Elliptic Curves - Spring 2025 Problem Set 2
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Problem 1.

Suppose that there exists at least one i such that $f_i(p) \neq 0$. Now for any $\lambda \in K$ we have:

$$f(\lambda p) = f_0(p) + \lambda f_1(p) + \cdots + \lambda^d f_d(p)$$

Since λp p for any $\lambda \in K$, then we have $f(\lambda p) = 0$. Consider the polynomial $G(x)$:

$$G(x) = f_d(p)x^d + \cdots + f_1(p)x + f_0(p)$$

Note that since $f_i(p) \neq 0$, then G is not the zero polynomial. Hence has a finite number of roots. But any $\lambda \in K$ is a root of this polynomial since:

$$G(\lambda) = \lambda^d f_d(p) + \cdots + \lambda f_1(p) + f_0(p) = f(\lambda p) = 0$$

Which is a contradiction. Thus there exists no such i, and we have $f_i(p) = 0$ for any $0 \leq i \leq d$.

Problem 2.

(i) If f of degree d and g is of degree e, then we have:

$$f^* = x_n^d f(\frac{x_1}{x_n}, \frac{x_2}{x_n}, \ldots, 1)$$
$$g^* = x_n^e g(\frac{x_1}{x_n}, \frac{x_2}{x_n}, \ldots, 1)$$

And since fg is of degree $e + d$, then we have:

$$(fg)^* = x_n^{d+e} fg(\frac{x_1}{x_n}, \frac{x_2}{x_n}, \ldots, 1)$$

Now it is easy to see that $(fg)^* = f^* g^*$.
As for F_* and G_* we have:

$$F_* = F(x_1, x_2, \ldots, x_{n-1}, 1)$$
$$G_* = G(x_1, x_2, \ldots, x_{n-1}, 1)$$
$$(FG)* = FG(x_1, x_2, \ldots, x_{n-1}, 1)$$

Thus we have: $(FG)_* = F_* G_*$.

(ii) Suppose f is of degree d:

$$(f^*)_* = (x_n^d f(\frac{x_1}{x_n}, \frac{x_2}{x_n}, \ldots, \frac{x_{n-1}}{x_n}, 1))_*$$

(iii)