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Problem 1.

(i) We have: $x^4 - 1 = 2y^4$, Therefore x is odd. Which implies that $x^2 \overset{4}{\equiv} 1$. This shows that $x^2 + 1 \overset{4}{\equiv} 2$.

$$x^4 - 1 = (x^2 - 1)(x^2 + 1) = 2y^4$$
$$\implies (x^2 - 1)(\frac{x^2 + 1}{2}) = y^4$$

Note that gcd of $x^2 + 1$ and $x^2 - 1$ can only be a power of 2. And since $\frac{x^2+1}{2}$ is an odd number, thus we have: $(x^2 - 1, \frac{x^2+1}{2}) = 1$. And since y^4 is squared, thus both $x^2 - 1$ and $\frac{x^2+1}{2}$ are squared.

$$x^2 - 1 = z^2 \implies x^2 - z^2 = 1 \implies (x - z)(x + z) = 1$$

Which only leaves us with $x = \pm 1$. And therefore $y = 0$.

(ii) Again x is odd. Therefore we have $x^2 \overset{4}{\equiv} 1$. Now we have:

$$(x^2 - 1)^2 = x^4 - 2x^2 + 1 = 2y^2 - 2x^2 = 2(y^2 - x^2)$$
$$(x^2 + 1)^2 = x^4 + 2x^2 + 1 = 2y^2 + 2x^2 = 2(y^2 + x^2)$$
$$(x^4 - 1)^2 = 4(y^4 - x^4)$$
$$\implies (\frac{x^4 - 1}{2})^2 = y^4 - x^4$$

Let $z = \frac{x^4-1}{2}$. Now we know all the answers to this type of equations, for some $(m, n) = 1$ we have:

$$x^2 = 2mn \overset{(m,n)=1}{\implies} m = u^2, n = 2v^2$$
$$z^2 = m^2 - n^2 = u^4 - v^4$$

Now we have $z^2 < u^4 = m^2 < m^2 + n^2 = y^2 < y^4$.

$$z^2y^2 = m^4 - n^4$$

Which gives us a smaller positive solution to the equation. Thus this equation doesn't have any solutions.

Problem 2.

We can write:

$$\begin{aligned}
\left|p_k^2 - dq_k^2\right| = \left|p_k - \sqrt{d}q_k\right|\left|p_k + \sqrt{d}q_k\right| &= \left|q_k^2\right|\left|\frac{p_k}{q_k} - \sqrt{d}\right|\left|\frac{p_k}{q_k} + \sqrt{d}\right| \\
&< q_k^2\frac{1}{q_k^2}\left|\frac{p_k}{q_k} + \sqrt{d}\right| = \left|\frac{p_k}{q_k} + \sqrt{d}\right| \\
&< \left|\frac{p_k}{q_k}\right| + \sqrt{d} \\
&< 1 + \sqrt{d} + \sqrt{d} = 1 + 2\sqrt{d}
\end{aligned}$$

Problem 3.

(i) We have to find the biggest k where $q_k < 40$.

$$\begin{aligned}
&q_0 = 1 \qquad q_1 = 1 \\
&q_2 = 2 \times 1 + 1 = 3 \\
&q_3 = 1 \times 3 + 1 = 4 \\
&q_4 = 1 \times 4 + 3 = 7 \\
&q_5 = 4 \times 7 + 4 = 32 \\
&q_6 = 1 \times 32 + 7 = 39
\end{aligned}$$

This shows that C_6 is the best apporoximation for e with the desired condition.

$$\begin{aligned}
&p_0 = 2 \qquad p_1 = 3 \\
&p_2 = 2 \times 3 + 2 = 8 \\
&p_3 = 1 \times 8 + 3 = 11 \\
&p_4 = 1 \times 11 + 8 = 19 \\
&p_5 = 4 \times 19 + 11 = 87 \\
&p_6 = 1 \times 87 + 19 = 106
\end{aligned}$$

Thus best apporoximation is $\frac{106}{39}$.

Problem 4.

First we find a single rational point. Namely $p = (1, 0)$. Now any rational line passing through p meets the curve in another rational point. And this way we can find all rational points on the curve. Thus we only have to find all rational lines passing through p. Let $y = mx + b$ where $m, b \in \mathbb{Q}$. we have:

$$0 = m \times 1 + b \implies b = -m$$

Now we have to find the other point of intersection:

$$\begin{aligned}
x^2 - 2(mx - m)^2 = 1 &\implies x^2 - 2m^2x^2 + 4m^2x - 2m^2 - 1 = 0 \\
&\implies (1 - 2m^2)x^2 + 4m^2x - 2m^2 - 1 = 0
\end{aligned}$$

And we know that $\frac{2m^2+1}{2m^2-1}$ is the product of roots of this polynomial, and since 1 is one of the roots, then the other is $\frac{2m^2+1}{2m^2-1}$. Whic is the x of the other intersection of the line with the curve. Now since $m \in \mathbb{Q}$, then we have: $m = \frac{a}{b}$, and rewrite:

$$\frac{2m^2+1}{2m^2-1} = 1 + \frac{2}{2m^2-1} = 1 + \frac{2}{\frac{2a^2-b^2}{b^2}} = 1 + \frac{2b^2}{2a^2-b^2}$$

As for the y of this intersection, it is $\frac{2m}{2m^2-1}$:

$$\frac{2m}{2m^2-1} = \frac{\frac{2a}{b}}{\frac{2a^2-b^2}{b^2}} = \frac{2ab}{2a^2-b^2}$$

Thus for any $a \in \mathbb{Z}$ and $b \in \mathbb{N}$, $(1 + \frac{2b^2}{2a^2-b^2}, \frac{2ab}{2a^2-b^2})$ is a rational point on the curve, and it's all of the rational points.

Problem 5.

If $c = 0$ then we have:

$$ax^2 + by^2 = 0$$

And since we have we have a rational point of this curve, (x_1, y_1), then for any rational line $y = mx + b$, $m, b \in \mathbb{Q}$, passing through (x_1, y_1) we have:

$$y_1 = mx_1 + b \implies b = y_1 - mx_1$$

Now for the intersections of the line with the curve we have:

$$ax^2 + b(mx + y_1 - mx_1)^2 = 0$$
$$\implies ax^2 + bm^2x^2 + 2bmy_1x - 2bmx_1x + by_1^1 + bm^2x_1^2 - 2bmy_1x_1 = 0$$

Thus the other root is $\frac{b(y_1^2+m^2x_1^2-2my_1x_1)}{(a+bm^2)x_1}$. The answer would be all points where this root is (x of the point) is an integer and also the corresponding y is also integer.

Now If $c \neq 0$ then if $z = 0$ in any answer, it would be like the case above, and if $z \neq 0$ then we can divide the equation by z^2.

$$ax^2 + by^2 + cz^2 = 0 \implies a(\frac{x}{z})^2 + b(\frac{y}{z})^2 + c = 0$$

Which again is exactly like the part above, and we can find all integer points on this curve. Thus If we find all the integer points on this curve, then we have a $(x/z, y/z)$ that works, and therefore for (x, y, z) the equation holds.

Problem 6.

Note that $d < \sqrt{d^2+1} < d+1$:

$$\sqrt{d^2+1} = d + \sqrt{d^2+1} - d = d + \frac{d^2+1-d^2}{\sqrt{d^2+1}+d} = d + \frac{1}{\sqrt{d^2+1}+d}$$
$$= d + \frac{1}{2d + \sqrt{d^2+1} - d}$$

Thus we have:

$$\sqrt{d^2+1} - d = \frac{1}{2d + (\sqrt{d^2+1} - d)} \implies \sqrt{d^2+1} - d = [\overline{2d}]$$

Therefore $\sqrt{d^2+1} = d + (\sqrt{d^2+1} - d) = [d; \overline{2d}]$.