Graph Theory - Fall 2025 Problem Set 4

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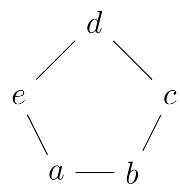
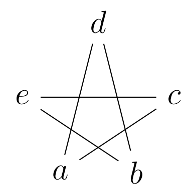
Problem 1.

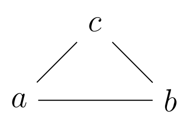
(a) G:



$\overline{G}$:



(b) G:



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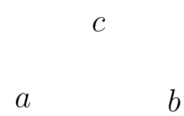
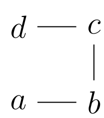
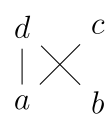


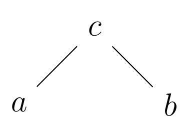
(c) G:



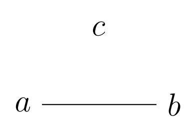
$\overline{G}$:



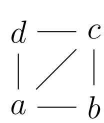
(d) G:



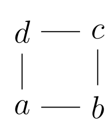
$\overline{G}$:



(e) G:



$G - e$:



Problem 3.

(a) Let there be $2n+1$ teams. If they all want to have the same score, then the score must be:

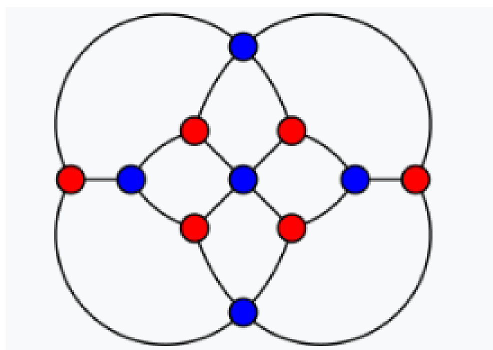
$$\frac{(2n+1)(2n)}{2}/(2n+1) = n$$

In order to achive this, put all teams around a cricle, and each team wins agains n teams in its right and loses agains n teams in its left. This way each team scores exactly n and all teams have the same score.

(b) Each game played produces a score. We have a total of $\binom{2n}{2} = n(2n-1)$ games hence a total of $n(2n-1)$ score to distribute among $2n$ teams. Since $2n \nmid n(2n-1)$, then it is not possible for all the teams to have the same score.

Problem 5.

Below is the fiure of Herschel graph:



It is easy to see that with the removal of *blue* vertices, the remaining graph is a disconnected and has 6 components (all *red* vertices). By Theorem 6.5 in any Hamiltonian graph and any nontrivial subset of vertices S, we have that $k(G-S) \leq |S|$, while in this graph if let S be the set of *blue* vertices, then $k(G-S) = 6 > 5 = |S|$, hence this graph is not Hamiltonian.

Problem 7.

($\Rightarrow$) If the graph is strongly connected, then there exists some vertex $s \in S$ and some vertex $t \in T$, by strongly connectedness we have that there exists a path from s to t, thus at some point the path leaves the set S and enters the set T, hence there eixsts an edge from S to T.

($\Leftarrow$) Conversely; Let u be a vertex of the graph and suppose there exist some vertices that u doesn't have a path to. Let these vertices be the set T, and let all the vertices that u has a path to them be in S. Note that $u \in S$ therefore both S and T are nonempty. Now by the condition of the problem there exists some edge from S to T. Suppose that this edge is from vertex $s \in S$ to $t \in T$. Since tehre exists a path from u to s and there is an edge from s to t, then there exists a path from u to t, and t must have been in S but was in T which is a contradiction. This shows that our assumption was wrong and u has a path to all other vertices.

Problem 9.

Suppose G is a directed graph and $G_1, G_2, \ldots, G_k$ be strongly connected components of this graph. Suppose that all these components have outgoing edges. Then you can consider these components as vertices and we have a graph that each vertex has an outgoing edge. Let P be the longest path in this graph which is formed by $v_1, \ldots, v_i$. We know that v_i has an outgoing edge. Since P was the longest path, then v_i can't be connected to a vertex out of P. Therefore we have a cylce since v_i is connected to a vetex in P. Therefore there is a cycle between the strongly connected components of the graph. This means that from each of the components of the cycle there is a path to any other component in the cylce, which proves that all of them are a single component. Which is a contradiction. This shows that there exists a component that doesn't have an outgoing edge. The proof for the other part is exacly the same.

Problem 11.

$$\begin{aligned}(m+1)^4 - (m+1) &= m^4 + 4m^3 + 6m^2 + 4m + 1 - m - 1 \\ &= (m^4 - m) + 4m^3 + 6m^2 + 4m\end{aligned} \tag{1}$$

$$\frac{m(m-1)(m-2)(m-3)}{4!} = \frac{m^4 - 6m^3 + 11m^2 - 6m}{24}$$

TODO!