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Problem 1.

Let G be of order $75 = 5^2 \cdot 3$. Let n_5 be the number of 5-Sylow subgroups of G. By Sylow's theorem we know that $n_5 \overset{5}{\equiv} 1$ and $n_5 \mid 3$. This shows that $n_5 = 1$, which shows that the only 5-Sylow subgroup of G is normal in G. Now we can write G with semi-direct product of P the 5-Sylow subgroup:

$$G = P \rtimes_\varphi H$$

Where H is a subgroup of order 3. Since there only exists one group with order 3, therefore $H \cong \mathbb{Z}_3$:

$$G = P \rtimes_\varphi \mathbb{Z}_3$$

Now P is of order 25, which means that there only exists two groups with this order (up to isomorphism), $\mathbb{Z}_{25}$ and $\mathbb{Z}_5 \times \mathbb{Z}_5$. First suppose $P = \langle \mathbb{Z}_{25}, + \rangle$:

$$\varphi : \mathbb{Z}_3 \to Aut(\mathbb{Z}_{25})$$

Since $Aut(\mathbb{Z}_{25}) \equiv (\mathbb{Z}_{25})^*$, and $(\mathbb{Z}_{25})*$ has no element of order 3, then φ can only be trivial. Therefore in case of $P = \langle \mathbb{Z}_{25}, + \rangle$ we have:

$$G = \mathbb{Z}_{25} \times \mathbb{Z}_3$$

Now suppose $P = \langle \mathbb{Z}_5 \times \mathbb{Z}_5, + \rangle$:

$$\varphi : \mathbb{Z}_3 \to Aut(\mathbb{Z}_5 \times \mathbb{Z}_5)$$

$Aut(\mathbb{Z}_5 \times \mathbb{Z}_5)$ is a group with $24 \times 20 = 480$ elements. Where $3 \mid 480$ and $9 \nmid 480$. Therefore every subgroup of order 3 in this group is a 3-sylow subgroup. If φ is to be a nontrivial homomorphism, then r where $\langle r \rangle = \mathbb{Z}_3$ must be mapped to an element of order 3. In other words we should have $\varphi(3)$ to be an element of order 3. In this case $\varphi(\mathbb{Z}_3) \equiv \mathbb{Z}_3$. Therefore image of $\mathbb{Z}_3$ is a 3-sylow subgroup. Since $\mathbb{Z}_3$ is cyclic and all 3-sylows are conjugate, therefore all such φs will produce the same group (up to isomorphism). Therefore we only need to find one such group. Since all elements of $Aut(\mathbb{Z}_5 \times \mathbb{Z}_5)$ are matrices from $GL_2(F_5)$, Consider the matrix below which has order 3:

$$\begin{pmatrix} 0 & 1 \\ 4 & 4 \end{pmatrix}$$

Now if we let $a = \begin{pmatrix} 1 & 0 \end{pmatrix}$ and $b = \begin{pmatrix} 1 & 0 \end{pmatrix}$, we have that:

$$ara^{-1} = a \cdot r = \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 4 & 4 \end{pmatrix} = \begin{pmatrix} 0 & 1 \end{pmatrix} = b$$

$$brb^{-1} = b \cdot r = \begin{pmatrix} 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 4 & 4 \end{pmatrix} = \begin{pmatrix} 4 & 4 \end{pmatrix} = a^4 b^4$$

$$G = (\mathbb{Z}_5 \times \mathbb{Z}_5) \rtimes_\varphi \mathbb{Z}_3$$

$$G = \langle r, a, b | a^5 = b^5 = r^3 = 1, rar^{-1} = b, rbr^{-1} = a^4 b^4 \rangle$$

And for trivial φ we have:

$$G = (\mathbb{Z}_5 \times \mathbb{Z}_5) \times \mathbb{Z}_3$$

This concludes all the cases:

$$G = \mathbb{Z}_{25} \times \mathbb{Z}_3$$
$$G = \mathbb{Z}_5 \times \mathbb{Z}_5 \times \mathbb{Z}_3$$
$$G = \langle r, a, b | a^5 = b^5 = r^3 = 1, rar^{-1} = b, rbr^{-1} = a^4b^4 \rangle$$

Problem 2.

First we need to find the $Aut(\mathbb{Z})$. Let $\delta : \mathbb{Z} \to \mathbb{Z}$ be an automorphism. Then we have $\delta(0) + \delta(0) = \delta(0 + 0) = \delta(0)$, which shows that $\delta(0) = 0$. Now there exists some a such that $\delta(a) = 1$. Then we have:

$$1 = \delta(a) = \delta(1 + 1 \cdots + 1) = a\delta(1)$$
$$\implies \delta(1) = \frac{1}{a} \in \mathbb{Z}$$
$$\implies a = \pm 1$$

If $\delta(1) = 1$ then it is the identity automorphism.
If $\delta(-1) = 1$ then we have:

$$0 = \delta(1 - 1) = \delta(1) + \delta(-1) = \delta(1) + 1$$
$$\implies \delta(1) = -1$$
$$\forall a \in \mathbb{Z}, a > 0 : \delta(a) = a\delta(1) = -a$$
$$\delta(-a) = a\delta(-1) = a$$

Which is the inversion automorphism with order 2. Now we have: $Aut(\mathbb{Z}) = \mathbb{Z}_2$. To identify $\mathbb{Z} \rtimes \mathbb{Z}$ we need to find all homomorphisms φ such that:

$$\varphi : \mathbb{Z} \to Aut(\mathbb{Z}) \equiv \mathbb{Z}_2$$

Since 1 is the generator of $\mathbb{Z}$ we only need to find $\varphi(1)$ to uniquely find φ. Since 1 has infinite order in $\mathbb{Z}$ then it can be mapped to any element in $Aut(\mathbb{Z})$. If $\varphi(1) = id$ then we have the direct product $\mathbb{Z} \times \mathbb{Z}$. If $\varphi(1) = inv$ then if a and b are both generators of $\mathbb{Z}$ and $\mathbb{Z}$ in $\mathbb{Z} \rtimes \mathbb{Z}$:

$$G = \langle a, b \mid aba^{-1} = b^{-1} \rangle$$

As for center of each group, since $\mathbb{Z} \times \mathbb{Z}$ is abelian, then all of elements are in center. And for G, suppose $s = a^i b^j \in Z(G)$, and since $ab = b^{-1}a$:

$$aa^ib^j = a^ib^ja = a^iab^{-j} \implies b^j = b^{-j} \implies j = 0 \implies s = a^i$$
$$ba^i = a^ib = b^{(-1)^i}a^i \implies b = b^{(-1)^i} \implies 1 = (-1)^i \implies \exists k : i = 2k$$

Therefore for any $s = a^{2k}$, s commutes with a and b and since they are generators of G therefore $s \in Z(G)$. Thus we have:

$$Z(G) = \{a^{2k} | k \in \mathbb{Z}\}$$

Problem 3.

(i) Let $r = (g_1, g_2, \dots, g_n)$ and $s = (h_1, g_2, \dots, g_n)$:

$$\begin{aligned}\phi(rs) = (rs)^p = (g_1h_1, g_2h_2, \dots, g_nh_n)^p &= (g_1^p h_1^p, g_2^p h_2^p \dots, g_n^p h_n^p) \\ &= (g_1^p, g_2^p, \dots, g_n^p)(h_1^p, h_2^p, \dots, h_n^p) \\ &= \phi(r)\phi(s)\end{aligned}$$

This shows that ϕ is a homomorphism.

(ii) If $x = (x_1^{\beta_1}, x_2^{\beta_2}, \dots, x_n^{\beta_n}) \in ker(\phi)$ with $\beta_i \leq p^{\alpha_i}$, then we have:

$$\phi(x) = (x_1^{\beta_1}, x_2^{\beta_2}, \dots, x_n^{\beta_n})^p = (x_1^{p\beta_1}, x_2^{p\beta_2}, \dots, x_n^{p\beta_n}) = (x_1^{p^{\alpha_1}k_1}, x_2^{p^{\alpha_2}k_2}, \dots, x_n^{p^{\alpha_n}k_n})$$
$$\implies p\beta_i = p^{\alpha_i}k_i \implies \beta_i = p^{\alpha_i - 1}k_i$$

And since $\beta_i \leq p^{\alpha_i}$, we have $k_i \leq p$. Therefore we have:

$$ker(\phi) = \{(x_1^{p^{\alpha_1-1}k_1}, x_2^{p^{\alpha_2-1}k_2}, \dots, x_n^{p^{\alpha_n-1}k_n}) | 1 \leq k_i \leq p\}$$

And for image we have:

$$\phi(x) = (x_1^{\beta_1}, x_2^{\beta_2}, \dots, x_n^{\beta_n})^p = (x_1^{p\beta_1}, x_2^{p\beta_2}, \dots, x_n^{p\beta_n})$$
$$im(\phi) = \{(x_1^{p\beta_1}, x_2^{p\beta_2}, \dots, x_n^{p\beta_n}) | \beta_i \in \mathbb{Z}\}$$

(iii) We only need to show that both of these are isomorphic (as vector spaces) to E_{p^n} and prove that E_{p^n} has rank n. And to prove that they are isomorphic as vector spaces, we only need to show that they are isomorphic as groups, since scalar product can be difined accordingly. For $ker\phi$:

$$\varphi : ker(\phi) \to \langle \mathbb{Z}_p \rangle \times \dots \times \langle \mathbb{Z}_p \rangle$$
$$(x_1^{p^{\alpha_1-1}k_1}, \dots, x_n^{p^{\alpha_n-1}k_n}) \to (k_1, k_2, \dots, k_n)$$

To check the homomorphism we have:

$$\begin{aligned}\phi((x_1^{p^{\alpha_1-1}k_1}, \dots, x_n^{p^{\alpha_n-1}k_n}) \cdot (x_1^{p^{\alpha_1-1}h_1}, \dots, x_n^{p^{\alpha_n-1}h_n})) &= \\ &= \phi((x_1^{p^{\alpha_1-1}k_1+h_1}, \dots, x_n^{p^{\alpha_n-1}k_n+h_n})) \\ &= (k_1 + h_1, \dots, k_n + h_n) \\ &= (k_1, \dots, k_n) + (h_1, \dots, h_n) \\ &= \phi((x_1^{p^{\alpha_1-1}k_1}, \dots, x_n^{p^{\alpha_n-1}k_n}))\phi((x_1^{p^{\alpha_1-1}h_1}, \dots, x_n^{p^{\alpha_n-1}h_n}))\end{aligned}$$

It is also trivial that φ is injective and surjective, therefore $ker(\phi) \cong E_{p^n}$.
As for $A/im(\phi)$ we have that for any δ_i, σ_i such that with $1 \leq i \leq n$, $\sigma_i \overset{p}{\equiv} \delta_i$:

$$zv^{-1} = (x_1^{\sigma_1}, \dots, x_n^{\sigma_n})(x_1^{\delta_1}, \dots, x_n^{\delta_n})^{-1} = (x_1^{\sigma_1-\delta_1}, \dots, x_n^{\sigma_n-\delta_n}) = (x_1^{pk_1}, \dots, x_n^{pk_n}) \in im(\phi)$$

Thus z and v are in the same coset: $A/im(\phi) = \{(x_1^{\alpha_1}, \ldots, x_n^{\alpha_n}) | 1 \leq \alpha_i \leq p\}$. Now it is trivial that $A/im(\phi)$ is isomorphic to E_{p^n}. Now it only remains to show that E_{p^n} has rank n, which is easy since the set below is the basis for it:

$$\begin{gathered}(1, 0, \ldots, 0)\\(0, 1, \ldots, 0)\\\vdots\\(0, 0, \ldots, 1)\end{gathered}$$

And all are linearly independent.

Problem 4.

(i) Since 5-sylow is not normal then we have more than 1 subgroup of order 5. Now we know that $n_5 \overset{5}{\equiv} 1$ and $n_5 \mid 12$. Since $n_5 \neq 1$ then we have $n_5 = 6$. We want to prove that G is simple. Suppose that $H \lhd G$ and $H \neq 1$.

Case 1) $5 \mid |H|$:
This shows that there exists some subgroup of order 5 in H, and since H is normal then all of its conjugates are also in H. Therefore all 6 5-sylow subgroups are in H. And since they all have different elements, then H has exactly 24 distinct elements of order 5. With addition of 1, we have $|H| \geq 25$. And also $|H| \mid 60$ therefore $|H| = 30$ or 60. But we proved earlier in the course that any group with order 30 has only one 5-sylow. Which is a contradiction, therefore $|H| = 60$. This proves that G is simple.

Case 2) $5 \nmid |H|$:
Therefore $|H| \big| 12$. If $|H| = 12$:

$$\begin{aligned}n_3 &\overset{3}{\equiv} 1 \quad n_3 \mid 4 \implies n_3 = 1 \text{ or } 4\\ n_2 &\overset{2}{\equiv} 1 \quad n_2 \mid 3 \implies n_2 = 1 \text{ or } 3\end{aligned}$$

Now if $n_3 = 4$ then there exists 8 distinct elements with order 3. Which leaves only 4 other elements in H, that forces $n_2 = 1$. This shows that either $n_2 = 1$ or $n_3 = 1$. Sicne either 2-sylow or 3-sylow in H is unique and H is normal in G therefore one of them is normal in G. Thus in this case G has a normal subgroup of order 3 or 2. Let that subgroup be H.
If $|H| = 6$:

$$n_3 \overset{3}{\equiv} 1 \quad n_3 \mid 2 \implies n_3 = 1$$

Since H has a unique subgroup of order 3 and H is normal in G then G has a normal subgroup of order 3. Let H be that subgroup.
Now if $|H| = 4$:

$$\left|\frac{G}{H}\right| = 15$$

Since group of order 15 has a unique subgroup of order 5 then by forth isomorphism theorem we have:

$$\left|\frac{N}{H}\right| = 5\ ,\ \frac{N}{H} \lhd \frac{G}{H} \implies N \lhd G\ ,\ 5\big||N|$$

But we solved this in case 1.
If $|H| = 3$:

$$\left|\frac{G}{H}\right| = 20$$

but in this group of order 20 we have:

$$n_5 \overset{5}{\equiv} 1 \quad n_5 \mid 4 \implies n_5 = 1$$

There is a subgroup of order 5:

$$\left|\frac{N}{H}\right| = 5\ ,\ \frac{N}{H} \lhd \frac{G}{H} \implies N \lhd G\ ,\ 5\big||N|$$

Which is solved in case 1.
If $|H| = 2$ then:

$$\left|\frac{G}{H}\right| = 30$$

We proved in the class that every group of order 30 has a normal subgroup of order 5:

$$\left|\frac{N}{H}\right| = 5\ ,\ \frac{N}{H} \lhd \frac{G}{H} \implies N \lhd G\ ,\ 5\big||N|$$

Which is also solved in case 1.

This proves that in both cases G is ineed simple. Now we prove that G has no subgroup of index 1, 2, 3, 4 or 5. Since if $H < G$ and $[G : H] = n$ then there exists some homomorphism π such that:

$$\pi : G \to S_n$$

For $n = 1, 2, 3, 4$ we have: $|S_n| \leq 24 < 60 = |G|$.This shows that $|ker\pi| > 1$ and thus G has a non-trivial normal subgroup, but G is simple. This is a contradiction. Therefore G can't have a subgroup of index 1, 2, 3, 4. Now if $H < G$ and $[G : H] = 5$ then we have:

$$\pi : G \to S_5$$

But since $|S_5| = 120$, then we could have $|ker\pi| = 1$. In this case we have: $\pi(G) < S_5$ and $|\pi(G)| = 60$. Now we have $A_5 \lhd S_5$ and $|A_5| = 60$, Therefore:

$$\left.\begin{array}{l} A_5 < \pi(G)A_5 < S_5 \\ [S_5 : A_5] = 2 \end{array}\right\} \implies A_5 = \pi(G)A_5 \text{ or } S_5 = \pi(G)A_5$$

If $A_5 = \pi(G)A_5$ therefore $G < A_5$ and since they are of the same size then $\pi(G) = A_5$ therefore $G \cong A_5$.
If $S_5 = \pi(G)A_5$ therefore by second isomorphism theorem we have:

$$2 = \frac{|S_5|}{|A_5|} = \frac{|\pi(G)A_5|}{|A_5|} = \frac{|\pi(G)|}{|A_5 \cap \pi(G)|} \implies |A_5 \cap \pi(G)| = 30$$
$$A_5 \cap \pi(G) < \pi(G)$$

This shows that $\pi(G)$ has a subgroup with index 2, which is normal, but this can't happen since G is simple, therfore $\pi(G)$ is simple and can't have a normal subgroup. This contradiction shows that G also can't have a subgroup of index 5, unless $G \cong A_5$.
Now consider in G consider the 2-sylow subgroups:

$$n_2 \overset{2}{\equiv} 1 \quad n_2 \mid 15 \implies n_2 = 1, 3, 5, 15$$

Since all 2-sylows are conjugate therefore they lie in the same orbit, thus we have:

$$\pi : G \to S_{n_2}$$

And with privious parts, we can't have $n_2 = 3, 5$, unless $G \cong A_5$. $n_2 = 1$ is also not an option since G is simple. Therefore we have $n_2 = 15$. If these 15 subgroups of order 5 have no interesection, then we have $15 \times 3 = 45$ distinct elements with order 2 or 4. And since we have $n_5 = 6$ then we have 24 elements of order 5. $45 + 24 = 67 > 60$. which is a contradiction. Therefore there exists some two of these 2-sylow gruops that have a non-trivial intersection. Let P and Q be those two:

$$H = P \cap Q \quad |H| = 2$$

Now consider $N_G(H)$. We have $P < N_G(H)$, and $Q < N_G(H)$ since both P and Q are abelian (All groups or order 4 are abelian). Which implies $|N_G(H)| \geq 6$. We also have $|N_G(H)| \big| 60$. And also $|N_G(H)| \neq 60$ since H is not normal in G since G is simple. Also $P < N_G(H)$ implies that $4 = |P| \big| |N_G(H)|$. Thus we have:

$$|N_G(H)| = 12 \text{ or } 20$$

If $|N_G(H)| = 20$ then it is a subgroup of index 3, which we proved to be impossible earlier. If $|N_G(H)| = 12$ then it is a subgroup of order 5, which can only happen if $G \cong A_5$. therfore $G \cong A_5$.

(ii)

(iii)

Problem 5.

First assume that $k > 0$ ($p > 3$) since we classified all groups of order 12 earlier in the class. By Cauchy's theorem we know that any group with $4p$ elements, has a subgroup of order 4 and a subgroup of order p. Now let n_p be the number of p-sylow subgroups in G:

$$n_p \overset{p}{\equiv} 1 \quad n_p \mid 4 \implies n_p = 1, 2, 4$$

If $n_2 = 4$ then $4 \overset{p}{\equiv} 1$ which implies $p \mid 3$ which is a contradiction knowing $p > 3$. If $n_2 = 2$ then $p \mid 1$ which is also a contradiction. Therefore we have $n_p = 1$. Which shows that p-sylow subgroup of G is normal in G. Now to find all semi-direct products of these two we have to find all homomorphisms:

$$\varphi : \mathbb{Z}_4 \to Aut(\mathbb{Z}_p) \cong \mathbb{Z}_{p-1}$$

If we let r to be the generator of $\mathbb{Z}_4$ then we should map r to an element of order 4, 2 or 1 in $\mathbb{Z}_{p-1}$. But since $4 \nmid p - 1 = 4k + 2$ then $\mathbb{Z}_{p-1}$ has no element of order 4. And since all 2-sylow subgroups are conjugate to each other, therefore we only need to find one element of order 2 in $\mathbb{Z}_{p-1}$. Now consider $\alpha \in Aut(\mathbb{Z}_p)$ such that:

$$\alpha(1) = p - 1 \implies \alpha(i) = p - i = i^{-1}$$
$$\alpha(\alpha(i)) = p - (p - i) = i$$

Therefore α is an element of order 2 in $Aut(\mathbb{Z}_p)$. Let $\varphi(r) = \alpha$ and $\langle s \rangle = \mathbb{Z}_p$:

$$G = \mathbb{Z}_p \rtimes \mathbb{Z}_4$$
$$G = \langle r, s | r^4 = s^p = 1, rsr^{-1} = s^{-1} \rangle$$

The other φ is where r is mapped to and element of order 1, where we have the direct product of these groups:

$$G = \mathbb{Z}_p \times \mathbb{Z}_4$$

Therefore there are only 2 non-isomorphic groups with order $4p$.

Problem 6.

(i) Suppose for all $q \in Q$, $\theta'(q) = \alpha \circ \theta(q) \circ \alpha^{-1}$, where $\alpha, \theta(q), \theta'(q) \in Aut(N)$. We show that:

$$N \rtimes_\theta Q \equiv N \rtimes_{\theta'} Q$$

Let $\phi : N \rtimes_\theta Q \to N \rtimes_{\theta'} Q$, where $\phi(n, q) = (\alpha(n), q)$. To show that ϕ is homomorphism:

$$\begin{aligned}
\phi((n, q).(n', q')) &= \phi(n\theta(q)(n'), qq') \\
&= (\alpha(n\theta(q)(n')), qq') \\
&= (\alpha(n)\alpha(\theta(q)(n')), qq') \\
&= (\alpha(n)\alpha \circ \theta(q) \circ \alpha^{-1} \circ \alpha(n'), qq') \\
&= (\alpha(n)\theta'(\alpha(n')), qq') \\
&= (\alpha(n), q)(\alpha(n'), q') \\
&= \phi(n, q)\phi(n', q')
\end{aligned}$$

And since $\alpha \in Aut(N)$ then α^{-1} is also in $Aut(N)$. And since ϕ' with $\phi'(n, q) = (\alpha^{-1}(n), q)$ is the inverse of ϕ then ϕ is an isomorphism, therfore $N \rtimes_\theta Q \equiv N \rtimes_{\theta'} Q$.

(ii) Suppose we have $\theta = \theta' \circ \alpha$ for some $\alpha \in Aut(Q)$. Let ϕ:

$$\begin{aligned} \phi : N \rtimes_\theta Q &\to N \rtimes_{\theta'} Q \\ (n, q) &\to (n, \alpha(q)) \end{aligned}$$

To check the homomorphism:

$$\begin{aligned} \phi((n, q)(n', q')) &= \phi((n\theta(q)(n'), qq')) \\ &= (n\theta(q)(n'), \alpha(qq')) \\ &= (n\theta'(q) \circ \alpha(q)(n'), \alpha(q)\alpha(q')) \\ &= (n, \alpha(q))(n', \alpha(q')) \\ &= \phi(n, q)\phi(n', q') \end{aligned}$$

And since α^{-1} is also in $Aut(Q)$ and $\phi'(n, q) = (n, \alpha^{-1}(q))$ is the inverse of ϕ therefore ϕ is an isomorphism, and we have $N \rtimes_\theta Q \cong N \rtimes_{\theta'} Q$

(iii) Suppose Q is cyclic and the subgroup $\theta(Q)$ of $Aut(N)$ is conjugate to $\theta'(Q)$. Sicne $\theta(Q)$ is conjugate with $\theta'(Q)$ therefore there exists some $\alpha \in Aut(N)$ such that:

$$\alpha \circ \theta(a) \circ \alpha^{-1} \in \theta'(Q)$$

Where a is the generator of Q. And since Q is cyclic then there exists some i such that:

$$\begin{aligned} \alpha \circ \theta(a) \circ \alpha^{-1} &= \theta'(a^i) \\ \alpha \circ \theta(a^j) \circ \alpha^{-1} &= \theta'(a^i)^j = \theta'(a^{ij}) \end{aligned}$$

Now consider:

$$\begin{aligned} \phi : N \rtimes_\theta Q &\to N \rtimes_{\theta'} Q \\ (n, q) &\to (\alpha(n), q^i) \end{aligned}$$

To prove the homomorphism:

$$\begin{aligned} \phi((n, a^j)(n', a^k)) &= \phi(n\theta(a^j)(n'), a^{j+k}) \\ &= (\alpha(n)\alpha(\theta(a^j)(n')), a^{i(j+k)}) \\ &= (\alpha(n)\alpha(\theta(a^j)(\alpha^{-1}\alpha(n'))), a^{ij}a^{ik}) \\ &= (\alpha(n)\theta'(a^{ij})\alpha(n'), a^{ij}a^{ik}) \\ &= (\alpha(n), a^{ij})(\alpha(n'), a^{ik}) \\ &= \phi(n, a^j)\phi(n', a^k) \end{aligned}$$

Therfore ϕ is a homomorphism. Similarly we have:

$$\begin{aligned} \alpha^{-1} \circ \theta'(a) \circ \alpha &\in \theta(Q) \\ \exists j : \alpha^{-1} \circ \theta'(a) \circ \alpha &= \theta(a^j) \end{aligned}$$

Therefore $\phi'(n, q) = (\alpha^{-1}(n), q^j)$ is the inverse of ϕ. Therefore ϕ is an isomorphism and we have $N \rtimes_\theta Q \cong N \rtimes_{\theta'} Q$.