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**Problem 1.**

***(i)*** Let $x, y \in Tor(M)$ and $r \in R$. We need to show that $rx + y \in Tor(M)$ in order to show that $Tor(M)$ is a submodule of $M$. Since both $x, y \in Tor(M)$, then there exists nonzero $r_x, r_y \in R$ such that $r_x x = 0$ and $r_y y = 0$. Now we show that $r_x r_y(rx+y) = 0$:

$$\begin{aligned} r_x r_y(rx + y) &= r_x r_y rx + r_x r_y y \\ &= r_x r_y rx + r_x(r_y y) \\ &= r_x r_y rx \\ &= r_y r(r_x x) = 0 \end{aligned}$$

And since both $r_x$ and $r_y$ are nonzero therefore $r_x r_y$ is nonzero as well. This proves that $rx + y \in Tor(M)$ and thus $Tor(M)$ is a submodule of $M$.

***(ii)*** Consider $\mathbb{Z}_6$ as a $\mathbb{Z}_6$ module. By definition we have:

$$Tor(\mathbb{Z}_6) = \{2, 3, 4\}$$

Which doesn't even form a group, let alone a submodule of $M$.

***(iii)*** Let $m \neq 0 \in M$, where $M$ is a $R$-module. And let $r_1 r_2 = 0$ where $r_1, r_2 \in R$ and $r_1, r_2 \neq 0$. If $r_2 m = 0$ then we are done since $m \in Tor(M)$. Otherwise consider $(r_1 r_2)m = r_1(r_2 m) = 0$ Since $r_2 m \neq 0$ then $r_2 m \in Tor(M)$. This completes the proof.

**Problem 2.**

Let $x, y \in \bigcup\limits_{k=1}^{\infty} N_k$ and $r \in R$. Therefore there exists $i, j \in \mathbb{N}$ such that $x \in N_i$ and $y \in N_j$. WLOG we can assume $i \leq j$. Since $N_i \subseteq N_j$ therefore we have $x, y \in N_j$. Since $N_j$ is a submodule of $M$ then we have:

$$\begin{aligned} rx + y \in N_j &\subseteq \bigcup_{k=1}^{\infty} N_k \\ \implies rx + y &\in \bigcup_{k=1}^{\infty} N_k \end{aligned}$$

Which shows that it is a submodule of $M$.

**Problem 3.**

Let $M$ be an $R$-module, where scalar product is non-trivial. Now consider the $M \times M$ as a $R$-module. With componentwise addition and multiplication.

$$\begin{aligned} M_1 &= \{(m, 0) | m \in M\} \\ M_2 &= \{(0, m) | m \in M\} \end{aligned}$$

$M_1$ and $M_2$ are both sub-modules of $M \times M$ since:

$$\begin{aligned}\forall x, y \in M_1, \forall r \in R : rx + y &= r(m_x, 0) + (m_y, 0)\\ &= (rm_x, 0) + (m_y, 0)\\ &= (rm_x + m_y, 0) \in M_1\end{aligned}$$

Last lines is a direct result of $M$ being a $R$-module. Now consider $M_1 \cup M_2$. And $x \neq 0 \in M_1$, $y \neq 0 \in M_2$ and $r \in R$ such that $rx \neq 0$:

$$rx + y = r(m_x, 0) + (0, m_y) = (rm_x, m_y)$$

Since both $rm_x \neq 0$ and $m_y \neq 0$, then $(rm_x, m_y) \neq M_1 \cup M_2$. This shows that $M_1 \cup M_2$ is not a sub-module of $M \times M$.

**Problem 4.**

Again consider $M = \mathbb{Z}_6$ as a $\mathbb{Z}_6$-module. And let $I = N = M$ We have $Tor_I(N) = \{2, 3, 4\}$ and $Ann_I(N) = \{0\}$ Thus $Tor_I(N) \neq Ann_I(N)$.

**Problem 5.**

**(a)** Let $r \in Ann_{\mathbb{Z}}(M)$. Then for any $(a, b, c) \in M$ we have $r(a, b, c) = (0, 0, 0)$. Let $a = 1_{\mathbb{Z}_{24}}, b = 1_{\mathbb{Z}_{15}}, c = 1_{\mathbb{Z}_{50}}$. Therfore we have:

$$r(a, b, c) = (r, r, r) = (0, 0, 0)$$

This shows that:

$$\left.\begin{aligned}24 \mid r\\ 15 \mid r\\ 50 \mid r\end{aligned}\right\} \implies 2^3 \times 3 \times 5^2 = 600 \mid r \implies r \in 600\mathbb{Z}$$

And since for any $r \in 600\mathbb{Z}$ we have:

$$600(a, b, c) = (600a, 600b, 600c) = (0, 0, 0)$$

Which implies $r \in Ann_{\mathbb{Z}}(M)$, then we have $Ann_{\mathbb{Z}}(M) = 600\mathbb{Z}$.

**(b)** If $(a, b, c) \in Ann_M(2\mathbb{Z})$ then we have:

$$\begin{aligned}2 \in 2\mathbb{Z} : 2(a, b, c) &= (2a, 2b, 2c) = (0, 0, 0)\\ \implies & a \in \{0, 12\} \cong \mathbb{Z}_2\\ & b \in \{0\} \cong \mathbb{Z}_1\\ & c \in \{0, 25\} \cong \mathbb{Z}_2\end{aligned}$$

This shows that $Ann_M(2\mathbb{Z}) \cong \mathbb{Z}_2 \times \mathbb{Z}_2$. It is easy to see that for any $2i \in 2\mathbb{Z}$ it works.

**Problem 6.**

Let $\varphi \in Hom_{\mathbb{Z}}(\mathbb{Z}_{30}, \mathbb{Z}_{21})$. Then $\varphi$ is a group homomorphism from $\mathbb{Z}_{30}$ to $\mathbb{Z}_{21}$. Let $r$ be the generator of $\mathbb{Z}_{30}$ having order 30. Then it must mapped to an element with order $s$ where $s \mid 30$. And since $s \mid 21$ then we can only have $s = 1, 3$. If $s = 3$, there are only two elements with order 3 in $\mathbb{Z}_{21}$, namely 7 and 14. Also since $R = \mathbb{Z}$ then if $\varphi$ is a homomorphism we have:

$$\varphi(rm) = \varphi(m + \cdots + m) = r\varphi(m)$$

Thus $\varphi$ is also a module homomorphism. Therefore any module homomorphism from $\mathbb{Z}_{30}$ to $\mathbb{Z}_{21}$ is uniquely identified with $\varphi(1)$. And we have only 3 such homomorphism:

$$\begin{gathered}\varphi(1) = 7 \implies \varphi(m) = 7m \pmod{21} \\ \varphi(1) = 14 \implies \varphi(m) = 14m \pmod{21} \\ \varphi(1) = 0 \implies \varphi(m) = m \pmod{21}\end{gathered}$$

Which form a group of order 3, $\mathbb{Z}_3$.
Now since any $\mathbb{Z}$-module homomorphism is uniquely identifined with $\varphi(1)$, and we have:

$$\left.\begin{array}{l} Ord_{\mathbb{Z}_n}(\varphi(1)) \mid n \\ Ord_{\mathbb{Z}_n}(\varphi(1)) \mid Ord_{\mathbb{Z}_m}(1) = m \end{array}\right\} \implies Ord_{\mathbb{Z}_n}(\varphi(1)) \mid (n, m)$$

We only need to show that homomorphisms that map 1 to a members with order $r$ in $\mathbb{Z}_n$ where $r \mid (n, m)$ form the group $\mathbb{Z}_{(m,n)}$. Let $\sigma \in Hom_{\mathbb{Z}}(\mathbb{Z}_m, \mathbb{Z}_n)$, where $\sigma(1) = s$ and $Ord(s) = (m, n)$. Now it is easy to see that $\sigma^r(1) = \sigma^{r-1}(s) = \cdots = s^r$. This shows that $Ord(\sigma) = Ord(s) = (m, n)$. On the otherhand the number of elements in $\mathbb{Z}_n$ such as $s$ where $Ord_{\mathbb{Z}_n}(s) \mid (n, m)$ is exactly $(m, n)$. This proves that $\sigma$ generates the whole group and thus we have:

$$Hom_{\mathbb{Z}}(\mathbb{Z}_m, \mathbb{Z}_n) \cong \mathbb{Z}_{(m,n)}$$

**Problem 7.**

First we show that $eM$ and $(1 - e)M$ is submodule of $M$:

$$\begin{gathered} x, y \in M \implies ex, ey \in eM, r \in R \\ r(ex) + ey = rex + ey = erx + ey = e(rx + y) \in eM \end{gathered}$$

Note that we used $re = er$ and also that since $M$ is module then $rx+y \in M$. Similarly for $(1 - e)M$ we have:

$$\begin{aligned} r((1 - e)x) + (1 - e)y &= rx - rex + y - ey \\ &= rx + y - (erx - ey) \\ &= rx + y - e(rx - y) \in (1 - e)M \end{aligned}$$

It is obvious that since both $eM$ and $(1-e)M$ are $R$ module, so is $eM \oplus (1-e)M$. It remains to show an isomorphism:

$$\begin{aligned}
\varphi : M &\to eM \oplus (1-e)M \\
m &\mapsto (em, (1-e)m) \\
\varphi(m+m') &= (e(m+m'), (1-e)(m+m')) \\
&= (em, (1-e)m) + (em', (1-e)m') \\
&= \varphi(m) + \varphi(m'). \\
\varphi(rm) &= (erm, (1-e)rm) \\
&= (rem, (r-re)m) \\
&= (rem, r(1-e)m) \\
&= r(em, (1-e)m) = r\varphi(m).
\end{aligned}$$

This shows that $\varphi$ is a homomorphism. Now to show that it is an isomorphism we have to show that it is both surjective and injective. Suppose $em+(1-e)m' \in eM \oplus (1-e)M$. For any $m, m' \in M$ we know $em + (1-e)m' \in M$. Thus we have:

$$\begin{aligned}
\varphi(em + (1-e)m') &= \big(e(em + (1-e)m'), (1-e)(em + (1-e)m')\big) \\
&= \big((e^2m + (e-e^2)m'), (e-e^2)m + (1-2e+e^2)m'\big)
\end{aligned}$$

Sicne $e^2 = e$:

$$= \big(em, (1-e)m'\big)$$

This gives us that $\varphi$ is surjective. For injective suppose $m, m' \in M$ where $m \neq m'$:

$$\begin{aligned}
&\varphi(m) = \varphi(m') \\
\implies &(em, (1-e)m) = (em', (1-e)m') \\
\implies &em = em', (1-e)m = (1-e)m' \\
\implies &e(m-m') = 0, (1-e)(m-m') = 0 \\
\implies &e(m-m') + (1-e)(m-m') = 0 \\
\implies &m - m' = 0 \implies m = m'
\end{aligned}$$

Which is a contradiction. This gives us that $\varphi$ is bijective as well. Therefore $\varphi$ is an isomorphism and thus we have:

$$M \cong eM \oplus (1-e)M$$

**Problem 8.**

***(i)*** Suppose that $R/I$ is free module with basis $E \subseteq R/I$ and $I$ is non-trivial. Let $r+I \in E$. And let $s_1, s_2 \in I$ such that $s_1 \neq s_2$ and $s_1 r + I = s_2 r + I$. This always exists since $I$ is non-trivial. Since $r + I \in E$ then we must have: $s_1(r+I) \neq s_2(r+I)$. But we have:

$$s_1(r+I) = s_1 r + I = s_2 r + I = s_2(r+I)$$

Which is a contradiction. Thus $R/I$ is not a free module.

***(ii)*** ($\Leftarrow$) If $I$ is a principal ideal of $R$ then we have $I = \langle a \rangle$ for some $a \in R$. Since $I$ is an ideal of $R$ then it's also a submodule of $R$. We show that $\{a\}$ is a basis for $I$ over $R$. Any element in $I$ is of the form $ra$ for some $r \in R$. We have to show that this is unique. Since $\{a\}$ is linearly independent. Suppose there exists some $r_1 \neq r_2 \in R$ such that $r_1 a = r_2 a$:

$$r_1 a = r_2 a \implies ar_1 = ar_2 \implies a(r_1 - r_2) = 0$$

And since $R$ is a integral domain then either $a = 0$ or $r_1 = r_2$. And since $a = 0$ makes a trivial ideal $I$ then we have $r_1 = r_2$. Therefore $I$ is a free module over $R$.
($\Rightarrow$) If $I$ is a free module over $R$ with basis $E$. If $E$ has more than 1 element such $a$ and $b$, then we have:

$$ba + (-a)b \in R$$

Since $R$ is integral domain then we have: $ba - ab = 0$. And since $a, b \in E$ then they are linearly independent. Which means that $b = -a = 0$. Which is a contradiction supposing $a$ and $b$ are different elements in $E$. This shows that $E$ has exactly 1 element $a$. And each element in $I$ can be expressed as $ra$ for some $r \in R$. This shows that $I = \langle a \rangle$. And thus $I$ is a principal ideal of $R$.

**Problem 9.**

We prove that it is a module homomorphism. Consider $\varphi$:

$$\begin{aligned} \varphi : Hom_R(M_1 \times M_2, N) &\to Hom_R(M_1, N) \times Hom_R(M_2, N) \\ \sigma(-,-) &\mapsto \big(\sigma(-,0), \sigma(0,-)\big) \end{aligned}$$

It is obvious that both $\sigma(-,0)$ and $\sigma(0,-)$ are homomorphism. It remains to show that $\varphi$ is a homomorphism:

$$\begin{aligned} \varphi(\sigma + \delta)(a,b) &= \big((\sigma+\delta)(a,0), (\sigma+\delta)(0,b)\big) \\ &= \big(\sigma(a,0) + \delta(a,0), \sigma(0,b) + \delta(0,b)\big) \\ &= \big(\sigma(a,0), \sigma(0,b)\big) + \big(\delta(a,0), \delta(0,b)\big) \\ &= \varphi(\sigma)(a,b) + \varphi(\delta)(a,b) \end{aligned}$$

Also

$$\begin{aligned} \varphi(r\sigma)(a,b) &= \big(r\sigma(a,0), r\sigma(0,b)\big) \\ &= r\big(\sigma(a,0), \sigma(0,b)\big) \\ &= r\varphi(\sigma)(a,b) \end{aligned}$$

Thus $\varphi$ is homomorphism. To show that it is an isomorphism we show that $\varphi$ has an inverse.

$$\begin{aligned} \psi : Hom_R(M_1, N) \times Hom_R(M_2, N) &\to Hom_R(M_1 \times M_2, N) \\ \big(\sigma, \delta\big) &\mapsto \pi \end{aligned}$$

Where $\pi(a,b) = (\sigma(a), \delta(b))$. It is obvious that $\varphi^{-1} = \psi$. First we have to show that $\pi$ is a homomorphism.

$$\begin{aligned}
\pi((a_1,b_1)+(a_2,b_2)) &= \pi(a_1+a_2, b_1+b_2) \\
&= (\sigma(a_1+a_2), \delta(b_1+b_2)) \\
&= (\sigma(a_1)+\sigma(a_2), \delta(b_1)+\delta(b_2)) \\
&= (\sigma(a_1), \delta(b_1)) + (\sigma(a_2), \delta(b_2)) \\
&= \pi(a_1,b_1) + \pi(a_2,b_2). \\
\pi(r(a_1,b_1)) &= \pi((ra_1, rb_1)) \\
&= (\sigma(ra_1), \delta(rb_1)) \\
&= (r\sigma(a_1), r\delta(b_1)) \\
&= r(\sigma(a_1), \delta(b_1)) \\
&= r\pi(a_1,b_1).
\end{aligned}$$

This shows that $\pi$ is a homomorphism. Now to show that $\psi$ is homomorphism.

$$\begin{aligned}
\psi((\sigma_1,\delta_1)+(\sigma_2,\delta_2))(a,b) &= \psi(\sigma_1+\sigma_2, \delta_1+\delta_2)(a,b) \\
&= ((\sigma_1+\sigma_2)(a), (\delta_1+\delta_2)(b)) \\
&= (\sigma_1(a)+\sigma_2(a), \delta_1(b)+\delta_2(b)) \\
&= (\sigma_1(a), \delta_1(b)) + (\sigma_2(a), \delta_2(b)) \\
&= \psi(\sigma_1,\delta_1)(a,b) + \psi(\sigma_2,\delta_2)(a,b)
\end{aligned}$$

Also:

$$\begin{aligned}
\psi(r(\sigma,\delta))(a,b) &= \psi(r\sigma, r\delta)(a,b) \\
&= (r\sigma(a), r\delta(b)) \\
&= r(\sigma(a), \delta(b)) \\
&= r\psi(\sigma,\delta)(a,b)
\end{aligned}$$

This shows that $\psi$ is a homomorphism. And since it is the inverse of $\varphi$, therefore $\varphi$ is an module isomorphism. Thus we have:

$$Hom_R(M_1 \times M_2, N) \cong_R Hom_R(M_1, N) \times Hom_R(M_2, N)$$

The second part is really similar and quite long.

**Problem 10.**

***(i)*** Let $\psi_1(a_1, a_2, \dots) = (a_1, 0, a_2, 0, \dots)$, and $\psi_2(a_1, a_2, \dots) = (0, a_1, 0, a_2, \dots)$. Now we have:

$$\begin{aligned}
\varphi_1\psi_1(a_1, a_2, a_3, \dots) = \varphi_1(a_1, 0, a_2, 0, \dots) = (a_1, a_2, a_3, \dots) \\
\implies \varphi_1 \circ \psi_1 = 1
\end{aligned}$$

Similarly $\varphi_2\psi_2 = 1$. And also:

$$\begin{aligned}\varphi_2\psi_1(a_1, a_2, \dots) &= \varphi_2(a_1, 0, a_2, \dots) = (0, 0, 0, \dots)\\ &\implies \varphi_2 \circ \psi_1 = 0\end{aligned}$$

And similarly we $\varphi_1\psi_2 = 0$. And also:

$$\begin{aligned}(\psi_1\varphi_1 + \psi_2\varphi_2)(a_0, a_1, a_2, \dots) &= \psi_1(a_0, a_2, a_4, \dots) + \psi_2(a_1, a_3, a_5, \dots)\\ &= (a_0, 0, a_2, 0, a_4, \dots) + (0, a_1, 0, a_3, 0, \dots)\\ &= (a_0, a_1, a_2, \dots)\end{aligned}$$

Thus we have $\psi_1\varphi_1 + \psi_2\varphi_2 = 1$. With this we can generate any element in $R$. Consider $\sigma \in R$:

$$\sigma = \sigma(\psi_1\varphi_1 + \psi_2\varphi_2) = (\sigma\psi_1)\varphi_1 + (\sigma\psi_2)\varphi_2$$

To show that this representation is unique suppose that $\sigma_1\varphi_1 + \sigma_2\varphi_2 = \delta_1\varphi_1 + \delta_2\varphi_2$, then we have:

$$\begin{aligned}\sigma_1\varphi_1\psi_1 + \sigma_2\varphi_2\psi_1 &= \delta_1\varphi_1\psi_1 + \delta_2\varphi_2\psi_1\\ \implies \sigma_1(1)\sigma_2(0) &= \delta_1(1) = \delta_2(0) \implies \sigma_1 = \delta_1\end{aligned}$$

Similarly with $\psi_2$ we can conclude that $\sigma_2 = \delta_2$. Which proves the uniqueness of representation. And since for $0\varphi_1 + 0\varphi_2 = 0$, the uniqueness of representation shows that if for some $\sigma_1, \sigma_2 \in R$ we have $\sigma_1\varphi_1 + \sigma_2\varphi_2 = 0$ then $\sigma_1 = \sigma_2 = 0$. Which proves that $\varphi_1$ and $\varphi_2$ are linearly independent. This proves that $\{\varphi_1, \varphi_2\}$ is a basis for $R$.

***(ii)*** Let us define a homomorphism from $R \times R$ to $R$:

$$\begin{aligned}\phi : R \times R &\to R\\ (f, g) &\mapsto f\varphi_1 + g\varphi_2\end{aligned}$$

Since $\varphi_1, \varphi_2$ form a basis, thus $\phi$ is both injective and surjective. Checking the homomorphism is easy:

$$\begin{aligned}\phi((f_1, g_1) + (f_2, g_2)) &= \phi(f_1 + f_2, g_1 + g_2)\\ &= (f_1 + f_2)\varphi_1 + (g_1 + g_2)\varphi_2\\ &= f_1\varphi_1 + g_1\varphi_2 + f_2\varphi_1 + g_2\varphi_2\\ &= \phi(f_1, g_1) + \phi(f_2, g_2)\end{aligned}$$

Therefore $\phi$ is a homomorphism. This proves that $R \times R \cong R$. And by induction we can see that $R^n \cong R$.

**Problem 11.**

**Problem 12.**

**Problem 13.**

***(i)*** ($\Rightarrow$) Suppose that $M$ is irreducible. We know by definition $M \neq 0$. Taking some nonzero $m \in M$, we see that $Rm$ is a nonzero submodule of $M$, and so $Rm = M$. This proves that $M$ is generated by any nonzero element.

($\Leftarrow$) Suppose $M$ is a cyclic $R$-module with generator $a \neq 0$. Then if $I \subseteq R$ is an ideal of $R$, then $Ia \subseteq Ra = M$ is a proper submodule of $M$. I think for this statement to hold, we need to have that $M$ is cyclic if for any element $m \in M$, we have $M = Rm$. Then see that for some submodule $N \subseteq M$, take $b \in N$, then $M = Rb \subseteq N$. Which shows that $N = M$, and thus $M$ is irreducible.

For all irreducible $\mathbb{Z}$-modules we have all $\mathbb{Z}/p\mathbb{Z}$ as $\mathbb{Z}$-modules. Which is easy to see that they are irreducible since they are all cylic and don't have any subgroup.

***(ii)*** Take $R = Z[x, y]$. It is easy to see that $R$ is a cyclic $R$-submodule, since its generated with 1. And we know that $\langle x, y\rangle$ is an ideal of $R$ which can't be generated with only 1 element. Thus is not cyclic.

***(iii)*** Let $N$ be a submodule of $M$, a cyclic $R$-module generated with $a \neq 0$. Then $N = \Big(r_i a\Big)_{i\in J}$ for some index set $J$. Now suppose $i, j \in J$ and $r \in R$, since $N$ is a submodule, then we have that $r_i a + r r_j a = (r_i + r r_j)a \in N$. Therefore $r_i + r r_j = r_k$ for some $k \in J$. Which is describing and ideal of $R$. Now if we have some $r_i a = 0$ and $r_j a = 0$, then it is easy to see that for any $r \in R$, $r r_i a + r_j a = 0$. which implies that $(r r_i + r_j)a = 0$. Therefore all coefficients of $a$ such that their product is 0, form another ideal. This proves that we can write $N = \frac{I}{K}a$ for some $K \subseteq I \subseteq R$ ideals of $R$.

**Problem 14.**

***(i)*** Let $r_1, r_2 \in \Big(M_1 : M_2\Big)$ and $r \in R$ and $m \in M_2$. Then an element of $(rr_1 - r_2)M_2$ is of the form $(rr_1 - r_2)m$. Since $r_1 m \in M_1$ and $r_2 m \in M_1$. And also since $r_1 m \in M_1$, then $rr_1 m \in M_1$. And therefore $(rr_1 - r_2)m \in M_1$. This shows that $rr_1 - r_2 \in \Big(M_1 : M_2\Big)$, Which shows that it is an ideal of $R$.

Since for any $r \in \Big(M_1 : M_2\Big)$ and any $m_1 \in M_1$ and $m_2 \in M_2$ we have that $r(m_1 + m_2) + M_1 = M_1$, then we have $\Big(M_1 : M_2\Big) \subseteq Ann(\frac{M_1+M_2}{M_1})$. Now take $r \in Ann(\frac{M_1+M_2}{M_1})$. Then we have that for any $m_2 \in M$ and $0 \in M_1$, we have that $r(0 + m_2) + M_1 = M_1$, which implies that $rm_2 \in M_1$. Which shows that $r \in \Big(M_1 : M_2\Big)$. This shows that $\Big(M_1 : M_2\Big) = Ann(\frac{M_1+M_2}{M_1})$.

***(ii)*** To show that $I \subseteq \Big(I : J\Big)$, let $r \in I$, and $j \in J$, we have:

$$rj = jr \in I$$

This proves that $I \subseteq \Big(I : j\Big)$.

As for the isomorphism we describe two homomorphisms that are inverse of each other:

$$\begin{aligned} \phi : \frac{\big(I : j\big)}{I} &\to Hom_R(\frac{R}{J}, \frac{R}{I}) \\ k + I &\mapsto \varphi_k \end{aligned}$$

Where

$$\begin{aligned} \varphi_k : \frac{R}{J} &\to \frac{R}{I} \\ r + J &\mapsto kr + I \end{aligned}$$

And also

$$\begin{aligned} \psi : Hom_R(\frac{R}{J}, \frac{R}{I}) &\to \frac{\big(I : J\big)}{I} \\ \varphi &\mapsto \varphi(1) \end{aligned}$$

It is easy to see that $\phi \circ \psi = 1 = \psi \circ \phi$. Thus this is an isomorphism.

**Problem 15.**

***(i)*** Suppose $\mathbb{Q}$ is finitelely generated, by $\{\frac{a_1}{b_1}, \dots, \frac{a_m}{b_m}\}$. Then for any element in $q \in \mathbb{Q}$ there exists $r_1, r_2, \dots, r_m \in \mathbb{Z}$ such that $r_1 \frac{a_1}{b_1} + \dots r_m \frac{a_m}{b_m} = q$. It is easy to see that if $q$ is of the form $\frac{r}{s}$ where $s$ is some number such that $(s, b_i) = 1$ for any $1 \leq i \leq m$, then it is not possible. Thus $\mathbb{Q}$ is not finitelely generated.

If $\mathbb{Q}$ is a free $\mathbb{Z}$-module then it has a basis (inifite) $\{\frac{a_1}{b_1}, \frac{a_2}{b_2}, \dots\}$. But it is easy to see that $a_2 b_1 \frac{a_1}{b_1} + (-a_1) b_2 \frac{a_2}{b_2} = 0$, and since $a_2 b_1 \neq 0$ and $-a_1 b_2 \neq 0$ then it is not a basis, and thus $\mathbb{Q}$ is not free.

***(ii)*** Suppose $rm = 0$. And let $\{a_1, a_2, \dots\}$ be a basis for $M$ (It can be finite as well). Thus there exists $r_1, r_2, \dots, r_n \in R$, where $m = r_1 a_1 + \dots + r_n a_n$. Thus we have:

$$rm = r r_1 a_1 + \dots r r_n a_n = 0$$

Since $a_1, \dots, a_n$ are in basis, then all $r r_i = 0$ for all $i$. If $r = 0$ then we are done. Otherwise for all $i$, $r r_i = 0$, and since $R$ doesn't have any zero-divisors, then $r_i = 0$ for all $i$. This means that $m = 0$.

**Problem 16.**

**Problem 17.**

**Problem 18.**

***(i)*** Let $\phi : I \cap J \to I \oplus J$, where $\phi(i) = (i, -i)$. And $\psi : I \oplus J \to I + J$, where $\psi(i, j) = i + j$. Then it is easy to see that $\phi$ is 1-1 and $\psi$ is onto. And also $Im(\phi) = ker(\psi)$ since we have:

$$(i, j) \in Ker(\psi) \implies \psi(i, j) = i + j = 0 \implies j = -i \implies (i, j) = (i, -i) = \phi(i)$$
$$i \in Im(\phi) \implies \phi(i) = (i, -i) \implies \psi(i, -i) = 0 \implies (i, -i) \in Ker(\psi)$$

Thus the sequence is exact.

***(ii)***