Elliptic Curves - Spring 2025 Problem Set 5
Amin Zolfagharian - 610002072

Problem 1.

For absolute value functions (multiplicative valuations) we have the Ostrowski's Theorem. For additive valuation, let v be some additive valuation on $\mathbb{Q}$. Then we know that $v(1) = v(1) + v(1)$, and $v(1) = 0$. Then $|\ |_v$ is a non-archimedean absolute value function:

$$|x|_v = (\frac{1}{e})^{v(x)}$$

By Ostrowski's Theorem this function is equivalent to some p-adic valueation, meaning there exists some $\alpha > 0$ such that $|\ |_v^\alpha = |\ |_p$ for some prime p. Then if we use ln in both sides of the equation above we get:

$$\ln |x|_p^\alpha = -v(x) \implies v(x) = -\alpha \ln |x|_p$$

We will show that for any α this v is a valuation function:

$$\begin{aligned} v(xy) = -\alpha \ln |xy|_p = -\alpha \ln |x|_p|y|_p &= -\alpha(\ln |x|_p + \ln |y|_p) \\ &= -\alpha \ln |x|_p - \alpha \ln |y|_p \\ &= v(x) + v(y) \end{aligned}$$

WLOG suppose $|x|_p \leq |y|_p$, then $|x + y|_p \geq \max\{|x|_p, |y|_p\} = |y|_p$:

$$v(x + y) = -\alpha \ln |x + y|_p \leq -\alpha \ln |y|_p = \min\{-\alpha \ln |y|_p, -\alpha \ln |x|_p\} = \min\{v(y), v(x)\}$$

And lastly:

$$v(x) = \infty \implies 0 = (\frac{1}{e})^{v(x)} = |x|_p^\alpha \implies |x|_p = 0 \implies x = 0$$
$$x = 0 \implies v(x) = -\alpha \ln |0|_p = -\alpha \ln 0 = \infty$$
$$v(x) = \infty \iff x = 0$$

This proves that any additive valuation is of the form $-\alpha \ln |x|_p$ for some prime p and positive α.

Problem 2.

Problem 3.

Problem 4.

Problem 5.

(*i*)

(*ii*)

(*iii*)