Graph Theory - Fall 2025 Problem Set 3

Amin Zolfagharian - 610002072

Problem 1.

We use the fact that in every tree G, for any two vertices $u, v \in V(G)$, there exists a unique path from u to v. Now since $e \in E(\overline{G})$, then u and v are not connected in G. Let P be the path between u and v in G. We know that e is not in P, hence $P + e$ is a cycle in $G + e$.

Now suppose that C is a cycle in $G + e$. Since G does not contain any cycle, then e is a part of C. Therefore u and v are two consecutive vertices in C. Since $C - e$ is contained in G, then it is the unique path from u to v, and C is the same path we introduced earlier. This shows that exists only 1 cylce in $G + e$.

Problem 3.

We prove this by induction on n. For $n = 1, 2$ this is trivial.

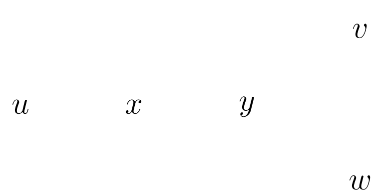
Lemma 1. *$\overline{C_{n+1}}$ is a subgraph of $\overline{C_{n+2}}$.*

Proof. Suppose that v and u are connected in $\overline{C_{n+1}}$. This means that u and v are not connected in C_{n+1}. This means that u and v are not adjacent in the cylce C_{n+1}. Let this cycle be $v_1, v_2, \ldots, v_{n+1}$ such that $u = v_i$ and $v = v_j$ and $|i - j| \neq 1$. Now if we add any vertex to this cycle, u and v still are going to be disconnected in C_{n+2} meaning that they are connected in $\overline{C_{n+2}}$. This shows that $E(\overline{C_{n+1}}) \subseteq E(\overline{C_{n+2}})$. $\square$

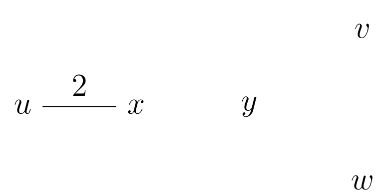
Now let T_n be a tree of order n. There exists some vertex w such that $deg(w) = 1$. Then $T_n - w$ is a tree of order $n - 1$. By induction hypothesis, $T_n - w$ is a subgraph of $\overline{C_{n+1}}$. Now just add w in the cycle in a way that it is not adjacent with its neighbor in T_n. Therefore T_n is a subgraph of $\overline{C_{n+2}}$ and we are done!

Problem 5.

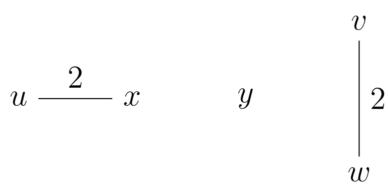
Kruskal's algorithm: Each step we add the smallest valid edge:



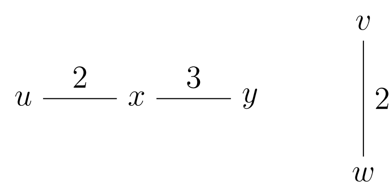
(1)



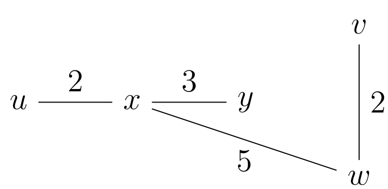
(2)



(3)

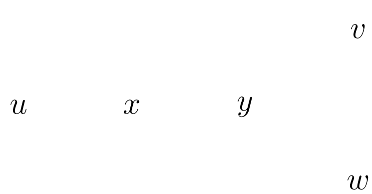


(4)

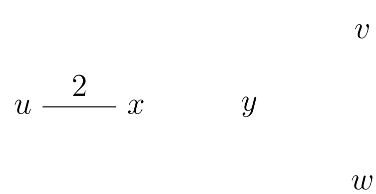


(5)

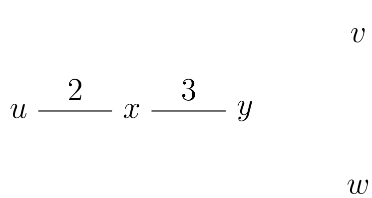
Prim's algorithm: Start from a random vertex, at each step add an edge with the least weight which is connected to previous vertices.



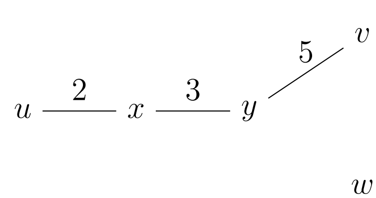
(1)



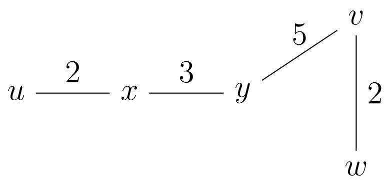
(2)



(3)



(4)



(5)

Problem 7.

We prove that thte answer is $n^{n-3}(n-2)$. Suppose that the edge e is removed from K_n. And let two ends of e be vertices u and v. We know that K_n has n^{n-2} spanning trees. Now we count the number of spanning trees that have e as an edge. Note that a tree containing e, is actually two trees, one having v as a vertex and one having u as a vertex. Thus we can partition the rest of the $n-2$ vertices into to groups in $\sum_{k=1}^{n-1} \binom{n-1}{k-1}$ way. And for each partition of i vertices we have i^{i-2} spanning trees. This means that the number of spanning trees including e is:

$$\sum_{k=1}^{n-1} \binom{k-1}{n-1} k^{k-2}(n-k)^{n-k-2} = 2n^{n-3}$$

Giving us the remainig $n^{n-2} - 2n^{n-3}$ spanning trees in K_n not including e. Therefore K_n without a single edge has exactly $n^{n-3}(n-2)$ spanning trees.