Amin Zolfagharian - 610002072

Problem 1.

Suppose that there exists at least one i such that $f_i(p) \neq 0$. Now for any $\lambda \in K$ we have:

$$f(\lambda p) = f_0(p) + \lambda f_1(p) + \cdots + \lambda^d f_d(p)$$

Since λp p for any $\lambda \in K$, then we have $f(\lambda p) = 0$. Consider the polynomial $G(x)$:

$$G(x) = f_d(p)x^d + \cdots + f_1(p)x + f_0(p)$$

Note that since $f_i(p) \neq 0$, then G is not the zero polynomial. Hence has a finite number of roots. But any $\lambda \in K$ is a root of this polynomial since:

$$G(\lambda) = \lambda^d f_d(p) + \cdots + \lambda f_1(p) + f_0(p) = f(\lambda p) = 0$$

Which is a contradiction. Thus there exists no such i, and we have $f_i(p) = 0$ for any $0 \leq i \leq d$.

Problem 2.

(i) If f of degree d and g is of degree e, then we have:

$$f^* = x_n^d f(\frac{x_1}{x_n}, \frac{x_2}{x_n}, \ldots, \frac{x_{n-1}}{x_n})$$
$$g^* = x_n^e g(\frac{x_1}{x_n}, \frac{x_2}{x_n}, \ldots, \frac{x_{n-1}}{x_n})$$

And since fg is of degree $e + d$, then we have:

$$(fg)^* = x_n^{d+e} fg(\frac{x_1}{x_n}, \frac{x_2}{x_n}, \ldots, \frac{x_{n-1}}{x_n})$$

Now it is easy to see that $(fg)^* = f^* g^*$.
As for F_* and G_* we have:

$$F_* = F(x_1, x_2, \ldots, x_{n-1}, 1)$$
$$G_* = G(x_1, x_2, \ldots, x_{n-1}, 1)$$
$$(FG)* = FG(x_1, x_2, \ldots, x_{n-1}, 1)$$

Thus we have: $(FG)_* = F_* G_*$.

(ii) Suppose f is of degree d:

$$(f^*)_* = (x_{n+1}^d f(\frac{x_1}{x_{n+1}}, \frac{x_2}{x_{n+1}}, \dots, \frac{x_n}{x_{n+1}}))_*$$

Now if we let $g(x_1, x_2, \dots, x_n, x_{n+1}) = x_{n+1}^d f(\frac{x_1}{x_{n+1}}, \frac{x_2}{x_{n+1}}, \dots, \frac{x_n}{x_{n+1}})$, Then we have:

$$(f^*)_* = g_* = g(x_1, x_2, \dots, x_n, 1) = 1^d f(x_1, \dots, x_n) = f$$

Also:

$$x_{n+1}^r (F_*)^* = x_{n+1}^r (F(x_1, x_2, \dots, x_n, 1))^* = x_{n+1}^r (x_{n+1}^s F(\frac{x_1}{x_{n+1}}, \dots, \frac{x_n}{x_{n+1}}, 1))$$

Where s is the degree of $F(x_1, x_2, \dots, x_n, 1)$. Now each term of F is of the form $kx_1^{r_1} x_2^{r_2} \dots x_n^{r_n} x_{n+1}^{r_{n+1}}$, for some $k \in K$, Which in $x_{n+1}^r (F_*)^*$ is transformed to:

$$x_{n+1}^{r+s} \frac{x_1^{r_1} \dots x_n^{r_n}}{x_{n+1}^{r_1 + \dots + r_n}} = x_1^{r_1} x_2^{r_2} \dots x_n^{r_n} x_{n+1}^{r+s-r_1-r_2-\dots-r_n}$$

Now since F is homogeneous, then there is some fixed d such that $r_1 + \dots + r_n + r_{n+1} = d$. Therefore the term with the biggest degree in F_* is the term with the least r_{n+1}, which is r. This shows us that $d = s + r$. Therefore $r + s - r_1 - r_2 - \dots - r_n = r_{n+1}$. Therefore this term is the same in both forms, and since this was an arbitrary term, then we can conclude that $x_{n+1}^r (F_*)^* = F$.

(iii) If we have both $F, G \in K[x_1, x_2, \dots, x_n]$, then we have:

$$\begin{aligned}(F+G)_* &= (F+G)(x_1, x_2, \dots, x_{n-1}, 1) \\ &= F(x_1, x_2, \dots, x_{n-1}, 1) + G(x_1, x_2, \dots, x_{n-1}, 1) \\ &= F_* + G_*\end{aligned}$$

And if f is of degree d and g is of degree e, and without loss of generality suppose that $d \leq e$, then we have:

$$\begin{aligned}(f+g)^* &= x_n^e (f+g)(\frac{x_1}{x_n}, \frac{x_2}{x_n}, \dots, \frac{x_{n-1}}{x_n}) \\ &= x_n^e f(\frac{x_1}{x_n}, \frac{x_2}{x_n}, \dots, \frac{x_{n-1}}{x_n}) + x_n^e g(\frac{x_1}{x_n}, \frac{x_2}{x_n}, \dots, \frac{x_{n-1}}{x_n}) \\ &= x_n^{e-d} f^* + g^*\end{aligned}$$

Problem 3.

Suppose we have $f = hg$ and f is of degree d, where $h = h_0 + h_1 + \dots + h_n$ and $g = g_0 + g_1 + \dots + g_m$ and $m + n = d$, And g_i and h_i are homogeneous form of degree i, or 0. We know that product of homogeneous forms is also homogeneous. Suppose x and y are the smallest number that h_x and g_y are nonzero. Thus the smallest homogeneous for in gh is of degree $x + y$. But since f is homogeneous, then all terms in f are of degree $x+y$. Since g_m and h_n are both nonzero (since we supposed h is of degree n and g is of degree m), then we have a form with degree $m+n$, and thus $m+n = x+y$. This shows that m and n are the smallest numbers that h_n and g_m are nonzero, therefore $g = g_m$ and $h = h_n$, and both f and g are homogeneous.

Problem 4.

(i)