Algebra III Fall 2024 Problem Set 1

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Problem 1.

Suppose G acts on set A from right:

$$. : A \times G \to A$$
$$\forall a \in A, g \in G : a.g \in A \tag{1}$$
$$\forall a \in A : a.1 = a \tag{2}$$
$$\forall g_1, g_2 \in G, a \in A : (a.g_2).g_1 = a.(g_2 g_1) \tag{3}$$

Now let $\star : G \times A \to A$ where $g \star a = ag^{-1}$.

$$\forall g \in G, a \in A : g \star a = a.g^{-1} \overset{(1)}{\in} A$$

$$\forall a \in A : 1 \star a = a.1^{-1} = a.1 \overset{(2)}{=} a$$

$$\forall g_1, g_2 \in G, a \in A : g_1 \star (g_2 \star a) = (a.g_2^{-1}).g_1^{-1} \overset{(3)}{=} a.(g_2^{-1} g_1^{-1}) = a.(g_1 g_2)^{-1} = (g_1 g_2) \star a$$

This shows that $\star$ is an action of G on A from left.

Problem 2.

(i) Suppose $g_1 \in g.\mathrm{stab}(x).g^{-1}$.

$$\exists g_2 \in \mathrm{stab}(x) : g_1 = g g_2 g^{-1}$$
$$g_1 g x = g g_2 g^{-1}(gx) = g g_2 x = gx \implies g_1 \in \mathrm{stab}(gx)$$
$$\implies g.\mathrm{stab}(x).g^{-1} \subset \mathrm{stab}(gx)$$

Now suppose $g_1 \in \mathrm{stab}(gx)$.

$$g_1 g x = gx \implies g^{-1}(g_1 g x) = g^{-1}(gx)$$
$$\implies g^{-1} g_1 g x = x \implies g^{-1} g_1 g \in \mathrm{stab}(x)$$
$$\implies g(g^{-1} g_1 g) g^{-1} = g_1 \in g.\mathrm{stab}(x).g^{-1}$$
$$\implies \mathrm{stab}(gx) \in g.\mathrm{stab}(x).g^{-1}$$

Therefore $\mathrm{stab}(gx) = g.\mathrm{stab}(x).g^{-1}$. Let two members of orbit of x be $g_1 x$ and $g_2 x$. We know that conjugation is surjective since:

$$x, y \in G : gxg^{-1} = gyg^{-1} \implies x = y$$

This shows that if H is a subset of G then $|gHg^{-1}| = |H|$.

$$|\mathrm{stab}(g_1 x)| = |g_1.\mathrm{stab}(x).g_1^{-1}| = |\mathrm{stab}(x)| = |g_2.\mathrm{stab}(x).g_2^{-1}| = |\mathrm{stab}(g_2 x)|$$

Thus $|\mathrm{stab}(g_1 x)| = |\mathrm{stab}(g_2 x)|$.

(ii) We saw in class that: $|O_{x_0}| = |G/G_{x_0}|$. Where O_{x_0} is orbit of x_0 and G_{x_0} is stabilizer of x_0. Now consider the set of equations where $g \in G, x \in X : gx = x$, denoted by $N(G \to X)$.

$$N(G \to X) = \underset{g \in G}{\cup} \text{fix}(g)$$

And since $\text{fix}(g_1)$ and $\text{fix}(g_2)$ represent different equations therefore:

$$|N(G \to X)| = \sum_{g \in G} \text{fix}(g)$$

And also we have:

$$N(G \to X) = \underset{x \in X}{\cup} G_x$$

And since G_{x_1} and G_{x_2} represent different equations therefore:

$$|N(G \to X)| = \sum_{x \in X} |G_x|$$

By part (i) we know that elements in the same orbit have stabilizers with the same size therefore:

$$|N(G \to X)| = \sum_{x \in I} |G_x|.|O_x| = \sum_{x \in I} |G_x|.|G|/|G_x| = \sum_{x \in I} |G| = N|G|$$

Where I is a set with exactly one element from each orbit and N number of orbits. Therefore:

$$\sum_{x \in G} \text{fix}(g) = N|G|$$

(iii) Assuming $|X| > 1$. Since it is transitive there is only one orbit thus:

$$|G| = \sum_{g \in G} \text{fix}(g)$$

Since $1 \in G$ fixes everything in X, if every element in G fixes at least one memeber then we would have:

$$\sum_{g \in G} \text{fix}(g) = |G| - 1 + \text{fix}(1) > |G|$$

Therefore there exists $g \in G$ where $\text{fix}(g) = 0$.

Problem 3.

(i) We can rewrite the group action:

$$a \in \mathbb{Z}, x \in \mathbb{Z}/6\mathbb{Z}$$
$$a.x = a + x \pmod 6$$

Then it is easy to see that this action is transitive. Since for any two $x, y \in \mathbb{Z}/6\mathbb{Z}$ we can use $x - y \in \mathbb{Z}$ to see that they are in the same orbit.

$$(x-y).y = x - y + y = x \overset{6}{\equiv} x$$
$$x \in O_y$$

And it is easy to see that if $x \in G_0$:

$$0 + x \overset{6}{\equiv} 0 \implies x \overset{6}{\equiv} 0$$

Thus $G_0 = \{6k | k \in \mathbb{Z}\}$.

(ii) Let $A \in \mathrm{stab}(I_n)$. Therefore $AI_n = I_n$ and also we know that for any A we have $AI_n = A$. This shows that $A = I_n$ and $\mathrm{stab}(I_n) = I_n$.
As for orbit if A and B have the same orbit then there exists a $C \in GL_n(\mathbb{R})$ such that $CA = B$. Now to show that they have the same kernel space:

$$Ax = 0 \implies CAx = 0 \implies Bx = 0 \implies ker(A) \subset ker(B)$$
$$Bx = 0 \implies CAx = 0 \implies C^{-1}CAx = 0 \implies Ax = 0 \implies ker(B) \subset ker(A)$$
$$\implies ker(B) = ker(A)$$

Therefore if two matrices have the same orbit then they have the same kernel space. We also know that if two matrices have the same kernel space Therefore There exists an invertible matrix Q such that $QA = B$. This shows that if two matrices have the same kernel space then they are in the same orbit.

Problem 4.

(i)

$$1(B) = B \implies 1 \in G_B$$
$$g \in G_B : g^{-1}(B) = g^{-1}(g(B)) = 1(B) = B \implies g^{-1} \in G_B$$
$$g_1, g_2 \in G_B : g_1 g_2(B) = g_1(g_2(B)) = g_1(B) = B \implies g_1 g_2 \in G_B$$

Therefore G_B is a sub-group of G.

(ii) For S_4 on $\{1,2,3,4\}$ we just have to check subsets with 2 and 3 members. If B has two elements, $B = \{1,2\}$ then for $\sigma = (1\ 3) \in S_4$, $\sigma(B) = \{2,3\}$. Which proves that B is not a block. If B has three elements, $B = \{1,2,3\}$. then for $\sigma = (3\ 4) \in S_4$, $\sigma(B) = \{1,2,4\}$. Which again proves that B is not a block. Therefore this action is primitive.
For D_8 on square. We can see that $A = \{A_1, A_3\}$ is a block.

$$\begin{tikzcd} A_1 \arrow[r, dash] \arrow[d, dash] & A_2 \arrow[d, dash] \\ A_4 \arrow[r, dash] & A_3 \end{tikzcd}$$

Since generators of D_8 are r and s we just have to check that these two act on A correctly. Since $\overline{A} = \{A_2, A_4\}$ is an identical block to A.

$$r(A) = \{A_2, A_4\} \implies \overline{A} \cap A = \varnothing$$
$$s(A) = A$$

We can also say the same about $\overline{A}$ as well.
This proves that A is a non-trivial block. Therefore this action is not primitive.

Problem 5.

(i) We will prove a stronger argument. for any $n \in \mathbb{N}$ and $m \in \mathbb{Q}$ there exists finite number of n-tuples $(i_1, i_2, \ldots, i_n)$ such that $i_j \in \mathbb{N}$ and $\sum_{j=1}^{n} \frac{1}{i_j} = m$.
We prove this by induction on n. for $n = 1$ it is trivial that there exists at most 1 i_1 such that $\frac{1}{i_1} = m$. Now suppose for any $m \in \mathbb{Q}$ there exists finite number of $(n-1)$-tuples like $(i_1, \ldots, i_{n-1})$, such that $\sum_{j=1}^{n-1} \frac{1}{i_j} = m$. Now for any n-tuple $(i_1, \ldots, i_n)$ that satisfies the equation let $i = \min\{i_j\}$.

$$m = \sum_{j=1}^{n} \frac{1}{i_j} \leq n\frac{1}{i}$$
$$\implies i \leq \frac{n}{m}$$

Without loss of generality suppose $i = i_1$. we know that i_1 has limited choices since $0 \leq i_1 \leq \frac{n}{m}$. And for any fixed i_1 we have:

$$m = \sum_{j=1}^{n} \frac{1}{i_j}$$
$$\implies m - \frac{1}{i_1} = \sum_{j=2}^{n} \frac{1}{i_j}$$

Which is a $(n-1)$-tuple and by induction hypothesis we know there are finite number of these $(n-1)$-tuples and since any n-tuple consists of union of finite number of these $(n-1)$-tuples (since i_1 have limited choices) then the statement is also true for n as well.

(ii) for any group G with n conjugacy class we have:

$$|G| = 1 + 1 \cdots + 1 + \frac{|G|}{|C(g_1)|} + \cdots + \frac{|G|}{|C(g_t)|}$$
$$\implies 1 = \frac{1}{|G|} + \cdots + \frac{1}{|G|} + \sum_{i=1}^{t} \frac{1}{|C(g_i)|}$$

This gives us a n-tuple with $(|G|, \ldots, |G|, |C(g_1)|, \ldots, |C(g_t)|)$. by part (i) we know there are finite number of these n-tuples. And in each of them $|G|$ must appear in the n-tuple since there exists at least one element in $Z(G)$ (The identity element) and it has a conjugacy class with one element. Since number of n-tuples are finite and in any of them size of $|G|$ is limited since it is in n-tuple therefore number of these groups is finite.

Problem 6.

We can write Q_8 with order $\{1, -1, i, -i, j, -j, k, -k\}$. Since i and j are generators of Q_8 then we just have to check these two elements:

$$i(1) = i.1 = i = 3$$
$$i(2) = i.(-1) = -i = 4$$
$$i(3) = i.(i) = -1 = 2$$
$$i(4) = i.(-i) = 1 = 1$$
$$i(5) = i.(j) = k = 7$$
$$i(6) = i.(-j) = -k = 8$$
$$i(7) = i.(k) = -j = 6$$
$$i(8) = i.(-k) = j = 5$$

This shows that i operates like $\delta = (1\ 3\ 2\ 4)(5\ 7\ 6\ 8)$ in S_8. Similarly:

$$j(1) = j.(1) = j = 5$$
$$j(2) = j.(-1) = -j = 6$$
$$j(3) = j.(i) = -k = 8$$
$$j(4) = j.(-i) = k = 7$$
$$j(5) = j.(j) = -1 = 2$$
$$j(6) = j.(-j) = 1 = 1$$
$$j(7) = j.(k) = i = 3$$
$$j(8) = j.(-k) = -i = 4$$

This shows that j operates like $\sigma = (1\ 5\ 2\ 6)(3\ 8\ 4\ 7)$ in S_8.
Therefore $Q_8 = \langle i, j \rangle$ is isomorphic to $\langle \delta, \sigma \rangle$ in S_8.