

**Problem 1.**

For absolute value functions (multiplicative valuations) we have the Ostrowski's Theorem. For additive valuation, let  $v$  be some additive valuation on  $\mathbb{Q}$ . Then we know that  $v(1) = v(1) + v(1)$ , and  $v(1) = 0$ . Then  $|\cdot|_v$  is a non-archimedean absolute value function:

$$|x|_v = \left(\frac{1}{e}\right)^{v(x)}$$

By Ostrowski's Theorem this function is equivalent to some  $p$ -adic valuation, meaning there exists some  $\alpha > 0$  such that  $|\cdot|_v^\alpha = |\cdot|_p$  for some prime  $p$ . Then if we use  $\ln$  in both sides of the equation above we get:

$$\ln |x|_p^\alpha = -v(x) \implies v(x) = -\alpha \ln |x|_p$$

We will show that for any  $\alpha$  this  $v$  is a valuation function:

$$\begin{aligned} v(xy) &= -\alpha \ln |xy|_p = -\alpha \ln |x|_p |y|_p = -\alpha (\ln |x|_p + \ln |y|_p) \\ &= -\alpha \ln |x|_p - \alpha \ln |y|_p \\ &= v(x) + v(y) \end{aligned}$$

WLOG suppose  $|x|_p \leq |y|_p$ , then  $|x + y|_p \geq \max\{|x|_p, |y|_p\} = |y|_p$ :

$$v(x + y) = -\alpha \ln |x + y|_p \leq -\alpha \ln |y|_p = \min\{-\alpha \ln |y|_p, -\alpha \ln |x|_p\} = \min\{v(y), v(x)\}$$

And lastly:

$$\begin{aligned} v(x) = \infty &\implies 0 = \left(\frac{1}{e}\right)^{v(x)} = |x|_p^\alpha \implies |x|_p = 0 \implies x = 0 \\ x = 0 &\implies v(x) = -\alpha \ln |0|_p = -\alpha \ln 0 = \infty \\ v(x) = \infty &\iff x = 0 \end{aligned}$$

This proves that any additive valuation is of the form  $-\alpha \ln |x|_p$  for some prime  $p$  and positive  $\alpha$ .

**Problem 2.**

First suppose that there exists  $a \neq b$  such that  $v(x - a) \neq 0$  and  $v(x - b) \neq 0$ . Then we have:

$$\begin{aligned} v(ax - ab) &= v(a) + v(x - b) = v(x - b) \neq 0 \\ v(bx - ba) &= v(b) + v(x - a) = v(x - a) \neq 0 \end{aligned}$$

Then we can write:

$$0 = v(b - a) = v((x - a) - (x - b)) \geq \min\{v(x - a), v(x - b)\}$$

**Problem 3.**

**Problem 4.**

**Problem 5.**

**(i)**

**(ii)**

**(iii)**