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Problem 1.

We prove this with induction. Base case is trivial as every group with prime order has only $\{id\}$ and itself as subgrouop.
Now suppose for any $n < N$, converse of Lagrange holds. Now let $n = N$.
Let G be a group with n elements. And let $d \mid n$. If d is a prime, then by Cauchy theorem we know that there exists a subgroup $H < G$ with $|H| = d$. If d is not a prime it follows that there exists a prime p where $p \mid d$ and $d = pd_1$. By Cauchy theorem There exists a subgroup $H < G$ with $|H| = p$. And since G is abelian, H is normal in G. Now we know that $\frac{G}{H}$ is group with $\frac{n}{p} < N$ elements. Since $d_1 \mid \frac{n}{p}$ by induction hypothesis we know there exists a subgroup $\frac{K}{H} < \frac{G}{H}$ where $\frac{|K|}{|H|} = d_1$. But since G is abelian then $\frac{G}{H}$ is also abelian which means that $\frac{K}{H} \triangleleft \frac{G}{H}$. By forth isomorphism theorem we know that $K \triangleleft G$ with $|K| = pd_1 = d$. This proves the converse of Lagrange.

Problem 2.

Suppose p is an odd prime. By Cauchy theorem we know there exists an element a with order 2 and an element b with order p. Since $|G : \langle b \rangle| = 2$ we know that $\langle b \rangle$ is normal in G. Therefore $\langle a, b \rangle$ is a group that is bigger than $\langle b \rangle$. and since $|G : \langle b \rangle| = 2$ then $G = \langle a, b \rangle$. Since $\langle b \rangle$ is normal then we have:

$$\exists_{0 \le k < p} \; aba^{-1} = b^k \in \langle b \rangle$$
$$b = a^2ba^{-2} = a(aba^{-1})a^{-1} = ab^ka^{-1} = (aba^{-1})\dots(aba^{-1}) = (b^k)^k = b^{k^2}$$
$$\implies b^{k^2-1} = 1$$

And since order of b was p we have $p \mid k^2 - 1 = (k-1)(k+1)$.

(a) If $p \mid k - 1$ then $k = 1$ which shows that $ab = ba$. It is easy to see that ab is the generator of G since it generates both a and b:

$$(ab)^p = a^pb^p = a^p = a$$
$$(ab)^{p+1} = a^{p+1}b^{p+1} = b^{p+1} = b$$

Therefore $G = \langle ab \rangle$. Thus $G \cong \mathbb{Z}_{2p}$.

(b) If $p \mid k + 1$ and since $0 \le k < p$ then $k = p - 1$. Thus G is a group with $G = \langle a, b \rangle$ where $aba^{-1} = b^{p-1}$. We can rewrite this as:

$$G = \langle a, b | a^2 = b^p = 1, ba = ab^{p-1} \rangle \cong D_{2p}$$

Problem 3.

Consider the action of G over G with conjugation. We know that x and y are in the same orbit. Let it be O_x. then we know that G also acts on O_x. Now we can write:

$$|O_x| = \frac{|G|}{|\text{stab}_G(x)|}$$

Now consider the action of H over O_x with conjugation. This time O_x^H and O_y^H show the orbit of x and y under this action. Now we have:

$$|O_x^H| = \frac{|H|}{|\text{stab}_H(x)|}$$

But we also know that $\text{stab}_H(x) = H \cap \text{stab}_G(x)$. Let $K = \text{stab}_G(x)$. And since H is normal in G then KH is a subgroup of G and $|KH| = \frac{|K||H|}{|K \cap H|}$. Then we have:

$$|O_x^H| = \frac{|H|}{\frac{|H||K|}{|KH|}} = \frac{|KH|}{|K|}$$

Also since $H < HK < G$ and $|G : H| = p$ then either $HK = H$ or $HK = G$.

(a) If $HK = H$ Then we have $|H \cap K| = |K|$. In other words we have:

$$|\text{stab}_H(x)| = |\text{stab}_G(x) \cap H| = |\text{stab}_G(x)|$$

Which cannot happen since we know $\text{stab}_H(x) \underset{\neq}{<} \text{stab}_G(x)$.

(b) If $HK = G$ then we have:

$$|O_x^H| = \frac{|KH|}{|K|} = \frac{|G|}{|\text{stab}_G(x)|} = |O_x|$$

Thus orbit of x under action of H has the same size as orbit of x under action of G and since $O_x^H \subset O_x$ then $y \in O_x^H$ and y and x have the same orbit under actoin of H. Which shows that x and y are conjugate in H.

Problem 4.

Let H be a subgroup of $G = S_n$ such that $|G : H| = i$ where $2 \leq i \leq n-1$. This shows that number of left cosets of H is i. Let $\frac{G}{H}$ be the set of left cosets of H. Therefore $sym(\frac{G}{H}) \cong S_i$. We also know that G acts on $\frac{G}{H}$ with left product on cosets. This shows that There exists a homomorphism φ from G to $sym(\frac{G}{H})$.

$$\varphi : G \to sym(\frac{G}{H})$$

$ker(\varphi)$ must be a subgroup of H. Since $\alpha \in ker(\varphi)$.

$$\varphi(\alpha) = id \implies \alpha H = H \implies \alpha \in H$$

Also $ker(\varphi)$ cannot be trivial since $|G| > |sym(\frac{G}{H})| = |S_i|$. Thus we have a normal subgroup of G which is a subgroup of H. But we know that the only normal subgroup of S_n is A_n, Which is not a subgroup of H (Since H is not A_n and A_n has biggest size possible as a non-trivial subgroup). Contradiction. And the reason for this contradiction is that we supposed $|sym(\frac{G}{H})| < n$ and therefore $ker(\varphi)$ had to be a proper normal subgroup of G. This shows that S_n doesn't have a subgroup with index less than n. Note that if $H = A_n$ there wouldn't be a contradiction since $A_n < A_n$.

Problem 5.

For any $x \in K$ we have $K = O_x^G$. And we know that:

$$|O_x^G| = \frac{|G|}{|C_G(x)|}$$

And orbit of x under action of H is:

$$|O_x^H| = \frac{|H|}{|C_H(x)|}$$

Where $C_H(x) = C_G(x) \cap H$. And since H is normal then $HC_G(x)$ is a subgroup of G. And we know that $|HC_G(x)| = |H||C_G(x)|/|H \cap C_G(x)|$.

$$|O_x^H| = \frac{|H|}{|C_H(x)|} = \frac{|H||HC_G(x)|}{|H||C_G(x)|} = \frac{|HC_G(x)|}{|C_G(x)|}$$

This shows that for any $x \in K$ size of every orbit under action of H is:

$$|O_x^H| = \frac{|HC_G(x)|}{|G|}|O_x^G|$$

To prove all of this orbits have the same size we use the fact that conjugation by $g \in G$ over a set is a 1-1 function. Suppose $x = gyg^{-1}$:

$$|O_x^H| = |gO_x^H g^{-1}| = |\{ghxh^{-1}g^{-1}|h \in H\}| = |\{ghg^{-1}gxg^{-1}gh^{-1}g^{-1}|h \in H\}|$$
$$= |\{ghg^{-1}ygh^{-1}g^{-1}|h \in H\}|$$

Since H is normal in G we know that $gHg^{-1} = H$:

$$|\{ghg^{-1}ygh^{-1}g^{-1}|h \in H\}| = |\{h'yh'^{-1}|h' \in H\}| = |O_y^H|$$

Thus size of all conjugacy classes under action of H is the same. And since set $\{O_x^H\}_{x \in K}$ is a partition of O_x^G. And size of each of conjugacy classes is $\frac{|HC_G(x)|}{|G|}|O_x^G|$ then we have exactly $|G : HC_G(x)|$ conjugacy class.
With this it is easy to see that a conjugacy class in A_n under action of S_n is union of $k = |G : A_nC_G(x)|$ conjugacy classes under action of A_n. And since $A_nC_G(x)$ is between A_n and S_n and $|S_n : A_n| = 2$ then either $A_nC_G(x) = A_n$ or $A_nC_G(x) = S_n$. Thus $k = 1$ or $k = 2$. Which proves the problem.

Problem 6.

Since we know that two elements of S_n are conjugate iff they have the same cycle type. It only remains to see how many differenct cycle types there are with order p. Consider $\sigma \in S_n$ with cylce type $(a_{i_1}, \dots, a_{i_{k_1}})(b_{j_1}, \dots, b_{j_{k_2}})\dots$. If size of these cylces are $k_1 \le k_2 \le \dots, k_t$, then it is easy to see that order of σ is $\text{lcm}(k_1, \dots, k_t)$. Now if σ has order p then $\text{lcm}(k_1, \dots, k_t) = p$ which means that each of k_is are either p or 1. This shows that if an element has order p then it consists of some p-cycles. Thus number of p-cycles uniquely determines the cycle type. And we have $\lfloor \frac{n}{p} \rfloor$ conjugacy class with elements of order p.