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Problem 1.

We have:

$$\alpha = a_0 + \frac{1}{a_1 + \frac{1}{a_2 + \dots}}$$

Thus we have:

$$\begin{aligned} -\alpha = -a_0 - \frac{1}{a_1 + \frac{1}{a_2+\dots}} &= -(a_0+1) + 1 - \frac{1}{a_1 + \frac{1}{a_2+\dots}} \\ &= -(a_0+1) + \frac{a_1 - 1 + \frac{1}{a_2+\dots}}{a_1 + \frac{1}{a_2+\dots}} \end{aligned}$$

Let $\beta = \frac{1}{a_2+\dots}$. No if $a_1 = 1$ then we have:

$$= -(a_0+1) + \frac{\beta}{1+\beta} = -(a_0+1) + \frac{1}{1+\frac{1}{\beta}} = -(a_0+1) + \frac{1}{1 + a_2 + \frac{1}{a_3+\dots}}$$

Thus we have: $-\alpha = [-a_0 - 1; a_2 + 1, a_3, \dots]$. And if $a_1 > 1$ then we have:

$$\begin{aligned} = -(a_0+1) + \frac{a_1 - 1 + \beta}{a_1 + \beta} &= -(a_0+1) + \frac{1}{\frac{a_1+\beta}{a_1-1+\beta}} \\ &= -(a_0+1) + \frac{1}{1 + \frac{1}{a_1 - 1 + \beta}} \end{aligned}$$

Thus we have: $-\alpha = [-a_0 - 1; 1, a_1 - 1, a_2, a_3, \dots]$.

Problem 2.

(i) First note that since $\alpha > 1$ then if we have $\alpha = [a_0; a_1, a_2, \dots]$, then $a_0 > 0$. Now consider $\frac{1}{\alpha}$:

$$\begin{aligned} \frac{1}{\alpha} &= \frac{1}{a_0 + \frac{1}{a_1+\dots}} \\ \implies \frac{1}{\alpha} &= [0; a_0, a_1, \dots] \end{aligned}$$

And since $a_0 > 0$ this is valid. Now if we have $[0; a_1, a_2, \dots, a_k]$ which is the $(k+1)$-th fraction for $\frac{1}{\alpha}$, we can write:

$$0 + \frac{1}{a_0 + \frac{1}{\ddots \frac{}{a_k}}} = \frac{1}{\frac{p_k}{q_k}}$$

Where $\frac{p_k}{q_k}$ is the k-th fraction for α. This completes the proof.

(ii) Similar to the last part, we have $\alpha = [a_0; a_1, \ldots, a_k]$ where $a_0 > 0$ since $\alpha > 1$. Note that the fraction is finite since α is a rational number. Now we have:

$$\frac{1}{\alpha} = \frac{1}{a_0 + \frac{1}{a_1 + \ldots}}$$

Which shows that $\frac{1}{\alpha} = [0; a_0, a_1, \ldots, a_k]$.

Problem 3.

We know that either $\frac{p_k}{q_k} < \alpha < \frac{p_{k+1}}{q_{k+1}}$ or $\frac{p_{k+1}}{q_{k+1}} < \alpha < \frac{p_k}{q_k}$. This means that $\alpha - \frac{p_k}{q_k}$ and $\frac{p_{k+1}}{q_{k+1}} - \alpha$ have the same sign. Suppose that both $\left|\alpha - \frac{p_k}{q_k}\right| \geq \frac{1}{2q_k^2}$ and $\left|\alpha - \frac{p_{k+1}}{q_{k+1}}\right| \geq \frac{1}{2q_{k+1}^2}$. Thus we have:

$$\begin{aligned}
\left|\frac{1}{q_k q_{k+1}}\right| = \left|\frac{p_{k+1}}{q_{k+1}} - \frac{p_k}{q_k}\right| &= \left|(\frac{p_{k+1}}{q_{k+1}} - \alpha) + (\alpha - \frac{p_k}{q_k})\right| \\
&= \left|\frac{p_{k+1}}{q_{k+1}} - \alpha\right| + \left|\alpha - \frac{p_k}{q_k}\right| \\
&\geq \frac{1}{2q_k^2} + \frac{1}{2q_{k+1}^2} \\
\implies 0 &\geq \frac{1}{2}(\frac{1}{q_k} - \frac{1}{q_{k+1}})^2
\end{aligned}$$

Which means that $q_k = q_{k+1}$, which is a contradiction, thus one of the inequalities must be true.

Problem 4.

We use the fact that $2 < \sqrt{7} < 3$:

$$\begin{aligned}
\frac{\sqrt{7}+1}{2} &= 1 + \frac{\sqrt{7}-1}{2} = 1 + \frac{1}{\frac{2}{\sqrt{7}-1}} \\
&= 1 + \frac{1}{\frac{2(\sqrt{7}+1)}{7-1}} = 1 + \frac{1}{\frac{\sqrt{7}+1}{3}} \\
&= 1 + \frac{1}{1 + \frac{\sqrt{7}-2}{3}} = 1 + \frac{1}{1 + \frac{1}{\frac{3}{\sqrt{7}-2}}}
\end{aligned}$$

And we have:

$$\begin{aligned}
\frac{3}{\sqrt{7}-2} &= \frac{3(\sqrt{7}+2)}{7-4} = \sqrt{7}+2 = 4 + \sqrt{7} - 2 \\
&= 4 + \frac{7-4}{\sqrt{7}+2} = 4 + \frac{1}{\frac{\sqrt{7}+2}{3}} = 4 + \frac{1}{1 + \frac{\sqrt{7}-1}{3}}
\end{aligned}$$

Which we can write:

$$\frac{\sqrt{7}-1}{3} = \frac{1}{\frac{3}{\sqrt{7}-1}} = \frac{1}{\frac{3(\sqrt{7}+1)}{7-1}} = \frac{1}{\frac{\sqrt{7}+1}{2}}$$

Which is what we started with, thus we can write:

$$\frac{\sqrt{7}+1}{2} = [\overline{1,1,4,1}]$$

Problem 5.

First we have to find the $\overline{1,3}$ part.

$$\begin{aligned} x = 1 + \frac{1}{3+\frac{1}{x}} &\implies x = 1 + \frac{x}{3x+1} \\ \implies x = \frac{4x+1}{3x+1} &\implies 3x^2 - 3x - 1 = 0 \\ &\implies x = \frac{3 \pm \sqrt{9+12}}{6} \end{aligned}$$

And since $x > 0$ then we have $x = \frac{3+\sqrt{21}}{6}$. Now we can find the value of the fraction:

$$\alpha = 1 + \frac{1}{5+\frac{1}{2+\frac{1}{3+\frac{1}{x}}}} = 1 + \frac{1}{5+\frac{1}{2+\frac{x}{3x+1}}} = 1 + \frac{1}{5+\frac{3x+1}{7x+2}} = 1 + \frac{7x+2}{38x+11} = \frac{45x+13}{38x+11}$$

And if we let $x = \frac{3+\sqrt{21}}{6}$:

$$\alpha = \frac{45(\frac{3+\sqrt{21}}{6})+13}{38(\frac{3+\sqrt{21}}{6})+11} = \frac{2430+6\sqrt{21}}{2074}$$

Problem 6.

Note that $d < \sqrt{d^2+1} < d+1$:

$$\begin{aligned} \sqrt{d^2+1} = d + \sqrt{d^2+1} - d = d + \frac{d^2+1-d^2}{\sqrt{d^2+1}+d} &= d + \frac{1}{\sqrt{d^2+1}+d} \\ &= d + \frac{1}{2d+\sqrt{d^2+1}-d} \end{aligned}$$

Thus we have:

$$\sqrt{d^2+1} - d = \frac{1}{2d+(\sqrt{d^2+1}-d)} \implies \sqrt{d^2+1}-d = [\overline{2d}]$$

Therefore $\sqrt{d^2+1} = d + (\sqrt{d^2+1}-d) = [d;\overline{2d}]$.