Amin Zolfagharian - 610002072

Problem 1.

(i) We only need to check the submodule criterion. First of all note that $Tor(M) \neq \varnothing$ since $0_M \in Tor(M)$. Now for any $x, y \in Tor(M)$ and $r \in R$ we have to show that $x + ry \in Tor(M)$. Now since $x, y \in Tor(M)$, there there exists $r_1, r_2 \in R$ such that $r_1 \neq 0$ and $r_2 \neq 0$ and $r_1 x = 0$ and $r_2 y = 0$. Now consider the nonzero element $r_1 r_2$:

$$r_1 r_2(x + ry) = r_1 r_2 x + r_1 r_2 r y = r_2(r_1 x) + r_1 r(r_2 y) = 0 + 0 = 0$$

This shows that $x + ry \in Tor(M)$. Note that we used the fact that R is commutative since it is an integral domain.

(ii) Consider $\mathbb{Z}_6$ as a $\mathbb{Z}_6$ module. It is clear that $Tor(M) = \{0, 2, 3, 4\}$. But this is clearly not a submodule of M since it is not a subgroup of M.

(iii) Since $M \neq 0$, then there exists $m \neq 0 \in M$. And since R has a zero-divisor, then we have nonzero $r_1, r_2 \in R$ such that $r_1 r_2 = 0$. Now since M is a R-module, then $r_2 m \in M$. Now if $r_2 m = 0$, then $m \in Tor(M)$ proving that $Tor(M) \neq 0$, otherwise, then $r_2 m \in Tor(M)$, since $r_1(r_2 m) = 0m = 0$. Therefore in either case $Tor(M) \neq 0$.

Problem 2.

($\Rightarrow$) If $I \subset Ann_R(M)$, then we have to show that M is a R/I module. Note that it has all properties of modules since I is an ideal of R. The only thing that we have to check is it being well-defined. For this purpose suppose that $r + I = r' + I$. Then we have to show that for any $m \in M$, $rm = r'm$. But this is obvious since we can write $r = r' + i$ for some $i \in I$. Now we have:

$$rm = (r' + i)m = r'm + im = r'm$$

The last part follows from the fact that $I \subset Ann_R(M)$.

($\Leftarrow$) Now suppose that M is a R/I module. Let $i \in I$. Then for any $r \in R$, we have that $\overline{r} = \overline{r + i}$. Then by definition for any $m \in M$ we have:

$$rm = (r + i)m \implies im = 0$$

This shows that for any $m \in M$, we have $im = 0$, thus $i \in Ann_R(M)$. Since i was an arbitrary element of I, then we showed that $I \subset Ann_R(M)$.

Problem 3.

(i) Consider $\mathbb{Z}$-modules, $\mathbb{Z}$ and $\mathbb{Z}_2$. The homomorphism $\varphi : \mathbb{Z} \to \mathbb{Z}_2$, such that

$$\varphi(x) = x \pmod 2$$

It is clear that $ker\varphi = 2\mathbb{Z}$. Which follows that:

$$\frac{\mathbb{Z}}{ker\varphi} \cong \mathbb{Z}_2$$

On the other hand to show that $\mathbb{Z} \ncong \mathbb{Z}_2 \oplus 2\mathbb{Z}$, it suffices to show that some element on the right hand side has order 2, while no element on the left hand side is of finite order. The element (1, 0) in $\mathbb{Z}_2 \oplus 2\mathbb{Z}$ has order two. This concludes the problem.

(ii) Consider the following homomorphism:

$$\varphi : \frac{S_1 \oplus S_2 \oplus \cdots \oplus S_n}{T_1 \oplus T_2 \oplus \cdots \oplus T_n} \to \frac{S_1}{T_1} \oplus \cdots \oplus \frac{S_n}{T_n}$$
$$(s_1, s_2, \ldots, s_n) + T_1 \oplus T_2 \oplus \cdots \oplus T_n \mapsto (s_1 + T_1, s_2 + T_2, \ldots, s_n + T_n)$$

To show that this is indeed a module homomorphism we have to show that for any x, y and $r \in R$ we have $\varphi(x + y) = \varphi(x) + \varphi(y)$ and $\varphi(rx) = r\varphi(x)$:

$$\begin{aligned}
\varphi\Bigg(\big((s_1, s_2, \ldots, s_n) + T_1 \oplus T_2 \oplus \cdots \oplus T_n\big) + \big((r_1, r_2, \ldots, r_n) + T_1 \oplus T_2 \oplus \cdots \oplus T_n\big) \Bigg) \\
= \varphi\Bigg((s_1 + r_1, s_2 + r_2, \ldots, s_n + r_n) + T_1 \oplus \cdots \oplus T_n \Bigg) \\
= (s_1 + r_2 + T_1, \ldots, s_n + r_n + T_n) \\
= (s_1 + T_1, \ldots, s_n + T_n) + (r_1 + T_1, \ldots, r_n + T_n) \\
= \varphi\Bigg((s_1, s_2, \ldots, s_n) + T_1 \oplus \cdots \oplus T_n \Bigg) + \varphi\Bigg((r_1, r_2, \ldots, r_n) + T_1 \oplus \cdots \oplus T_n \Bigg)
\end{aligned}$$

And for $\varphi(rx) = r\varphi(x)$:

$$\begin{aligned}
\varphi\Bigg(r\big((s_1, s_2, \ldots, s_n) + T_1 \oplus \ldots, \oplus T_n\big) \Bigg) &= \varphi\Bigg((rs_1, rs_2, \ldots, rs_n) + T_1 \oplus \cdots \oplus T_n \Bigg) \\
&= (rs_1 + T_1, \ldots, rs_n + T_n) \\
&= r(s_1 + T_1, \ldots, s_n + T_n) \\
&= r\varphi\Bigg((s_1, \ldots, s_n) + T_1 \oplus \cdots \oplus T_n \Bigg)
\end{aligned}$$

Now we have to show that φ is an isomorphism. For this we show that it is injective and surjective. The surjectivity is clear since for any $s_1 \in S_1, \ldots, s_n \in S_n$, and element b:

$$b = (s_1 + T_1, s_2 + T_2, \ldots, s_n + T_n) \in \frac{S_1}{T_1} \oplus \cdots \oplus \frac{S_n}{T_n}$$

There exist the a element:

$$a = (s_1, s_2, \dots, s_n) + T_1 \oplus \cdots \oplus T_n \in \frac{S_1 \oplus S_2 \oplus \cdots \oplus S_n}{T_1 \oplus T_2 \oplus \cdots \oplus T_n}$$

such that $\varphi(a) = b$. This proves that φ is surjective. To show that it is injective suppose that $\varphi(a) = \varphi(b)$, where $a = (a_1, \dots, a_n) + T_1 \oplus \cdots \oplus T_n$ and $b = (b_1, \dots, b_n) + T_1 \oplus \cdots \oplus T_n$. From the equality $\varphi(a) = \varphi(b)$ we have that:

$$(a_1 + T_1, \dots, a_n + T_n) = (b_1 + T_1, \dots, b_n + T_n)$$
$$\implies \forall_{1 \le i \le n} : a_i - b_i \in T_i$$

Which proves that $a = b$. Thus φ is also injective, making it an isomorphism and we are done.

Problem 4.

(i) Let $R = \mathbb{Z}_6$ and $M = \mathbb{Z}_6$. Then number 3 is not linearly independent since there exists a nonzero solution to the equation:

$$r3 = 0$$

Which is $r = 2$. This shows that the statement is not true for modules.

(ii) Again we can use the example from the previous part for $\mathbb{Z}_6$ over $\mathbb{Z}_6$ as module. The set $\{2, 3\}$ is not linearly independent since:

$$0 \times 2 + 4 \times 3 = 0$$

But there is no element $x \in \mathbb{Z}_6$ such that $2x = 3$.

(iii) Take $\mathbb{Z}$ as a $\mathbb{Z}$ module. Notice that $\{2\}$ is a linearly independent set, since there is no nonzero element $x \in \mathbb{Z}$ such that $2x = 0$. To show that this set is also maximal, suppose that the set $\{2, a\}$ is linearly independent for some $a \in \mathbb{Z}$. Then it can be seen that:

$$a \times 2 + (-2) \times a = 0$$

reaching to a contradiction. Now that $\{2\}$ is a maximal independent set, it can be seen that it doesn't generate the whole $\mathbb{Z}$ since it only generates even numbers.

(iv) A basis must be linearly independent, but in the previous part, we showed an example of a maximal linearly independent set that did not generate $\mathbb{Z}$. Since it was maximal, then it can't be extended to another linearly independent set that generates the space.

(v) Consider $A = R[x_1, x_2, \dots]$ as a A-module where R has 1. Then it is finitely generated by 1_R. But consider the submodule $\langle x_1, x_2, \dots \rangle$. This is not finitely generated.

Problem 5.

Todo!

Problem 6.

(i) Any $\mathbb{Z}$-module homomorphism from $\mathbb{Z}_m$ to $\mathbb{Z}_n$ is uniquely identified with $\varphi(1)$ since for any $k \in \mathbb{Z}_m$:

$$\varphi(k) = \varphi(k \times 1) = k\varphi(1)$$

Now note that $\varphi(1) \in \mathbb{Z}_n$, hence $Ord(\varphi(1)) \mid n$. On the other hand since 1_m is of order m in $\mathbb{Z}_m$, then we have $Ord(\varphi(1)) \mid m$. This means that:

$$Ord(\varphi(1)) \mid (m, n)$$

Now we know that the number elements in $\mathbb{Z}_n$ such that their order divides (n, m) is exactly (m, n). So we only need to show that the group is cyclic. To show this we introduce an element with order (m, n). This proves the statement. For this consider an element s of order (m, n). Let $\sigma : \mathbb{Z}_m \to \mathbb{Z}_n$ such that $\sigma(1) = s$.

$$\begin{aligned} \sigma(1) &= s \\ \sigma^2(1) &= \sigma(s) = s\sigma(1) = s^2 \\ &\vdots \\ \sigma^i(1) &= s^i \end{aligned}$$

This shows that order of σ is the same as order of s in $\mathbb{Z}_n$ which is (m, n). Therefore we found an element of order (m, n) in $Hom(\mathbb{Z}_m, \mathbb{Z}_n)$, thus we are done:

$$Hom(\mathbb{Z}_m, \mathbb{Z}_n) = \mathbb{Z}_{(m,n)}$$

(ii) First note that $(ker f, i)$ is one of these pairs, where i is the inclusion map from $ker f$ to X. To show that this is a valid pair, first we have to show that $ker f$ is a submodule of X. Suppose $x, y \in ker f$ and $r \in R$. We have to show that $x + ry \in ker f$:

$$\begin{gathered} f(x + ry) = f(x) + f(ry) = f(x) + rf(y) = 0 + r \cdot 0 = 0 \\ \implies x + ry \in ker f \end{gathered}$$

Since the map is inclusion, it is obvious that the diagram commutes. We will prove that this pair in fact has the universal property. Suppose (A, α) is another pair with stated properties. Since $f \circ \alpha = 0$ we get that $im(\alpha) \subset ker f$, meaning that $\alpha : A \to ker f$:

$$\begin{tikzcd} A \arrow[r, "a"] \arrow[rr, bend left=50, "\alpha"] \arrow[rr, phantom, bend left=25, "\circlearrowright"] \arrow[drr, "0"'] & ker f \arrow[r, "i"] \arrow[dr, "0"] & X \arrow[d, "f"] \\ & & Y \end{tikzcd}$$

We must have $\alpha = i \circ a$. Since i is inclusion and both sides are from A to $kerf$, then a is uniquely identified, which is α:

$$\begin{aligned} a : A &\to kerf \\ x &\mapsto \alpha(x) \end{aligned}$$

Now since α is exactly a, since $im(\alpha) \subset kerf$, we don't have to check for a being a homomorphism. And we are done!