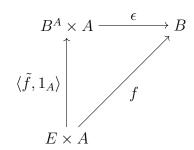
## Problem 1. Is the category of monoids cartesian closed? (Exercise 6.8.4)

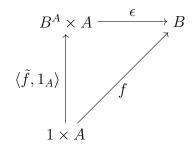
*Proof.* We know that the category of monoids, has initial object  $E = \langle \{e\}, \star, e \rangle$ . Now consider two monoids A and B such that there exists more than 1 homomorphism from A to B. (We can find two monoids with this property). We want to show that the object  $B^A$  cannot exists. Assume the opposite:



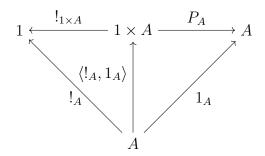
Now we know that there exists more than 1 homomorphism from A to B, but there only exists one  $\tilde{f}$  from E to  $B^A$  since E is initial. And since for any f there must exists a unique morphism  $\tilde{f}$  from E to  $B^A$ , we get a contradition, and therefore object  $B^A$  cannot exists. This shows that category of monoids is not CCC.

Problem 2. Show that for any objects A,B in a CCC, there is a bijective correspondence between points of the exponential  $1 \to B^A$  and arrows  $A \to B$ .(Exercise 6.8.9)

*Proof.* By definition of exponential object we have:



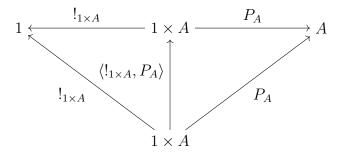
For any f there exists a unique  $\tilde{f}$  and for any  $\tilde{f}$  there exists a unique f, therefore there exists a bijection between points in  $1 \to B^A$  and  $1 \times A \to B$ . Thus we only have to show that there is a bijection between arrows in  $1 \times A \to B$  and arrows in  $A \to B$ . For this we show that  $A \cong 1 \times A$ .



Now it is only enough to show that  $P_A \circ \langle !_A, 1_A \rangle = 1_A$  and  $\langle !_A, 1_A \rangle \circ P_A = 1_{1 \times A}$ . The first one is obvious since  $1 \times A$  is the product of 1 and A. For the second one:

$$\langle !_A, 1_A \rangle \circ P_A = \langle !_A \circ P_A, 1_A \circ P_A \rangle = \langle !_{1 \times A}, P_A \rangle$$

Where we can observe:



Since  $\langle !_{1\times A}, P_A \rangle$  is unique and also  $1_{1\times A}$  commutes with the diagram, then we have  $1_{1\times A} = \langle !_{1\times A}, P_A \rangle$ . This proves that  $A \cong 1\times A$ . Thus we have  $A \rightleftharpoons_{f^{-1}} 1\times A$ . Now we can have our bijection with:

$$g:A\to B\implies f^{-1}\circ g:1\times A\to B\implies f\circ f^{-1}\circ g=g:A\to B$$
 
$$\overline{g}:1\times A\to B\implies f\circ \overline{g}:A\to B\implies f^{-1}\circ f\circ \overline{g}=\overline{g}:1\times A\to B$$

Therefore there is a bijection between arrows in  $A \to B$  and  $1 \times A \to B$ .

Problem 3. Show that the category of  $\omega$ CPOs is cartesian closed, but that the category of strict  $\omega$ CPOs is not.(Exercise 6.8.10)

*Proof.* First we show that category of  $\omega$ CPOs is cartesian closed. Given two  $\omega$ CPOs P and Q, the  $\omega$ CPO  $P \times Q$  has elements of the form (p,q) with  $p \in P$  and  $q \in Q$ . And relations as below:

$$(p,q) \le (p',q') \iff p \le p' \text{ and } q \le q'$$

To check if this really is an  $\omega$ CPO let:

$$(p_0, q_0) \leq (p_1, q_1) \leq \dots$$

$$\implies p_0 \leq p_1 \leq \dots \implies \lim_{\stackrel{\longrightarrow}{}} p_i = p_\omega$$

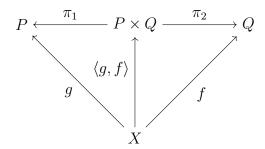
$$\implies q_0 \leq q_1 \leq \dots \implies \lim_{\stackrel{\longrightarrow}{}} q_i = q_\omega$$

$$\implies \lim_{\stackrel{\longrightarrow}{}} (p_i, q_i) = (p_\omega, q_\omega)$$

Now let (x, y) such that for any i,  $(p_i, q_i) \leq (x, y)$ :

$$\begin{cases} \forall i : p_i \le x \implies p_\omega \le x \\ \forall i : q_i \le y \implies q_\omega \le y \end{cases} \implies (p_\omega, q_\omega) \le (x, y)$$

Therefore  $P \times Q$  is indeed an  $\omega$ CPO. Now let  $\pi_1$  and  $\pi_2$  be projections. Let  $(p_0, q_0) \leq (p_1, q_1) \leq \ldots$ , with colimit  $(p_\omega, q_\omega)$ . Therefore for any  $i, p_i \leq p_\omega$ . Now suppose there is an element  $x \in P$  such that for any  $i, p_i \leq x$ . This shows that for any i we have:  $(p_i, q_i) \leq (x, q_\omega)$ . And since  $(p_\omega, q_\omega)$  is colimit, then we have  $(p_\omega, q_\omega) \leq (x, q_\omega)$  which means that  $p_\omega \leq x$ . This shows that  $p_\omega$  is the colimit of  $p_0, p_1, \ldots$ . Therefore  $\pi_1$  (and similarly  $\pi_2$ ) is continuous.



Since both f and g are monotone, and  $\langle g, f \rangle(x) = (g(x), f(x))$  with  $x \in X$ , then  $\langle g, f \rangle$  is also monotone. Also since f and g are continuous, suppose suppose  $x_{\omega}$  is the colimit of  $x_0 \leq x_1 \leq x_2 \leq \ldots$ , then  $f(x_{\omega})$  and  $g(x_{\omega})$  are both colimit of the respective diagrams. It is obvious that  $\langle g(x_{\omega}), f(x_{\omega}) \rangle$  is the colimit of  $\langle g(x_0), f(x_0) \rangle \leq \langle g(x_1), f(x_1) \rangle \leq \ldots$ . Thus  $\langle g, f \rangle$  is continuous.

For exponentials, consider:

$$Q^P = \{f : P \to Q | f \text{ is monotone and } \omega\text{-continuous.}\}$$

First we show that this object is an  $\omega$ CPO. Note that order in functions is pointwise. Let  $\forall i: f_i \in Q^P$  such that:

$$f_0 \le f_1 \le f_2 \le \dots$$

$$\implies \forall p \in P : f_0(p) \le f_1(p) \le f_2(p) \le \dots$$

Since all  $f_i(p)$ s are in Q, then there exists a colimit for them,  $p_{\omega}$ . Let  $g(p) = p_{\omega}$ . It is easy to see that for any  $0 \le i$ ,  $f_i \le g$ . Now consider  $h \in Q^P$  such that for any  $0 \le i$ ,  $f_i \le h$ .

$$\forall p \in P : f_i(p) \le h(p) \implies p_w \le h(p) \implies g(p) \le h(p)$$
  
 $\implies g \le h$ 

Therfore g is the colimit of this sequence and therefore  $Q^P$  is an  $\omega$ CPO. The next step is to show that  $Q^P$  has the properties of an exponential object. define  $\epsilon: Q^P \times P \to Q$  as  $\epsilon(f,p) = f(p)$ . We need to show that  $\epsilon$  is monotone and continuous. Suppose  $(f,p) \leq (f',p')$ :

$$\epsilon(f,p) = f(p) \le f'(p) \le f'(p') = \epsilon(f',p')$$

Thus  $\epsilon$  is monotone. Now suppose  $(f_{\omega}, p_{\omega})$  is the colimit of  $(f_0, p_0) \leq (f_1, p_0) \leq \ldots$ . Since  $\epsilon$  is monotone, then  $f_i(p_i) \leq f_{\omega}(p_{\omega})$ . Now let  $x \in Q$  such that for any  $i, f_i(p_i) \leq x$ . First we prove that for any  $i, f_i(p_{\omega}) \leq x$ .

$$f_i(p_0) \le f_i(p_1) \le \dots \le f_i(p_i) \le x$$
  
 $\forall_{j>i} : f_i(p_j) \le f_j(p_j) \le x$ 

Since  $f_i$  is  $\omega$ -continuous, then since  $p_{\omega}$  is the colimit of  $p_0 \leq p_1 \leq \ldots$ , then  $f_i(p_{\omega})$  is the colimit of  $f_i(p_0) \leq f_i(p_1) \leq \ldots$ , which means that  $f_i(p_{\omega}) \leq x$ . Now consider:

$$f_0(p_\omega) \leq f_1(p_\omega) \leq \dots$$

We introduce  $g: P \to Q$  as below:

$$g(p) = \begin{cases} x & \text{if } p \le p_0 \text{ or } p \ge p_0 \\ f_{\omega}(p) & O.W. \end{cases}$$

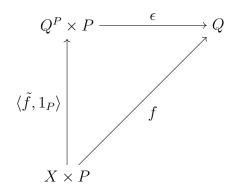
We can see that g is monotone. And also  $\omega$ -continuous, since for any  $r_0 \leq r_1 \leq \ldots$ , where  $r_i \in P$ , either  $g(r_i) = x$  for all of  $r_i$ s, or  $g(r_i) = f_{\omega}(r_i)$ . First case is obvious. The second case is  $\omega$ -continuous since  $f_{\omega}$  is  $\omega$ -continuous. Now we can see:

$$p \le p_0 \text{ or } p \ge p_0 \implies f_i(p) = x = g(p)$$
  
 $O.W. \implies f_i(p) \le f_\omega(p) = g(p)$ 

This proves that for all  $i, f_i \leq g$ . Now we know that  $f_{\omega}$  is the colimit of  $f_0 \leq f_1 \leq \ldots$ , therefore we have  $f_{\omega} \leq g$ :

$$f_i(p_\omega) \le f_\omega(p_\omega) \le g(p_\omega) = x$$

This proves that  $f_{\omega}(p_{\omega})$  is the colimit of  $f_0(p_0) \leq f_1(p_1) \leq \ldots$ . Thus  $\epsilon$  is  $\omega$ -continuous. Now let:



 $\tilde{f}(x) \in Q^P$  where for any  $p \in P$  we have  $(\tilde{f}(x))(p) = f(x,p)$ . We know that in case of existing such function, it is unique, Now we only have to show that if f is monotone and  $\omega$ -continuous, then  $\tilde{f}$  is monotone and  $\omega$ -continuous as well. Suppose  $x, x' \in X$  such that  $x \leq x'$ . We need to show that  $\tilde{f}(x) \leq \tilde{f}(x')$ . And for this we need to show that for any  $p \in P$ ,  $\tilde{f}(x)(p) \leq \tilde{f}(x')(p)$ :

$$\tilde{f}(x)(p) = f(x,p) \le f(x',p) = \tilde{f}(x')(p)$$

Thus  $\tilde{f}$  is monotone. Note that since  $(x,p) \leq (x',p)$  and f is monotone we conclude that  $f(x,p) \leq f(x',p)$ . As for  $\omega$ -continuous, consider  $x_0 \leq x_1 \leq \ldots$ , with colimit  $x_\omega$  in X. Since  $\tilde{f}$  is monotone, therefore  $\tilde{f}(x_i) \leq \tilde{f}(x_\omega)$ . Now Consider the function  $g: P \to Q$  such that for any  $i, \tilde{f}(x_i) \leq g$ .

$$\tilde{f}(x_i) \le g \implies \forall p \in P : \tilde{f}(x_i)(p) \le g(p)$$
  
 $\implies \forall p \in P : f(x_i, p) \le g(p)$ 

Since  $x_{\omega}$  is the colimit of  $x_0 \leq x_1 \leq \ldots$ , therefore  $(x_{\omega}, p)$  is the colimit of  $(x_0, p) \leq (x_1, p) \leq \ldots$ . And since f is  $\omega$ -continuous, therefore:

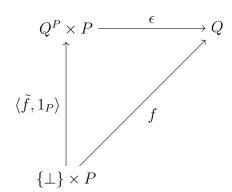
$$\forall p \in P : f(x_i, p) \le f(x_\omega, p) \le g(p)$$

$$\implies \forall p \in P : \tilde{f}(x_\omega)(p) \le g(p)$$

$$\implies \tilde{f}(x_\omega) \le g$$

This shows that  $\tilde{f}$  preserves limit and is  $\omega$ -continuous. This concludes that the category of  $\omega$ CPOs is indeed CCC.

To show that the category of strict  $\omega$ CPOs is not CCC we show that some exponential objects cannot exist. Let P and Q be two  $\omega$ CPOs with more than 1 element. We know that  $\{\bot\}$  is also an  $\omega$ CPO:



Now there exists many  $f: \{\bot\} \times P \cong P \to Q$ . But there exists only one  $\tilde{f}: \{\bot\} \to Q^P$  since  $\tilde{f}$  preserves  $\bot$ . This shows that such object  $Q^P$  doesn't exist. And therefore the category of strict  $\omega$ CPOs is not CCC.

## Problem 4. Consider the forgetful functors

$$oldsymbol{Groups} \overset{U}{
ightarrow} oldsymbol{Monoids} \overset{V}{
ightarrow} oldsymbol{Sets}$$

Say whether each is faithful, full, injective on arrows, surjective on arrows, injective on objects, and surjective on objects. (Exercise 7.11.4)

Proof. U:

## $U: \mathbf{Groups} \to \mathbf{Monoids}$

$$\langle X, \star, e \rangle \to \langle X, \star, e \rangle$$

$$f \downarrow \qquad \qquad \downarrow f$$

$$\langle Y, *, e' \rangle \to \langle Y, *, e' \rangle$$

(i) Full: We have to see that whether

$$F: Hom_{Grp}(A, B) \to Hom_{Mon}(UA, UB)$$

is surjective. Let  $g \in Hom_{Mon}(UA, UB)$  be a monoid homomorphism. Therefore for any  $x, y \in UA$  we have:

$$g(xy) = g(x)g(y)$$
$$g(1_{UA}) = 1_{UB}$$

Since U is a forgetful functor, therefore UA (similarly UB) is exactly A (similarly B). This shows that g also a group homomorphism between A and B and since both A and B have group structure as well, therefore F is surjective, thus U is full.

(ii) Faithful: We have to see that whether

$$F: Hom_{Grp}(A, B) \to Hom_{Mon}(UA, UB)$$

is injective. Let  $f, g \in Hom_{Grp}(A, B)$ ,  $f \neq g$  be two group homomorphisms. Since any group homomorphism is also a monoid homomorphism thus Fg = g and Ff = f. Now if Ff = Fg then we would have for any  $a \in A$ , f(a) = g(a). But this cannot happen since  $f \neq g$  in the first place.

- (iii) Surjective on objects: Since any monoid is not a group, thus U is not surjective on objects.
- (iv) Surjective on arrows: Since U is not surjective on objects, it can't be surjective on arrows as well.
- (v) Injective on objects: Let  $G = \langle X, \star, e \rangle$  and  $H = \langle Y, *, e' \rangle$  where  $H \neq G$ . If  $Y \neq X$  it is easy to see that after U, they are both different monoids. If Y = X, therefore there exists some  $a, b \in X$  such that  $a \star b \neq a * b$ . This shows that after U they are two different monoids. Therefore U is injective on objects.

(vi) Injective on arrows: Since U is injective on objects, then for any  $f: A \to B$  and  $g: C \to D$  where  $f \neq g$ , if one of  $A \neq C$  or  $B \neq D$  happens, then  $Uf \neq Ug$  because of different objects on domain or codomain. And if A = C and B = D then since U is faithful, we have  $Uf \neq Ug$ . Therefore U is injective on arrows as well.

 $\mathbf{V}$ :

## $V: \mathbf{Monoids} \to \mathbf{Sets}$

$$\langle X, \star, e \rangle \longrightarrow X$$

$$f \downarrow \qquad \qquad \downarrow f$$

$$\langle Y, \star, e' \rangle \longrightarrow Y$$

(i) Full: We have to see whether

$$V: Hom_{Mon}(A, B) \to Hom_{Set}(VA, VB)$$

is surjective. Consider A and B non-trivial monoids, then for any morphism f between A and B we have:  $f(1_A) = 1_B$ . Now consider some function between A and B such that it doesn't preserve identity. It can't be the image of any morphism in Monoids. Therefore V here is not surjective and therefore is not full.

(ii) Faithful: We have to see whether

$$V: Hom_{Mon}(A, B) \to Hom_{Set}(VA, VB)$$

is injective. Let  $f, g \in Hom_{Mon}(A, B)$  such that  $f \neq g$ . Therefore there exists some  $a \in A$  such that  $f(a) \neq g(a)$ . Now since V is forgetful, then Vf = f and Vg = g as functions. Therefore  $Vf \neq Vg$ , thus V here is injective and faithful.

- (iii) Surjective on objects: There is no moniod M such that VM =, since for any monoid M there exists some identity element  $e \in M$ , Thus it can't be empty. Therefore V is not surjective on objects.
- (iv) Surjective on arrows: Since V is not surjective on objects, it can't be surjective on arrows.
- (v) Injective on objects: Consider the monoids  $M = \langle \{e, a\}, \star, e \rangle$  and  $M' = \langle \{e, a\}, \star, e \rangle$  where  $a \star e = a \star e = a \star a = a$  and  $a \star a = e$ . Clearly  $M \neq M'$ , but  $VM = \{1, 2\} = VM'$ , thus V is not injective on objects.
- (vi) Injective on arrows: Consider M and M' from the previous part. Functions below are different in Monoids:

$$1_M: M \to M$$
  $1_{M'}: M' \to M'$   $m \mapsto m$   $m' \mapsto m'$ 

But are the same in Sets since they are both  $1_{\{a,e\}}:\{a,e\}\to\{a,e\}$ . Therefore V is not injective on arrows.

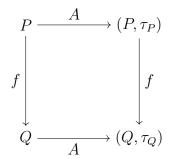
Problem 5. Make every poset  $(X, \leq)$  into a topological space by letting  $U \subset X$  be open just if  $x \in U$  and  $x \leq y$  implies  $y \in U$  (U is "closed upwards"). This is called Alexandroff topology on X. Show that it gives a functor

$$A: Pos \rightarrow Top$$

from posets and monotone maps to spaces and continuous maps by showing that any monotone map of posets  $f: P \to Q$  is continuous with respect to this topology on P and Q (the inverse image of an open set must be open). Is A faithful? is it full?

How would the situation change if instead one took as open sets those subsets that are closed downwards? (Exercise 7.11.5)

*Proof.* First we show that any monotone map between P and Q is also a continuous map between topology on P and Q.



Let  $V \subset Q$  be an open in  $\tau_Q$ . And let  $U = f^{-1}(V)$ . Let  $x \in U$  and  $y \in P$  such that  $x \leq y$ . Since f is monotone:

$$\left. \begin{array}{l} x \leq y \implies f(x) \leq f(y) \\ f(x) \in V \\ V \text{ is open} \end{array} \right\} \implies f(y) \in V \implies y \in f^{-1}(V) = U$$

This shows that U is open in  $\tau_P$ . Therefore inverse image of any open, is open, thus f is continuous. Two other properties of functors is easy to check since for any map f in Pos, Af = f.

Sicne Af = f then for any two  $f, g \in Hom_{Pos}(P, Q)$  where  $f \neq g$ , then  $Af \neq Ag$ , therefore A is faithful. For full, consider the continuous function  $f: (P, \tau_P) \to (Q, \tau_Q)$ . Let  $x, y \in P$  such that  $x \leq y$ . We want to show that f is also monotone. If  $f(x) \leq f(y)$  then we are done. Otherwise suppose f(y) < f(x). Now in  $(Q, \tau_Q)$  consider the open set U with  $f(x) \in U$  and  $z \in U$  iff  $f(x) \leq z$ . It is easy to see that f(y) is not in U. Since f is continuous, then  $V = f^{-1}(U)$  is open in  $(P, \tau_P)$ . And since  $f(y) \notin U$  then  $y \notin V$ . And also since  $f(x) \in U$  then  $x \in V$ . Since V is open we have:

$$\left. \begin{array}{l} x \in V \\ x \le y \end{array} \right\} \implies y \in V$$

But we showed that  $y \notin V$ . The contradiction shows that the assumption f(y) < f(x) was wrong and we have  $f(x) \leq f(y)$ . Therefore if f is a continuous map between  $(P, \tau_P)$  and  $(Q, \tau_Q)$ , then f is also monotone between P and Q. This shows that A is full, since any continuous map is image of some monotone map.

For open sets such that they are closed downwards, is exactly the same, A is functor, full, and faithful.

Problem 6. Prove that every functor  $F: C \to D$  can be factored as  $D \circ E = F$ ,

$$C \stackrel{E}{\longrightarrow} E \stackrel{D}{\longrightarrow} D$$

in the following two ways:

- (a)  $E: C \rightarrow E$  is bijective on objects and full, and  $D: E \rightarrow D$  is faithful;
- (b)  $E: C \to E$  is surjective on objects and  $D: E \to D$  is injective on objects and full and faithful.

When do the two factorizations agree? (Exercise 7.11.6)

*Proof.* For each part, we construct  $\mathbf{E}$ :

(a) Let **E** be a category with  $\mathbf{E_0} = \mathbf{C_0}$  and for any  $X, Y \in \mathbf{E_0}$  we have:

$$Hom_{\mathbf{E}}(X,Y) = Hom_{F(\mathbf{C})}(FX,FY)$$

In other words, each morphism between X and Y in  $\mathbf{E}$ , is representing a morphism in image of F, between FX and FY. For any  $X \in \mathbf{E}$ ,  $1_X \in Hom_{\mathbf{E}}(X,X)$  is the equivalent of  $1_{F(\mathbf{C})}(FX,FX)$ . And composition of two morphisms f and g is as below:

$$\exists f', g' : f = Ff', g = Fg'$$
$$f \circ g = F(f') \circ F(g') = F(f' \circ g')$$

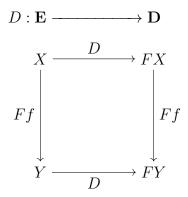
 $E: \mathbf{C} \longrightarrow \mathbf{E}$ 

It is obvious that **E** is a category. Now consider the functor:

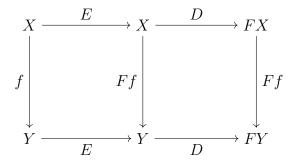
$$\begin{array}{ccc}
X & \xrightarrow{E} & X \\
\downarrow & & \downarrow \\
f \downarrow & & \downarrow \\
Y & \xrightarrow{E} & Y
\end{array}$$

To show that E is functor, we only need to show that composition and Identity are preserved, which is trivial since both are preserved with functor F.

Note that any morphism in **E** is of the form Ff' for some  $f' \in C_1$ . Now consider the functor:



To see that D is functor, we have to show that it preserves Identity and composition, which is trivial since F is a functor. To see that  $F = D \circ E$ :

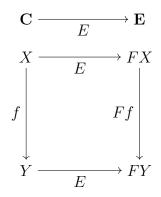


This shows that the composition of D and E is indeed F. Now E is injective on objects, since for any  $X \in \mathbb{C}$ , we have E(X) = X. And is also full, since by construction, any morphism in  $\mathbb{E}$  is of the form Ff for some  $f \in \mathbb{C}_1$ . And also D is faithful, since for any morphism  $Ff \in \mathbb{E}_1$ , we have D(Ff) = Ff.

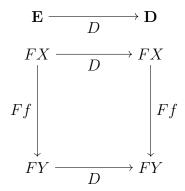
(b) Let **E** be a category with  $E_0 = F(C_0)$  and for any  $FX, FY \in \mathbf{E_0}$  we have:

$$Hom_{\mathbf{E}}(FX, FY) = Hom_{\mathbf{D}}(FX, FY)$$

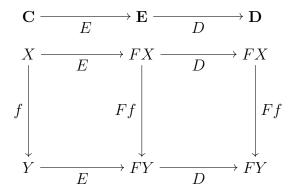
In other words, the set of morphisms in  $\mathbf{E}$  between FX and FY is exactly the set of morphism in  $\mathbf{D}$  between FX and FY. For any  $FX \in \mathbf{E_0}$ , let  $1_{FX}$  be the equivalent of  $1_{FX}$  in  $\mathbf{D}$ . And composition is defined exactly like composition for morphisms in  $\mathbf{D}$ . It is obvious that E is a category. Consider the functors:



To check if E is a functor we have to check that it preserves identity and composition. Identity is preserved since  $E(1_X) = F(1_X) = 1_{FX}$ . And composition is preserved since F is a functor. The other functor is:



Where D is the inclusion functor. Now to see that the composition of these two is F:



Now E is surjective on objects, since by definition we had  $\mathbf{E_0} = \mathbf{F}(\mathbf{C_0})$ . Now D is injective on objects since for any  $FX \in \mathbf{E_0}$  we have D(FX) = FX. Also D is full and faithful since by definition we had:

$$Hom_{\mathbf{E}}(FX, FY) = Hom_{\mathbf{D}}(FX, FX)$$

Problem 7. Show that the map of sets

$$\eta_A: A \to PP(A)$$

$$a \mapsto \{U \subseteq A | a \in U\}$$

is the component at A of a natural transformation  $\eta: 1_{Sets} \to PP$ , where  $P: Sets^{op} \to Sets$  is the (contraviariant) powerset functor. (Exercise 7.11.9)

*Proof.* First to understand the functor PP, since P is contraivariant:

$$A \xrightarrow{P} P(A) \xrightarrow{P} PP(A)$$

$$f \downarrow \qquad \qquad \uparrow \\ P(f) = f^{-1} \qquad \downarrow PP(f)$$

$$B \xrightarrow{PB} P(B) \xrightarrow{P} PP(B)$$

Let  $X \in P(B)$ , Then  $P(f)(X) = f^{-1}(X) = \{x \in A | f(x) \in X\}$ , similarly for PP(f), if  $U \in PP(A)$ :

$$PP(f)(U) = P(f)^{-1}(U) = (f^{-1})^{-1}(U) = \{X \in P(B)|f^{-1}(X) \in U\}$$

Now we only need to show that for any sets A and B, the following diagram commutes:

$$A \xrightarrow{\eta_A} PP(A)$$

$$f \downarrow \qquad \qquad \downarrow PP(f)$$

$$B \xrightarrow{\eta_B} PP(B)$$

And since we are in **Sets**, let  $a \in A$ , we need to show that  $\eta_B(f(a)) = PP(f)(\eta_A(a))$ .

$$PP(f)(\eta_{A}(a)) = PP(f)(\{U \subseteq A | a \in U\})$$

$$= \{X \in P(B) | f^{-1}(X) \in \{U \subseteq A | a \in U\}\}\}$$

$$= \{X \in P(B) | a \in f^{-1}(X)\}$$

$$= \{X \in P(B) | f(a) \in X\}$$

$$= \{X \subseteq B | f(a) \in X\}$$

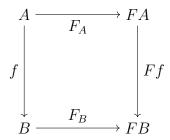
$$= \eta_{B}(f(a))$$

This completes the proof.

Problem 8. A category is skeletal if isomorphic objects are always identical. Show that every category is equivalent to a skeletal subcategory. (Every category has a "skeleton.") (Exercise 7.11.16)

*Proof.* Let C be a category. Since isomorphism of objects in a category is an equivalence relationship, then with the help of AC we choose a subcategory D, that from each equivalence class there is exactly one object. And all the morphisms between the chosen objects are also chosen. Now let F be a functor from C to D such that it maps every object to its equivalent in D:

$$F: C \longrightarrow D$$



Since A and FA are isomorphic to each other, therefore there exist  $F_A^{-1}$ . And we have  $Ff = F_B \circ f \circ F_A^{-1}$ . It is easy to see that  $F(1_A) = F_A \circ 1_A \circ F_A^{-1} = 1_{FA}$ . And for composition, let  $g: A \to B$  and  $f: B \to C$ :

$$F(f \circ g) = F_C \circ (f \circ g) \circ F_A^{-1} = F_c \circ f \circ F_B \circ F_B^{-1} \circ g \circ F_A^{-1} = F(f) \circ F(g)$$

Thus F is indeed a functor. Now we have to show that F is full, faithful and essensially surjective on objects.

For full consider any  $h: FA \to FB$ . Then  $k = F_B^{-1} \circ h \circ F_A : A \to B$  is in  $Hom_C(A, B)$  and F(k) = h.

For faithful, if F(f) = F(g) then we have:

$$F(f) = F_B \circ f \circ F_A^{-1} \implies f = F_B^{-1} \circ F(f) \circ F_A$$

$$F(g) = F_B \circ g \circ F_B^{-1} \implies g = F_B^{-1} \circ F(g) \circ F_A$$

$$\implies f = g$$

For essensially surjective on objects, for any  $X \in D$ , we have  $X \in C$ . And we have  $FX = X \cong X$ .

This shows that C and D are equivalent, and since D by construction was skeletal, then the proof is complete.