
Problem 1. ***Is the category of monoids cartesian closed?(Exercise 6.8.4)***

Proof. We know that the category of monoids, has initial object $E = \langle \{e\}, \star, e\rangle$. Now consider two monoids A and B such that there exists more than 1 homomorphism from A to B. (We can find two monoids with this property). We want to show that the object B^A cannot exists. Assume the opposite:

$$\begin{tikzcd}
B^A \times A \arrow[r, "\epsilon"] & B \\
E \times A \arrow[u, "\langle \tilde{f}, 1_A\rangle"] \arrow[ur, "f"'] &
\end{tikzcd}$$

Now we know that there exists more than 1 homomorphism from A to B, but there only exists one $\tilde{f}$ from E to B^A since E is initial. And since for any f there must exists a unique morphism $\tilde{f}$ from E to B^A, we get a contradition, and therefore object B^A cannot exists. This shows that category of monoids is not CCC. $\square$

Problem 2. ***Show that for any objects A, B in a CCC, there is a bijective correspondence between points of the exponential $1 \to B^A$ and arrows $A \to B$.(Exercise 6.8.9)***

Proof. By definition of exponential object we have:

$$\begin{tikzcd}
B^A \times A \arrow[r, "\epsilon"] & B \\
1 \times A \arrow[u, "\langle \tilde{f}, 1_A\rangle"] \arrow[ur, "f"'] &
\end{tikzcd}$$

For any f there exists a unique $\tilde{f}$ and for any $\tilde{f}$ there exists a unique f, therefore there exists a bijection between points in $1 \to B^A$ and $1 \times A \to B$. Thus we only have to show that there is a bijection between arrows in $1 \times A \to B$ and arrows in $A \to B$. For this we show that $A \cong 1 \times A$.

$$
\begin{tikzcd}
1 & 1 \times A \arrow[l, "!_{1\times A}"'] \arrow[r, "P_A"] & A \\
& A \arrow[ul, "!_A"] \arrow[u, "\langle !_A, 1_A \rangle"] \arrow[ur, "1_A"'] &
\end{tikzcd}
$$

Now it is only enough to show that $P_A \circ \langle !_A, 1_A \rangle = 1_A$ and $\langle !_A, 1_A \rangle \circ P_A = 1_{1\times A}$. The first one is obvious since $1 \times A$ is the product of 1 and A. For the second one:

$$\langle !_A, 1_A \rangle \circ P_A = \langle !_A \circ P_A, 1_A \circ P_A \rangle = \langle !_{1\times A}, P_A \rangle$$

Where we can observe:

$$
\begin{tikzcd}
1 & 1 \times A \arrow[l, "!_{1\times A}"'] \arrow[r, "P_A"] & A \\
& 1 \times A \arrow[ul, "!_{1\times A}"] \arrow[u, "\langle !_{1\times A}, P_A \rangle"] \arrow[ur, "P_A"'] &
\end{tikzcd}
$$

Since $\langle !_{1\times A}, P_A \rangle$ is unique and also $1_{1\times A}$ commutes with the diagram, then we have $1_{1\times A} = \langle !_{1\times A}, P_A \rangle$. This proves that $A \cong 1 \times A$. Thus we have $A \underset{f^{-1}}{\overset{f}{\rightleftarrows}} 1 \times A$. Now we can have our bijection with:

$$g : A \to B \implies f^{-1} \circ g : 1 \times A \to B \implies f \circ f^{-1} \circ g = g : A \to B$$
$$\overline{g} : 1 \times A \to B \implies f \circ \overline{g} : A \to B \implies f^{-1} \circ f \circ \overline{g} = \overline{g} : 1 \times A \to B$$

Therefore there is a bijection between arrows in $A \to B$ and $1 \times A \to B$. $\square$

Problem 3. ***Show that the category of ωCPOs is cartesian closed, but that the category of strict ωCPOs is not.(Exercise 6.8.10)***

Proof. First we show that category of ωCPOs is cartesian closed. Given two ωCPOs P and Q, the ωCPO $P \times Q$ has elements of the form (p, q) with $p \in P$ and $q \in Q$. And relations as below:

$$(p, q) \leq (p', q') \iff p \leq p' \text{ and } q \leq q'$$

To check if this really is an ωCPO let:

$$
\begin{gathered}
(p_0, q_0) \leq (p_1, q_1) \leq \dots \\
\implies p_0 \leq p_1 \leq \cdots \implies \varinjlim p_i = p_\omega \\
\implies q_0 \leq q_1 \leq \cdots \implies \varinjlim q_i = q_\omega \\
\implies \varinjlim (p_i, q_i) = (p_\omega, q_\omega)
\end{gathered}
$$

Now let (x, y) such that for any i, $(p_i, q_i) \leq (x, y)$:

$$\left.\begin{array}{l}\forall i : p_i \leq x \implies p_\omega \leq x \\ \forall i : q_i \leq y \implies q_\omega \leq y\end{array}\right\} \implies (p_\omega, q_\omega) \leq (x, y)$$

Therefore $P \times Q$ is indeed an ωCPO. Now let π_1 and π_2 be projections. Let $(p_0, q_0) \leq (p_1, q_1) \leq \ldots$, with colimit (p_ω, q_ω). Therefore for any i, $p_i \leq p_\omega$. Now suppose there is an element $x \in P$ such that for any i, $p_i \leq x$. This shows that for any i we have: $(p_i, q_i) \leq (x, q_\omega)$. And since (p_ω, q_ω) is colimit, then we have $(p_\omega, q_\omega) \leq (x, q_\omega)$ which means that $p_\omega \leq x$. This shows that p_ω is the colimit of $p_0, p_1, \ldots$. Therefore π_1 (and similarly π_2) is continuous.

$$\begin{tikzcd} P & P \times Q \arrow[l, "\pi_1"'] \arrow[r, "\pi_2"] & Q \\ & X \arrow[ul, "g"] \arrow[u, "\langle g, f \rangle"] \arrow[ur, "f"'] & \end{tikzcd}$$

Since both f and g are monotone, and $\langle g, f \rangle (x) = (g(x), f(x))$ with $x \in X$, then $\langle g, f \rangle$ is also monotone. Also since f and g are continuous, suppose suppose x_ω is the colimit of $x_0 \leq x_1 \leq x_2 \leq \ldots$, then $f(x_\omega)$ and $g(x_\omega)$ are both colimit of the respective diagrams. It is obvious that $\langle g(x_\omega), f(x_\omega) \rangle$ is the colimit of $\langle g(x_0), f(x_0) \rangle \leq \langle g(x_1), f(x_1) \rangle \leq \ldots$. Thus $\langle g, f \rangle$ is continuous.
For exponentials, consider:

$$Q^P = \{f : P \to Q | f \text{ is monotone and } \omega\text{-continuous.}\}$$

First we show that this object is an ωCPO. Note that order in functions is pointwise. Let $\forall i : f_i \in Q^P$ such that:

$$\begin{gathered} f_0 \leq f_1 \leq f_2 \leq \ldots \\ \implies \forall p \in P : f_0(p) \leq f_1(p) \leq f_2(p) \leq \ldots \end{gathered}$$

Since all $f_i(p)$s are in Q, then there exists a colimit for them, p_ω. Let $g(p) = p_\omega$. It is easy to see that for any $0 \leq i$, $f_i \leq g$. Now consider $h \in Q^P$ such that for any $0 \leq i$, $f_i \leq h$.

$$\begin{aligned} \forall p \in P : f_i(p) \leq h(p) &\implies p_w \leq h(p) \implies g(p) \leq h(p) \\ &\implies g \leq h \end{aligned}$$

Therfore g is the colimit of this sequence and therefore Q^P is an ωCPO. The next step is to show that Q^P has the properties of an exponential object. define $\epsilon : Q^P \times P \to Q$ as $\epsilon(f, p) = f(p)$. We need to show that ϵ is monotone and continuous. Suppose $(f, p) \leq (f', p')$:

$$\epsilon(f, p) = f(p) \leq f'(p) \leq f'(p') = \epsilon(f', p')$$

Thus ϵ is monotone. Now suppose (f_ω, p_ω) is the colimit of $(f_0, p_0) \leq (f_1, p_0) \leq \dots$. Since ϵ is monotone, then $f_i(p_i) \leq f_\omega(p_\omega)$. Now let $x \in Q$ such that for any i, $f_i(p_i) \leq x$. First we prove that for any i, $f_i(p_\omega) \leq x$.

$$f_i(p_0) \leq f_i(p_1) \leq \dots \leq f_i(p_i) \leq x$$
$$\forall_{j>i} : f_i(p_j) \leq f_j(p_j) \leq x$$

Since f_i is ω-continuous, then since p_ω is the colimit of $p_0 \leq p_1 \leq \dots$, then $f_i(p_\omega)$ is the colimit of $f_i(p_0) \leq f_i(p_1) \leq \dots$, which means that $f_i(p_\omega) \leq x$. Now consider:

$$f_0(p_\omega) \leq f_1(p_\omega) \leq \dots$$

We introduce $g : P \to Q$ as below:

$$g(p) = \begin{cases} x & \text{if } p \leq p_0 \text{ or } p \geq p_0 \\ f_\omega(p) & O.W. \end{cases}$$

We can see that g is monotone. And also ω-continuous, since for any $r_0 \leq r_1 \leq \dots$, where $r_i \in P$, either $g(r_i) = x$ for all of r_is, or $g(r_i) = f_\omega(r_i)$. First case is obvious. The second case is ω-continuous since f_ω is ω-continuous. Now we can see:

$$p \leq p_0 \text{ or } p \geq p_0 \implies f_i(p) = x = g(p)$$
$$O.W. \implies f_i(p) \leq f_\omega(p) = g(p)$$

This proves that for all i, $f_i \leq g$. Now we know that f_ω is the colimit of $f_0 \leq f_1 \leq \dots$, therefore we have $f_\omega \leq g$:

$$f_i(p_\omega) \leq f_\omega(p_\omega) \leq g(p_\omega) = x$$

This proves that $f_\omega(p_\omega)$ is the colimit of $f_0(p_0) \leq f_1(p_1) \leq \dots$. Thus ϵ is ω-continuous. Now let:

$$\begin{tikzcd} Q^P \times P \arrow[r, "\epsilon"] & Q \\ X \times P \arrow[u, "\langle \tilde{f}, 1_P \rangle"] \arrow[ur, "f"'] & \end{tikzcd}$$

$\tilde{f}(x) \in Q^P$ where for any $p \in P$ we have $(\tilde{f}(x))(p) = f(x, p)$. We know that in case of existing such function, it is unique, Now we only have to show that if f is monotone and ω-continuous, then $\tilde{f}$ is monotone and ω-continuous as well. Suppose $x, x' \in X$ such that $x \leq x'$. We need to show that $\tilde{f}(x) \leq \tilde{f}(x')$. And for this we need to show that for any $p \in P$, $\tilde{f}(x)(p) \leq \tilde{f}(x')(p)$:

$$\tilde{f}(x)(p) = f(x, p) \leq f(x', p) = \tilde{f}(x')(p)$$

Thus $\tilde{f}$ is monotone. Note that since $(x,p) \leq (x',p)$ and f is monotone we conclude that $f(x,p) \leq f(x',p)$. As for ω-continuous, consider $x_0 \leq x_1 \leq \ldots$, with colimit x_ω in X. Since $\tilde{f}$ is monotone, therefore $\tilde{f}(x_i) \leq \tilde{f}(x_\omega)$. Now Consider the function $g : P \to Q$ such that for any i, $\tilde{f}(x_i) \leq g$.

$$\begin{aligned} \tilde{f}(x_i) \leq g &\implies \forall p \in P : \tilde{f}(x_i)(p) \leq g(p) \\ &\implies \forall p \in P : f(x_i, p) \leq g(p) \end{aligned}$$

Since x_ω is the colimit of $x_0 \leq x_1 \leq \ldots$, therefore (x_ω, p) is the colimit of $(x_0, p) \leq (x_1, p) \leq \ldots$. And since f is ω-continuous, therefore:

$$\begin{gathered} \forall p \in P : f(x_i, p) \leq f(x_\omega, p) \leq g(p) \\ \implies \forall p \in P : \tilde{f}(x_\omega)(p) \leq g(p) \\ \implies \tilde{f}(x_\omega) \leq g \end{gathered}$$

This shows that $\tilde{f}$ preserves limit and is ω-continuous. This concludes that the category of ωCPOs is indeed CCC.

To show that the category of strict ωCPOs is not CCC we show that some exponential objects cannot exist. Let P and Q be two ωCPOs with more than 1 element. We know that $\{\perp\}$ is also an ωCPO:

$$\begin{tikzcd} Q^P \times P \arrow[r, "\epsilon"] & Q \\ \{\perp\} \times P \arrow[u, "\langle \tilde{f}, 1_P \rangle"] \arrow[ur, "f"'] & \end{tikzcd}$$

Now there exists many $f : \{\perp\} \times P \cong P \to Q$. But there exists only one $\tilde{f} : \{\perp\} \to Q^P$ since $\tilde{f}$ preserves $\perp$. This shows that such object Q^P doesn't exist. And therefore the category of strict ωCPOs is not CCC. $\square$

Problem 4. ***Consider the forgetful functors***

$$\textbf{\textit{Groups}} \xrightarrow{U} \textbf{\textit{Monoids}} \xrightarrow{V} \textbf{\textit{Sets}}$$

Say whether each is faithful, full, injective on arrows, surjective on arrows, injective on objects, and surjective on objects.(Exercise 7.11.4)

Proof. **U**:

$$U : \mathbf{Groups} \to \mathbf{Monoids}$$

$$\begin{tikzcd}
\langle X, \star, e\rangle \arrow[r, mapsto] \arrow[d, "f"'] & \langle X, \star, e\rangle \arrow[d, "f"] \\
\langle Y, *, e'\rangle \arrow[r, mapsto] & \langle Y, *, e'\rangle
\end{tikzcd}$$

(i) Full: We have to see that whether

$$F : Hom_{Grp}(A, B) \to Hom_{Mon}(UA, UB)$$

is surjective. Let $g \in Hom_{Mon}(UA, UB)$ be a monoid homomorphism. Therefore for any $x, y \in UA$ we have:

$$g(xy) = g(x)g(y)$$
$$g(1_{UA}) = 1_{UB}$$

Since U is a forgetful functor, therefore UA (similarly UB) is exactly A (similarly B). This shows that g also a group homomorphism between A and B and since both A and B have group structure as well, therfore F is surjective, thus U is full.

(ii) Faithful: We have to see that whether

$$F : Hom_{Grp}(A, B) \to Hom_{Mon}(UA, UB)$$

is injective. Let $f, g \in Hom_{Grp}(A, B)$, $f \neq g$ be two group homomorphisms. Since any group homomorphism is also a monoid homomorphism thus $Fg = g$ and $Ff = f$. Now if $Ff = Fg$ then we would have for any $a \in A$, $f(a) = g(a)$. But this cannot happen since $f \neq g$ in the first place.

(iii) Surjective on objects: Since any monoid is not a group, thus U is not surjective on objects.

(iv) Surjective on arrows: Since U is not surjective on objects, it can't be surjective on arrows as well.

(v) Injective on objects: Let $G = \langle X, \star, e\rangle$ and $H = \langle Y, *, e'\rangle$ where $H \neq G$. If $Y \neq X$ it is easy to see that after U, they are both different monoids. If $Y = X$, therefore there exists some $a, b \in X$ such that $a \star b \neq a * b$. This shows that after U they are two different monoids. Therefore U is injective on objects.

(*vi*) Injective on arrows: Since U is injective on objects, then for any $f : A \to B$ and $g : C \to D$ where $f \neq g$, if one of $A \neq C$ or $B \neq D$ happens, then $Uf \neq Ug$ because of different objects on domain or codomain. And if $A = C$ and $B = D$ then since U is faithful, we have $Uf \neq Ug$. Therefore U is injective on arrows as well.

V:

$$V : \textbf{Monoids} \to \textbf{Sets}$$

$$\begin{tikzcd}
\langle X, \star, e \rangle \arrow[r] \arrow[d, "f"'] & X \arrow[d, "f"] \\
\langle Y, *, e' \rangle \arrow[r] & Y
\end{tikzcd}$$

(*i*) Full: We have to see whether

$$V : Hom_{Mon}(A, B) \to Hom_{Set}(VA, VB)$$

is surjective. Consider A and B non-trivial monoids, then for any morphism f between A and B we have: $f(1_A) = 1_B$. Now consider some function between A and B such that it doesn't preserve identity. It can't be the image of any morphism in Monoids. Therefore V here is not surjective and therefore is not full.

(*ii*) Faithful: We have to see whether

$$V : Hom_{Mon}(A, B) \to Hom_{Set}(VA, VB)$$

is injective. Let $f, g \in Hom_{Mon}(A, B)$ such that $f \neq g$. Therefore there exists some $a \in A$ such that $f(a) \neq g(a)$. Now since V is forgetful, then $Vf = f$ and $Vg = g$ as functions. Therefore $Vf \neq Vg$, thus V here is injective and faithful.

(*iii*) Surjective on objects: There is no moniod M such that $VM =$, since for any monoid M there exists some identity element $e \in M$, Thus it can't be empty. Therefore V is not surjective on objects.

(*iv*) Surjective on arrows: Since V is not surjective on objects, it can't be surjective on arrows.

(*v*) Injective on objects: Consider the monoids $M = \langle \{e, a\}, \star, e \rangle$ and $M' = \langle \{e, a\}, *, e \rangle$ where $a \star e = a * e = a * a = a$ and $a \star a = e$. Clearly $M \neq M'$, but $VM = \{1, 2\} = VM'$, thus V is not injective on objects.

(*vi*) Injective on arrows: Consider M and M' from the previous part. Functions below are different in Monoids:

$$1_M : M \to M \qquad\qquad 1_{M'} : M' \to M'$$
$$m \mapsto m \qquad\qquad m' \mapsto m'$$

But are the same in Sets since they are both $1_{\{a,e\}} : \{a, e\} \to \{a, e\}$. Therefore V is not injective on arrows.

$\square$

Problem 5. ***Make every poset*** $(X, \leq)$ ***into a topological space by letting*** $U \subset X$ ***be open just if*** $x \in U$ ***and*** $x \leq y$ ***implies*** $y \in U$ ***(***U ***is "closed upwards"). This is called Alexandroff topology on*** X***. Show that it gives a functor***

$$A : \textbf{\textit{Pos}} \to \textbf{\textit{Top}}$$

from posets and monotone maps to spaces and continuous maps by showing that any monotone map of posets $f : P \to Q$ ***is continuous with respect to this topology on*** P ***and*** Q ***(the inverse image of an open set must be open).***
Is A ***faithful? is it full?***
How would the situation change if instead one took as open sets those subsets that are closed downwards? (Exercise 7.11.5)

Proof. First we show that any monotone map between P and Q is also a continiuous map between topology on P and Q.

$$\begin{tikzcd} P \arrow[r, "A"] \arrow[d, "f"'] & (P, \tau_P) \arrow[d, "f"] \\ Q \arrow[r, "A"'] & (Q, \tau_Q) \end{tikzcd}$$

Let $V \subset Q$ be an open in τ_Q. And let $U = f^{-1}(V)$. Let $x \in U$ and $y \in P$ such that $x \leq y$. Since f is monotone:

$$\left.\begin{array}{l} x \leq y \implies f(x) \leq f(y) \\ f(x) \in V \\ V \text{ is open} \end{array}\right\} \implies f(y) \in V \implies y \in f^{-1}(V) = U$$

This shows that U is open in τ_P. Therefore inverse image of any open, is open, thus f is continiuous. Two other properties of functors is easy to check since for any map f in Pos, $Af = f$.
Sicne $Af = f$ then for any two $f, g \in Hom_{Pos}(P, Q)$ where $f \neq g$, then $Af \neq Ag$, therefore A is faithful. For full, consider the continiuous function $f : (P, \tau_P) \to (Q, \tau_Q)$. Let $x, y \in P$ such that $x \leq y$. We want to show that f is also monotone. If $f(x) \leq f(y)$ then we are done. Otherwise suppose $f(y) < f(x)$. Now in (Q, τ_Q) consider the open set U with $f(x) \in U$ and $z \in U$ iff $f(x) \leq z$. It is easy to see that $f(y)$ is not in U. Since f is continiuous, then $V = f^{-1}(U)$ is open in (P, τ_P). And since $f(y) \notin U$ then $y \notin V$. And also since $f(x) \in U$ then $x \in V$. Since V is open we have:

$$\left.\begin{array}{l} x \in V \\ x \leq y \end{array}\right\} \implies y \in V$$

But we showed that $y \notin V$. The contradiction shows that the assumption $f(y) < f(x)$ was wrong and we have $f(x) \leq f(y)$. Therefore if f is a continiuous map between (P, τ_P) and (Q, τ_Q), then f is also monotone between P and Q. This shows that A is full, since any continiuous map is image of some monotone map.
For open sets such that they are closed downwards, is exactly the same, A is functor, full, and faithful. $\square$

Problem 6. ***Prove that every functor*** $F : \boldsymbol{C} \to \boldsymbol{D}$ ***can be factored as*** $D \circ E = F$***,***

$$\boldsymbol{C} \xrightarrow{E} \boldsymbol{E} \xrightarrow{D} \boldsymbol{D}$$

in the following two ways:

(a) $E : \boldsymbol{C} \to \boldsymbol{E}$ ***is bijective on objects and full, and*** $D : \boldsymbol{E} \to \boldsymbol{D}$ ***is faithful;***

(b) $E : \boldsymbol{C} \to \boldsymbol{E}$ ***is surjective on objects and*** $D : \boldsymbol{E} \to \boldsymbol{D}$ ***is injective on objects and full and faithful.***

When do the two factorizations agree? (Exercise 7.11.6)

Proof. For each part, we construct $\mathbf{E}$:

(a) Let $\mathbf{E}$ be a category with $\mathbf{E_0} = \mathbf{C_0}$ and for any $X, Y \in \mathbf{E_0}$ we have:

$$Hom_{\mathbf{E}}(X, Y) = Hom_{F(\mathbf{C})}(FX, FY)$$

In other words, each morphism between X and Y in $\mathbf{E}$, is representing a morphism in image of F, between FX and FY. For any $X \in \mathbf{E}$, $1_X \in Hom_{\mathbf{E}}(X, X)$ is the equivalent of $1_{F(\mathbf{C})}(FX, FX)$. And composition of two morphisms f and g is as below:

$$\exists f', g' : f = Ff', g = Fg'$$
$$f \circ g = F(f') \circ F(g') = F(f' \circ g')$$

It is obvious that $\mathbf{E}$ is a category. Now consider the functor:

$$E : \mathbf{C} \longrightarrow \mathbf{E}$$

$$\begin{tikzcd} X \arrow[r, "E"] \arrow[d, "f"'] & X \arrow[d, "Ff"] \\ Y \arrow[r, "E"'] & Y \end{tikzcd}$$

To show that E is functor, we only need to show that composition and Identity are preserved, which is trivial since both are preserved with functor F.
Note that any morphism in $\mathbf{E}$ is of the form Ff' for some $f' \in \mathbf{C_1}$. Now consider the functor:

$$D : \mathbf{E} \longrightarrow \mathbf{D}$$

$$\begin{tikzcd} X \arrow[r, "D"] \arrow[d, "Ff"'] & FX \arrow[d, "Ff"] \\ Y \arrow[r, "D"'] & FY \end{tikzcd}$$

To see that D is functor, we have to show that it preserves Identity and composition, which is trivial since F is a functor. To see that $F = D \circ E$:

$$\begin{tikzcd} X \arrow[r, "E"] \arrow[d, "f"'] & X \arrow[r, "D"] \arrow[d, "Ff"'] & FX \arrow[d, "Ff"] \\ Y \arrow[r, "E"'] & Y \arrow[r, "D"'] & FY \end{tikzcd}$$

This shows that the composition of D and E is indeed F. Now E is injective on objects, since for any $X \in \mathbf{C}$, we have $E(X) = X$. And is also full, since by construction, any morphism in $\mathbf{E}$ is of the form Ff for some $f \in \mathbf{C_1}$. And also D is faithful, since for any morphism $Ff \in \mathbf{E_1}$, we have $D(Ff) = Ff$.

(b) Let $\mathbf{E}$ be a category with $E_0 = F(C_0)$ and for any $FX, FY \in \mathbf{E_0}$ we have:

$$Hom_{\mathbf{E}}(FX, FY) = Hom_{\mathbf{D}}(FX, FY)$$

In other words, the set of morphisms in $\mathbf{E}$ between FX and FY is exactly the set of morphism in $\mathbf{D}$ between FX and FY. For any $FX \in \mathbf{E_0}$, let 1_{FX} be the equivalent of 1_{FX} in $\mathbf{D}$. And composition is defined exactly like composition for morphisms in $\mathbf{D}$. It is obvious that E is a category. Consider the functors:

$$\mathbf{C} \xrightarrow[E]{} \mathbf{E}$$

$$\begin{tikzcd} X \arrow[r, "E"'] \arrow[d, "f"'] & FX \arrow[d, "Ff"'] \\ Y \arrow[r, "E"'] & FY \end{tikzcd}$$

To check if E is a functor we have to check that it preserves identity and composition. Identity is preserved since $E(1_X) = F(1_X) = 1_{FX}$. And composition is preserved since F is a functor. The other functor is:

$$\begin{tikzcd}
\mathbf{E} \arrow[r, "D"'] & \mathbf{D} \\
FX \arrow[r, "D"'] \arrow[d, "Ff"'] & FX \arrow[d, "Ff"] \\
FY \arrow[r, "D"'] & FY
\end{tikzcd}$$

Where D is the inclusion functor. Now to see that the composition of these two is F:

$$\begin{tikzcd}
\mathbf{C} \arrow[r, "E"'] & \mathbf{E} \arrow[r, "D"'] & \mathbf{D} \\
X \arrow[r, "E"'] \arrow[d, "f"'] & FX \arrow[r, "D"'] \arrow[d, "Ff"'] & FX \arrow[d, "Ff"] \\
Y \arrow[r, "E"'] & FY \arrow[r, "D"'] & FY
\end{tikzcd}$$

Now E is surjective on objects, since by definition we had $\mathbf{E_0} = \mathbf{F}(\mathbf{C_0})$. Now D is injective on objects since for any $FX \in \mathbf{E_0}$ we have $D(FX) = FX$. Also D is full and faithful since by definition we had:

$$Hom_{\mathbf{E}}(FX, FY) = Hom_{\mathbf{D}}(FX, FX)$$

$\square$

Problem 7. ***Show that the map of sets***

$$\begin{aligned} \eta_A : A &\to PP(A) \\ a &\mapsto \{U \subseteq A | a \in U\} \end{aligned}$$

is the component at A ***of a natural transformation*** $\eta : 1_{Sets} \to PP$***, where*** $P : Sets^{op} \to Sets$ ***is the (contraviariant) powerset functor. (Exercise 7.11.9)***

Proof. First to understand the functor PP, since P is contraivariant:

$$\begin{tikzcd}
A \arrow[r, "P"] \arrow[d, "f"'] & P(A) \arrow[r, "P"] & PP(A) \arrow[d, "PP(f)"] \\
B \arrow[r, "PB"'] & P(B) \arrow[u, "P(f)=f^{-1}"'] \arrow[r, "P"'] & PP(B)
\end{tikzcd}$$

Let $X \in P(B)$, Then $P(f)(X) = f^{-1}(X) = \{x \in A | f(x) \in X\}$, similarly for $PP(f)$, if $U \in PP(A)$:

$$PP(f)(U) = P(f)^{-1}(U) = (f^{-1})^{-1}(U) = \{X \in P(B) | f^{-1}(X) \in U\}$$

Now we only need to show that for any sets A and B, the following diagram commutes:

$$\begin{tikzcd}
A \arrow[r, "\eta_A"] \arrow[d, "f"'] & PP(A) \arrow[d, "PP(f)"] \\
B \arrow[r, "\eta_B"'] & PP(B)
\end{tikzcd}$$

And since we are in **Sets**, let $a \in A$, we need to show that $\eta_B(f(a)) = PP(f)(\eta_A(a))$.

$$\begin{aligned}
PP(f)(\eta_A(a)) &= PP(f)(\{U \subseteq A | a \in U\}) \\
&= \{X \in P(B) | f^{-1}(X) \in \{U \subseteq A | a \in U\}\} \\
&= \{X \in P(B) | a \in f^{-1}(X)\} \\
&= \{X \in P(B) | f(a) \in X\} \\
&= \{X \subseteq B | f(a) \in X\} \\
&= \eta_B(f(a))
\end{aligned}$$

This completes the proof. □

Problem 8. ***A category is skeletal if isomorphic objects are always identical. Show that every category is equivalent to a skeletal subcategory. (Every category has a "skeleton.") (Exercise 7.11.16)***

Proof. Let C be a category. Since isomorphism of objects in a category is an equivalence relationship, then with the help of **AC** we choose a subcategory D, that from each equivalence class there is exactly one object. And all the morphisms between the chosen objects are also chosen. Now let F be a functor from C to D such that it maps every object to its equivalent in D:

$$F : C \longrightarrow D$$

$$
\begin{tikzcd}
A \arrow[r, "F_A"'] \arrow[d, "f"'] & FA \arrow[d, "Ff"] \\
B \arrow[r, "F_B"] & FB
\end{tikzcd}
$$

Since A and FA are isomorphic to each other, therefore there exist F_A^{-1}. And we have $Ff = F_B \circ f \circ F_A^{-1}$. It is easy to see that $F(1_A) = F_A \circ 1_A \circ F_A^{-1} = 1_{FA}$. And for composition, let $g : A \to B$ and $f : B \to C$:

$$F(f \circ g) = F_C \circ (f \circ g) \circ F_A^{-1} = F_c \circ f \circ F_B \circ F_B^{-1} \circ g \circ F_A^{-1} = F(f) \circ F(g)$$

Thus F is indeed a functor. Now we have to show that F is full, faithful and essensially surjective on objects.
For full consider any $h : FA \to FB$. Then $k = F_B^{-1} \circ h \circ F_A : A \to B$ is in $Hom_C(A, B)$ and $F(k) = h$.
For faithful, if $F(f) = F(g)$ then we have:

$$
\begin{aligned}
F(f) = F_B \circ f \circ F_A^{-1} &\implies f = F_B^{-1} \circ F(f) \circ F_A \\
F(g) = F_B \circ g \circ F_B^{-1} &\implies g = F_B^{-1} \circ F(g) \circ F_A \\
&\implies f = g
\end{aligned}
$$

For essensially surjective on objects, for any $X \in D$, we have $X \in C$. And we have $FX = X \cong X$.
This shows that C and D are equivalent, and since D by construction was skeletal, then the proof is complete. $\square$