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Problem 1.

(i) (a) Suppose $a = (a_1, a_2, \dots, a_n) \in V(J_1) \cup V(J_2)$. WLOG let $a \in V(J_1)$. This means that for any $P \in K[x_1, \dots, x_n]$ we have $P(a) = 0$. Now for any $Q \in J_1 \cap J_2$, we have $Q \in J_1$, thus $Q(a) = 0$ which implies $a \in V(J_1 \cap J_2)$. This shows that :

$$V(J_1) \cup V(J_2) \subseteq V(J_1 \cap J_2)$$

Now let $a \in V(J_1 \cap J_2)$. And for the sake of contradiction suppose there exists $P \in J_1$ and $Q \in J_2$ such that $P(a) \neq 0$ and $Q(a) \neq 0$. Note that $PQ \in J_1$ and $PQ \in J_2$. Therefore $PQ \in J_1 \cap J_2$. Since $a \in V(J_1 \cap J - 2)$, then we have $PQ(a) = P(a)Q(a) = 0$. This shows that either $P(a) = 0$ or $Q(a) = 0$, which is a contradiction. Tehrefore at least for one of the J_1 and J_2, a is a root for all polynomials in that ideal. Let that be J_1, thus we have $a \in V(J_1) \subseteq V(J_1) \cup V(J_2)$. Now we have $V(J_1 \cap J_2) \subseteq V(J_1) \cup V(J_2)$, hence:

$$V(J_1) \cup V(J_2) = V(J_1 \cap J_2)$$

Again suppose $a \in V(J_1) \cup V(J_2)$, WLOG $a \in V(J_1)$. Consider a member of $J_1 J_2$:

$$\begin{aligned} &P_i \in J_1, Q_i \in J_2 \\ &R = P_1Q_1 + P_2Q_2 + \dots P_lQ_l \in J_1J_2 \\ \implies &R(a) = P_1(a)Q_1(a) + \cdots + P_l(a)Q_l(a) \\ &= 0Q_1(a) + \cdots + 0Q_l(a) = 0 \end{aligned}$$

And since R was an arbitrary member of J_1J_2, then $a \in V(J_1J_2)$. Then $V(J_1) \cup V(J_2) \subseteq V(J_1J_2)$. Conversely suppose $a \in V(J_1J_2)$. And again for the sake of contradiction suppose $P \in J_1$ and $Q \in J_2$ such that $P(a) \neq 0$ and $Q(a) \neq 0$. Since $PQ \in J_1J_2$, then $P(a)Q(a) = 0$. Which is a contradiction. Therefore at least one of J_1 and J_2 have a as a root. Thus $a \in V(J_1)$ or $a \in V(J_2)$, which means $a \in V(J_1) \cup V(J_2)$. This shows that $V(J_1J_2) \subseteq V(J_1) \cup V(J_2)$. This proves that:

$$V(J_1) \cup V(J_2) = V(J_1J_2)$$

(b) Let $a \in V(\sum_{\lambda \in I} J_\lambda)$. Now for any $P \in J_i$, since $P + 0 + 0 + \cdots + 0 \in \sum_{\lambda \in I} J_\lambda$ then $P(a) = 0$. Therefore for any $P \in J_i$ we have $P(a) = 0$, Therefore $a \in V(J_i)$, hence $a \in \bigcap_{\lambda \in I} V(J_\lambda)$, therefore $V(\sum_{\lambda \in I} J_\lambda) \subseteq \bigcap_{\lambda \in I} V(J_\lambda)$. Now suppose $a \in \bigcap_{\lambda \in I} V(J_\lambda)$. Now any element in $\sum_{\lambda \in I} J_\lambda$ is of the form $\sum_{\lambda \in L} P_l$ where $P_\lambda \in J_l$ and L is a finite subset of I. Now since $a \in V(J_\lambda)$ for any λ, then $P_\lambda(a) = 0$ for any $\lambda \in L$. Then we can see that $\sum_{\lambda \in L} P_\lambda(a) = 0$. Thus $a \in V(\sum_{\lambda \in I} J_\lambda)$:

$$V(\sum_{\lambda \in I} J_\lambda) = \bigcap_{\lambda \in I} V(J_\lambda)$$

(ii) We know that union of two algebraic sets is algebraic, and intersection of any family of algebraic sets is also algebraic. Now since these are the closed sets, then complements of these sets are open sets in this topology. To show that indeed is a topology we need to show that $A^n(K)$ and $\emptyset$ are in T. This is trivial since complements of these sets are $\emptyset$ and $A^n(K)$ which both are algebraic sets. Also union of any collections of open sets is also open since for complements we must have intersection of any family of closed sets is also closed, which we talked about earlier. Also intersection of any two open sets must be open, which means union of any two closed sets must be closed, which is true for algebraic sets. Hence this is a topology.

Problem 2.

(i) Let $P \in \sqrt{J}$. This means that $P^r \in J$ for some $r \in \mathbb{N}$.

$$\begin{aligned} \forall a \in V(J) : P^r(a) = 0 &\implies P(a) = 0 \\ &\implies P \in I(V(J)) \end{aligned}$$

This shows that $\sqrt{J} \subseteq I(V(J))$.

(ii) Let $P \in \sqrt{IJ}$. Thus $P^r \in IJ$ for some $r \in \mathbb{N}$. This means that $P^r = P_1Q_1 + \cdots + P_kQ_k$ for some $P_i \in I$ and $Q_i \in J$. Now for any i, $P_iQ_i \in I$ and also $P_iQ_i \in J$. Therefore $P^r \in I$ and $P^r \in J$. Thus $P^r \in I \cap J$, which means that $P \in \sqrt{I \cap J}$. Hence $\sqrt{IJ} \subseteq \sqrt{I \cap J}$. Now suppose $P \in \sqrt{I \cap J}$. Then $P^r \in I \cap J$ for some $r \in \mathbb{N}$. Which means $P^r \in I$ and $P^r \in J$. Then $P^r \times P^r \in IJ$. Therefore $P \in \sqrt{IJ}$. Hence we have $\sqrt{IJ} = \sqrt{I \cap J}$.

Problem 3.

(i) Suppose $p = (p_1, p_2, \ldots, p_n)$. Now consider the ideal below:

$$J = \langle (x_1 - p_1), (x_2 - p_2), \ldots, (x_n - p_n) \rangle$$

Let $X = V(J)$. If $y = (y_1, \ldots, y_n) \in X$ then we have for any $i \in \{1, 2, \ldots, n\}$ we have $y_i - p_i = 0$, which implies $y_i = p_i$. This shows that $y = p$. And since y was an arbitrary element of X, then we have $V(J) = X = \{p\}$, which shows that $\{p\}$ is an algebraic set.

(ii) Let X be an algebraic set, this means that there exists some ideal J such that $X = V(J)$. Now since $K[x]$ is noetherian ring, then there exists some $P \in K[x]$ such that $J = \langle P \rangle$. Now if P has no root in K then it is obvious that $V(J) = \emptyset$. If $P = 0$ then $V(J) = A^1(K)$. And otherwise it has a finite number of roots Y. therefore $V(J) = Y$. This shows that an algebraic set is either $\emptyset$ or $A^1(K)$, or is a finite set of points.

Problem 4.

($\Leftarrow$) Suppose X is irreducible. And for the sake of contradiction suppose there exists $PQ \in I(X)$, where $P \notin I(X)$ and $Q \notin I(X)$. This means that there exists some $x_1 \in X$ such that $P(x_1) \neq 0$, and also $x_2 \in X$ such that $Q(x_2) \neq 0$. But also since $PQ \in I(X)$, then for any $x \in X$, we have that $PQ(x) = 0$, this means that for any $x \in X$ at least one of the $P(x) = 0$ or $Q(x) = 0$ happens. Let all roots of P which are in X be X_P, similarly we define X_Q. It is obvious that $X_p \cup X_Q = X$. And $X_P \subsetneq X$ and $X_Q \subsetneq X$. It only remains to show that X_P and X_Q are algebraic sets. Now consider the ideal $\langle I(X), P\rangle$. Sicne $X_P \subset X$, then $V(\langle I(X), P\rangle) = X_P$. Similarly X_Q is also algebraic. This shows that X is reducible, which is a contradiction. Then either $P \in I(X)$ or $Q \in I(X)$, which shows that $I(X)$ is prime.

($\Rightarrow$) Suppose $I(X)$ is prime. And for the sake of contradiction suppose that $X = X_1 \cup X_2$ where $X_1, X_2 \neq X$, and both X_1 and X_2 are algebraic sets. Consider $I(X_1)$. It has some element P where for some $y \in X - X_1$, $P(y) \neq 0$. Similarly we can find $Q \in I(X_2)$ where for some $z \in X - X_2$, $Q(z) \neq 0$. Now consider PQ. Since $X_1 \cup X_2 = X$, then $PQ \in I(X)$. But because of y and z it is obvious that $P, Q \notin I(X)$. Which shows that $I(X)$ is not prime, which is a contradiction, thus X is irreducible.

Problem 5.

($a \Rightarrow b$) Let $f = f_0 + f_1 + \dots f_d$. And let $\{P_1, \dots, P_k\}$ be a generator for this ideal. Then we have $f = P_1Q_1 + \dots P_kQ_k$ for some $Q_i \in K[x_1, \dots, x_n]$. Now if we write all Q_i in terms of homogeneous polynomial summation, we have:

$$Q_i = q_{0i} + q_{1i} + \dots + q_{ri}$$

It is obvious that if $w \neq v$, then $P(q_{wi} + q_{vi})$ is not homogeneous. This shows that any f_i is created with the form $P_1 q_{n_1 1} + \dots + P_k q_{n_k k}$. Therefore $f_i \in I$ since we can obtain it with generators.

($a \Leftarrow b$) Since $K[x_1, \dots, x_n]$ is noetherian, then we know that any ideal of it is finitely generated, then I has generator $\{P_1, \dots, P_k\}$. Now for any i, P_i can be written as $P_i = p_{0i} + \dots + p_{di}$. We know that $p_{ji} \in I$. Now if we put all these homogeneous parts of generators in one set, then that set is trivially finite, and can generate the generator, hence can generate I.