

Problem 1.

(i) Let $\langle s_1, s_2, \dots, s_n \rangle$ be the generators of M_1 .

$$0 \rightarrow M_1 \xrightarrow{\varphi} M \xrightarrow{\psi} M_2 \rightarrow 0$$

We know that $\ker\psi = \text{im } \varphi$. Therefore we can generate $\ker\psi$ with $\varphi(s_1), \varphi(s_2), \dots, \varphi(s_n)$.

Now since ψ is surjective, we have that:

$$\frac{M}{\ker\psi} = M_2$$

Since M_2 is finitely generated, then $M/\ker\psi$ is also finitely generated. Therefore the set of cosets is finitely generated. Let the generators be $\langle \bar{t}_1, \bar{t}_2, \dots, \bar{t}_k \rangle$. We argue that $\langle \varphi(s_1), \dots, \varphi(s_n), t_1, \dots, t_k \rangle$ generates M . Let $m \in M$. m belongs to one of cosets, thus we can write:

$$\begin{aligned} m + \ker\psi &= r_1 \bar{t}_1 + \dots + r_k \bar{t}_k + \ker\psi \\ \implies m - r_1 t_1 - \dots - r_k t_k &= i \end{aligned}$$

for some i in $\ker\psi$. Since $\ker\psi = \text{im } \varphi$, then we can write:

$$i = u_1 \varphi(s_1) + \dots + u_n \varphi(s_n)$$

This gives us that m is generated by the generator we introduced:

$$m = r_1 t_1 + \dots + r_k t_k + u_1 \varphi(s_1) + \dots + u_n \varphi(s_n)$$

Thus M is finitely generated.

However it is not necessarily true that if M is finitely generated, then M_1 and M_2 are finitely generated as well. An example of this could be the R -modules below:

$$0 \rightarrow R[x_1, x_2, \dots] \xrightarrow{\sigma_0} R \xrightarrow{id} R \rightarrow 0$$

Where R has 1, and σ_0 is valuation at 0 for all x_i s.

(ii)

Problem 2.

(i) Consider Φ :

$$\Phi : \text{Hom}_R\left(\bigoplus_{i \in I} M_i, N\right) \rightarrow \bigotimes_{i \in I} \text{Hom}_R(M_i, N)$$

Then for each $\sigma \in \text{Hom}_R(\bigoplus_{i \in I} M_i, N)$, we have that for any $m_i \in M_i$ for all $i \in I$, there exists some $n \in N$ such that:

$$\sigma(m_1, m_2, \dots) = n$$

Now define $\sigma_i = \sigma(0, 0, \dots, 0, m_i, 0, 0, \dots)$. Meaning it only one nonzero entry in i th place. Now we map:

$$\Phi : \sigma \mapsto \bigotimes_{i \in I} \sigma_i$$

Now it is easy to see that for any $\sigma, \delta \in \text{Hom}_R(\bigoplus_{i \in I} M_i, N)$, we have:

$$\Phi(\sigma + \delta) = \bigotimes_{i \in I} (\sigma + \delta)_i = \bigotimes_{i \in I} (\sigma_i + \delta_i) = \bigotimes_{i \in I} \sigma_i + \bigotimes_{i \in I} \delta_i = \Phi(\sigma) + \Phi(\delta)$$

Thus Φ is a homomorphism. Now to see that it is surjective as well suppose $\bigotimes_{i \in I} \psi_i \in \bigotimes_{i \in I} \text{Hom}_R(M_i, N)$, then let us introduce $\psi \in \text{Hom}_R(\bigotimes_{i \in I} M_i, N)$ such that for any $\bigotimes_{i \in I} m_i$ we have:

$$\psi\left(\bigotimes_{i \in I} m_i\right) = \sum_{i \in I} \psi_i(m_i)$$

This proves the surjectivity, and thus these are isomorphic as groups.

(ii)