

Problem 1.

For absolute value functions (multiplicative valuations) we have the Ostrowski's Theorem. For additive valuation, let v be some additive valuation on \mathbb{Q} . Then we know that $v(1) = v(1) + v(1)$, and $v(1) = 0$. Then $| \cdot |_v$ is a non-archimedean absolute value function:

$$|x|_v = \left(\frac{1}{e}\right)^{v(x)}$$

By Ostrowski's Theorem this function is equivalent to some p -adic valuation, meaning there exists some $\alpha > 0$ such that $| \cdot |_v^\alpha = | \cdot |_p$ for some prime p . Then if we use \ln in both sides of the equation above we get:

$$\ln |x|_p^\alpha = -v(x) \implies v(x) = -\alpha \ln |x|_p$$

We will show that for any α this v is a valuation function:

$$\begin{aligned} v(xy) &= -\alpha \ln |xy|_p = -\alpha \ln |x|_p |y|_p = -\alpha (\ln |x|_p + \ln |y|_p) \\ &= -\alpha \ln |x|_p - \alpha \ln |y|_p \\ &= v(x) + v(y) \end{aligned}$$

WLOG suppose $|x|_p \leq |y|_p$, then $|x+y|_p \geq \max\{|x|_p, |y|_p\} = |y|_p$:

$$v(x+y) = -\alpha \ln |x+y|_p \leq -\alpha \ln |y|_p = \min\{-\alpha \ln |y|_p, -\alpha \ln |x|_p\} = \min\{v(y), v(x)\}$$

And lastly:

$$\begin{aligned} v(x) = \infty &\implies 0 = \left(\frac{1}{e}\right)^{v(x)} = |x|_p^\alpha \implies |x|_p = 0 \implies x = 0 \\ x = 0 &\implies v(x) = -\alpha \ln |0|_p = -\alpha \ln 0 = \infty \\ v(x) = \infty &\iff x = 0 \end{aligned}$$

This proves that any additive valuation is of the form $-\alpha \ln |x|_p$ for some prime p and positive α .

Problem 2.

Frist suppose that there exists $a \neq b$ such that $v(x-a) \neq 0$ and $v(x-b) \neq 0$. Then we have:

$$\begin{aligned} v(ax-ab) &= v(a) + v(x-b) = v(x-b) \neq 0 \\ v(bx-ba) &= v(b) + v(x-a) = v(x-a) \neq 0 \end{aligned}$$

Then we can write:

$$0 = v(b-a) = v((x-a) - (x-b)) \geq \min\{v(x-a), v(x-b)\}$$

Problem 3.

Problem 4.

Problem 5.

(*i*)

(*ii*)

(*iii*)