Elliptic Curves - Spring 2025 Problem Set 5
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Problem 1.

For absolute value functions (multiplicative valuations) we have the Ostrowski's Theorem. For additive valuation, let v be some additive valuation on $\mathbb{Q}$. Then we know that $v(1) = v(1) + v(1)$, and $v(1) = 0$. Then $|\ |_v$ is a non-archimedean absolute value function:

$$|x|_v = (\frac{1}{e})^{v(x)}$$

By Ostrowski's Theorem this function is equivalent to some p-adic valueation, meaning there exists some $\alpha > 0$ such that $|\ |_v^\alpha = |\ |_p$ for some prime p. Then if we use ln in both sides of the equation above we get:

$$\ln |x|_p^\alpha = -v(x) \implies v(x) = -\alpha \ln |x|_p$$

We will show that for any α this v is a valuation function:

$$\begin{aligned} v(xy) = -\alpha \ln |xy|_p = -\alpha \ln |x|_p |y|_p &= -\alpha(\ln |x|_p + \ln |y|_p) \\ &= -\alpha \ln |x|_p - \alpha \ln |y|_p \\ &= v(x) + v(y) \end{aligned}$$

WLOG suppose $|x|_p \leq |y|_p$, then $|x + y|_p \geq \max\{|x|_p, |y|_p\} = |y|_p$:

$$v(x+y) = -\alpha \ln |x+y|_p \leq -\alpha \ln |y|_p = \min\{-\alpha \ln |y|_p, -\alpha \ln |x|_p\} = \min\{v(y), v(x)\}$$

And lastly:

$$\begin{gathered} v(x) = \infty \implies 0 = (\frac{1}{e})^{v(x)} = |x|_p^\alpha \implies |x|_p = 0 \implies x = 0 \\ x = 0 \implies v(x) = -\alpha \ln |0|_p = -\alpha \ln 0 = \infty \\ v(x) = \infty \iff x = 0 \end{gathered}$$

This proves that any additive valuation is of the form $-\alpha \ln |x|_p$ for some prime p and positive α.

Problem 2.

Suppose there exists some $a \in \mathbb{C}$ such that $v(x - a) > 0$. Then for any other $b \in \mathbb{C}$ we have that $v(x - b) = 0$, since $a - b \in \mathbb{C}$:

$$0 = v(a-b) = v(x - b - (x-a)) \geq \min\{v(x-b), v(x-a)\}$$

Now if $v(x-a) = v(x-b)$, then this is impossible since $0 > v(x-a)$ is wrong. And if $v(x-a) \neq v(x-b)$, then we know that:

$$v(x-b-(x-a)) = \min\{v(x-a), v(x-b)\}$$

Then again if $v(x-b) > v(x-a)$, then $0 > v(x-a)$ which can not happen, hence $v(x-b) < v(x-a)$ and therefore $v(a-b) = v(x-b) = 0$. This valuation gives us the R_v such that:

$$R_v = \{\frac{f}{g}\big|f, g \in \mathbb{C}[x], \gcd(f,g) = 1, x-a \mid f\}$$

Or in other words, all elements of $\mathbb{C}(x)$, such that $x-a$ divides it, since we can write any f by the form of multiplication of $x-\alpha$ such that α is a root of f:

$$f = (x-\alpha_1)\dots(x-\alpha_n)$$
$$v(f) = v(x-\alpha_1) + v(x-\alpha_2) + \dots + v(x-\alpha_n)$$

This means that only we have to find value of v in $x-a$ for all $a \in \mathbb{C}$. Now suppose there is no $x-a$ such that $v(x-a) > 0$. If there is some element a such that $v(x-a) < 0$, then with a similar argument we have that for any $b \in \mathbb{C}$ that $v(x-b) \neq v(x-a)$, then:

$$0 = v(a-b) = v(x-b-(x-a)) = \min\{v(x-b), v(x-a)\}$$

But knowing that $v(x-a) < 0$, then we know that $\min\{v(x-a), v(x-b)\} < 0$. This is a contradiction. And the only case remaining is that if for any $a \in \mathbb{C}$, we have the same value for $v(x-a)$. This is indeed a valuation, which counts the number of roots (not distinct) in a polynomial in $\mathbb{C}[x]$, for $\mathbb{C}(x)$ simply we have $v(\frac{f}{g}) = v(f) - v(g)$. And the corresponding R_v is all $\frac{f}{g}$ such that $f, g \in \mathbb{C}[x]$ and g has equal or more number of roots than f:

$$R_v = \{\frac{f}{g}\big|f = (x-\alpha_1)\dots(x-\alpha_n), g = (x-\beta_1)\dots(x-\beta_m), m \geq n\}$$

And of course the trivial valuation, which for any $a \in \mathbb{C}$, we have $v(x-a) = 0$. This gives us that $R_v = \mathbb{C}(x)$, as valuation for any element is 0.

Problem 3.

We know that $B = \{x \in \mathbb{Q}_p\big||x|_p \leq 1\}$ is the disjoint union of open balls around $0, 1, \dots, p-1$, with radius 1. Using this fact:

$$f(x) = \begin{cases} 0 & |x|_p > 1 \\ i & x \in B(i,1) \end{cases}$$

is the desired function. It is not constant, so we only need to check that if it is localy constant. Let $x \in \mathbb{Q}_p$. If $|x|_p > 1$, then $x \notin B$. Since B is both closed and open, then $\overline{B}$ is also closed and open. Then there x is contained in an open $(\overline{B})$ such that the function is constant over that open. Similarly for any $x \in B(i,1)$, we have that x is contained in an open that f is contant over it. And we are done. **Bugggg**

Problem 4.

It is enough to show that for any $\epsilon > 0$, we have a finite covering with balls of radius ϵ. Now ϵ can only acquire values of the form p^{α}. with integer $\alpha \leq 0$. Let $\epsilon = p^{-r}$, then we have:

$$\mathbb{Z}_p = B(0, \epsilon) \cup \cdots \cup B(p^r - 1, \epsilon)$$

Problem 5.

(i)

(ii)

(iii)