $\langle f, g \rangle i = \langle id, id \rangle j$ . Where  $\langle f, g \rangle$  and  $\langle id, id \rangle$  are defined below.



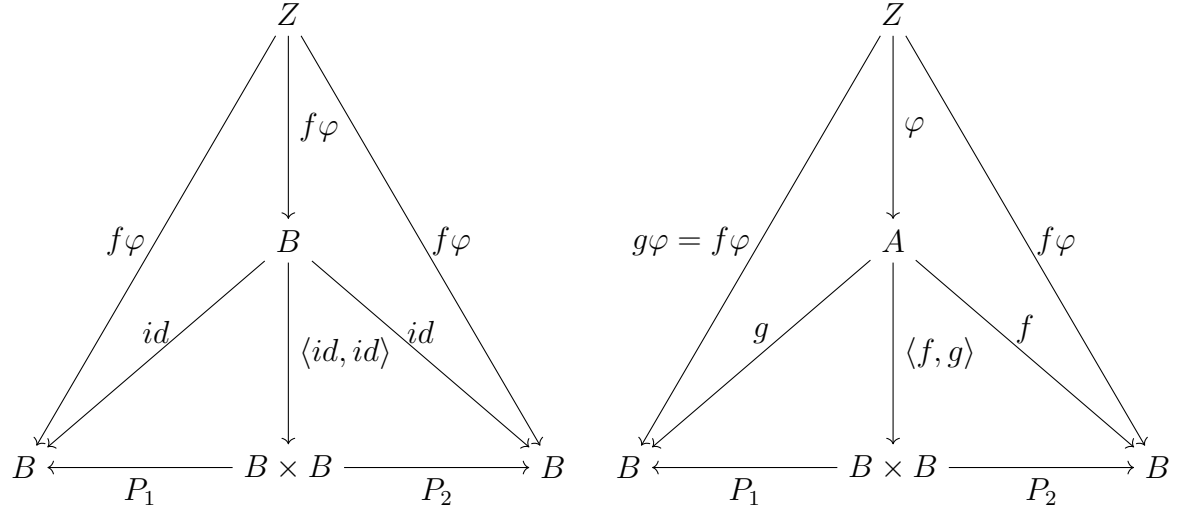
First we show that  $fi = gi$ . Since both of these diagrams commute then we know

that  $j = P_1 \langle id, id \rangle j = P_2 \langle id, id \rangle j$ . Now we can replace  $\langle id, id \rangle j$  with  $\langle f, g \rangle i$ .

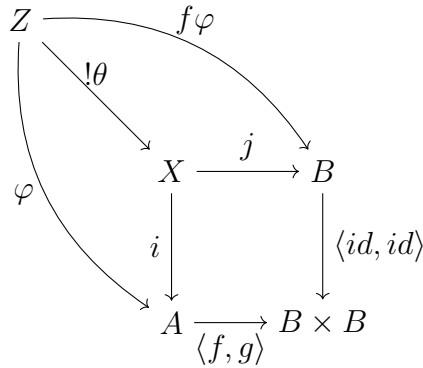
$$P_1 \langle f, g \rangle i = P_2 \langle f, g \rangle i$$

But we know  $gi = P_1 \langle f, g \rangle i$  and  $fi = P_2 \langle f, g \rangle i$ . This shows that  $gi = fi$ .

Now let  $(Z, \varphi)$  in a way that  $Z \xrightarrow{\varphi} A$  and  $f\varphi = g\varphi$  where  $f\varphi$  is a morphism from  $Z$  to  $B$ . we want to show that  $\langle id, id \rangle f\varphi = \langle f, g \rangle \varphi$ . since both are morphisms from  $Z$  to  $B \times B$  and they both commute with the diagram below, and since product is unique then they are equal.



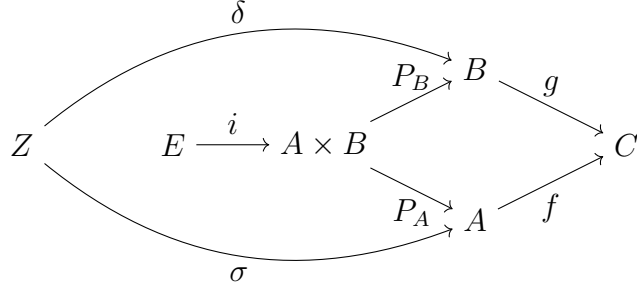
Since they are the same diagram then  $\langle id, id \rangle f\varphi = \langle f, g \rangle \varphi$ . then  $Z$  commutes with pull-back diagram and therefore there exists a unique morphism  $(\theta)$  from  $Z$  to  $X$  in a way that  $i\theta = \varphi$ .



this shows that  $(X, i)$  is the equalizer of  $f, g$ .

(iii) Product, Equalizer  $\Rightarrow$  Pull-back

This part is similar to the proof of limit, with equalizer and product. Suppose  $f : A \rightarrow C$  and  $g : B \rightarrow C$ . Let  $X = A \times B$ . and let  $(E, i)$  be the equalizer of  $fP_A$  and  $gP_B$ .



$E$  is the pull-back. it is easy to see that  $gP_Bi = fP_Ai$  since  $E$  is the equalizer. this shows that the diagram commutes. Now for any  $Z$  such that commutes with this diagram, There exists a unique morphism ( $\varphi$ ) from  $Z$  to  $A \times B$  such that the diagram commutes.

$$\begin{aligned}
 g\delta &= f\sigma \\
 \implies gP_B\varphi &= fP_A\varphi
 \end{aligned}$$

this shows that  $\varphi$  also makes  $gP_B$  and  $fP_A$  equal. since  $E$  is the equalizer then there exists a unique morphism from  $Z$  to  $E$  such that the diagram commutes. Therefore  $E$  is the pull-back for  $f$  and  $g$ .