

Problem 1.

We have:

$$\alpha = a_0 + \frac{1}{a_1 + \frac{1}{a_2 + \dots}}$$

Thus we have:

$$\begin{aligned} -\alpha &= -a_0 - \frac{1}{a_1 + \frac{1}{a_2 + \dots}} = -(a_0 + 1) + 1 - \frac{1}{a_1 + \frac{1}{a_2 + \dots}} \\ &= -(a_0 + 1) + \frac{a_1 - 1 + \frac{1}{a_2 + \dots}}{a_1 + \frac{1}{a_2 + \dots}} \end{aligned}$$

Let $\beta = \frac{1}{a_2 + \dots}$. No if $a_1 = 1$ then we have:

$$= -(a_0 + 1) + \frac{\beta}{1 + \beta} = -(a_0 + 1) + \frac{1}{1 + \frac{1}{\beta}} = -(a_0 + 1) + \frac{1}{1 + a_2 + \frac{1}{a_3 + \dots}}$$

Thus we have: $-\alpha = [-a_0 - 1; a_2 + 1, a_3, \dots]$. And if $a_1 > 1$ then we have:

$$\begin{aligned} &= -(a_0 + 1) + \frac{a_1 - 1 + \beta}{a_1 + \beta} = -(a_0 + 1) + \frac{1}{\frac{a_1 + \beta}{a_1 - 1 + \beta}} \\ &= -(a_0 + 1) + \frac{1}{1 + \frac{1}{a_1 - 1 + \beta}} \end{aligned}$$

Thus we have: $-\alpha = [-a_0 - 1; 1, a_1 - 1, a_2, a_3, \dots]$.

Problem 2.

- (i) First note that since $\alpha > 1$ then if we have $\alpha = [a_0; a_1, a_2, \dots]$, then $a_0 > 0$. Now consider $\frac{1}{\alpha}$:

$$\begin{aligned} \frac{1}{\alpha} &= \frac{1}{a_0 + \frac{1}{a_1 + \dots}} \\ \implies \frac{1}{\alpha} &= [0; a_0, a_1, \dots] \end{aligned}$$

And since $a_0 > 0$ this is valid. Now if we have $[0; a_1, a_2, \dots, a_k]$ which is the $(k+1)$ -th fraction for $\frac{1}{\alpha}$, we can write:

$$0 + \frac{1}{a_0 + \frac{1}{\frac{\vdots}{a_k}}} = \frac{1}{\frac{p_k}{q_k}}$$

Where $\frac{p_k}{q_k}$ is the k -th fraction for α . This completes the proof.

(ii) Similar to the last part, we have $\alpha = [a_0; a_1, \dots, a_k]$ where $a_0 > 0$ since $\alpha > 1$. Note that the fraction is finite since α is a rational number. Now we have:

$$\frac{1}{\alpha} = \frac{1}{a_0 + \frac{1}{a_1 + \dots}}$$

Which shows that $\frac{1}{\alpha} = [0; a_0, a_1, \dots, a_k]$.

Problem 3.

Problem 4.

We use the fact that $2 < \sqrt{7} < 3$:

$$\begin{aligned} \frac{\sqrt{7}+1}{2} &= 1 + \frac{\sqrt{7}-1}{2} = 1 + \frac{1}{\frac{2}{\sqrt{7}-1}} \\ &= 1 + \frac{1}{\frac{2(\sqrt{7}+1)}{7-1}} = 1 + \frac{1}{\frac{\sqrt{7}+1}{3}} \\ &= 1 + \frac{1}{1 + \frac{\sqrt{7}-2}{3}} = 1 + \frac{1}{1 + \frac{1}{\frac{3}{\sqrt{7}-2}}} \end{aligned}$$

And we have:

$$\begin{aligned} \frac{3}{\sqrt{7}-2} &= \frac{3(\sqrt{7}+2)}{7-4} = \sqrt{7}+2 = 4 + \sqrt{7}-2 \\ &= 4 + \frac{7-4}{\sqrt{7}+2} = 4 + \frac{1}{\frac{\sqrt{7}+2}{3}} = 4 + \frac{1}{1 + \frac{\sqrt{7}-1}{3}} \end{aligned}$$

Which we can write:

$$\frac{\sqrt{7}-1}{3} = \frac{1}{\frac{3}{\sqrt{7}-1}} = \frac{1}{\frac{3(\sqrt{7}+1)}{7-1}} = \frac{1}{\frac{\sqrt{7}+1}{2}}$$

Which is what we started with, thus we can write:

$$\frac{\sqrt{7}+1}{2} = [1, 1, 4, 1]$$

Problem 5.

First we have to find the $\overline{1, 3}$ part.

$$\begin{aligned} x &= 1 + \frac{1}{3 + \frac{1}{x}} \implies x = 1 + \frac{x}{3x+1} \\ \implies x &= \frac{4x+1}{3x+1} \implies 3x^2 - 3x - 1 = 0 \\ \implies x &= \frac{3 \pm \sqrt{9+12}}{6} \end{aligned}$$

And since $x > 0$ then we have $x = \frac{3+\sqrt{21}}{6}$. Now we can find the value of the fraction:

$$\alpha = 1 + \frac{1}{5 + \frac{1}{2 + \frac{1}{3 + \frac{1}{x}}}} = 1 + \frac{1}{5 + \frac{1}{2 + \frac{x}{3x+1}}} = 1 + \frac{1}{5 + \frac{3x+1}{7x+2}} = 1 + \frac{7x+2}{38x+11} = \frac{45x+13}{38x+11}$$

And if we let $x = \frac{3+\sqrt{21}}{6}$:

$$\alpha = \frac{45(\frac{3+\sqrt{21}}{6}) + 13}{38(\frac{3+\sqrt{21}}{6}) + 11} = \frac{2430 + 6\sqrt{21}}{2074}$$

Problem 6.

Note that $d < \sqrt{d^2 + 1} < d + 1$:

$$\begin{aligned} \sqrt{d^2 + 1} &= d + \sqrt{d^2 + 1} - d = d + \frac{d^2 + 1 - d^2}{\sqrt{d^2 + 1} + d} = d + \frac{1}{\sqrt{d^2 + 1} + d} \\ &= d + \frac{1}{2d + \sqrt{d^2 + 1} - d} \end{aligned}$$

Thus we have:

$$\sqrt{d^2 + 1} - d = \frac{1}{2d + (\sqrt{d^2 + 1} - d)} \implies \sqrt{d^2 + 1} - d = [\overline{2d}]$$

Therefore $\sqrt{d^2 + 1} = d + (\sqrt{d^2 + 1} - d) = [d; \overline{2d}]$.