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Problem 1.

(i) (a) Suppose $a = (a_1, a_2, \dots, a_n) \in V(J_1) \cup V(J_2)$. WLOG let $a \in V(J_1)$. This means that for any $P \in K[x_1, \dots, x_n]$ we have $P(a) = 0$. Now for any $Q \in J_1 \cap J_2$, we have $Q \in J_1$, thus $Q(a) = 0$ which implies $a \in V(J_1 \cap J_2)$. This shows that :

$$V(J_1) \cup V(J_2) \subseteq V(J_1 \cap J_2)$$

Now let $a \in V(J_1 \cap J_2)$. And for the sake of contradiction suppose there exists $P \in J_1$ and $Q \in J_2$ such that $P(a) \neq 0$ and $Q(a) \neq 0$. Note that $PQ \in J_1$ and $PQ \in J_2$. Therefore $PQ \in J_1 \cap J_2$. Since $a \in V(J_1 \cap J - 2)$, then we have $PQ(a) = P(a)Q(a) = 0$. This shows that either $P(a) = 0$ or $Q(a) = 0$, which is a contradiction. Tehrefore at least for one of the J_1 and J_2, a is a root for all polynomials in that ideal. Let that be J_1, thus we have $a \in V(J_1) \subseteq V(J_1) \cup V(J_2)$. Now we have $V(J_1 \cap J_2) \subseteq V(J_1) \cup V(J_2)$, hence:

$$V(J_1) \cup V(J_2) = V(J_1 \cap J_2)$$

Again suppose $a \in V(J_1) \cup V(J_2)$, WLOG $a \in V(J_1)$. Consider a member of $J_1 J_2$:

$$\begin{aligned} P_i \in J_1, Q_i \in J_2 \\ R = P_1Q_1 + P_2Q_2 + \dots P_lQ_l \in J_1J_2 \\ \implies R(a) = P_1(a)Q_1(a) + \dots + P_l(a)Q_l(a) \\ = 0Q_1(a) + \dots + 0Q_l(a) = 0 \end{aligned}$$

And since R was an arbitrary member of J_1J_2, then $a \in V(J_1J_2)$. Then $V(J_1) \cup V(J_2) \subseteq V(J_1J_2)$. Conversely suppose $a \in V(J_1J_2)$. And again for the sake of contradiction suppose $P \in J_1$ and $Q \in J_2$ such that $P(a) \neq 0$ and $Q(a) \neq 0$. Since $PQ \in J_1J_2$, then $P(a)Q(a) = 0$. Which is a contradiction. Therefore at least one of J_1 and J_2 have a as a root. Thus $a \in V(J_1)$ or $a \in V(J_2)$, which means $a \in V(J_1) \cup V(J_2)$. This shows that $V(J_1J_2) \subseteq V(J_1) \cup V(J_2)$. This proves that:

$$V(J_1) \cup V(J_2) = V(J_1J_2)$$

(b) Let $a \in V(\sum_{\lambda \in I} J_\lambda)$. Now for any $P \in J_i$, since $P + 0 + 0 + \dots + 0 \in \sum_{\lambda \in I} J_\lambda$ then $P(a) = 0$. Therefore for any $P \in J_i$ we have $P(a) = 0$, Therefore $a \in V(J_i)$, hence $a \in \bigcap_{\lambda \in I} V(J_\lambda)$, therefore $V(\sum_{\lambda \in I} J_\lambda) \subseteq \bigcap_{\lambda \in I} V(J_\lambda)$. Now suppose $a \in \bigcap_{\lambda \in I} V(J_\lambda)$. Now any element in $\sum_{\lambda \in I} J_\lambda$ is of the form $\sum_{\lambda \in L} P_l$ where $P_\lambda \in J_l$ and L is a finite subset of I. Now since $a \in V(J_\lambda)$ for any λ, then $P_\lambda(a) = 0$ for any $\lambda \in L$. Then we can see that $\sum_{\lambda \in L} P_\lambda(a) = 0$. Thus $a \in V(\sum_{\lambda \in I} J_\lambda)$:

$$V(\sum_{\lambda \in I} J_\lambda) = \bigcap_{\lambda \in I} V(J_\lambda)$$