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Problem 1.

1. $x \in E_x$
   Let f be the characteristic function of the problem.

$$f(x) = \begin{cases} 1 & \text{if } x \in E_x \\ 0 & \text{if } x \notin E_x \end{cases}$$

   Suppose f is computable. Now define g by:

$$g(x) = \begin{cases} x & \text{if } x \notin E_x \text{ or } f(x) = 0 \\ \uparrow & \text{if } x \in E_x \text{ or } f(x) = 1 \end{cases}$$

   $g(x)$ is computable by Church's thesis. Therefore there exists e such that $g(x) \cong \phi_e$. Now we have:

$$\phi_e(x) \downarrow \iff x \in W_x \iff x \in E_x$$
$$\phi_e(x) \downarrow \iff x \in W_x \iff x \notin E_x$$

   This gives us the contradiction. Which shows that the assumption of f being a computable function was wrong and $x \in E_x$ is not decidable.

2. $W_x = W_y$
   Let $f(x) \equiv 0$. Since f is computable then there exists e such that $f \cong \phi_e$. Since f is total, then $W_e = \mathbb{N}$. Now suppose $W_x = W_y$ is decidable. Then the function g is computable.

$$g(x, y) = \begin{cases} 1 & \text{if } W_x = W_y \\ 0 & O.W. \end{cases}$$

   Now let $g(e, y) = g_e(y)$ for fixed e. Since g is computable then g_e is also computable.

$$g_e(y) = 1 \iff W_y = \mathbb{N} \iff \phi_y \text{ is total}$$
$$g_e(y) = 0 \iff W_y \neq \mathbb{N} \iff \phi_y \text{ is not total}$$

   This means that "ϕ_x is total" is decidable iff g_e is computable. And since we know that "ϕ_x is total" is not decidable, then g_e is not computable which is a contradiction. Therefore $W_x = W_y$ is not decidable.

3. $\phi_x(x) = 0$
   Suppose it is decidable. Let g:

$$g(x) = \begin{cases} 1 & \text{if } \phi_x(x) = 0 \\ 0 & \text{if } \phi_x(x) \neq 0 \end{cases}$$

Since $\phi_x(x) = 0$ is decidable, then g is computable. Thus there exists e such that $g(x) \cong \phi_e(x)$. Now we have:

$$\phi_e(e) = 0 \iff g(e) = 0 \iff \phi_e(e) \neq 0$$

Which gives us the contradiction, Therefore $\phi_x(x) = 0$ is undecidable.

4. $\phi_x(y) = 0$
   If $\phi_x(y) = 0$ is decidable then with $y = x$ we have $\phi_x(x) = 0$ is decidable, which by last part we know is undecidable. Therefore $\phi_x(y) = 0$ is undecidable.

5. $x \in E_y$
   If it is decidable, by substituting y with x we have that $x \in E_x$ is decidable, which by part 1 we know is undecidable, Therefore $x \in E_y$ is undecidable.

6. ϕ_x is total and constant

$$f(x,y) = \begin{cases} 1 & \text{if } x \in W_x \\ \uparrow & \text{if } x \notin W_x \end{cases}$$

By Church's thesis f is computable, therefore by *s-m-n* theorem there exists total computable function k such that $f(x,y) = \phi_{k(x)}(y)$.

$$\begin{aligned} x \in W_x &\implies f(x,y) \equiv 1 \implies \phi_{k(x)} \text{ is total and constant} \\ x \notin W_x &\implies f(x,y) \uparrow \implies \phi_{k(x)} \text{ is not total and constant} \end{aligned}$$

Thus $\phi_{k(x)}$ is total and constant if and only if $x \in W_x$. Since $x \in W_x$ is undecidable, then "$\phi_{k(x)}$ is total and constant" is undecidable, which means "ϕ_x is total and constant" is undecidable.

7. $W_x = \emptyset$

$$f(x,y) = \begin{cases} x & \text{if } x \in W_x \\ \uparrow & \text{if } x \notin W_x \end{cases}$$

By Church's thesis f is computable, therefore by *s-m-n* theorem there exists a total computable function k such that $f(x,y) \cong \phi_{k(x)}(y)$.

$$\begin{aligned} x \in W_x &\implies f(x,y) \equiv 1 \implies \phi_{k(x)}(y) \equiv 1 \implies W_{k(x)} = \mathbb{N} \\ x \notin W_x &\implies \forall y(f(x,y)\uparrow) \implies \forall y(\phi_{k(x)}(y)\uparrow) \implies W_{k(x)} = \emptyset \end{aligned}$$

Thus we have $W_{k(x)} = \emptyset$ iff $x \notin W_x$. Since $x \notin W_x$ is decidable, then $W_{k(x)} = \emptyset$ is undecidable, which means $W_x = \emptyset$ is undecidable.

8. E_x is an infinite set

$$f(x,y) = \begin{cases} y & x \in W_x \\ \uparrow & x \notin W_x \end{cases}$$

By Church's thesis f is computable, therefore by *s-m-n* theorem there exists a total computable function k such that $f(x,y) \cong \phi_{k(x)}(y)$.

$$x \in W_x \implies \forall y(f(x,y) = y) \implies W_{k(x)} = \mathbb{N}$$
$$x \notin W_x \implies \forall y(f(x,y)\uparrow) \implies W_{k(x)} = \emptyset$$

This means that the set $W_{k(x)}$ is infinite iff $x \in W_x$. And since $x \in W_x$ is undecidable, then "$W_{k(x)}$ is an infinite set" is undecidable, which means "W_x is an infinite set" is undecidable.

9. $\phi_x = g$

$$f(x,y) = \begin{cases} g(y) & x \in W_x \\ \uparrow & x \notin W_x \end{cases}$$

By Church's thesis f is computable, therefore by *s-m-n* theorem there exists a total computable function k such that $f(x,y) = \phi_{k(x)}(y)$.

$$x \in W_x \implies f(x,y) = g(y) \implies \phi_{k(x)} = g$$
$$x \notin W_x \implies f(x,y) \neq g(y) \implies \phi_{k(x)} \neq g$$

Therefore $\phi_{k(x)} = g$ iff $x \in W_x$, since $x \in W_x$ is undecidable, then "$\phi_{k(x)} = g$ is undecidable, which means $\phi_x = g$ is undecidable.

Problem 2.

(i) We know that $x \in W_x$ iff $\neg M(k(x))$. Since $x \in W_x$ is partial decidable, then $\neg M(k(x))$ is also partial decidable. Now $M(k(x))$ cannot be partial decidable, since if it was partial decidable then $M(k(x))$ would be decidable (if both $M(x)$ and $\neg M(x)$ are partial decidable, then $M(x)$ is decidable.), which means that $x \in W_x$ is decidable, which is not. Now suppose $M(x)$ is partial decidable. This means that $M(k(x))$ is also partial decidable, and since we know that $M(k(x))$ is not partial decidable, we get a contradiction, showing that $M(x)$ is not partial decidable.

(ii)

$$f(x,y) = \begin{cases} 1 & x \in W_x \\ \uparrow & x \notin W_x \end{cases}$$

By Church's thesis f is computable. By *s-m-n* theorem there exists total computable k such that $f(x,y) = \phi_{k(x)}(y)$. Let $M(x)$ be the predicate "ϕ_x is total".

$$x \in W_x \iff \forall y(\phi_{k(x)}(y) = 1) \iff \phi_{k(x)} \text{ is total} \iff M(k(x)) = 1$$
$$\implies x \in W_x \longleftrightarrow M(k(x))$$

By part (i) we have $\neg M(k(x))$ is not partial decidable, which means that "ϕ_x is not total" is not partial decidable.

(iii)

$$f(x,y) = \begin{cases} \uparrow & \text{if } P_x(x) \text{ converges in at most } y \text{ steps.} \\ 0 & O.W. \end{cases}$$

Since f is computable there exists a total computable function k such that $f(x,y) = \phi_{k(x)}(y)$.

$$x \in W_x \iff \exists y(\phi_{k(x)}(y)\uparrow) \iff \phi_{k(x)} \text{ is not total}$$

If let M(x) = "ϕ_x is total". Therefore $x \in W_x \longleftrightarrow \neg M(k(x))$. By part (i) we know that $M(x)$ is not partial decidable.