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Problem 1.

(i) Let $\langle s_1, s_2, \dots, s_n \rangle$ be the generators of M_1.

$$0 \to M_1 \xrightarrow{\varphi} M \xrightarrow{\psi} M_2 \to 0$$

We know that $ker\psi = im\varphi$. Therefore we can generate $ker\psi$ with $\varphi(s_1), \varphi(s_2), \dots, \varphi(s_n)$. Now since ψ is surjective, we have that:

$$\frac{M}{ker\psi} = M_2$$

Since M_2 is finitely generated, then $M/ker\psi$ is also finitely generated. Therefore the set of cosets is finitely generated. Let the generators be $\langle \overline{t_1}, \overline{t_2}, \dots, \overline{t_k} \rangle$. We argue that $\langle \varphi(s_1), \dots, \varphi(s_n), t_1, \dots, t_k \rangle$ generates M. Let $m \in M$. m belongs to one of cosets, thus we can write:

$$m + ker\psi = r_1\overline{t_1} + \cdots + r_k\overline{t_k} + ker\psi$$
$$\implies m - r_1t_1 - \cdots - r_kt_k = i$$

for some i in $ker\psi$. Since $ker\psi = im\varphi$, then we can write:

$$i = u_1\varphi(s_1) + \cdots + u_n\varphi(s_n)$$

This gives us that m is generated by the generator we introduced:

$$m = r_1t_1 + \cdots + r_kt_k + u_1\varphi(s_1) + \cdots + u_n\varphi(s_n)$$

Thus M is finitely generated.

However it is not necessarily true that if M is finitely generated, then M_1 and M_2 are finitely generated as well. An example of this could be the R-modules below:

$$0 \to R[x_1, x_2, \dots] \xrightarrow{\sigma_0} R \xrightarrow{id} R \to 0$$

Where R has 1, and σ_0 is valuation at 0 for all x_is.

(ii)

Problem 2.

(i) Consider Φ:

$$\Phi : \mathrm{Hom}_R\Big(\bigoplus_{i\in I} M_i, N\Big) \to \bigotimes_{i\in I} \mathrm{Hom}_R(M_i, N)$$

Then for each $\sigma \in \mathrm{Hom}_R\big(\bigoplus_{i\in I} M_i, N\big)$, we have that for any $m_i \in M_i$ for all $i \in I$, there exists some $n \in N$ such that:

$$\sigma(m_1, m_2, \dots) = n$$

Now define $\sigma_i = \sigma(0, 0, \dots, 0, m_i, 0, 0, \dots)$. Meaning it only one nonzero entry in ith place. Now we map:

$$\Phi : \sigma \mapsto \bigotimes_{i\in I} \sigma_i$$

Now it is easy to see that for any $\sigma, \delta \in \mathrm{Hom}_R\big(\bigoplus_{i\in I} M_i, N\big)$, we have:

$$\Phi(\sigma + \delta) = \bigotimes_{i\in I}(\sigma + \delta)_i = \bigotimes_{i\in I}(\sigma_i + \delta_i) = \bigotimes_{i\in I}\sigma_i + \bigotimes_{i\in I}\delta_i = \Phi(\sigma) + \Phi(\delta)$$

Thus Φ is a homomorphism. Now to see that it is surjective as well suppose $\bigotimes_{i\in I}\psi_i \in \bigotimes_{i\in I}\mathrm{Hom}_R(M_i, N)$, then let us introduce $\psi \in \mathrm{Hom}_R\big(\bigotimes_{i\in I} M_i, N\big)$ such that for any $\bigotimes_{i\in I} m_i$ we have:

$$\psi(\bigotimes_{i\in I} m_i) = \sum_{i\in I}\psi_i(m_i)$$

This proves the surjectivity, and thus these are isomorphic as <u>groups</u>.

(ii)

Problem 3.

(i)

Problem 4.

(i)

Problem 5.

Problem 6.

(i) We need to introduce a homomorphism from V to $U \oplus W$.

$$\begin{tikzcd}
0 \arrow[r] & U \arrow[r, "\varphi"] \arrow[d, "id_U"'] & V \arrow[r, "\sigma"] \arrow[d] & W \arrow[r] \arrow[d, "id_W"] & 0 \\
0 \arrow[r] & U \arrow[r, "p_1"'] & U \oplus W \arrow[r, "p_2"'] & W \arrow[r] & 0
\end{tikzcd}$$

by exactness in the sequence we know that:

$$ker(\varphi) = 0 \qquad Im(\varphi) = ker(\sigma) \qquad Im(\sigma) = W$$

Each of these homomorphisms are uniquely identified with how they behave on a the basis. Define:

$$\begin{aligned} \psi : V &\to U \oplus W \\ v &\mapsto \langle \varphi^{-1}(v), \sigma(v) \rangle \end{aligned}$$

Since $ker(\varphi) = 0$, then φ^{-1} is well defined. To show that This means that for any element in $v \in V$, there exist $f_1, f_2, \ldots, f_n \in \mathbb{F}$ such that:

$$v = f_1 v_1 + f_2 v_2 + \cdots + f_n v_n$$