Theory of Computation Fall 2024 Problem Set 3

Amin Zolfagharian - 610300118

Problem 1.

By characterisation of r.e. sets we know that if A is r.e. then A is the range of a unary total computable function. Let that function be f. Thus we have:

$$A = \{f(0), f(1), f(2), \dots\}$$

But f might have repetitions. Now we define total computable g such that A is the range of g, and g is injective.

$$g(x) = \mu_z(\forall_{x<z}(f(x) \neq f(z)))$$

Since f is computable and both μ and $\forall$ are computable, then g is also computable. Also g is total since A is infinite and range of f is A hence there will always exists such z, that $\forall_{x<z}(f(x) \neq f(z))$. Also $Ran(g) = A$, since for any $a \in A$, there exists i such that $f(i) = a$. Let j be such that $f(j) = a$ and for any $f(i) = a$, we have $j < i$. Then we have $g(j) = a$. This completes the proof as g is a total unary function that enumerates A without repetitions.

Problem 2.

Assume A is decidable. We define the function g as below:

$$g(x) = \begin{cases} x & x \notin A \\ x+1 & x \in A \end{cases}$$

Since A is decidable, then g is also decidable. Therefore there exists some $e \in \mathbb{N}$ such that $g \cong \phi_e$.

$$\phi_e(e) = e \iff g(e) = e \iff e \notin A \iff \phi_e(e) \neq e$$

Which is a contradiction. This shows that A cannot be a decidable set.

Problem 3.