Category Theory Fall 2024 Problem Set -

Amin Zolfagharian

Problem 1. *Let $\mathcal{C}$ be a category. Prove that for any finite diagram $\mathcal{D} : \mathcal{I} \to \mathcal{C}$ there exists $\lim_{\leftarrow I} \mathcal{D}$ iff category $\mathcal{C}$ has a Terminal object and for any two morphisms with the same domain and codomain, there exists an Equalizer and also for any $A, B \in \mathcal{C}_0$, $A \times B \in \mathcal{C}_0$.*

($\Rightarrow$) Suppose for any finite diagram limit exists.

(i) let $\mathcal{I}_0 = \{\}$. since $X = \lim_{\leftarrow \mathcal{I}_0} \mathcal{D}_0$ exists, and the image of $\mathcal{I}$ in $\mathcal{C}$ is empty, therefore any object of $\mathcal{C}$ like Y with any morphisms, commutes with this emtpy Image of $\mathcal{I}$. Which means that there exists a unique morphism $Y \xrightarrow{\varphi} X$. Therefore we can say that there exists a unique morphism from any object of $\mathcal{C}$ to X. This shows that X is terminal object of the cateogry $\mathcal{C}$.

(ii) Let $A, B \in \mathcal{C}_0$. Let $\mathcal{I}$ be the category of 2 with only identity morphisms.

$$\begin{tikzcd} \cdot \arrow[loop above, "\mathrm{id}"] & * \arrow[loop above, "\mathrm{id}"] \end{tikzcd}$$

Where $\mathcal{D}$ maps . to A and $*$ to B. Let $X = \lim_{\leftarrow \mathcal{I}} \mathcal{D}$ with morphisms P_A and P_B be the limit of this diagram. Now for any f and g we know that there exists a unique morphism φ from Y to X such that the diagram commutes.

$$\begin{tikzcd}
 & Y \arrow[ddl, "f"'] \arrow[d, "\exists!\varphi"] \arrow[ddr, "g"] & \\
 & X = \lim_{\leftarrow \mathcal{I}} \mathcal{D} \arrow[dl, "P_A"'] \arrow[dr, "P_B"] & \\
A \arrow[loop below, "\mathrm{id}"] & & B \arrow[loop below, "\mathrm{id}"]
\end{tikzcd}$$

Since for any f and g, Y commutes with the image of $\mathcal{I}$, therefore we can say that for any f and g there exists a unique morphism from Y to X such that the diagram above commutes. With this description X is the product of A and B.

(iii) As for equalizer of two morphisms f and g from A to B, let $\mathcal{I}$ be the cateogry of 2 with 2 non-trivial morphisms.

$$\begin{tikzcd} \cdot \arrow[r, bend left] \arrow[r, bend right] & * \end{tikzcd}$$

And let functor $\mathcal{D}$ be in a way that maps . to A and $*$ to B and two non-trivial morphisms to f and g. Let $X = \lim_{\leftarrow \mathcal{I}} \mathcal{D}$ and morphisms P_A and P_B be the limit of this diagram $\mathcal{D}$.

$$
\begin{tikzcd}
 & & Y \arrow[ddll, "\sigma"'] \arrow[dd, "\exists!\varphi"] \arrow[ddrr, "\delta"] \arrow[ddl, phantom, "="] \arrow[ddr, phantom, "="] & & \\
 & & & & \\
 & & X = \lim_{\leftarrow \mathcal{I}} \mathcal{D} \arrow[dll, "P_A"'] \arrow[drr, "P_B"] \arrow[d, phantom, "="] & & \\
A \arrow[rrrr, bend right=20, "f"] \arrow[rrrr, bend right=35, "g"'] & & {} & & B
\end{tikzcd}
$$

Since X commutes with image of $\mathcal{I}$ we have $fP_A = gP_A = P_B$. Now let Y be in a way such that it commutes with Image of $\mathcal{I}$. Then we have: $f\sigma = g\sigma = \delta$. Now since X is the limit of this diagram we know that there exists a unique φ from Y to X such that it commutes. This shows that (X, P_A) is exactly the equalizer of f and g. Since we have $fP_A = gP_A$. And also for any other Y and σ such that $f\sigma = g\sigma$ there exists a unique morphism from Y to X such that $\sigma = P_A\varphi$.

($\Leftarrow$) Now suppose we have all 3 properties in $\mathcal{C}$. We want to prove that for any arbitrary finite index category $\mathcal{I}$ and diagram $\mathcal{D}$ there exists a limit. Let all of the objects in image of $\mathcal{I}$ be $D_1, D_2, \ldots, D_n$. Since product of any number of objects is available in $\mathcal{C}$ consider $X = \Pi_{i=1}^n D_i$. With morphisms π_i. But this diagram doesn't necessarily commute. For any $(Z, f_1, \ldots, f_n)$ where $Z \xrightarrow{f_i} D_i$, that commutes with image of $\mathcal{I}$ there exists a unique morphism from Z to X such that the diagram commutes.

$$
\begin{tikzcd}
 & & Z \arrow[ddll, bend right, "f_1"'] \arrow[d, "\exists!\varphi"'] \arrow[ddrr, bend left, "f_n"] \arrow[ddl, phantom, "="] \arrow[ddr, phantom, "="] & & \\
 & & X = \Pi_{i=1}^n D_i \arrow[dll, "\pi_1"'] \arrow[drr, "\pi_n"] & & \\
D_1 & & \cdots & & D_n
\end{tikzcd}
\tag{1}
$$

Now suppose object Y with morphisms $\pi_1, \ldots, \pi_n$ to $D_1, \ldots, D_n$ where for the set U of morphisms in image of $\mathcal{I}$ diagram commutes. in other words:

$$
\begin{gathered}
\forall k \in U \quad s.t \quad D_i \xrightarrow{k} D_j \\
\implies \pi_j = k\pi_i
\end{gathered}
$$

Also for any $(Z, f_1, \ldots, f_n)$ that commutes with image of $\mathcal{I}$ then there exists a unique morphism (φ) from Z to Y such that the diagram commutes or: $\forall i : f_i = \pi_i\varphi$. Let k be a morphism in image of $\mathcal{I}$ where $k : D_i \to D_j \notin U$. This means that $\pi_j \neq k\pi_i$. Now let (Y', σ) be the equalizer of π_j and $k\pi_i$. Therefore $\pi_j\sigma = k\pi_i\sigma$. We will show that Y'

has all the properties of Y with a bigger U.
Let $(Z, f_1, \dots, f_n)$ be in such a way that it commutes with image of $\mathcal{I}$. Then we know there exists a unique morphism from Z to Y such that the diagram commutes.

$$
\begin{tikzcd}
 & Z \arrow[ddl, bend right, "f_i"'] \arrow[d, "\exists!\delta"] \arrow[dd, bend right, "\exists!\varphi"'] \arrow[ddr, bend left, "f_j"] & \\
 & Y' \arrow[d, "\sigma"] & \\
 & Y \arrow[dl, "\pi_i"'] \arrow[dr, "\pi_j"] & \\
D_i \arrow[rr, bend right, "k"'] & & D_j
\end{tikzcd}
\tag{2}
$$

$$
\left.\begin{aligned}
f_j &= kf_i \\
f_i &= \pi_i\varphi \\
f_j &= \pi_j\varphi
\end{aligned}\right\} \implies \pi_j\varphi = k\pi_i\varphi
$$

And since Y' is equalizer of π_j and $k\pi_i$ then there exists a unique morphism (δ) from Z to Y' such that $\varphi = \sigma\delta$. If we name $\pi'_i = \pi_i\sigma$ it is easy to see that $\pi'_j = k\pi'_i$. And also for any $r \in U$ that $D_x \xrightarrow{r} D_y$ then we know $\pi_y = r\pi_x$. therefore $\pi'_y = \pi_y\sigma = r\pi_x\sigma = r\pi'_x$. Therefore $(Y', \pi'_1, \dots, \pi'_n)$ commutes with a bigger set of morphisms in image of $\mathcal{I}$. and also δ commutes with the diagram since:

$$\pi'_i\delta = \pi_i\sigma\delta = \pi_i\varphi = f_i$$

Also uniqueness of δ follows from uniqueness of φ. This shows that we can obtain the object with a U consisting of all of morphisms in image of $\mathcal{I}$. Which shows that $\varprojlim_{\mathcal{I}} \mathcal{D}$ exists.

Problem 2. *Let $\mathcal{C}$ be a category. Prove that for any finite diagram $\mathcal{D} : \mathcal{I} \to \mathcal{C}$ there exists $\varprojlim_{\mathcal{I}} \mathcal{D}$ iff category $\mathcal{C}$ has terminal object and for any two morphisms with the same co-domain, there exists pull-back.*

We just have to show that the set of pull-back and terminal is the same as the set of terminal, product and equalizer.

(i) Pull-back, Terminal $\Rightarrow$ Product
If $A, B \in \mathcal{C}_0$ we can obtain $A \times B$ with this pull-back:

$$
\begin{tikzcd}
Y \arrow[drr, bend left, "f"] \arrow[dr, "!\varphi"] \arrow[ddr, bend right, "g"'] & & \\
& X \arrow[r, "P_A"] \arrow[d, "P_B"'] & A \arrow[d, "!t_A"] \\
& B \arrow[r, "!t_B"'] & 1
\end{tikzcd}
$$

Since for any f and g the diagram commutes since 1 is terminal object and there is only one morphism from Y to 1. And since X is the pull-back then there exists a unique morphism from Y to X such that it commutes with f and g. This shows that X is the product of A and B.

(ii) Pull-back $\Rightarrow$ Equalizer
Consider the product $B \times B$. We want to find the equalizer of $f, g : A \to B$. Let X be the pull-back of $\langle f, g \rangle$ and $\langle id, id \rangle$.

$$
\begin{tikzcd}
X \arrow[r, "j"] \arrow[d, "i"'] & B \arrow[d, "\langle id, id \rangle"] \\
A \arrow[r, "\langle f, g \rangle"'] & B \times B
\end{tikzcd}
$$

With $\langle f, g \rangle\, i = \langle id, id \rangle\, j$. Where $\langle f, g \rangle$ and $\langle id, id \rangle$ are defined below.

$$
\begin{tikzcd}
& X \arrow[d, "j"] \arrow[ddl, "j"'] \arrow[ddr, "j"] & \\
& B \arrow[dl, "id"'] \arrow[d, "\langle id, id \rangle"] \arrow[dr, "id"] & \\
B & B \times B \arrow[l, "P_1"] \arrow[r, "P_2"'] & B
\end{tikzcd}
\qquad
\begin{tikzcd}
& X \arrow[d, "i"] \arrow[ddl, "gi"'] \arrow[ddr, "fi"] & \\
& A \arrow[dl, "g"'] \arrow[d, "\langle f, g \rangle"] \arrow[dr, "f"] & \\
B & B \times B \arrow[l, "P_1"] \arrow[r, "P_2"'] & B
\end{tikzcd}
$$

First we show that $fi = gi$. Since both of these diagrams commute then we know that $j = P_1 \langle id, id \rangle j = P_2 \langle id, id \rangle j$. Now we can replace $\langle id, id \rangle j$ with $\langle f, g \rangle i$.

$$P_1 \langle f, g \rangle i = P_2 \langle f, g \rangle i$$

But we know $gi = P_1 \langle f, g \rangle i$ and $fi = P_2 \langle f, g \rangle i$. This shows that $gi = fi$.
Now let (Z, φ) in a way that $Z \xrightarrow{\varphi} A$ and $f\varphi = g\varphi$ where $f\varphi$ is a morphism from Z to B. we want to show that $\langle id, id \rangle f\varphi = \langle f, g \rangle \varphi$. since both are morphisms from Z to $B \times B$ and they both commute with the diagram below, and since product is unique then they are equal.

$$\begin{tikzcd}
 & Z \arrow[d, "f\varphi"] \arrow[ddl, "f\varphi"'] \arrow[ddr, "f\varphi"] & \\
 & B \arrow[dl, "id"'] \arrow[dr, "id"] \arrow[d, "\langle id, id \rangle"] & \\
B & B \times B \arrow[l, "P_1"] \arrow[r, "P_2"'] & B
\end{tikzcd}
\qquad
\begin{tikzcd}
 & Z \arrow[d, "\varphi"] \arrow[ddl, "g\varphi = f\varphi"'] \arrow[ddr, "f\varphi"] & \\
 & A \arrow[dl, "g"'] \arrow[dr, "f"] \arrow[d, "\langle f, g \rangle"] & \\
B & B \times B \arrow[l, "P_1"] \arrow[r, "P_2"'] & B
\end{tikzcd}$$

Since they are the same diagram then $\langle id, id \rangle f\varphi = \langle f, g \rangle \varphi$. then Z commutes with pull-back diagram and therefore there exists a unique morphism (θ) from Z to X in a way that $i\theta = \varphi$.

$$\begin{tikzcd}
Z \arrow[dr, "!\theta"] \arrow[drr, bend left, "f\varphi"] \arrow[ddr, bend right, "\varphi"'] & & \\
 & X \arrow[r, "j"] \arrow[d, "i"'] & B \arrow[d, "\langle id, id \rangle"] \\
 & A \arrow[r, "\langle f, g \rangle"'] & B \times B
\end{tikzcd}$$

this shows that (X, i) is the equalizer of f, g.

(iii) Product, Equalizer $\Rightarrow$ Pull-back
This part is similar to the proof of limit, with equalizer and product. Suppose $f : A \to C$ and $g : B \to C$. Let $X = A \times B$. and let (E, i) be the equalizer of fP_A and gP_B.

$$
\begin{tikzcd}
 & & & B \arrow[dr, "g"] & \\
Z \arrow[urrr, bend left, "\delta"] \arrow[drrr, bend right, "\sigma"'] & E \arrow[r, "i"] & A \times B \arrow[ur, "P_B"] \arrow[dr, "P_A"'] & & C \\
 & & & A \arrow[ur, "f"'] &
\end{tikzcd}
$$

E is the pull-back. it is easy to see that $gP_Bi = fP_Ai$ since E is the equalizer. this shows that the diagram commutes. Now for any Z such that commutes with this diagram, There exists a unique morphism (φ) from Z to $A \times B$ such that the diagram commutes.

$$
\begin{aligned}
& g\delta = f\sigma \\
\implies & gP_B\varphi = fP_A\varphi
\end{aligned}
$$

this shows that φ also makes gP_B and fP_A equal. since E is the equalizer then there exists a unique morphism from Z to E such that the diagram commutes. Therefore E is the pull-back for f and g.