

Problem 1.

Let M be an R -module, where scalar product is non-trivial. Now consider the $M \times M$ as a R -module. With componentwise addition and multiplication.

$$\begin{aligned} M_1 &= \{(m, 0) | m \in M\} \\ M_2 &= \{(0, m) | m \in M\} \end{aligned}$$

M_1 and M_2 are both sub-modules of $M \times M$ since:

$$\begin{aligned} \forall x, y \in M_1, \forall r \in R : rx + y &= r(m_x, 0) + (m_y, 0) \\ &= (rm_x, 0) + (m_y, 0) \\ &= (rm_x + m_y, 0) \in M_1 \end{aligned}$$

Last lines is a direct result of M being a R -module. Now consider $M_1 \cup M_2$. And $x \neq 0 \in M_1, y \neq 0 \in M_2$ and $r \in R$ such that $rx \neq 0$:

$$rx + y = r(m_x, 0) + (0, m_y) = (rm_x, m_y)$$

Since both $rm_x \neq 0$ and $m_y \neq 0$, then $(rm_x, m_y) \notin M_1 \cup M_2$. This shows that $M_1 \cup M_2$ is not a sub-module of $M \times M$.

Problem 2.

(b) Consider \mathbb{Z}_6 as a \mathbb{Z}_6 -module. It is easy to see that $Tor(\mathbb{Z}_6) = \{0, 2, 3, 4\}$. But this is not a subgroup of \mathbb{Z}_6 and can't be a sub-module.

(c) Let $m \neq 0 \in M$, where M is a R -module. And let $r_1 r_2 = 0$ where $r_1, r_2 \in R$ and $r_1, r_2 \neq 0$: If $r_2 m = 0$ then we are done since $m \in Tor(M)$. Otherwise consider $(r_1 r_2)m = r_1(r_2 m) = 0$ Since $r_2 m \neq 0$ then $r_2 m \in Tor(M)$. This completes the proof.

Problem 3.

Again consider $M = \mathbb{Z}_6$ as a \mathbb{Z}_6 -module. And let $I = N = M$ We have $Tor_I(N) = \{0, 2, 3, 4\}$ and $Ann_I(N) = \{0\}$ Thus $Tor_I(N) \neq Ann_I(N)$.

Problem 4.

(a) Let $r \in Ann_{\mathbb{Z}}(M)$. Then for any $(a, b, c) \in M$ we have $r(a, b, c) = (0, 0, 0)$. Let $a = 1_{\mathbb{Z}_{24}}, b = 1_{\mathbb{Z}_{15}}, c = 1_{\mathbb{Z}_{50}}$. Therefore we have:

$$r(a, b, c) = (r, r, r) = (0, 0, 0)$$

This shows that:

$$\left. \begin{array}{l} 24 \mid r \\ 15 \mid r \\ 50 \mid r \end{array} \right\} \implies 2^3 \times 3 \times 5^2 = 600 \mid r \implies r \in 600\mathbb{Z}$$

And since for any $r \in 600\mathbb{Z}$ we have:

$$600(a, b, c) = (600a, 600b, 600c) = (0, 0, 0)$$

Which implies $r \in \text{Ann}_{\mathbb{Z}}(M)$, then we have $\text{Ann}_{\mathbb{Z}}(M) = 600\mathbb{Z}$.

(b) If $(a, b, c) \in \text{Ann}_M(2\mathbb{Z})$ then we have:

$$\begin{aligned} 2 \in 2\mathbb{Z} : 2(a, b, c) &= (2a, 2b, 2c) = (0, 0, 0) \\ \implies a &\in \{0, 12\} \cong \mathbb{Z}_2 \\ b &\in \{0\} \cong \mathbb{Z}_1 \\ c &\in \{0, 25\} \cong \mathbb{Z}_2 \end{aligned}$$

This shows that $\text{Ann}_M(2\mathbb{Z}) \cong \mathbb{Z}_2 \times \mathbb{Z}_2$. It is easy to see that for any $2i \in 2\mathbb{Z}$ it works.

Problem 5.

Let $\varphi \in \text{Hom}_{\mathbb{Z}}(\mathbb{Z}_{30}, \mathbb{Z}_{21})$. Then φ is a group homomorphism from \mathbb{Z}_{30} to \mathbb{Z}_{21} . Let r be the generator of \mathbb{Z}_{30} having order 30. Then it must mapped to an element with order s where $s \mid 30$. And since $s \mid 21$ then we can only have $s = 1, 3$. If $s = 3$, there are only two elements with order 3 in \mathbb{Z}_{21} , namely 7 and 14. Also since $R = \mathbb{Z}$ then if φ is a homomorphism we have:

$$\varphi(rm) = \varphi(m + \cdots + m) = r\varphi(m)$$

Thus φ is also a module homomorphism. Therefore any module homomorphism from \mathbb{Z}_{30} to \mathbb{Z}_{21} is uniquely identified with $\varphi(1)$. And we have only 3 such homomorphism:

$$\begin{aligned} \varphi(1) = 7 &\implies \varphi(m) = 7m \pmod{21} \\ \varphi(1) = 14 &\implies \varphi(m) = 14m \pmod{21} \\ \varphi(1) = 0 &\implies \varphi(m) = m \pmod{21} \end{aligned}$$

Which form a group of order 3, \mathbb{Z}_3 .

Problem 6.

First we show that eM and $(1 - e)M$ is submodule of M :

$$\begin{aligned} x, y \in M &\implies ex, ey \in eM, r \in R \\ r(ex) + ey &= rex + ey = erx + ey = e(rx + y) \in eM \end{aligned}$$

Note that we used $re = er$ and also that since M is module then $rx + y \in M$. Similarly for $(1 - e)M$ we have:

$$\begin{aligned} r((1 - e)x) + (1 - e)y &= rx - rex + y - ey \\ &= rx + y - (erx - ey) \\ &= rx + y - e(rx - y) \in (1 - e)M \end{aligned}$$

It is obvious that since both eM and $(1 - e)M$ are R module, so is $eM \oplus (1 - e)M$. It remains to show an isomorphism:

$$\begin{aligned} \varphi : M &\rightarrow eM \oplus (1 - e)M \\ m &\mapsto (em, (1 - e)m) \\ \varphi(m + m') &= (e(m + m'), (1 - e)(m + m')) \\ &= (em, (1 - e)m) + (em', (1 - e)m') \\ &= \varphi(m) + \varphi(m'). \\ \varphi(rm) &= (erm, (1 - e)rm) \\ &= (rem, (r - re)m) \\ &= (rem, r(1 - e)m) \\ &= r(em, (1 - e)m) = r\varphi(m). \end{aligned}$$

This shows that φ is a homomorphism. Now to show that it is an isomorphism we have to show that it is both surjective and injective. Suppose $em + (1 - e)m' \in eM \oplus (1 - e)M$. For any $m, m' \in M$ we know $em + (1 - e)m' \in M$. Thus we have:

$$\begin{aligned} \varphi(em + (1 - e)m') &= (e(em + (1 - e)m'), (1 - e)(em + (1 - e)m')) \\ &= ((e^2m + (e - e^2)m'), (e - e^2)m + (1 - 2e + e^2)m') \end{aligned}$$

Since $e^2 = e$:

$$= (em, (1 - e)m')$$

This gives us that φ is surjective. For injective suppose $m, m' \in M$ where $m \neq m'$:

$$\begin{aligned} \varphi(m) &= \varphi(m') \\ \implies (em, (1 - e)m) &= (em', (1 - e)m') \\ \implies em &= em', (1 - e)m = (1 - e)m' \\ \implies e(m - m') &= 0, (1 - e)(m - m') = 0 \\ \implies e(m - m') + (1 - e)(m - m') &= 0 \\ \implies m - m' &= 0 \implies m = m' \end{aligned}$$

Which is a contradiction. This gives us that φ is bijective as well. Therefore φ is an isomorphism and thus we have:

$$M \cong eM \oplus (1 - e)M$$

Problem 7.

We prove that it is a module homomorphism. Consider φ :

$$\begin{aligned}\varphi : \text{Hom}_R(M_1 \times M_2, N) &\rightarrow \text{Hom}_R(M_1, N) \times \text{Hom}_R(M_2, N) \\ \sigma(-, -) &\mapsto (\sigma(-, 0), \sigma(0, -))\end{aligned}$$

It is obvious that both $\sigma(-, 0)$ and $\sigma(0, -)$ are homomorphism. It remains to show that φ is a homomorphism:

$$\begin{aligned}\varphi(\sigma + \delta)(a, b) &= ((\sigma + \delta)(a, 0), (\sigma + \delta)(0, b)) \\ &= (\sigma(a, 0) + \delta(a, 0), \sigma(0, b) + \delta(0, b)) \\ &= (\sigma(a, 0), \sigma(0, b)) + (\delta(a, 0), \delta(0, b)) \\ &= \varphi(\sigma)(a, b) + \varphi(\delta)(a, b)\end{aligned}$$

Also

$$\begin{aligned}\varphi(r\sigma)(a, b) &= (r\sigma(a, 0), r\sigma(0, b)) \\ &= r(\sigma(a, 0), \sigma(0, b)) \\ &= r\varphi(\sigma)(a, b)\end{aligned}$$

Thus φ is homomorphism. To show that it is an isomorphism we show that φ has an inverse.

$$\begin{aligned}\psi : \text{Hom}_R(M_1, N) \times \text{Hom}_R(M_2, N) &\rightarrow \text{Hom}_R(M_1 \times M_2, N) \\ (\sigma, \delta) &\mapsto \pi\end{aligned}$$

Where $\pi(a, b) = (\sigma(a), \delta(b))$. It is obvious that $\varphi^{-1} = \psi$. First we have to show that π is a homomorphism.

$$\begin{aligned}\pi((a_1, b_1) + (a_2, b_2)) &= \pi(a_1 + a_2, b_1 + b_2) \\ &= (\sigma(a_1 + a_2), \delta(b_1 + b_2)) \\ &= (\sigma(a_1) + \sigma(a_2), \delta(b_1) + \delta(b_2)) \\ &= (\sigma(a_1), \delta(b_1)) + (\sigma(a_2), \delta(b_2)) \\ &= \pi(a_1, b_1) + \pi(a_2, b_2).\end{aligned}$$

$$\begin{aligned}\pi(r(a_1, b_1)) &= \pi((ra_1, rb_1)) \\ &= (\sigma(ra_1), \delta(rb_1)) \\ &= (r\sigma(a_1), r\delta(b_1)) \\ &= r(\sigma(a_1), \delta(b_1)) \\ &= r\pi(a_1, b_1).\end{aligned}$$

This shows that π is a homomorphism. Now to show that ψ is homomorphism.

$$\begin{aligned}\psi((\sigma_1, \delta_1) + (\sigma_2, \delta_2))(a, b) &= \psi(\sigma_1 + \sigma_2, \delta_1 + \delta_2)(a, b) \\ &= ((\sigma_1 + \sigma_2)(a), (\delta_1 + \delta_2)(b)) \\ &= (\sigma_1(a) + \sigma_2(a), \delta_1(b) + \delta_2(b)) \\ &= (\sigma_1(a), \delta_1(b)) + (\sigma_2(a), \delta_2(b)) \\ &= \psi(\sigma_1, \delta_1)(a, b) + \psi(\sigma_2, \delta_2)(a, b)\end{aligned}$$

Also:

$$\begin{aligned}\psi(r(\sigma, \delta))(a, b) &= \psi(r\sigma, r\delta)(a, b) \\ &= (r\sigma(a), r\delta(b)) \\ &= r(\sigma(a), \delta(b)) \\ &= r\psi(\sigma, \delta)(a, b)\end{aligned}$$

This shows that ψ is a homomorphism. And since it is the inverse of φ , therefore φ is an module isomorphism. Thus we have:

$$\text{Hom}_R(M_1 \times M_2, N) \cong_R \text{Hom}_R(M_1, N) \times \text{Hom}_R(M_2, N)$$