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Problem 1.

Let $\phi(x) = a_d x^d + a_{d-1}x^{d-1} + \cdots + a_1 x + a_0$ and $\psi(x) = b_e x^e + b_{e-1}x^{e-1} + \cdots + e_1 x + e_0$. Without the loss of generality suppose $d \geq e$. Now:

$$\begin{aligned} H(\frac{\phi(\frac{m}{n})}{\psi(\frac{m}{n})}) &\leq H(\frac{n^d\phi(\frac{m}{n})}{n^d\psi(\frac{m}{n})}) \\ &= \max\{|a_d m^d + \cdots + a_1 m n^{d-1} + a_0 n^d|, |b_e m^e n^{d-e} + \cdots + e_1 m n^{d-1} + e_0 n^d|\} \end{aligned}$$

Knowing $H(\frac{m}{n}) = \max\{|m|, |n|\}$, then $m^x n^{d-x} \leq H(\frac{m}{n})^d$:

$$\max\{|a_d m^d + \cdots + a_1 m n^{d-1} + a_0 n^d|, |b_e m^e n^{d-e} + \cdots + e_1 m n^{d-1} + e_0 n^d|\} \leq \\ \max\{|a_d + \cdots + a_1 + a_0|, |b_e + \cdots + b_1 + b_0|\} H(\frac{m}{n})^d$$

Putting $k = \max\{|a_d + \cdots + a_1 + a_0|, |b_e + \cdots + b_1 + b_0|\}$, we get:

$$H(\frac{\phi(\frac{m}{n})}{\psi(\frac{m}{n})}) \leq kH(\frac{m}{n})^d$$

$$\implies h(\frac{\phi(\frac{m}{n})}{\psi(\frac{m}{n})}) \leq dh(\frac{m}{n}) + log(k)$$

Problem 2.

Problem 3.

(i) We know that:

$$4h(P) - k \leq h(2P) \leq 4h(P) + k$$

Where k is not dependent on P. Thus:

$$|h(2P) - 4h(P)| \leq k \tag{1}$$

Now for every n we have that:

$$|\frac{1}{4^n}h(2^nP) - \frac{1}{4^{n-1}}h(2^{n-1}P)| = \frac{1}{4^n}|h(2^nP) - 4h(2^{n-1}P)|$$

Using (1) with $P = 2^{n-1}P$:

$$\frac{1}{4^n}|h(2^nP) - 4h(2^{n-1}P)| \leq \frac{1}{4^n}k$$

This means that for any $\epsilon > 0$, there exists some n such that $\frac{k}{4^n} \leq \epsilon$, meaning there exists some index n such that for any term with index $i > n$ in the sequence $\{a_n\} = \{\frac{1}{4^n}h(2^nP)\}_{n=1}^{\infty}$, we have that $|a_i - a_{i-1}| < \epsilon$, meaning that this sequence is Cauchy, which shows that it has a limit in $\mathbb{R}$, hence $\hat{h}(P)$ is well-defined.

(ii) Using $h(2P) \geq 4h(P) - k$ we get:

$$\begin{aligned} h(2^nP) \geq 4h(2^{n-1}P) - k &\geq 4^2h(2^{n-2}P) - (4+1)k \\ &\geq 4^3h(2^{n-3}P) - (4^2+4+1)k \\ &\vdots \\ &\geq 4^nh(P) - (4^{n-1}+4^{n-2}+\cdots+4+1)k \end{aligned}$$

On the other hand using $h(2P) \leq 4h(P) + k$ we get:

$$\begin{aligned} h(2^nP) \leq 4h(2^{n-1}P) + k &\leq 4^2h(2^{n-2}P) + (4+1)k \\ &\leq 4^3h(2^{n-3}P) + (4^2+4+1)k \\ &\vdots \\ &\leq 4^nh(P) + (4^{n-1}+4^{n-2}+\cdots+4+1)k \end{aligned}$$

Meaning:

$$\begin{aligned} \left|\hat{h}(P) - h(P)\right| &= \left|\lim_{n\to\infty} \frac{1}{4^n}h(2^nP) - h(P)\right| \\ &\leq \left|\lim_{n\to\infty} h(P) + \frac{4^{n-1}+\cdots+4+1}{4^n}k - h(P)\right| \\ &= \left|\lim_{n\to\infty} \frac{4^{n-1}+\cdots+4+1}{4^n}k\right| \end{aligned}$$

Now there eixists a $C > 0$ such that:

$$\left|\lim_{n\to\infty} \frac{4^{n-1}+\cdots+4+1}{4^n}k\right| < C$$

Then we have:

$$\left|\hat{h}(P) - h(P)\right| \leq C$$

(iii)

(iv) If P is of finite order, then the set of points 2^nP is finite, meaning that the set $\{h(2^nP)|n \in \mathbb{N}\}$ has finite elements. This means that with increase of n, $\frac{1}{4^n}h(2^nP)$ decreases and its limit is 0. Therefore $\hat{h}(P) = 0$. Conversely, if $\hat{h}(P) = 0$, then by part (iii), we have that $\hat{h}(mP) = m^2\hat{h}(P) = 0$ for any $m \in \mathbb{Z}$. Using part (ii) we get:

$$\left|\hat{h}(mP) - h(mP)\right| < C$$

for any P and any m. But we have:

$$\left|\hat{h}(mP) - h(mP)\right| = |-h(mP)| = h(mP) < C$$

For any $m \in \mathbb{Z}$. But we know that the set $\{Q \in E|h(Q) < C\}$ is finite. This means that the set $\{mP|m \in \mathbb{Z}\}$ is finite, which means that P is of finite order.