Normal Distribution

Q. Define normal distribution (NU-16)

Or, What is Normal distribution? (NU-18)

Ans: A continuous random variable X is said to have a normal distribution if its density function is given by

$$f\left(x;\ \ \mu,\ \sigma^2\right)=\frac{1}{\sigma\sqrt{2\pi}}\ e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2};\quad -\infty<x<\infty$$

where the parameters μ and σ^2 satisfy $-\infty<\mu<\infty$ and $\sigma^2>0$.

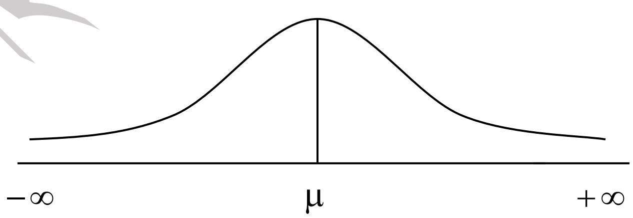
Remarks:

(i) The pdf of normal distribution with parameters μ and σ^2 is

$$f\left(x;\ \mu,\ \sigma^2\right)=\frac{1}{\sigma\sqrt{2\pi}}\ e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2};\ -\infty<x<\infty,\ -\infty<\mu<\infty,\ \sigma^2>0\text{..........(i)}$$

The variable X whose density function is (i), is called normal variate with parameters μ and σ^2 which is denoted by $N\left(\mu,\sigma^2\right)$.

(ii) The parameters μ and σ^2 are actually the mean and variance of the normal variate X. The graph of the normal curve is



Normal density curve

Standard Normal Variate:

A continuous random variable Z is said to have a standard normal variate if its density function is given by

$$f\left(z;\ \ 0,\ 1\right)=\frac{1}{\sqrt{2\pi}}\ e^{-\frac{1}{2}z^2};\qquad -\infty<z<\infty$$

Remarks:

(i) $x \sim N(\mu, \sigma^2)$

$E(x)=\mu$ and $V(x)=\sigma^2$

Let, $z=\dfrac{x-\mu}{\sigma}$

Now, $E(z)=E\left(\dfrac{x-\mu}{\sigma}\right)=\dfrac{1}{\sigma}E(x-\mu)=\dfrac{1}{\sigma}\left[E(x)-\mu\right]=\dfrac{1}{\sigma}[\mu-\mu]=0$

and $V(z)=V\left(\dfrac{x-\mu}{\sigma}\right)=\dfrac{1}{\sigma^2}V(x-\mu)=\dfrac{1}{\sigma^2}V(x)=\dfrac{1}{\sigma^2}\times\sigma^2=1$

So, z is a standard normal variate with mean zero and variance unity.

(ii) The pdf of normal variate x with mean μ and variance σ^2 is

$$f\left(x;\ \mu,\ \sigma^2\right)=\frac{1}{\sigma\sqrt{2\pi}}\ e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2};\qquad -\infty<x<\infty$$

Let, $z=\dfrac{x-\mu}{\sigma}$

$\Rightarrow x-\mu=\sigma z$

$\Rightarrow x=\mu+\sigma z$

Now, $\dfrac{dx}{dz}=\sigma$ i.e., $|J|=\sigma$

Hence the density function of z is

$$f(z)=f(x)\ |J|$$

$$=\frac{1}{\sigma\sqrt{2\pi}}\ e^{-\frac{1}{2}z^2}\times\ \sigma$$

$\therefore\ f(z)=\dfrac{1}{\sqrt{2\pi}}\ e^{-\frac{1}{2}z^2};\ -\infty<z<\infty$; which is the pdf of standard normal distribution.

Q. Mention the properties of normal distribution (NU-18)

Or, Write down the important characteristics of the normal distribution.

Ans: The properties of normal distribution are as follows-

i) The curve of normal distribution is symmetrical about the point $x = \mu$ and it is bell shaped.

ii) As the distribution is symmetrical about the point $x = \mu$, it divides the whole distribution into two equal parts. That is $x = \mu$ is the median of the distribution.

iii) Mean, median and mode of normal distribution coincides.

iv) As the curve is symmetrical, all odd moments about mean are zero i.e., $\mu_{2r+1} = 0 \ (r = 1, 2, \cdots)$. On the other hand, the even moments of the distribution are found from the relation, $\mu_{2r} = \dfrac{2^r \sigma^{2r}}{\sqrt{\pi}} \left(r - \dfrac{1}{2}\right)!; \quad r = 1, 2, \cdots$.

v) The mean deviation about mean is approximately equal to $\dfrac{4}{5}\sigma$.

vi) For normal distribution, skewness and kurtosis are $\beta_1 = 0$ and $\beta_2 = 3$ respectively. On the other hand, the coefficient of skewness and kurtosis are $\gamma_1 = \sqrt{\beta_1} = 0$ and $\gamma_2 = \beta_2 - 3 = 0$ respectively.

vii) If z is a standard normal variate, then the pdf of z is,

$$f(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2}; \qquad -\infty < z < \infty$$

viii) The quartile deviation of the distribution is $\dfrac{2}{3}\sigma$.

Q. Show that normal distribution is a limiting form of Binomial distribution.

Or, Stating the underlying assumptions, derive normal distribution from binomial distribution (NU-14)

Ans: Normal distribution is obtained from the binomial distribution under the following conditions:

(i) The probability of success ‘p’ or the probability of failure ‘q’ are not so small and

(ii) n, the number of trials is very large i.e., n tends to infinity.

We know the probability function of binomial distribution with parameters n and p is

$$f(x;\ n,\ p) = \binom{n}{x}\ p^x\ q^{n-x}; \qquad x = 0,\ 1,\ 2, \cdots, n \ \text{ and } \ p+q=1$$

$$= \frac{n!}{x!\ (n-x)!}\ p^x\ q^{n-x} \text{ ....(i)}$$

Here, x is a binomial variate with mean, $E(x) = n\,p$ and variance, $V(x) = n\,p\,q$.

Now, let us consider the standard binomial variate

$$z = \frac{x - E(x)}{\sqrt{V(x)}}$$

$$\therefore\ z = \frac{x - n\,p}{\sqrt{n\,p\,q}} \text{ ....(ii)}$$

When $x = 0$, then $z = \frac{-\,n\,p}{\sqrt{n\,p\,q}} = -\sqrt{\frac{n\,p}{q}}$

and $x = n$, then $z = \frac{n - n\,p}{\sqrt{n\,p\,q}} = \frac{n\,(1-p)}{\sqrt{n\,p\,q}} = \frac{n\,q}{\sqrt{n\,p\,q}} = \sqrt{\frac{n\,q}{p}}$

Thus, if $n \to \infty$, then z takes the values $-\infty$ to ∞.

Hence the distribution of z will be a continuous distribution over the range $-\infty$ to ∞ with mean zero and variance unity.

Now, under the conditions (i) and (ii), we shall find the limiting form of (i) by using Stirling's approximation for factorials.

That is for large n, we have

$$\lim_{n\to\infty} n! = \sqrt{2\pi}\ e^{-n}\ n^{n+\frac{1}{2}}$$

Now (i) can be written as

$$f(x;\ \ n,\ p) = \lim_{n\to\infty}\left[\frac{\sqrt{2\pi}\ \ e^{-n}\ \ n^{n+\frac{1}{2}}\ \ p^x\ \ q^{n-x}}{\sqrt{2\pi}\ \ e^{-x}\ \ x^{x+\frac{1}{2}}.\ \ \sqrt{2\pi}\ \ e^{-(n-x)}\ \left(n-x\right)^{n-x+\frac{1}{2}}}\right]$$

$$= \lim_{n\to\infty}\left[\frac{e^{-n}\ \ n^n\ \ n^{\frac{1}{2}}\ \ p^x\ \ q^{n-x}}{\sqrt{2\pi}\ \ e^{-x-n+x}\ \ x^{x+\frac{1}{2}}\ \ (n-x)^{n-x+\frac{1}{2}}}\right]$$

$$= \lim_{n\to\infty} \frac{(np)^{x+\frac{1}{2}} \; n^{-x} \; p^{-\frac{1}{2}} \; n^{n} \; q^{n-x}}{\sqrt{2\pi} \;\; x^{x+\frac{1}{2}} \; (n-x)^{n-x+\frac{1}{2}}}$$

$$= \lim_{n\to\infty} \frac{(np)^{x+\frac{1}{2}} \; n^{n-x} \; q^{n-x}}{\sqrt{2\pi} \; \sqrt{p} \;\; x^{x+\frac{1}{2}} \; (n-x)^{n-x+\frac{1}{2}}}$$

$$= \lim_{n\to\infty} \frac{(np)^{x+\frac{1}{2}} \; (nq)^{n-x+\frac{1}{2}} \; (nq)^{-\frac{1}{2}}}{\sqrt{2\pi} \; \sqrt{p}. \;\; x^{x+\frac{1}{2}} \; (n-x)^{n-x+\frac{1}{2}}}$$

$$= \lim_{n\to\infty} \frac{(np)^{x+\frac{1}{2}} \; (nq)^{n-x+\frac{1}{2}}}{\sqrt{2\pi} \; \sqrt{npq}. \;\; x^{x+\frac{1}{2}} \; (n-x)^{n-x+\frac{1}{2}}}$$

$$\therefore \; f(x;\; n,\; p) = \lim_{n\to\infty} \frac{1}{\sqrt{2\pi npq}} \left(\frac{np}{x}\right)^{x+\frac{1}{2}} \left(\frac{nq}{n-x}\right)^{n-x+\frac{1}{2}}$$

From (ii) we can write,

$$x - np = z\sqrt{npq}$$

$$\Rightarrow x = np + z\sqrt{npq} \;\ldots\ldots\ldots\text{(iii)}$$

$$\Rightarrow \frac{x}{np} = 1 + \frac{z\sqrt{npq}}{np}$$

$$\Rightarrow \frac{x}{np} = 1 + z\sqrt{\frac{q}{np}}$$

and $n - x$

$$= n - np - z\sqrt{npq} \quad \text{[putting the value of x from (iii)]}$$

$$= n(1-p) - z\sqrt{npq}$$

$$\Rightarrow n - x = nq - z\sqrt{npq}$$

$$\Rightarrow \frac{n-x}{nq} = 1 - \frac{z\sqrt{npq}}{nq}$$

$$\Rightarrow \frac{n-x}{nq} = 1 - z\sqrt{\frac{p}{nq}}$$

Differentiating (iii) w.r.t z, we get

$$\frac{dx}{dz} = \sqrt{npq}$$

$$\therefore \ |J_1| = \sqrt{npq}$$

Hence the pdf of z is

$$f(z) = f(x)\ |J_1|$$

$$= \lim_{n\to\infty} \frac{1}{\sqrt{2\pi\, npq}} \left(\frac{np}{x}\right)^{x+\frac{1}{2}} \left(\frac{nq}{n-x}\right)^{n-x+\frac{1}{2}} .\ \sqrt{npq}$$

$$= \lim_{n\to\infty} \frac{1}{\sqrt{2\pi}} \left(\frac{np}{x}\right)^{x+\frac{1}{2}} \left(\frac{nq}{n-x}\right)^{n-x+\frac{1}{2}}$$

$$= \lim_{n\to\infty} \frac{1}{\sqrt{2\pi}} \frac{1}{\left(\frac{x}{np}\right)^{x+\frac{1}{2}} \left(\frac{n-x}{nq}\right)^{n-x+\frac{1}{2}}}$$

$$\therefore\ f(z) = \lim_{n\to\infty} \frac{1}{\sqrt{2\pi}} . \frac{1}{N} \ldots\ldots\ldots\ldots\ldots\text{(iv)}$$

Where, $N = \left(\frac{x}{np}\right)^{x+\frac{1}{2}} \left(\frac{n-x}{nq}\right)^{n-x+\frac{1}{2}}$

$$\Rightarrow \log N = \left(x+\frac{1}{2}\right) \log\left(\frac{x}{np}\right) + \left(n-x+\frac{1}{2}\right) \log\left(\frac{n-x}{nq}\right)$$

$$\Rightarrow \log N = \left(np + z\sqrt{npq} + \frac{1}{2}\right) \log\left(1 + z\sqrt{\frac{q}{np}}\right) + \left(nq - z\sqrt{npq} + \frac{1}{2}\right) \log\left(1 - z\sqrt{\frac{p}{nq}}\right) \quad \text{[from (iii)]}$$

$$= \left(np + z\sqrt{npq} + \frac{1}{2}\right) \left[z\sqrt{\frac{q}{np}} - \frac{z^2}{2}\frac{q}{np} + \frac{z^3}{3}\left(\frac{q}{np}\right)^{\frac{3}{2}} - \cdots\right] + \left(nq - z\sqrt{npq} + \frac{1}{2}\right) \left[-z\sqrt{\frac{p}{nq}} - \frac{z^2}{2}\frac{p}{nq} - \frac{z^3}{3}\left(\frac{p}{nq}\right)^{\frac{3}{2}} - \cdots\right]$$

$$= \left(z\sqrt{npq} - \frac{1}{2}qz^2 + \frac{1}{3}z^3\frac{q^{\frac{3}{2}}}{\sqrt{np}} + z^2q - \frac{1}{2}z^3\frac{q^{\frac{3}{2}}}{\sqrt{np}} + \frac{1}{2}z\sqrt{\frac{q}{np}} - \frac{1}{4}z^2\frac{q}{np} + \cdots\right) + \left(-z\sqrt{npq} - \frac{1}{2}z^2p - \frac{1}{3}z^3\frac{p^{\frac{3}{2}}}{\sqrt{nq}} + z^2p + \frac{1}{2}z^3\frac{p^{\frac{3}{2}}}{\sqrt{nq}} - \frac{1}{2}z\sqrt{\frac{p}{nq}} - \frac{1}{4}z^2\frac{p}{nq} + \cdots\right)$$

Hence as limit $n \to \infty$, we get

$$\log N = -\frac{1}{2} q z^2 - \frac{1}{2} z^2 p + z^2 q + z^2 p$$

$$= -\frac{1}{2} z^2 (q+p) + z^2 (q+p)$$

$$= -\frac{1}{2} z^2 + z^2 \qquad [\therefore\ q+p=1]$$

$$\Rightarrow \log N = \frac{z^2}{2}$$

$$\Rightarrow N = e^{\frac{z^2}{2}}$$

Now substituting the value of N for $n \to \infty$ in (iv), we have

$$f(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}; \quad -\infty < z < \infty \qquad \left[\because\ N = e^{\frac{z^2}{2}} \Rightarrow \frac{1}{N} = e^{-\frac{z^2}{2}}\right]$$

This is the pdf of standard normal distribution.

If we put $z = \dfrac{x - np}{\sqrt{npq}}$, then the pdf of x is

$$f(x) = f(z)\ |J_2|; \text{ where } \frac{dz}{dx} = \frac{1}{\sqrt{npq}} \quad \text{i.e.,} \quad |J_2| = \frac{1}{\sqrt{npq}}$$

$$= \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} . \frac{1}{\sqrt{npq}}$$

$$\Rightarrow f(x) = \frac{1}{\sqrt{2\pi\, npq}}\ e^{-\frac{1}{2}\left(\frac{x-np}{\sqrt{npq}}\right)^2} \quad \text{.................(v)}$$

If x is normal variate with mean, $E(x) = \mu$ and variance, $V(x) = \sigma^2$, then let us consider the standard normal variate $z = \dfrac{x - E(x)}{\sqrt{V(x)}} = \dfrac{x-\mu}{\sqrt{\sigma^2}}$. So, for $z = \dfrac{x-\mu}{\sqrt{\sigma^2}}$, the pdf of x can be written as (v) in the following way-

$$f\left(x;\quad \mu, \sigma^2\right) = \frac{1}{\sqrt{2\pi\sigma^2}}\ e^{-\frac{1}{2}\left(\frac{x-\mu}{\sqrt{\sigma^2}}\right)^2}$$

$$= \frac{1}{\sigma\sqrt{2\pi}}\; e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}; \qquad -\infty < x < \infty$$

which is the pdf of normal distribution.

Remarks:

(i) The pdf of normal distribution with parameters μ and σ^2 is

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}}\; e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}; \qquad -\infty < x < \infty;$$

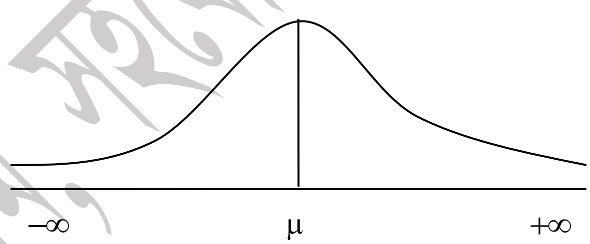
We know, $\int_{-\infty}^{\infty} f(x)\,dx = 1$

$$\Rightarrow \int_{-\infty}^{\mu} f(x)\,dx + \int_{\mu}^{\infty} f(x)\,dx = 1$$

Since normal distribution is symmetric at the point $x = \mu$, so

$$\int_{-\infty}^{\mu} f(x)\,dx = \int_{\mu}^{\infty} f(x)\,dx = \frac{1}{2}$$

$$\Rightarrow \int_{-\infty}^{\mu} \frac{1}{\sigma\sqrt{2\pi}}\; e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx = \int_{\mu}^{\infty} \frac{1}{\sigma\sqrt{2\pi}}\; e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx = \frac{1}{2}$$



$$P(-\infty < x < \mu) = P(\mu < x < \infty) = \frac{1}{2}$$

(ii) If z is the standard normal variate with mean zero and variance unity, then the pdf of z is

$$f(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2}; \qquad -\infty < z < \infty$$

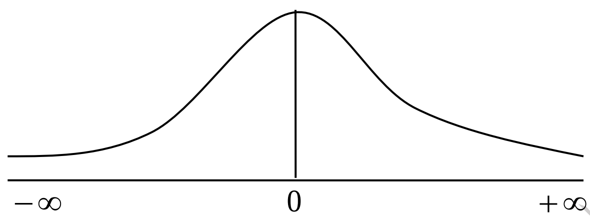
We know, $\int_{-\infty}^{\infty} f(z)\,dz = 1$

$$\Rightarrow \int_{-\infty}^{0} f(z)\, dz + \int_{0}^{\infty} f(z)\, dz = 1$$

Since z is symmetric at the point $z = 0$, so

$$\int_{-\infty}^{0} f(z)\, dz = \int_{0}^{\infty} f(z)\, dz = \frac{1}{2}$$

$$\Rightarrow \int_{-\infty}^{0} \frac{1}{\sqrt{2\Pi}}\, e^{-\frac{1}{2}z^2}\, dz = \int_{\mu}^{\infty} \frac{1}{\sigma\ \sqrt{2\Pi}}\, e^{-\frac{1}{2}z^2}\, dz = \frac{1}{2}$$



$$P(-\infty < z < 0) = P(0 < z < \infty) = \frac{1}{2} = 0.5$$

(iii) $\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$

and $\log(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \cdots$

Q Derive normal distribution from Poisson distribution (NU-11, 13, 15)

Ans: If the mean of Poisson distribution tends to infinity, then Poisson distribution turns into normal distribution.

Let X be a Poisson variate with parameter λ, then the probability function of x is

$$f(x;\ \lambda) = \frac{e^{-\lambda}\ \lambda^x}{x!}; \qquad x = 0, 1,\ 2, \cdots, \infty \quad \text{...........(i)}$$

The Stirling's approximation for $x!$ is

$$x! = \sqrt{2\pi}.\ \ x^{x+\frac{1}{2}}.\ \ e^{-x}\text{; when x is large ....(ii)}$$

By using the result (ii) in (i), we have

$$f(x;\ \lambda) = \frac{e^{-\lambda}\ \lambda^x}{\sqrt{2\pi}.\ \ x^{x+\frac{1}{2}}.\ \ e^{-x}}$$

$$= \frac{\sqrt{\lambda}.\ e^{x}\ e^{-\lambda}\ \lambda^{x}}{\sqrt{\lambda}\ \sqrt{2\pi}.\ x^{x+\frac{1}{2}}}$$

$$= \frac{e^{x-\lambda}\ \lambda^{x+\frac{1}{2}}}{\sqrt{2\pi\lambda}.\ \ x^{x+\frac{1}{2}}}$$

$$= \frac{1}{\sqrt{2\pi\lambda}}\ e^{x-\lambda}\ \left(\frac{x}{\lambda}\right)^{-\left(x+\frac{1}{2}\right)}$$

Now, let us take the standard Poisson variate z as

$$z = \frac{x-\lambda}{\sqrt{\lambda}} \ \text{...............(iii)}$$

Where, λ and $\sqrt{\lambda}$ are the mean and standard deviation of the Poisson variate X.

When $x = 0$ then $z = -\sqrt{\lambda}$ and $x = \infty$ then $z = \infty$

Hence, when λ tends to infinity, then z turns into a continuous variate with mean zero and variance unity in the range $-\infty$ to ∞.

Thus, we have from (iii)

$$x - \lambda = z\ \sqrt{\lambda}$$

$$\Rightarrow x = \lambda + z\ \sqrt{\lambda} \ \text{.............(iv)}$$

Now, $\frac{dx}{dz} = \sqrt{\lambda}$

$$\therefore\ |J_1| = \sqrt{\lambda}$$

So, $f(z) = f(x)\ \ |J_1|$

$$= \frac{1}{\sqrt{2\pi\lambda}}\ e^{x-\lambda}\ \left(\frac{x}{\lambda}\right)^{-\left(x+\frac{1}{2}\right)} .\ \sqrt{\lambda}$$

$$\Rightarrow f(z) = \frac{1}{\sqrt{2\pi}}\ e^{z\sqrt{\lambda}} .\ \Phi \qquad \left[\because\ \ x - \lambda = z\ \sqrt{\lambda}\right] \text{...............(v)}$$

Where, $\Phi = \left(\frac{x}{\lambda}\right)^{-\left(x+\frac{1}{2}\right)} = \left(\frac{\lambda + z\sqrt{\lambda}}{\lambda}\right)^{-\left(\lambda + z\sqrt{\lambda}+\frac{1}{2}\right)} = \left(1+\frac{z}{\sqrt{\lambda}}\right)^{-\left(\lambda + z\sqrt{\lambda}+\frac{1}{2}\right)}$ [using (iv)]

Taking logarithm, we have

$$\log\Phi = -\left(\lambda + z\sqrt{\lambda} + \frac{1}{2}\right)\log\left(1+\frac{z}{\sqrt{\lambda}}\right)$$

$$= -\left(\lambda + z\sqrt{\lambda} + \frac{1}{2}\right)\left(\frac{z}{\sqrt{\lambda}} - \frac{z^2}{2\lambda} + \frac{z^3}{3\lambda^{\frac{3}{2}}} - \cdots\right)$$

$$= -\lambda\left(\frac{z}{\sqrt{\lambda}} - \frac{z^2}{2\lambda} + \frac{z^3}{3\lambda^{\frac{3}{2}}} - \cdots\right) - z\sqrt{\lambda}\left(\frac{z}{\sqrt{\lambda}} - \frac{z^2}{2\lambda} + \frac{z^3}{3\lambda^{\frac{3}{2}}} - \cdots\right) - \frac{1}{2}\left(\frac{z}{\sqrt{\lambda}} - \frac{z^2}{2\lambda} + \frac{z^3}{3\lambda^{\frac{3}{2}}} - \cdots\right)$$

$$= -z\sqrt{\lambda} + \frac{z^2}{2} - \frac{z^3}{3\sqrt{\lambda}} - z^2 + \frac{z^3}{2\sqrt{\lambda}} - \frac{z^4}{3\lambda} - \frac{1}{2}\left(\frac{z}{\sqrt{\lambda}} - \frac{z^2}{2\lambda} + \frac{z^3}{3\lambda^{\frac{3}{2}}} - \cdots\right)$$

$= -z\sqrt{\lambda} + \frac{z^2}{2} - z^2 +$ terms containing power of λ in the denominator (হর).

$$\therefore \quad \lim_{\lambda\to\infty} \log\Phi = -\frac{1}{2}z^2 - z\sqrt{\lambda}$$

Hence, $\Phi = e^{-\frac{1}{2}z^2 - z\sqrt{\lambda}}$

Putting the value of Φ in (v), we have

$$f(z) = \frac{1}{\sqrt{2\pi}}\, e^{z\sqrt{\lambda}} . \, e^{-\frac{1}{2}z^2 - z\sqrt{\lambda}}$$

$$= \frac{1}{\sqrt{2\pi}}\, e^{-\frac{1}{2}z^2}; \qquad -\infty < z < \infty$$

which is the pdf of a standard normal variate.

If x is normal variate with mean, $E(x) = \mu$ and variance, $V(x) = \sigma^2$, then let us consider the standard normal variate, $z = \frac{x - E(x)}{\sqrt{V(x)}} = \frac{x-\mu}{\sigma}$. Now pdf of x is

$f(x) = f(z)\ |J_2|;$ where $\frac{dz}{dx} = \frac{1}{\sigma}$ i.e., $|J_2| = \frac{1}{\sigma}$

$$= \frac{1}{\sqrt{2\pi}}\ e^{-\frac{z^2}{2}}.\ \frac{1}{\sigma}$$

$$= \frac{1}{\sigma\sqrt{2\pi}}\ e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}; \qquad -\infty < x < \infty$$

This is the pdf of normal variate with mean μ and variance σ^2.

NB: $\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots$

Q. Find the m.g.f of normal distribution and also find mean, variance, β_1 and β_2.

Or, Find the moment generating function of normal distribution and hence find β_1 and β_2 **(NU-12)**

Or, Find the mean and variance of normal distribution **(NU-11, 13)**

Ans: We know the pdf of normal variate x with mean μ and variance σ^2 is

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}}\ e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}; \qquad -\infty < x < \infty$$

The m.g.f of x is

$$M_x(t) = E\left(e^{tx}\right)$$

$$= \int_{-\infty}^{\infty} e^{tx}\ f(x)\ dx$$

$$\Rightarrow M_x(t) = \int_{-\infty}^{\infty} e^{tx}\ \frac{1}{\sigma\sqrt{2\pi}}\ e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}\ dx \text{ ....(i)}$$

Let, $z = \frac{x-\mu}{\sigma}$

$\Rightarrow x - \mu = \sigma z$

$\Rightarrow x = \mu + \sigma z$

If $x = -\infty$ then $z = -\infty$

and if $x = \infty$ then $z = \infty$

$\therefore\ \frac{dx}{dz} = \sigma$

$$\Rightarrow dx = \sigma\, dz$$

Now (i) can be written as

$$M_x(t) = \int_{-\infty}^{\infty} e^{t(\mu+\sigma z)} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}z^2} .\ \sigma\, dz \quad [\because\ x = \mu + \sigma z]$$

$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{\mu t} .\ e^{t\sigma z}\ e^{-\frac{1}{2}z^2}\ dz$$

$$= e^{\mu t} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2 + z\,t\,\sigma} dz$$

$$= e^{\mu t} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\left(z^2 - 2z.\,t\,\sigma\right)} dz$$

$$= e^{\mu t} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\left(z^2 - 2\,z.\,t\,\sigma\, +\, t^2\,\sigma^2 - t^2\,\sigma^2\right)} dz$$

$$= e^{\mu t} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(z - t\,\sigma)^2\, +\, \frac{1}{2}t^2\,\sigma^2} dz$$

$$= e^{\mu t} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(z - t\,\sigma)^2} .\ e^{\frac{1}{2}t^2\,\sigma^2}\ dz$$

$$\Rightarrow M_x(t) = e^{\mu t} .\ e^{\frac{1}{2}t^2\,\sigma^2} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(z - t\,\sigma)^2} dz \text{ ....(ii)}$$

Again, let $u = z - \sigma\, t$ | If $z = -\infty$ then $u = -\infty$

$\Rightarrow z = u + \sigma\, t$ | and if $z = \infty$ then $u = \infty$

$$\therefore \frac{dz}{du} = 1$$

$$\Rightarrow dz = du$$

So, (ii) can be written as

$$M_x(t) = e^{\mu t + \frac{1}{2}\sigma^2 t^2} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}u^2} du$$

$$= e^{\mu t + \frac{1}{2}\sigma^2 t^2} . \; 1$$

$$\therefore \quad M_x(t) = e^{\mu t + \frac{1}{2}\sigma^2 t^2}$$

Now cumulant generating function,

$$K_x(t) = \log \; M_x(t)$$

$$= \log \; e^{\mu t + \frac{1}{2}\sigma^2 t^2}$$

$$= \mu \; t + \frac{1}{2} \sigma^2 \; t^2$$

$$\therefore \quad K_x(t) = \mu . \; \frac{t}{1!} + \sigma^2 \; \frac{t^2}{2!}$$

$K_r =$ Coefficient of $\frac{t^r}{r!}$ in $K_x(t)$

Putting $r = 1, 2, 3, 4$, we get

$K_1 =$ Coefficient of $\frac{t}{1!}$ in $K_x(t) = \mu$

$K_2 =$ Coefficient of $\frac{t^2}{2!}$ in $K_x(t) = \sigma^2$

$K_3 =$ Coefficient of $\frac{t^3}{3!}$ in $K_x(t) = 0$

$K_4 =$ Coefficient of $\frac{t^4}{4!}$ in $K_x(t) = 0$

$\therefore$ Mean, $K_1 = \mu$

Variance, $\mu_2 = K_2 = \sigma^2$

$\mu_3 = K_3 = 0$

and $\mu_4 = K_4 + 3\,K_2^{\;2} = 0 + 3\left(\sigma^2\right)^2 = 3\,\sigma^4$

$$\therefore \quad \beta_1 = \frac{{\mu_3}^2}{{\mu_2}^3} = \frac{0}{{\mu_2}^3} = 0$$

and $$\beta_2 = \frac{\mu_4}{{\mu_2}^2} = \frac{3\,\sigma^4}{(\sigma^2)^2} = \frac{3\,\sigma^4}{\sigma^4} = 3$$

Since $\beta_1 = 0$, so the distribution is symmetric and since $\beta_2 = 3$, so the distribution is mesokurtic.

NB: We know the pdf for standard normal variate z is

$$f(z) = \frac{1}{\sqrt{2\pi}}\; e^{-\frac{1}{2}z^2}; \qquad -\infty < z < \infty$$

We can write,

$$\int_{-\infty}^{\infty} f(z)\; dz = 1$$

$$\Rightarrow \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}}\; e^{-\frac{1}{2}z^2}\; dz = 1$$

$$\Rightarrow \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}z^2}\; dz = 1$$

$$\Rightarrow \int_{-\infty}^{\infty} e^{-\frac{1}{2}z^2}\; dz = \sqrt{2\pi}$$

Similarly, $\int_{-\infty}^{\infty} e^{-\frac{1}{2}u^2} du = \sqrt{2\pi}$

$$\therefore \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}}\; e^{-\frac{1}{2}u^2}\; du = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}u^2}\; du = \frac{1}{\sqrt{2\pi}}.\;\sqrt{2\pi} = 1$$

Q. Prove that all even moments are $\mu_{2r} = \dfrac{\sigma^{2r}(2r)!}{2^r\; r!}$

Proof: Let $x \sim N(\mu, \sigma^2)$

We know the m.g.f of normal distribution about origin is

$$M_x(t) = e^{\mu t + \frac{1}{2}\sigma^2 t^2}$$

and Mean $= \mu$

Now the m.g.f. about mean is given by

$$M_{x-\mu}(t) = E\left[e^{t(x-\mu)}\right]$$

$$= E\left[e^{tx-\mu t}\right]$$

$$= E\left[e^{tx}.\ e^{-\mu t}\right]$$

$$= e^{-\mu t}\ E\left[e^{tx}\right]$$

$$= e^{-\mu t}\ M_x(t)$$

$$= e^{-\mu t}\ e^{\mu t + \frac{1}{2}\sigma^2 t^2}$$

$$= e^{\frac{1}{2}\sigma^2 t^2}$$

$$= 1 + \frac{1}{2}\sigma^2 t^2 + \frac{\left(\frac{1}{2}\sigma^2 t^2\right)^2}{2!} + \frac{\left(\frac{1}{2}\sigma^2 t^2\right)^3}{3!} + \cdots + \frac{\left(\frac{1}{2}\sigma^2 t^2\right)^r}{r!} + \cdots$$

$$= 1 + \frac{1}{2}\sigma^2 t^2 + \frac{1}{4}\sigma^4 \frac{t^4}{2!} + \frac{1}{8}\sigma^6 \frac{t^6}{3!} + \cdots + \frac{1}{2^r}\sigma^{2r}\frac{t^{2r}}{r!} + \cdots$$

$$= 1 + \sigma^2 \frac{t^2}{2!} + 3\sigma^4 \frac{t^4}{4!} + 15\sigma^6 \frac{t^6}{6!} + \cdots + \frac{1}{2^r}\sigma^{2r}\frac{(2r)!}{r!}\frac{t^{2r}}{(2r)!} + \cdots$$

$\therefore\ \mu_{2r} =$ Coefficient of $\dfrac{t^{2r}}{(2r)!}$ in $M_{x-\mu}(t)$

$$= \frac{\sigma^{2r}(2r)!}{2^r\ r!}$$

NB: $e^x = 1 + x + \dfrac{x^2}{2!} + \dfrac{x^3}{3!} + \cdots$

Q. Show that for normal distribution all odd moments are zero and even moments are

$$\mu_{2r} = \frac{2^r \sigma^{2r} \left(r - \frac{1}{2}\right)!}{\sqrt{\pi}}$$

$$= 1.\,3 \cdots (2r-3)\,(2r-1)\,\sigma^{2r}$$

Also find β_1 and β_2.

Or, Prove that the odd order central moments of a normal distribution is zero but the value of even order central moment is-

$$\mu_{2r} = 1,\ 3,\ 5, \cdots, (2n-1)\,\sigma^{2r}; \qquad r = 1,\ 2,\ 3, \cdots$$ **(NU-12, 14)**

Proof: Let $x \sim N(\mu,\ \sigma^2)$, so the pdf of x is

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}}\ e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}; \qquad -\infty < x < \infty$$

We know, $E(x) = \mu$

Now odd order moments about mean are given by

$$\mu_{2r+1} = E\left[x - E(x)\right]^{2r+1}$$

$$= E\left[x - \mu\right]^{2r+1}$$

$$= \int_{-\infty}^{\infty} (x-\mu)^{2r+1}\ f(x)\,dx$$

$$\Rightarrow \mu_{2r+1} = \int_{-\infty}^{\infty} (x-\mu)^{2r+1}\ \frac{1}{\sigma\sqrt{2\pi}}\ e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}\,dx \ ....(i)$$

Let, $z = \dfrac{x-\mu}{\sigma}$

$\Rightarrow x - \mu = \sigma z$

$\Rightarrow x = \mu + \sigma z$

$\therefore \dfrac{dx}{dz} = \sigma$

$\Rightarrow dx = \sigma\ dz$

If $x = -\infty$, then $z = -\infty$

and if $x = \infty$, then $z = \infty$

So (i) can be written as

$$\mu_{2r+1} = \int_{-\infty}^{\infty} (\sigma z)^{2r+1} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}z^2} \sigma \, dz$$

$$= \frac{\sigma^{2r+1}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} z^{2r+1} e^{-\frac{1}{2}z^2} dz$$

$$= \frac{\sigma^{2r+1}}{\sqrt{2\pi}} \left[\int_{-\infty}^{0} z^{2r+1} e^{-\frac{1}{2}z^2} dz + \int_{0}^{\infty} z^{2r+1} e^{-\frac{1}{2}z^2} dz \right]$$

$$= \frac{\sigma^{2r+1}}{\sqrt{2\pi}} \left[-\int_{0}^{\infty} z^{2r+1} e^{-\frac{1}{2}z^2} dz + \int_{0}^{\infty} z^{2r+1} e^{-\frac{1}{2}z^2} dz \right]$$

$$= \frac{\sigma^{2r+1}}{\sqrt{2\pi}} \times 0$$

$$\therefore \quad \mu_{2r+1} = 0$$

Hence, $\mu_1 = \mu_3 = \mu_5 = \cdots = \mu_{2r+1} = 0$ i.e., all odd moments are zero.

Again, even order moments about mean are given by

$$\mu_{2r} = E\left[x - E(x)\right]^{2r}$$

$$= E\left[x - \mu\right]^{2r}$$

$$= \int_{-\infty}^{\infty} (x-\mu)^{2r} f(x)\, dx$$

$$= \int_{-\infty}^{\infty} (x-\mu)^{2r} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx$$

$$= \int_{-\infty}^{\infty} (\sigma z)^{2r} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}z^2} . \sigma \, dz$$

$$= \frac{\sigma^{2r}}{\sqrt{2\pi}} \left[\int_{-\infty}^{0} z^{2r} e^{-\frac{1}{2}z^2} dz + \int_{0}^{\infty} z^{2r} e^{-\frac{1}{2}z^2} dz \right]$$

$$= \frac{\sigma^{2r}}{\sqrt{2\pi}} \left[\int_{0}^{\infty} z^{2r} e^{-\frac{1}{2}z^2} dz + \int_{0}^{\infty} z^{2r} e^{-\frac{1}{2}z^2} dz \right]$$

$$= \frac{\sigma^{2r}}{\sqrt{2\pi}} \cdot 2 \int_0^\infty z^{2r} \; e^{-\frac{1}{2} z^2} \, dz$$

$$= \frac{2\,\sigma^{2r}}{\sqrt{2\pi}} \int_0^\infty \left(\sqrt{2t}\right)^{2r} \; e^{-t} \; \frac{dt}{\sqrt{2t}}$$

$$= \frac{2\,\sigma^{2r}}{\sqrt{2\pi}} \int_0^\infty (2t)^{2r \times \frac{1}{2}} \; e^{-t} \; \frac{dt}{t^{\frac{1}{2}} \sqrt{2}}$$

$$= \frac{2\,\sigma^{2r}}{2\sqrt{\pi}} \int_0^\infty (2t)^r \; e^{-t} \; t^{-\frac{1}{2}} \, dt$$

$$= \frac{\sigma^{2r}}{\sqrt{\pi}} \int_0^\infty 2^r \; t^r \; e^{-t} \; t^{-\frac{1}{2}} \, dt$$

Let, $t = \frac{z^2}{2} \Rightarrow z^2 = 2t$

$$\Rightarrow \frac{dt}{dz} = \frac{1}{2} \cdot 2z = z$$

$$\Rightarrow \frac{dt}{z} = dz$$

$$\Rightarrow \frac{dt}{\sqrt{2t}} = dz$$

$$= \frac{2^r\,\sigma^{2r}}{\sqrt{\pi}} \int_0^\infty e^{-t} \; t^{r-\frac{1}{2}} \, dt$$

$$= \frac{2^r\,\sigma^{2r}}{\sqrt{\pi}} \int_0^\infty e^{-t} \; t^{\left(r+\frac{1}{2}\right)-1} \, dt$$

$$\Rightarrow \mu_{2r} = \frac{2^r\,\sigma^{2r}}{\sqrt{\pi}} \; \Gamma\!\left(r+\frac{1}{2}\right) \quad \left[\because \int_0^\infty e^{-x} \; x^{\alpha-1} \, dx = \Gamma(\alpha)\right] \ldots\ldots\ldots\ldots\text{(ii)}$$

$$= \frac{2^r\,\sigma^{2r}}{\sqrt{\pi}} \left(r+\frac{1}{2}-1\right)\left(r+\frac{1}{2}-2\right)\cdots\left\{r+\frac{1}{2}-(r-1)\right\}\left(r+\frac{1}{2}-r\right) \Gamma\!\left(r+\frac{1}{2}-r\right)$$

$$= \frac{2^r\,\sigma^{2r}}{\sqrt{\pi}} \left(r-\frac{1}{2}\right)\left(r-\frac{3}{2}\right)\cdots\frac{3}{2}\,\frac{1}{2}\,\Gamma\!\left(\frac{1}{2}\right)$$

$$= \frac{2^r\,\sigma^{2r}}{\sqrt{\pi}} \left(\frac{2r-1}{2}\right)\left(\frac{2r-3}{2}\right)\cdots\frac{3}{2}\,\frac{1}{2}\,\sqrt{\pi} \qquad \left[\because \Gamma\!\left(\frac{1}{2}\right) = \sqrt{\pi}\right]$$

$$= 2^r\,\sigma^{2r} \; \frac{(2r-1)(2r-3)\cdots 3.\,1}{2^r}$$

$$= 1.\,3.\,5\cdots(2r-1)\,\sigma^{2r}; \qquad r = 1, 2, 3, \cdots$$

Now putting $r = 1$ in (ii), we get

$$\mu_2 = \frac{2\,\sigma^2}{\sqrt{\pi}} \; \Gamma\!\left(1+\frac{1}{2}\right)$$

$$= \frac{2\,\sigma^2}{\sqrt{\pi}}\,\Gamma\!\left(\frac{3}{2}\right)$$

$$= \frac{2\,\sigma^2}{\sqrt{\pi}}\left(\frac{3}{2}-1\right)\Gamma\!\left(\frac{3}{2}-1\right)$$

$$= \frac{2\,\sigma^2}{\sqrt{\pi}}\,\frac{1}{2}\,\Gamma\!\left(\frac{1}{2}\right)$$

$$= \frac{\sigma^2}{\sqrt{\pi}}\,\sqrt{\pi}$$

$$\therefore\quad \mu_2 = \sigma^2$$

Again, if we put $r = 2$ in (ii), we get

$$\mu_4 = \frac{2^2\,\sigma^4}{\sqrt{\pi}}\,\Gamma\!\left(2+\frac{1}{2}\right)$$

$$= \frac{2^2\,\sigma^4}{\sqrt{\pi}}\,\Gamma\!\left(\frac{5}{2}\right)$$

$$= \frac{2^2\,\sigma^4}{\sqrt{\pi}}\left(\frac{5}{2}-1\right)\left(\frac{5}{2}-2\right)\Gamma\!\left(\frac{5}{2}-2\right)$$

$$= \frac{3\,\sigma^4}{\sqrt{\pi}}\,\sqrt{\pi}$$

$$\therefore\quad \mu_4 = 3\,\sigma^4$$

Now, $\beta_1 = \dfrac{\mu_3^{\,2}}{\mu_2^{\,3}} = \dfrac{0}{\mu_2^{\,3}} = 0$

and $\beta_2 = \dfrac{\mu_4}{\mu_2^{\,2}} = \dfrac{3\sigma^4}{(\sigma^2)^2} = \dfrac{3\sigma^4}{\sigma^4} = 3$

Since $\beta_1 = 0$, so the distribution is symmetric and since $\beta_2 = 3$, so the distribution is mesokurtic.

Remarks: For even function:

(a) $\displaystyle\int_{-3}^{3} x^2 dx = \int_{-3}^{0} x^2 dx + \int_{0}^{3} x^2 dx = \left[\frac{x^3}{3}\right]_{-3}^{0} + \left[\frac{x^3}{3}\right]_{0}^{3} = \frac{1}{3}\left[0-(-3)^3\right] + \frac{1}{3}\left[(3)^3 - 0\right] = \frac{27}{3} + \frac{27}{3} = 2\,\frac{27}{3}$

(b) $\int_{-3}^{3} x^2\ dx = \int_{-3}^{0} x^2\ dx + \int_{0}^{3} x^2\ dx = \int_{0}^{3} x^2\ dx + \int_{0}^{3} x^2\ dx = 2\ \int_{0}^{3} x^2\ dx = 2\ \frac{27}{3}$

For odd function:

(a) $\int_{-3}^{3} x^3\, dx = \int_{-3}^{0} x^3\, dx + \int_{0}^{3} x^3\, dx = \left[\frac{x^4}{4}\right]_{-3}^{0} + \left[\frac{x^4}{4}\right]_{0}^{3} = \frac{1}{4}\left[0-(-3)^4\right] + \frac{1}{4}\left[(3)^4 - 0\right] = -\frac{81}{4} + \frac{81}{4} = 0$

(b) $\int_{-3}^{3} x^3\ dx = \int_{-3}^{0} x^3\ dx + \int_{0}^{3} x^3\ dx = -\int_{0}^{3} x^3\ dx + \int_{0}^{3} x^3\ dx = 0$

For series:

$$\frac{1}{2}.\frac{3}{2}.\frac{5}{2}.\frac{7}{2}.\frac{9}{2} = \frac{1.3\cdots 9}{2.2\cdots 2} = \frac{1.3\cdots 9}{2^5} = \frac{1.3\cdots(2\times 5-1)}{2^5} = \frac{1.3\cdots(2r-1)}{2^r};\ r = 5$$

NB: (i) $\int_{-\infty}^{\infty} z^{2r+1}\ e^{-\frac{1}{2}z^2} dz = 0$ since the integrand $\int_{-\infty}^{\infty} z^{2r+1}\ e^{-\frac{1}{2}z^2}$ is an odd function of z

(P-8.25, S. C. Gupta)

(ii) $\int_{-\infty}^{0} z^{2r+1}\ e^{-\frac{1}{2}z^2} dz + \int_{0}^{\infty} z^{2r+1}\ e^{-\frac{1}{2}z^2} dz$ ....(i)

If we Put $z = -z$ in the first integral of (i), we get

$-\int_{0}^{\infty} z^{2r+1}\ e^{-\frac{1}{2}z^2} dz + \int_{0}^{\infty} z^{2r+1}\ e^{-\frac{1}{2}z^2} dz$ [P-547, M. K. Roy (New)]

(iii) The pdf of standard gamma distribution with parameter α is

$$f(x;\ \alpha) = \frac{e^{-x}\ x^{\alpha-1}}{\Gamma\alpha}; \qquad x > 0, \quad \alpha > 0$$

(iv) The r-th central or corrected moment about mean μ, usually denoted by μ_r is defined as

$$\mu_r = E[x - E(x)]^r = E[x-\mu]^r = \begin{cases} \sum (x-\mu)^r f(x); & \text{if x is discrete} \\ \int_{-\infty}^{\infty} (x-\mu)^r\ f(x)\ dx; & \text{if x is continuous} \end{cases}$$

Q. Find the mean deviation of normal distribution (NU-13)

Or, Find the mean deviation about mean of normal distribution.

Ans: Let $x \sim N(\mu,\ \sigma^2)$

We know the pdf of normal distribution is

$$f(x;\ \mu,\ \sigma^2) = \frac{1}{\sigma\sqrt{2\pi}}\ e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}; \qquad -\infty < x < \infty$$

and $E(x) = \mu$

Now, Mean deviation about mean is

$$MD_{(\text{about }\mu)} = E\left|x - E(x)\right|$$

$$= E\left|x-\mu\right|$$

$$= \int_{-\infty}^{\infty} \left|x-\mu\right| f(x)dx$$

$$\Rightarrow MD_{(\text{about }\mu)} = \int_{-\infty}^{\infty} \left|x-\mu\right|\ \frac{1}{\sigma\sqrt{2\pi}}\ e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx \ \ldots\ldots\ldots\text{(i)}$$

Let, $z = \frac{x-\mu}{\sigma}$

$\Rightarrow x - \mu = \sigma z$

If $x = -\infty$ then $z = -\infty$

and if $x = \infty$ then $z = \infty$

$\Rightarrow x = \mu + \sigma z$

$\therefore \ \frac{dx}{dz} = \sigma$

$\Rightarrow dx = \sigma\, dz$

So (i) can be written as

$$MD_{(\text{about }\mu)} = \int_{-\infty}^{\infty} \left|\sigma z\right|\ \frac{1}{\sigma\sqrt{2\pi}}\ e^{-\frac{1}{2}z^2}.\ \sigma\ dz$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left|\sigma\ z\right|\ e^{-\frac{1}{2}z^2}\ dz$$

$$= \frac{\sigma}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left|z\right|\ e^{-\frac{1}{2}z^2}\ dz$$

$$= \frac{2\sigma}{\sqrt{2\pi}} \int_0^\infty z \ e^{-\frac{1}{2}z^2} \ dz$$

$$\Rightarrow MD_{(about\ \mu)} = \sigma \sqrt{\frac{2}{\pi}} \int_0^\infty z \ e^{-\frac{1}{2}z^2} \ dz \ \ldots\ldots\ldots\text{(ii)}$$

Again, Let $u = \frac{1}{2}z^2$

$$\Rightarrow \frac{du}{dz} = \frac{1}{2}.\ 2\,z = z$$

$$\Rightarrow dz = \frac{du}{z} = \frac{du}{\sqrt{2u}}$$

When $z = 0$, then $u = 0$

$z = \infty$, then $u = \infty$

Now (ii) can be written as

$$MD_{(about\ \mu)} = \sigma \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-u} \ du$$

$$= \sigma \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-u} \ u^{1-1} \ du$$

$$= \sigma \sqrt{\frac{2}{\pi}}.\ \lceil 1 \quad \left[\because \int_0^\infty e^{-x} \ x^{\alpha-1} \ dx = \lceil\alpha \right]$$

$$\therefore \ MD_{(about\ \mu)} = \sigma \sqrt{\frac{2}{\pi}}$$

NB: (i) $\sigma \sqrt{\frac{2}{\pi}} = \sigma \sqrt{\frac{2}{\frac{22}{7}}} = \sigma \sqrt{\frac{14}{22}} = \frac{4}{5}\sigma$ (approx.)

(ii) The pdf of standard gamma distribution with parameter α is

$$f(x;\ \alpha) = \frac{e^{-x} \ x^{\alpha-1}}{\lceil\alpha}; \qquad x > 0, \quad \alpha > 0$$

Q. Show that in normal distribution Mean = Median = Mode.

Or, Show that mean, median and mode of normal distribution coincides (NU-15)

Q. Find the median and mode of a normal distribution and show that median = mode (NU-11, 13)

Proof: If $x \sim N\left(\mu,\ \sigma^2\right)$, then

$$f(x;\ \mu,\ \sigma^2) = \frac{1}{\sigma\ \sqrt{2\pi}}\ e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}; \qquad -\infty < x < \infty$$

By definition,

$$\text{Mean,} \quad E(x) = \int_{-\infty}^{\infty} x\ f(x)\ dx$$

$$\Rightarrow E(x) = \int_{-\infty}^{\infty} x\ \frac{1}{\sigma\sqrt{2\pi}}\ e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}\ dx \quad \text{..........(i)}$$

Let, $z = \dfrac{x-\mu}{\sigma}$

$\Rightarrow x - \mu = \sigma\ z$

$\Rightarrow x = \mu + \sigma\ z$

$\therefore\ \dfrac{dx}{dz} = \sigma$

$\Rightarrow dx = \sigma\ dz$

If $x = -\infty$, then $z = -\infty$

and if $x = \infty$, then $z = \infty$

Now (i) can be written as

$$E(x) = \int_{-\infty}^{\infty} \left(\mu + \sigma\ z\right)\ \frac{1}{\sigma\sqrt{2\pi}}\ e^{-\frac{1}{2}z^2}.\ \sigma\ dz$$

$$= \int_{-\infty}^{\infty} \mu \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2} dz + \int_{-\infty}^{\infty} \sigma z \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2} dz$$

$$= \mu \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}}\ e^{-\frac{1}{2}z^2} dz + \sigma \int_{-\infty}^{\infty} z \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2} dz$$

$$= \mu .\, 1 + \sigma \left[\int_{-\infty}^{0} z \frac{1}{\sqrt{2\pi}}\, e^{-\frac{1}{2}z^2} dz + \int_{0}^{\infty} z \frac{1}{\sqrt{2\pi}}\, e^{-\frac{1}{2}z^2} dz \right]$$

$$= \mu + \sigma \left[-\int_{0}^{\infty} z \frac{1}{\sqrt{2\pi}}\, e^{-\frac{1}{2}z^2} dz + \int_{0}^{\infty} z \frac{1}{\sqrt{2\pi}}\, e^{-\frac{1}{2}z^2} dz \right]$$

$$= \mu + \sigma .\, 0$$

$$= \mu$$

$\therefore$ Mean, $E(x) = \mu$

Let, M be the Median of the distribution, so

$$\int_{-\infty}^{M} f(x)\, dx = \int_{M}^{\infty} f(x)\, dx = \frac{1}{2}$$

Now, $\displaystyle\int_{-\infty}^{M} f(x)\, dx = \frac{1}{2}$

$$\Rightarrow \int_{-\infty}^{M} \frac{1}{\sigma\sqrt{2\pi}}\, e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx = \frac{1}{2} \quad \text{.............(ii)}$$

Let, $z = \dfrac{x-\mu}{\sigma}$

$\Rightarrow x - \mu = \sigma z$

$\Rightarrow x = \mu + \sigma z$

$\therefore \dfrac{dx}{dz} = \sigma$

$\Rightarrow dx = \sigma\, dz$

If $x = -\infty$, then $z = -\infty$

and if $x = M$, then $z = \dfrac{M-\mu}{\sigma}$

Now (ii) can be written as

$$\int_{-\infty}^{\frac{M-\mu}{\sigma}} \frac{1}{\sigma\sqrt{2\pi}}\, e^{-\frac{1}{2}z^2} .\ \sigma\ dz = \frac{1}{2}$$

$$\Rightarrow \int_{-\infty}^{\frac{M-\mu}{\sigma}} \frac{1}{\sqrt{2\pi}}\, e^{-\frac{1}{2}z^2} dz = \frac{1}{2} \quad \text{.............(iii)}$$

Again, we know in case of standard normal variate

$$\int_{-\infty}^{0} f(z)\, dz = \frac{1}{2}$$

$$\Rightarrow \int_{-\infty}^{0} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2} dz = \frac{1}{2} \ldots\ldots\ldots\ldots\text{(iv)}$$

Now comparing (iii) and (iv), we get

$$\frac{M-\mu}{\sigma} = 0$$

$$\Rightarrow M = \mu$$

$$\therefore \text{ Median, } M = \mu$$

The mode of the distribution will be the solution of $f'(x) = 0$, provided $f''(x) < 0$.

Here, we have

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} \; e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$$

$$\therefore \quad f'(x) = \frac{1}{\sigma\sqrt{2\pi}} \; e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} . \; -\frac{1}{2\sigma^2} . \quad 2(x-\mu)$$

$$\Rightarrow f'(x) = -f(x). \; \frac{x-\mu}{\sigma^2} \ldots\ldots\ldots\ldots\text{(v)}$$

According to the condition $f'(x) = 0$, equation (v) can be written as

$$-f(x). \; \frac{x-\mu}{\sigma^2} = 0$$

$$\Rightarrow \frac{x-\mu}{\sigma^2} = 0 \quad [\because \; f(x) \neq 0]$$

$$\therefore \quad x = \mu$$

It is evident $f''(x) < 0$ for $x = \mu$.

Therefore, $x = \mu$ is the mode of the distribution.

Hence, Mean = Median = Mode.

Q. Find the variance of normal distribution.

Ans: Let $x \sim N(\mu,\ \sigma^2)$, so the pdf of x is

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}}\ e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}; \qquad -\infty < x < \infty$$

and Mean, $E(x) = \mu$.

By definition, $V(x) = E\{x - E(x)\}^2$

$$= E(x-\mu)^2$$

$$\Rightarrow V(x) = \int_{-\infty}^{\infty} (x-\mu)^2 \frac{1}{\sigma\sqrt{2\pi}}\ e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx \ ....(i)$$

Let, $z = \frac{x-\mu}{\sigma}$	If $x = -\infty$, then $z = -\infty$
$\Rightarrow x - \mu = \sigma z$	and if $x = \infty$, then $z = \infty$
$\Rightarrow x = \mu + \sigma z$	
$\therefore \frac{dx}{dz} = \sigma$	
$\Rightarrow dx = \sigma\ dz$	

Now (i) can be written as

$$V(x) = \int_{-\infty}^{\infty} (\sigma z)^2 \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}z^2} \sigma\ dz$$

$$= \frac{\sigma^2}{\sqrt{2\pi}} \int_{-\infty}^{\infty} z^2\ e^{-\frac{1}{2}z^2} dz$$

$$= \frac{2\sigma^2}{\sqrt{2\pi}} \int_{0}^{\infty} z^2\ e^{-\frac{1}{2}z^2} dz \ ....(ii)$$

Let $u = \frac{1}{2}z^2$	If $z = 0$, then $u = 0$
$\therefore \frac{du}{dz} = \frac{1}{2}\, 2z = z$	and if $z = \infty$, then $u = \infty$
$\Rightarrow dz = \frac{du}{z} = \frac{du}{\sqrt{2u}}$	

Now (ii) can be written as

$$V(x)=\frac{2\sigma^2}{\sqrt{2\pi}}\int_0^\infty 2u\quad e^{-u}\frac{du}{\sqrt{2u}}$$

$$=\frac{2\sigma^2 .\, 2}{\sqrt{2\pi}\ \ \sqrt{2}}\int_0^\infty e^{-u}\ u^{1-\frac{1}{2}}\ du$$

$$=\frac{2\sigma^2}{\sqrt{\pi}}\int_0^\infty e^{-u}\ u^{\frac{3}{2}-1}\ du$$

$$=\frac{2\sigma^2}{\sqrt{\pi}}.\ \Gamma\frac{3}{2} \qquad \left[\because \int_0^\infty e^{-x}\ x^{\alpha-1}\,dx=\Gamma\alpha\right]$$

$$=\frac{2\sigma^2}{\sqrt{\pi}}.\ \left(\frac{3}{2}-1\right)\Gamma\left(\frac{3}{2}-1\right)$$

$$=\frac{2\sigma^2}{\sqrt{\pi}}\times\frac{1}{2}\times\sqrt{\pi} \qquad \left[\because \Gamma\frac{1}{2}=\sqrt{\pi}\right]$$

$$=\sigma^2$$

Q. State and prove the additive property of normal distribution.

Statement: The linear combination of a set of k independent normal variate is also a normal variate. That is if $x_1, x_2,\cdots,x_k$ are k independent normal variates with means $\mu_1, \mu_2,\cdots,\mu_k$ and variances $\sigma_1^2, \sigma_2^2,\cdots,\sigma_k^2$ respectively, then $X=\sum_{i=1}^{k}a_i x_i$ is also a normal variate with mean $\mu=\sum_{i=1}^{k}a_i\ \mu_i$ and variance $\sigma^2=\sum_{i=1}^{k}a_i^2\sigma_i^2$, where $a_1, a_2,\cdots,a_k$ are constants.

Proof: If $x_i\sim N\left(\mu_i,\sigma_i^2\right)$, then

$$M_{x_i}(t)=e^{\mu_i\,t+\frac{1}{2}\sigma_i^2\,t^2}; \qquad \left(i=1,\ 2,\cdots,k\right)$$

Now, $$M_{x_1}(t)=e^{\mu_1\,t+\frac{1}{2}\sigma_1^2\,t^2}$$

$$M_{x_2}(t)=e^{\mu_2\,t+\frac{1}{2}\sigma_2^2\,t^2}$$

$$\vdots$$

$$M_{x_k}(t) = e^{\mu_k t + \frac{1}{2}\sigma_k^2 t^2}$$

The m.g.f of $X = \sum_{i=1}^{k} a_i x_i$ is

$$M_X(t) = M_{\sum_{i=1}^{k} a_i x_i}(t)$$

$$= E\left(e^{t\sum_{i=1}^{k} a_i x_i}\right)$$

$$= E\left[e^{t(a_1 x_1 + a_2 x_2 + \cdots + a_k x_k)}\right]$$

$$= E\left[e^{a_1 x_1 t} . \ e^{a_2 x_2 t} \cdots e^{a_k x_k t}\right]$$

$$= E\left(e^{a_1 x_1 t}\right) \ E\left(e^{a_2 x_2 t}\right) \cdots E\left(e^{a_k x_k t}\right)$$

$$= M_{x_1}(a_1 t) \ M_{x_2}(a_2 t) \cdots M_{x_k}(a_k t)$$

$$= e^{\mu_1 a_1 t + \frac{1}{2}\sigma_1^2 a_1^2 t^2} . e^{\mu_2 a_2 t + \frac{1}{2}\sigma_2^2 a_2^2 t^2} \cdots e^{\mu_k a_k t + \frac{1}{2}\sigma_k^2 a_k^2 t^2}$$

$$= e^{\mu_1 a_1 t + \frac{1}{2}\sigma_1^2 a_1^2 t^2 + \mu_2 a_2 t + \frac{1}{2}\sigma_2^2 a_2^2 t^2 + \cdots + \mu_k a_k t + \frac{1}{2}\sigma_k^2 a_k^2 t^2}$$

$$= e^{\left(\sum_{i=1}^{k} a_i \mu_i\right) t + \frac{1}{2}\left(\sum_{i=1}^{k} a_i^2 \sigma_i^2\right) t^2}$$

which is the m.g.f of normal variate with mean, $\mu = \sum_{i=1}^{k} a_i \mu_i$ and variance, $\sigma^2 = \sum_{i=1}^{k} a_i^2 \sigma_i^2$.

Hence by uniqueness property of m.g.f, we have

$$\sum_{i=1}^{k} a_i x_i \sim N\left(\sum_{i=1}^{k} a_i \mu_i, \quad \sum_{i=1}^{k} a_i^2 \sigma_i^2\right)$$

Theorem: Show that mean and variance of standard normal variate are zero and one respectively.

Proof: Let $x \sim N(\mu, \sigma^2)$, then

$$E(x) = \mu \quad \text{and} \quad V(x) = \sigma^2$$

Now, standard normal variate

$$z = \frac{x-\mu}{\sigma}$$

We have to show that $E(z) = 0$ and $V(z) = 1$.

So, $z = \dfrac{x-\mu}{\sigma}$

$\Rightarrow x - \mu = \sigma\ z$

$\Rightarrow x = \mu + \sigma\ z$ ....(i)

$\Rightarrow E(x) = E(\mu + \sigma\ z)$

$\Rightarrow \mu = \mu + \sigma\ E(z)$

$\Rightarrow \sigma\ E(z) = \mu - \mu$

$\Rightarrow E(z) = \dfrac{0}{\sigma}$

$\therefore\ E(z) = 0$

Again, using (i) we can write

$V(x) = V(\mu + \sigma\ z)$

$\Rightarrow \sigma^2 = V(\mu) + \sigma^2\ V(z)$

$\Rightarrow \sigma^2 = 0 + \sigma^2\ V(z)$

$\Rightarrow V(z) = \dfrac{\sigma^2}{\sigma^2}$

$\therefore\ V(z) = 1$

So, it is proved that mean and variance of standard normal variate are zero and one respectively.

Theorem: If $x \sim N(0,\ 1)$ and $y \sim N(0,\ 1)$, then show that $x + y$ is independent of $x - y$. Also find their distribution.

Solution: Since $x \sim N(0, 1)$ and $y \sim N(0,\ 1)$, so

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}; \quad -\infty < x < \infty$$

and $$f(y) = \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}}; \quad -\infty < y < \infty$$

The joint density of x and y is

$$f(x,\, y) = \frac{1}{2\pi} e^{-\frac{1}{2}\left(x^2+y^2\right)}; \qquad -\infty < x < \infty, \quad -\infty < y < \infty$$

Let $u = x + y$

$v = x - y$

$\therefore \quad u + v = x + y + x - y = 2x$

$\Rightarrow x = \dfrac{u+v}{2}$

and $u - v = x + y - x + y = 2y$

$\Rightarrow y = \dfrac{u-v}{2}$

$$\therefore \quad J = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \dfrac{1}{2} & \dfrac{1}{2} \\ \dfrac{1}{2} & -\dfrac{1}{2} \end{vmatrix} = -\frac{1}{4} - \frac{1}{4} = 2\left(-\frac{1}{4}\right) = -\frac{1}{2}$$

$$\Rightarrow |J| = \left|-\frac{1}{2}\right| = \frac{1}{2}$$

The joint density of u and v are

$$f(u, v) = \frac{1}{2\pi}\; e^{-\frac{1}{2}\left\{\left(\frac{u+v}{2}\right)^2 + \left(\frac{u-v}{2}\right)^2\right\}}\; |J|$$

$$= \frac{1}{2\pi}\; e^{-\frac{1}{2}\left\{\frac{1}{4}\left(u^2+v^2+2uv\right) + \frac{1}{4}\left(u^2+v^2-2uv\right)\right\}} \cdot \frac{1}{2}$$

$$= \frac{1}{4\pi}\; e^{-\frac{1}{2}\left\{\frac{1}{4}\left(u^2+v^2+u^2+v^2\right)\right\}}$$

$$= \frac{1}{4\pi}\; e^{-\frac{1}{2}\left\{\frac{1}{4}\left(2u^2+2v^2\right)\right\}}$$

$$= \frac{1}{4\pi} e^{-\frac{1}{2}\left(\frac{u^2}{2}+\frac{v^2}{2}\right)}$$

$$= \frac{1}{4\pi}\; e^{-\frac{u^2}{4}} .\; e^{-\frac{v^2}{4}}$$

$$= \frac{1}{\sqrt{2}\;\sqrt{2\pi}}\; e^{-\frac{1}{2}\left(\frac{u-0}{\sqrt{2}}\right)^2} .\; \frac{1}{\sqrt{2}\;\sqrt{2\pi}}\; e^{-\frac{1}{2}\left(\frac{v-0}{\sqrt{2}}\right)^2}$$

$\therefore\;\; f(u, v) = f_1(u).\, f_2(v)$

So, $f_1(u)$ and $f_2(v)$ are independently distributed.

The pdf of u and v are

$$f_1(u) = \frac{1}{\sqrt{2}\;\sqrt{2\pi}}\; e^{-\frac{1}{2}\left(\frac{u-0}{\sqrt{2}}\right)^2}; \qquad u \sim N(0,\; 2)$$

and $$f_2(v) = \frac{1}{\sqrt{2}\;\sqrt{2\pi}}\; e^{-\frac{1}{2}\left(\frac{v-0}{\sqrt{2}}\right)^2}; \qquad v \sim N(0, 2)$$

NB: We know,

$$f(x;\; \mu, \sigma^2) = \frac{1}{\sigma\sqrt{2\pi}}\; e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} \text{ ....(i)}$$

and if $$f_1(u) = \frac{1}{\sqrt{2}\;\sqrt{2\pi}}\; e^{-\frac{1}{2}\left(\frac{u-0}{\sqrt{2}}\right)^2} \text{ ....(ii)}$$

In case of (i), $x \sim N(\mu, \sigma^2)$

Comparing (ii) with (i), we can write, $u \sim N(0, 2)$.

Theorem: If x and y are two independent standard normal variate, find the distribution of (i) x^2 and (ii) $x^2 + y^2$.

Solution: Since $x \sim N(0,1)$ and $y \sim N(0,1)$, so

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$$

$$f(y) = \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}}$$

(i) Now we have to find the distribution of x^2.

Let $u = x^2$

$$\Rightarrow \frac{du}{dx} = 2x$$

$$\Rightarrow dx = \frac{du}{2x}$$

$$\Rightarrow dx = \frac{du}{2x}$$

$$\therefore \frac{dx}{du} = \frac{1}{2\sqrt{u}} \qquad [\because u = x^2]$$

So, the distribution of u is

$$f(u) = f(x)\,|J|$$

$$= \frac{2}{\sqrt{2\pi}} e^{-\frac{u}{2}} . \frac{1}{2\sqrt{u}}; \qquad 0 < u < \infty$$

$$= \frac{1}{\sqrt{2\pi}} e^{-\frac{u}{2}} . u^{-\frac{1}{2}}$$

$$\therefore \quad f\left(x^2\right) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} . (x^2)^{-\frac{1}{2}}; \qquad 0 < x^2 < \infty$$

(ii) We have to find the distribution of $x^2 + y^2$

Now, $M_{x^2}(t) = E\left(e^{tx^2}\right)$

$$= \int_0^\infty e^{tx^2} f(x^2) dx^2$$

$$= \int_0^\infty e^{tx^2} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} . (x^2)^{-\frac{1}{2}} dx^2$$

$$= \frac{1}{\sqrt{2\pi}} \int_0^\infty e^{-\left(\frac{1}{2}-t\right)x^2} (x^2)^{-\frac{1}{2}} dx^2$$

$$= \frac{1}{\sqrt{2\pi}} \int_0^{\infty} \frac{\left(\frac{1}{2}-t\right)^{\frac{1}{2}} e^{-\left(\frac{1}{2}-t\right)x^2} \left(x^2\right)^{\frac{1}{2}-1}}{\left(\frac{1}{2}-t\right)^{\frac{1}{2}}} dx^2$$

$$= \frac{1}{\sqrt{2\pi}} \times \frac{\Gamma\frac{1}{2}}{\left(\frac{1}{2}-t\right)^{\frac{1}{2}}}$$

$$= \frac{1}{\sqrt{2}\sqrt{\pi}} \times \frac{\sqrt{\pi}}{\left(\frac{1-2t}{2}\right)^{\frac{1}{2}}}$$

$$= \frac{1}{\sqrt{2}} \times \frac{2^{\frac{1}{2}}}{(1-2t)^{\frac{1}{2}}}$$

$$\therefore \quad M_{x^2}(t) = \left(1-2t\right)^{-\frac{1}{2}}$$

Similarly, $M_{y^2}(t) = \left(1-2t\right)^{-\frac{1}{2}}$

Now, $M_{x^2+y^2}(t) = M_{x^2}(t).\ M_{y^2}(t) \quad (\because$ x and y are independent)

$$= \left(1-2t\right)^{-\frac{1}{2}}.\ \left(1-2t\right)^{-\frac{1}{2}}$$

$$= \left(1-2t\right)^{-1}$$

$$= \left(1-\frac{t}{\frac{1}{2}}\right)^{-1}$$

which is the m.g.f of gamma distribution with parameters 1 and $\frac{1}{2}$.

So, the distribution of x^2+y^2 is

$$f\left(x^2+y^2\right) = \frac{\left(\frac{1}{2}\right) e^{-\frac{1}{2}\left(x^2+y^2\right)} \left(x^2+y^2\right)^{1-1}}{\Gamma 1}$$

$$= \frac{1}{2} e^{-\frac{1}{2}\left(x^2+y^2\right)}; \quad 0 < x^2,\ y^2 < \infty$$

NB: (a) The pdf of Gamma distribution is

$$f(x;\ \alpha, \beta) = \frac{\beta^{\alpha}\ e^{-\beta x}\ x^{\alpha-1}}{\lceil\alpha};\qquad 0 < x < \infty$$

(b) $\int_0^{\infty} f(x;\ \alpha, \beta)\, dx = 1$

$$\Rightarrow \int_0^{\infty} \frac{\beta^{\alpha}\ e^{-\beta x}\ x^{\alpha-1}}{\lceil\alpha}\, dx = 1$$

$$\Rightarrow \int_0^{\infty} \beta^{\alpha}\ e^{-\beta x}\ x^{\alpha-1}\, dx = \lceil\alpha$$

(c) The m.g.f of Gamma distribution is

$$M_x(t) = \left(1 - \frac{t}{\beta}\right)^{-\alpha} \text{ with parameter } \alpha \text{ and } \beta.$$

(d) The pdf of standard Gamma distribution is

$$f(x;\alpha) = \frac{e^{-x}\ x^{\alpha-1}}{\lceil\alpha};\qquad 0 < x < \infty$$

Q. Discuss the importance of normal distribution in statistics (NU-11, 14)

Or, Write the uses of normal distribution (NU-16)

Ans: Normal distribution plays central role in the theory of statistics. Now we shall cite some of its important applications or uses such as-

(i) Under certain conditions most of the probability and sampling distributions can be approximated by normal distribution.

(ii) Central limit theorem proves that the distribution of most of the standardised variate turns into normal distribution as the sample size tends to infinity. This theorem plays a central role in the theorem of probability.

(iii) Normal distribution is the basis of all the sampling distribution. Without the assumption of normality, sampling distributions have no existence.

(iv) Assumption of normality is the basis of all the test of significance in applied statistics.

(v) Normal distribution finds its application in industrial statistics such as quality control.

Short questions and answers:

(1) Write down the probability density function of normal distribution (NU-11)

Ans: The probability density function of normal distribution with parameters μ and σ^2 is

$$f\left(x;\,\mu,\,\sigma^2\right)=\frac{1}{\sigma\sqrt{2\pi}}\;e^{-\frac{(x-\mu)^2}{2\sigma^2}}\;;\qquad -\infty<x<\infty$$

(2) What is the mean and variance of standard normal variate? (NU-11)

Ans: The mean and variance of standard normal variate are 0 and 1 respectively.

(3) What is the characteristic function of normal distribution? (NU-11)

Ans: The characteristic function of normal distribution is

$$Q_x(t)=e^{i\mu t+\frac{1}{2}i^2\,\sigma^2\,t^2}$$

(4) What is the median and mode of normal distribution? (NU-13)

Ans: The median and mode of normal distribution are μ.

(5) Write down the probability density function of standard normal distribution (NU-13)

Ans: The probability density function of standard normal distribution is

$$f\left(z;\;0,\,1\right)=\frac{1}{\sqrt{2\pi}}\;e^{-\frac{1}{2}z^2}\;;\qquad -\infty<z<\infty$$

(6) Write down the relation between mean deviation and standard deviation of normal distribution (NU-14)

Ans: The relation between mean deviation and standard deviation of normal distribution is

$$MD_{(\text{about }\mu)}=\sigma\,\sqrt{\frac{2}{\pi}}$$

(7) What is the mean deviation of normal distribution (NU-12)

Ans: The mean deviation of normal distribution is

$$MD_{(\text{about } \mu)} = \sigma \sqrt{\frac{2}{\pi}}$$

(8) Write down the moment generating function of normal distribution (NU-12)

Ans: The moment generating function for normal distribution is

$$M_x(t) = e^{\mu t + \frac{1}{2}\sigma^2 t^2}$$

(9) What is the value of 5th central moment of normal distribution? (NU-15)

Ans: The value of 5th central moment of normal distribution is zero.

(10) What is the relation among $\overline{X}$, Me and Mo of normal distribution? (NU-15)

Ans: The relation among $\overline{X}$, Me and Mo of normal distribution is

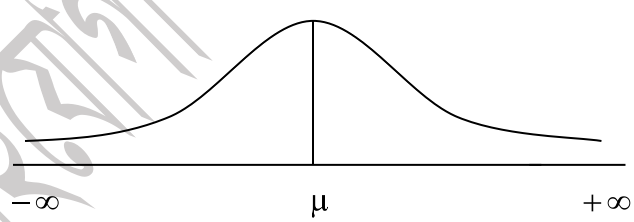
$$\overline{X} = Me = M_0 = \mu$$

(11) What are the values of mean, median and mode of normal distribution? (NU-16)

Ans: The values of mean, median and mode of normal distribution are μ.

(12) What is normal curve? (NU-15)

Ans: The following graph is known as normal curve.



Normal curve

Where μ indicates the mean, median and mode of normal distribution.

Mathematical Problems:

(1) Using normal probability table, find the probabilities (i) $P(z < 1.5)$ **and (ii)** $P(z > 2.4)$**.**

Solution: (i) $P(z < 1.5)$

$$= P(-\infty \le z \le 0) + P(0 \le z \le 1.5)$$

$= 0.5 + 0.4332$

$= 0.9332$

(ii) $P(z > 2.4)$

$= P(0 \le z \le \infty) - P(0 \le z \le 2.4)$

$= 0.5 - 0.4918$

$= 0.0082$

NB: We can find the value of $P(z < 1.5)$ **through calculator by the following way-**

For Calculator Model- $fx - 991$ ES PLUS **:**

(a) At first, we have to go to STAT Mode by pressing Mode $\rightarrow 3 \rightarrow$ AC

(b) Then press Shift $\rightarrow 1 \rightarrow 5 \rightarrow 1 \rightarrow P(1.5) \rightarrow =$, then we get the value 0.93319

Where, $P(z \le 1.5)$ indicates the summation of the probabilities of the values from $-\infty$ to 1.5.

(c) Similarly, we can get $P(0 \le z \le 1.5)$.

Just press Shift $\rightarrow 1 \rightarrow 5 \rightarrow 2 \rightarrow Q(1.5) \rightarrow =$, then we get the value 0.43319

(d) Again, we have to find $P(z > 2.4)$

Then press Shift $\rightarrow 1 \rightarrow 5 \rightarrow 3 \rightarrow R(2.4) \rightarrow =$, then we get the value 0.0081975

For Calculator Model- $fx - 100MS$ **:**

(a) At first, we have to go to SD Mode.

(b) Then Press Shift $\rightarrow 3 \rightarrow 1 \rightarrow P(1.5) \rightarrow =$, then we get the value 0.93319

Where, $P(z \le 1.5)$ indicates the summation of the probabilities of the values from $-\infty$ to 1.5.

Remarks: (i)P-191**,** উচ্চ মাধ্যমিক পরিসংখ্যান (২য় পত্র), খান মোহাম্মদ শরীফ and Page-675, Siddiqur Rahman.

(ii) চিত্রের সাহায্যে প্রাপ্ত ও প্রত্যাশিত গণসংখ্যা ঘটনসংখ্যা উপস্থাপন- Page-193**,** উচ্চ মাধ্যমিক পরিসংখ্যান (২য় পত্র), খান মোহাম্মদ শরীফ।

(2) Suppose that the growth in inches during the tenth year of life of Bangladeshi boy is a normal random variable with mean 2 inches and standard deviation 1 inch. Find the probability that a randomly selected boy will grow.

(a) between 1 and 2 inches in his tenth year.

(b) more than 3 inches in his tenth year.

(c) at least 1 inch in his tenth year and

(d) less than 1 inch in his tenth year.

Solution: Let x be the normal variate with mean 2 inches and variance 1 inch i.e.,

$$\mu = 2 \text{ and } \sigma = 1$$

(a) We have to find

$$P(1 \le x \le 2)$$

$$= P\left(\frac{1-\mu}{\sigma} \le \frac{x-\mu}{\sigma} \le \frac{2-\mu}{\sigma}\right)$$

$$= P\left(\frac{1-2}{1} \le z \le \frac{2-2}{1}\right)$$

$$= P(-1 \le z \le 0)$$

$$= P(0 \le z \le 1)$$

$$= 0.3413$$

(b) $P(x > 3)$

$$= P\left(\frac{x-\mu}{\sigma} > \frac{3-\mu}{\sigma}\right)$$

$$= P\left(z > \frac{3-2}{1}\right)$$

$$= P(z > 1)$$

$$= P(0 \le z \le \infty) - P(0 \le z \le 1)$$

$$= 0.5 - 0.3413$$

$$= 0.1587$$

(c) $P(x \ge 1)$

$$= P\left(\frac{x-\mu}{\sigma} \geq \frac{1-\mu}{\sigma}\right)$$

$$= P\left(z > \frac{1-2}{1}\right)$$

$$= P(z > -1)$$

$$= P(-1 \leq z \leq 0) + P(0 \leq z \leq \infty)$$

$$= P(0 \leq z \leq 1) + 0.5$$

$$= 0.3413 + 0.5$$

$$= 0.8413$$

(d) $P(x < 1)$

$$= P\left(\frac{x-\mu}{\sigma} < \frac{1-\mu}{\sigma}\right)$$

$$= P\left(z < \frac{1-2}{1}\right)$$

$$= P(z < -1)$$

$$= P(-\infty \leq z \leq 0) - P(-1 \leq z \leq 0)$$

$$= P(0 \leq z \leq \infty) - P(0 \leq z \leq 1)$$

$$= 0.5 - 0.3413$$

$$= 0.1587$$

Exponential (অস্কাভিসারী) Distribution

Q. Define exponential distribution (NU-13)

Ans: A continuous random variable X is said to have exponential distribution if its probability density function is given by

$$f(x;\alpha)=\alpha\ e^{-\alpha x};\ \ x>0,\ \ \alpha>0$$

where α is the only parameter of the distribution.

NB: The pdf of exponential distribution is

$$f(x;\alpha)=\alpha e^{-\alpha x};\ \ x>0,\ \ \alpha>0 \quad ....(i)$$

(i) If $\alpha=1$ in (i), then we get

$$f(x)=e^{-x};\ \ x>0$$

which is called the standard exponential distribution.

(ii) Sometimes exponential distribution is defined as

$$f(x;\ \alpha)=\frac{1}{\alpha}e^{-\frac{1}{\alpha}x};\qquad x>0,\ \ \alpha>0$$

$$\text{Or,}\quad f(x;\ \theta)=\frac{1}{\theta}\ e^{-\frac{1}{\theta}x};\qquad x>0,\ \ \theta>0$$

(iii) Write down the density of exponential distribution with mean $\frac{1}{\theta}$.

Ans: The density of exponential distribution with mean $\frac{1}{\theta}$ is

$$f(x;\ \theta)=\theta\ e^{-\theta x};\qquad x>0,\quad \theta>0$$

(iv) Write down the density of exponential distribution with mean θ.

Ans: The density of exponential distribution with mean θ is

$$f(x;\ \theta)=\frac{1}{\theta}\ e^{-\frac{1}{\theta}x};\qquad x>0,\ \ \theta>0$$

Q. Find the m.g.f of exponential distribution and hence find mean, variance, β_1 and β_2 .

Or, Find moment generating function of exponential distribution and hence or otherwise find its mean and variance **(NU-11)**

Or, Find the cumulant generating function of exponential distribution. Find the β_1 and β_2 of exponential distribution **(NU-12)**

Or, Find mean and variance of exponential distribution **(NU-16)**

Ans: The probability density function of exponential distribution with parameter α is

$$f(x) = \alpha\, e^{-\alpha x}; \qquad x > 0, \quad \alpha > 0$$

The m.g.f. is,

$$M_x(t) = E\left(e^{tx}\right)$$

$$= \int_0^\infty e^{tx}\, f(x)\, dx$$

$$= \int_0^\infty e^{tx}.\ \alpha\, e^{-\alpha x}\, dx$$

$$= \alpha \int_0^\infty e^{-(\alpha - t)x}\, dx$$

$$= \alpha \left[\frac{e^{-(\alpha-t)x}}{-(\alpha - t)}\right]_0^\infty$$

$$= -\frac{\alpha}{\alpha - t}\left[e^{-(\alpha-t)x}\right]_0^\infty$$

$$= -\frac{\alpha}{\alpha - t}\left[e^{-\infty} - e^{-0}\right]$$

$$= -\frac{\alpha}{\alpha - t}\left[\frac{1}{e^\infty} - \frac{1}{e^0}\right]$$

$$= -\frac{\alpha}{\alpha\left(1 - \frac{t}{\alpha}\right)}\left[\frac{1}{\infty} - \frac{1}{1}\right] \qquad [\because\ e^\infty = \infty]$$

$$= -\frac{1}{1 - \frac{t}{\alpha}}\ [0 - 1]$$

$$\therefore \quad M_x(t) = \left(1 - \frac{t}{\alpha}\right)^{-1}$$

Now, cumulant generating function

$$K_x(t) = \log M_x(t)$$

$$= \log\left(1 - \frac{t}{\alpha}\right)^{-1}$$

$$= -\log\left(1 - \frac{t}{\alpha}\right)$$

$$= \frac{t}{\alpha} + \frac{t^2}{2\,\alpha^2} + \frac{t^3}{3\,\alpha^3} + \frac{t^4}{4\,\alpha^4} + \cdots$$

$$= \frac{1}{\alpha} \cdot \frac{t}{1!} + \frac{1}{\alpha^2} \cdot \frac{t^2}{2!} + \frac{2}{\alpha^3} \cdot \frac{t^3}{3!} + \frac{6}{\alpha^4} \cdot \frac{t^4}{4!} + \cdots$$

$K_r =$ coefficient of $\frac{t^r}{r!}$ in $K_x(t)$

Putting $r = 1, 2, 3, 4$ we get respectively

$K_1 =$ coefficient of $\frac{t}{1!}$ in $K_x(t) = \frac{1}{\alpha}$

$K_2 =$ coefficient of $\frac{t^2}{2!}$ in $K_x(t) = \frac{1}{\alpha^2}$

$K_3 =$ coefficient of $\frac{t^3}{3!}$ in $K_x(t) = \frac{2}{\alpha^3}$

$K_4 =$ coefficient of $\frac{t^4}{4!}$ in $K_x(t) = \frac{6}{\alpha^4}$

Now, Mean, $\mu_1' = K_1 = \frac{1}{\alpha}$

Variance, $\mu_2 = K_2 = \frac{1}{\alpha^2}$

$$\mu_3 = K_3 = \frac{2}{\alpha^3}$$

and $\mu_4 = K_4 + 3\,K_2^{\,2} = \frac{6}{\alpha^4} + 3.\,\frac{1}{\alpha^4} = \frac{9}{\alpha^4}$

Hence, $\beta_1 = \dfrac{\mu_3^{\,2}}{\mu_2^{\,3}} = \dfrac{4}{\alpha^6} \times \dfrac{\alpha^6}{1} = 4$

$$\beta_2 = \frac{\mu_4}{\mu_2^{\,2}} = \frac{9}{\alpha^4} \times \frac{\alpha^4}{1} = 9$$

Since $\beta_1 = 4 > 0$, so the distribution is positively skewed and $\beta_2 = 9 > 3$, so the distribution is leptokurtic.

NB:

(i) Mean, $\mu_1' = \dfrac{1}{\alpha}$ and Variance, $\mu_2 = \dfrac{1}{\alpha^2} \Rightarrow SD = \dfrac{1}{\alpha}$. Therefore, Mean and Standard deviation are equal i.e., Mean $= SD = \dfrac{1}{\alpha}$.

(ii) $\log(1-x) = -x - \dfrac{x^2}{2} - \dfrac{x^3}{3} - \dfrac{x^4}{4} - \cdots$

(iii) $\int e^{-(\alpha - t)x} \, dx$ ....(i)

Let, $z = -(\alpha - t)x$

$$\Rightarrow \frac{dz}{dx} = -(\alpha - t) \Rightarrow \frac{dz}{-(\alpha - t)} = dx$$

So (i) can be written as,

$$\int e^{-(\alpha-t)x} dx = \int e^z . \frac{dz}{-(\alpha - t)} = \frac{1}{-(\alpha - t)} \int e^z dz = \frac{e^z}{-(\alpha - t)} = \frac{e^{-(\alpha-t)x}}{-(\alpha - t)}$$

Q. Find the Median of exponential distribution (NU-11, 13, 15)

Ans: The pdf of exponential distribution with parameter α is

$$f(x;\alpha) = \alpha\, e^{-\alpha x}; \qquad x > 0, \quad \alpha > 0$$

Let m be the median of the distribution, then by definition

$$\int_0^m f(x)\, dx = \frac{1}{2}$$

$$\Rightarrow \int_0^m \alpha\, e^{-\alpha x}\, dx = \frac{1}{2}$$

$$\Rightarrow \alpha \int_0^m e^{-\alpha x}\, dx = \frac{1}{2}$$

$$\Rightarrow \alpha \left[-\frac{e^{-\alpha x}}{\alpha} \right]_0^m = \frac{1}{2}$$

$$\Rightarrow -\left[e^{-\alpha x}\right]_0^m = \frac{1}{2}$$

$$\Rightarrow -\left[e^{-\alpha m} - e^{-0}\right] = \frac{1}{2}$$

$$\Rightarrow 1 - e^{-\alpha m} = \frac{1}{2}$$

$$\Rightarrow e^{-\alpha m} = 1 - \frac{1}{2} = \frac{1}{2}$$

$$\Rightarrow \log e^{-\alpha m} = \log \frac{1}{2}$$

$$\Rightarrow -\alpha\, m = \log \frac{1}{2}$$

$$\Rightarrow m = -\frac{1}{\alpha} \log \frac{1}{2}$$

$$= \frac{1}{\alpha} \log \left(\frac{1}{2}\right)^{-1}$$

$$= \frac{1}{\alpha} \log \left(\frac{1}{\frac{1}{2}} \right)$$

$$\therefore \quad m = \frac{1}{\alpha} \log 2$$

Therefore, Median is $\frac{1}{\alpha} \log 2$

Q. Find the mean deviation from mean of exponential distribution (NU-15)

Ans: The pdf of exponential distribution with parameter α is

$$f(x;\ \alpha)=\alpha\ e^{-\alpha x};\qquad x>0,\quad \alpha>0$$

Now the mean deviation from mean of the distribution is

$$\mathrm{MD}=E\left[\,|x-\text{mean}|\,\right]$$

$$=\int_0^\infty |x-\text{mean}|\ \ f(x)\,dx$$

$$=\int_0^\infty \left|x-\frac{1}{\alpha}\right|.\ \alpha\, e^{-\alpha x}\ dx \qquad \left[\because\ \text{Mean}=\frac{1}{\alpha}\right]$$

$$=\alpha\int_0^\infty \left|x-\frac{1}{\alpha}\right|\ e^{-\alpha x}\ dx$$

$$=\alpha\left[\int_0^{\frac{1}{\alpha}}\left|x-\frac{1}{\alpha}\right| e^{-\alpha x}\ dx+\int_{\frac{1}{\alpha}}^\infty \left|x-\frac{1}{\alpha}\right| e^{-\alpha x}\ dx\right]$$

$$=2\alpha\int_{\frac{1}{\alpha}}^\infty \left(x-\frac{1}{\alpha}\right)\ e^{-\alpha x}\ dx$$

$$=2\alpha\int_{\frac{1}{\alpha}}^\infty x\ e^{-\alpha x}\ dx-2\alpha\ \frac{1}{\alpha}\int_{\frac{1}{\alpha}}^\infty e^{-\alpha x}\ dx$$

$$=2\alpha\int_{\frac{1}{\alpha}}^\infty x\ e^{-\alpha x}\ dx-2\int_{\frac{1}{\alpha}}^\infty e^{-\alpha x}\ dx$$

$$=2\alpha\left[-\frac{1}{\alpha}\ x\ e^{-\alpha x}\right]_{\frac{1}{\alpha}}^\infty-2\alpha\int_{\frac{1}{\alpha}}^\infty -\frac{1}{\alpha}e^{-\alpha x}\ dx-2\int_{\frac{1}{\alpha}}^\infty e^{-\alpha x}\ dx$$

$$=-2\left[x\ e^{-\alpha x}\right]_{\frac{1}{\alpha}}^\infty+2\int_{\frac{1}{\alpha}}^\infty e^{-\alpha x}\ dx-2\int_{\frac{1}{\alpha}}^\infty e^{-\alpha x}\ dx$$

$$=-2\ \left[x\ e^{-\alpha x}\right]_{\frac{1}{\alpha}}^\infty$$

$$=-2\ \left[\lim_{x\to\infty} x\ e^{-\alpha x}-\lim_{x\to\frac{1}{\alpha}} x\ e^{-\alpha x}\right]$$

$$= -2\left[\lim_{x\to\infty} \frac{x}{e^{\alpha x}} - \frac{1}{\alpha}\, e^{-\alpha \frac{1}{\alpha}}\right]$$

$$= -2\left[\lim_{x\to\infty} \frac{x}{\sum_{r=0}^{\infty} \frac{(\alpha x)^r}{r!}} - \frac{1}{\alpha e}\right]$$

$$= -2\left[\lim_{x\to\infty} \frac{1}{\sum_{r=0}^{\infty} \frac{\alpha^r\ x^{r-1}}{r!}} - \frac{1}{\alpha e}\right]$$

$$= -2\left[\frac{1}{\infty} - \frac{1}{\alpha e}\right]$$

$$= 0 + \frac{2}{\alpha\, e}$$

$$= \frac{2}{\alpha e}$$

NB: P-756, Chapter: Gamma and Beta function, **Book**: Advanced calculus I.

Alternative method:

The pdf of exponential distribution with parameter α is

$$f(x;\ \alpha) = \alpha\ e^{-\alpha x};\quad x > 0,\quad \alpha > 0$$

Now the mean deviation from mean of the distribution is

$$MD = E\left[|x - \text{mean}|\right]$$

$$= \int_0^{\infty} |x - \text{mean}|\ f(x)\, dx$$

$$= \int_0^{\infty} \left|x - \frac{1}{\alpha}\right| .\ \alpha\, e^{-\alpha x} dx \qquad \left[\because \text{ Mean, } \mu_1' = \frac{1}{\alpha}\right]$$

$$= \alpha \int_0^{\infty} \left|x - \frac{1}{\alpha}\right|\ e^{-\alpha x}\ dx$$

$$= \alpha \int_0^{\frac{1}{\alpha}} \left(\frac{1}{\alpha} - x\right) e^{-\alpha x}\ dx + \alpha \int_{\frac{1}{\alpha}}^{\infty} \left(x - \frac{1}{\alpha}\right) e^{-\alpha x}\ dx$$

$$= \alpha \int_{0}^{\frac{1}{\alpha}} \frac{1}{\alpha} e^{-\alpha x} \, dx - \alpha \int_{0}^{\frac{1}{\alpha}} x \, e^{-\alpha x} \, dx + \alpha \int_{\frac{1}{\alpha}}^{\infty} x \, e^{-\alpha x} \, dx - \alpha \int_{\frac{1}{\alpha}}^{\infty} \frac{1}{\alpha} e^{-\alpha x} \, dx$$

$$= \int_{0}^{\frac{1}{\alpha}} e^{-\alpha x} dx - \alpha \left[-\frac{1}{\alpha} x e^{-\alpha x} \right]_{0}^{\frac{1}{\alpha}} + \alpha \int_{0}^{\frac{1}{\alpha}} -\frac{1}{\alpha} e^{-\alpha x} \, dx + \alpha \left[-\frac{1}{\alpha} x e^{-\alpha x} \right]_{\frac{1}{\alpha}}^{\infty} - \alpha \int_{\frac{1}{\alpha}}^{\infty} -\frac{1}{\alpha} e^{-\alpha x} \, dx - \int_{\frac{1}{\alpha}}^{\infty} e^{-\alpha x} \, dx$$

$$= \int_{0}^{\frac{1}{\alpha}} e^{-\alpha x} \, dx + \left[x \, e^{-\alpha x} \right]_{0}^{\frac{1}{\alpha}} - \int_{0}^{\frac{1}{\alpha}} e^{-\alpha x} \, dx - \left[x \, e^{-\alpha x} \right]_{\frac{1}{\alpha}}^{\infty} + \int_{\frac{1}{\alpha}}^{\infty} e^{-\alpha x} \, dx - \int_{\frac{1}{\alpha}}^{\infty} e^{-\alpha x} \, dx$$

$$= \left[x \, e^{-\alpha x} \right]_{0}^{\frac{1}{\alpha}} - \left[x \, e^{-\alpha x} \right]_{\frac{1}{\alpha}}^{\infty}$$

$$= \left[\frac{1}{\alpha} e^{-1} - 0 \right] - \left[0 - \frac{1}{\alpha} e^{-1} \right]$$

$$= \frac{1}{\alpha} e^{-1} + \frac{1}{\alpha} e^{-1}$$

$$= \frac{2}{\alpha e}$$

NB: (i) $MD = E\left[|x - \text{mean}| \right] = \begin{cases} \sum |x - \text{mean}| \, f(x); & \text{if x is discrete} \\ \int_{a}^{b} |x - \text{mean}| \, f(x) \, dx; & \text{if x is continuous} \end{cases}$

(ii) $\int uv dx = u \int v dx - \int \left\{ \frac{d}{dx} u \int v dx \right\} dx$

(iii) $\alpha \int x \, e^{-\alpha x} \, dx = \alpha \left[x \int e^{-\alpha x} \, dx - \int \left\{ \frac{d}{dx} x \int e^{-\alpha x} \, dx \right\} dx \right]$

$$= \alpha \left(-\frac{1}{\alpha} x \, e^{-\alpha x} \right) - \alpha \int \left(-\frac{1}{\alpha} \, e^{-\alpha x} \right) dx$$

P-504, 505, M. K. Roy (New book)

Q. The pdf of exponential distribution is $f(x;\,\theta)=\frac{1}{\theta}\,e^{-\frac{x}{\theta}};\quad x>0,\quad \theta>0$**. Find** $M_x(t),\ K_x(t)$**, Mean, Variance,** β_1 **and** β_2**.**

Or, If $f(x)=\frac{1}{\theta}e^{-\frac{x}{\theta}};\quad 0<x<\infty$ then what is the mean of variable x? (NU-13)

Solution: The given pdf of exponential distribution is

$$f(x;\,\theta)=\frac{1}{\theta}e^{-\frac{x}{\theta}};\quad x>0,\quad \theta>0$$

By definition,

$$M_x(t)=E\left(e^{tx}\right)$$

$$=\int_0^\infty e^{tx}\ f(x)\,dx$$

$$=\int_0^\infty e^{tx}.\ \frac{1}{\theta}\,e^{-\frac{x}{\theta}}\,dx$$

$$=\int_0^\infty \frac{1}{\theta}\,e^{-\frac{x}{\theta}+tx}dx$$

$$=\frac{1}{\theta}\int_0^\infty e^{-\left(\frac{1}{\theta}-t\right)x}\,dx$$

$$=\frac{1}{\theta}\left[\frac{e^{-\left(\frac{1}{\theta}-t\right)x}}{-\left(\frac{1}{\theta}-t\right)}\right]_0^\infty$$

$$=-\frac{1}{\theta\left(\frac{1}{\theta}-t\right)}\left[e^{-\left(\frac{1}{\theta}-t\right)x}\right]_0^\infty$$

$$=-\frac{1}{\theta\left(\frac{1-\theta t}{\theta}\right)}\left[e^{-\infty}-e^{-0}\right]$$

$$=-\frac{1}{1-\theta t}\left[0-1\right]$$

$$\therefore \quad M_x(t) = (1-\theta\, t)^{-1}$$

Now, cumulant generating function,

$$K_x(t) = \log M_x(t)$$

$$= \log(1-\theta\, t)^{-1}$$

$$= -\log(1-\theta\, t)$$

$$= \theta\, t + \frac{\theta^2 t^2}{2} + \frac{\theta^3 t^3}{3} + \frac{\theta^4 t^4}{4} + \cdots$$

$$= \theta.\, \frac{t}{1!} + \theta^2.\, \frac{t^2}{2!} + 2\theta^3.\, \frac{t^3}{3!} + 6\theta^4.\, \frac{t^4}{4!} + \cdots$$

$$K_r = \text{coefficient of } \frac{t^r}{r!} \text{ in } K_x(t)$$

Putting $r = 1, 2, 3, 4$ we get respectively

$$K_1 = \text{coefficient of } \frac{t}{1!} \text{ in } K_x(t) = \theta$$

$$K_2 = \text{coefficient of } \frac{t^2}{2!} \text{ in } K_x(t) = \theta^2$$

$$K_3 = \text{coefficient of } \frac{t^3}{3!} \text{ in } K_x(t) = 2\theta^3$$

$$K_4 = \text{coefficient of } \frac{t^4}{4!} \text{ in } K_x(t) = 6\theta^4$$

Now, Mean, $\mu_1' = K_1 = \theta$

Variance, $\mu_2 = K_2 = \theta^2$

$$\mu_3 = K_3 = 2\theta^3$$

and $\mu_4 = K_4 + 3\,K_2^{\,2} = 6\theta^4 + 3\theta^4 = 9\theta^4$

Hence, $\beta_1 = \dfrac{\mu_3^{\,2}}{\mu_2^{\,3}} = \dfrac{4\theta^6}{\theta^6} = 4$

$$\beta_2 = \frac{\mu_4}{\mu_2^{\,2}} = \frac{9\theta^4}{\theta^4} = 9$$

Since $\beta_1 = 4 > 0$, so the distribution is positively skewed and $\beta_2 = 9 > 3$, so the distribution is leptokurtic.

Theorem: Show that the exponential distribution "lacks memory" that is if X has an exponential distribution, then for every constant $a \geq 0$ one has

$$P\left[Y \leq x \mid X \geq a\right] = P[X \leq x] \textbf{ for all x, where } Y = X - a$$

Solution: The pdf of exponential distribution with parameter α is

$$f(x;\alpha) = \alpha\, e^{-\alpha x}; \qquad x > 0, \quad \alpha > 0$$

Now, $P\left[Y \leq x \middle| X \geq a\right] = \dfrac{P[Y \leq x \cap X \geq a]}{P[X \geq a]}$ ....(i)

We have, $P[Y \leq x \cap X \geq a] = P[X - a \leq x \cap X \geq a] \quad [\because Y = X - a]$

$$= P[X \leq a + x \cap X \geq a]$$

$$= P[a \leq X \leq a + x]$$

$$= \int_a^{a+x} f(x)\, dx$$

$$= \int_a^{a+x} \alpha\, e^{-\alpha x}\, dx$$

$$= \alpha \left[\frac{e^{-\alpha x}}{-\alpha}\right]_a^{a+x}$$

$$= -\left[e^{-\alpha x}\right]_a^{a+x}$$

$$= -\left[e^{-\alpha(a+x)} - e^{-\alpha a}\right]$$

$$= e^{-\alpha a} - e^{-\alpha a}\, e^{-\alpha x}$$

$$\because \quad P[Y \leq x \cap X \geq a] = e^{-\alpha a}\left(1 - e^{-\alpha x}\right)$$

and $P[X \geq a] = \int_a^{\infty} f(x)\, dx$

$$= \int_a^{\infty} \alpha\, e^{-\alpha x}\, dx$$

$$= \alpha \left[\frac{e^{-\alpha x}}{-\alpha} \right]_a^{\infty}$$

$$= -\left[e^{-\alpha x} \right]_a^{\infty}$$

$$= -\left[e^{-\infty} - e^{-\alpha a} \right]$$

$$= -\left[0 - e^{-\alpha a} \right]$$

$$= e^{-\alpha a}$$

$$\therefore \quad P[X \geq a] = e^{-\alpha a}$$

So (i) can be written as

$$P\left[Y \leq x \,\middle|\, X \geq a \right] = \frac{e^{-\alpha a}\left(1 - e^{-\alpha x}\right)}{e^{-\alpha a}}$$

$$= 1 - e^{-\alpha x} \quad \text{...............(ii)}$$

Also, $P[X \leq x] = \int_0^{x} f(x)\, dx$

$$= \int_0^{x} \alpha\, e^{-\alpha x}\, dx$$

$$= \alpha \left[\frac{e^{-\alpha x}}{-\alpha} \right]_0^{x}$$

$$= -\left[e^{-\alpha x} \right]_0^{x}$$

$$= -\left[e^{-\alpha x} - e^{-0} \right]$$

$$= -e^{-\alpha x} + 1$$

$$\therefore \quad P[X \leq x] = 1 - e^{-\alpha x} \quad \text{...............(iii)}$$

Hence from (ii) and (iii), we get

$$P\left[Y \leq x \,\middle|\, X \geq a \right] = P\left[X \leq x \right]$$

That is exponential distribution lacks memory.

Short questions and answers:

(1) Define exponential distribution (NU-11)

Ans: A continuous random variable x is said to have exponential distribution if its probability density function is given by

$$f(x;\ \alpha) = \alpha\ e^{-\alpha x}; \qquad x > 0,\ \ \alpha > 0$$

(2) If $f(x) = \frac{1}{\theta} e^{-\frac{x}{\theta}};\ 0 < x < \infty$ **then what is the mean of variable x? (NU-11)**

Ans: If $f(x) = \frac{1}{\theta} e^{-\frac{x}{\theta}};\ 0 < x < \infty$ then the mean of variable x is θ.

(3) Write down the probability density function of exponential distribution with mean 4 (NU-14)

Ans: The probability density function of exponential distribution with mean 4 is

$$f(x;\ 4) = \frac{1}{4}\ e^{-\frac{1}{4}x}; \qquad x > 0$$

Continuous Uniform Distribution

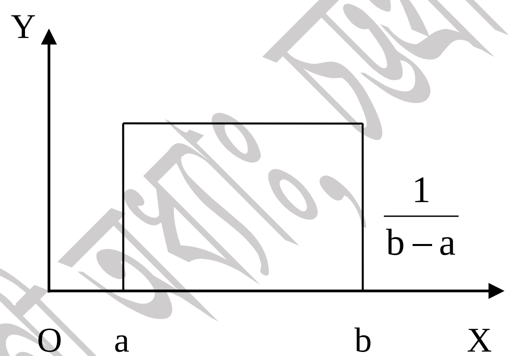
Q. Define continuous uniform distribution (NU-12)

Ans: A continuous random variable X is said to have a uniform distribution if its probability density function is defined by

$$f(x;\, a, b) = \frac{1}{b-a}; \quad a \le x \le b$$

Where a and b are the two parameters of the distribution such that $-\infty \le a \le b \le \infty$

The figure given below gives the probability density function of a uniform distribution over the interval $[a,\, b]$.



The distribution is known as uniform distribution since the probability density function over the whole interval $[a,\, b]$ is uniform or constant.

Again, the distribution is known as rectangular distribution, because the shape of the distribution is rectangular.

Q. Find the r-th raw moment about origin and hence find mean, variance, β_1 and β_2.

Or, Find mean and variance of continuous uniform distribution (NU-12)

Ans: We know the pdf of uniform distribution with parameters a and b is

$$f(x) = \frac{1}{b-a}; \quad a \le x \le b$$

By definition,

$$\mu_r' = E\left(x^r\right)$$

$$= \int_a^b x^r\ f(x)\ dx$$

$$= \int_a^b x^r.\ \frac{1}{b-a} dx$$

$$= \frac{1}{b-a} \int_a^b x^r\ dx$$

$$= \frac{1}{b-a} \left[\frac{x^{r+1}}{r+1}\right]_a^b$$

$$\therefore \quad \mu'_r = \frac{b^{r+1} - a^{r+1}}{(b-a)(r+1)} \ldots\ldots\ldots\ldots(i)$$

Now, substituting $r = 1, 2, 3, 4$ in (i), we get respectively

$$\mu'_1 = \frac{b^2 - a^2}{2(b-a)} = \frac{(b+a)(b-a)}{2(b-a)} = \frac{a+b}{2}$$

$$\mu'_2 = \frac{b^3 - a^3}{3(b-a)} = \frac{(b-a)\left(b^2 + ba + a^2\right)}{3(b-a)} = \frac{a^2 + ab + b^2}{3}$$

$$\mu'_3 = \frac{b^4 - a^4}{4(b-a)} = \frac{(b^2)^2 - (a^2)^2}{4(b-a)}$$

$$= \frac{(b^2 + a^2)(b^2 - a^2)}{4(b-a)}$$

$$= \frac{(b^2 + a^2)(b+a)(b-a)}{4(b-a)}$$

$$= \frac{(a+b)(a^2 + b^2)}{4}$$

and $\mu'_4 = \dfrac{b^5 - a^5}{5(b-a)}$

Now, Mean, $\mu'_1 = \dfrac{a+b}{2}$

Variance, $\mu_2 = \mu'_2 - \mu_1'^2$

$$= \frac{a^2 + ab + b^2}{3} - \frac{(a+b)^2}{4}$$

$$= \frac{a^2 + ab + b^2}{3} - \frac{a^2 + 2ab + b^2}{4}$$

$$= \frac{4a^2 + 4ab + 4b^2 - 3a^2 - 6ab - 3b^2}{12}$$

$$= \frac{a^2 - 2ab + b^2}{12}$$

$$= \frac{(a-b)^2}{12}$$

$$\mu_3 = \mu_3' - 3\,\mu_2'\;\mu_1' + 2\mu_1'^3$$

$$= \frac{(a+b)\,(a^2+b^2)}{4} - 3\,\frac{a^2 + ab + b^2}{3} \cdot \frac{a+b}{2} + 2 \cdot \frac{(a+b)^3}{8}$$

$$= \frac{a^3 + ab^2 + a^2b + b^3}{4} - \frac{a^3 + a^2b + ab^2 + a^2b + ab^2 + b^3}{2} + \frac{a^3 + 3a^2b + 3ab^2 + b^3}{4}$$

$$= \frac{1}{4}\left[a^3 + ab^2 + a^2b + b^3 - 2a^3 - 4a^2b - 4ab^2 - 2b^3 + a^3 + 3a^2b + 3ab^2 + b^3\right]$$

$$= \frac{1}{4} \times 0$$

$$= 0$$

$$\mu_4 = \mu_4' - 4\,\mu_3'\;\mu_1' + 6\,\mu_2'\;\mu_1'^2 - 3\mu_1'^4$$

$$= \frac{b^5 - a^5}{5\,(b-a)} - 4\,\frac{(a+b)\,(a^2+b^2)}{4} \cdot \frac{a+b}{2} + 6\,\frac{a^2 + ab + b^2}{3} \cdot \frac{(a+b)^2}{4} - 3 \cdot \frac{(a+b)^4}{16}$$

$$= \frac{b^5 - a^5}{5\,(b-a)} - \frac{(a+b)^2\,(a^2+b^2)}{2} + \frac{(a+b)^2\,\left(a^2 + ab + b^2\right)}{2} - \frac{3\,(a+b)^4}{16}$$

$$= \frac{16(b^5 - a^5) - 40(b-a)(a+b)(a+b)(a^2+b^2) + 40(b-a)(a+b)^2(a^2+ab+b^2) - 15(b-a)(a+b)^4}{80(b-a)}$$

$$= \frac{1}{80(b-a)}\left[16b^5 - 16a^5 - 40(b^2-a^2)(b^2+a^2)(a+b) + 40(b^2-a^2)(a+b)(a^2+ab+b^2) - 15(b^2-a^2)(a+b)^3\right]$$

$$= \frac{1}{80(b-a)}\left[16b^5 - 16a^5 - 40(b^4-a^4)(a+b) + 40(b^2-a^2)(a^3+a^2b+ab^2+a^2b+ab^2+b^3) - 15(b^2-a^2)(a+b)^3\right]$$

$$= \frac{1}{80(b-a)}(b-a)^5 \text{(on simplification)}$$

$$= \frac{(b-a)^4}{80}$$

So, $\beta_1 = \frac{\mu_3^{\ 2}}{\mu_2^{\ 3}} = \frac{0}{\mu_2^{\ 3}} = 0$

$$\beta_2 = \frac{\mu_4}{\mu_2^{\ 2}} = \frac{(b-a)^4}{80} \times \frac{144}{(b-a)^4} = \frac{9}{5}$$

Since $\beta_1 = 0$, so the distribution is symmetric and $\beta_2 = \frac{9}{5} < 3$, so the distribution is platykurtic.

Q. Find mean, variance, different moments, β_1 and β_2 of continuous uniform distribution.

Ans: We know the pdf of uniform distribution with parameters a and b is

$$f(x) = \frac{1}{b-a}; \qquad a \le x \le b$$

By definition,

$$\text{Mean, } E(x) = \int_a^b x\ f(x)\ dx$$

$$= \int_a^b x.\ \frac{1}{b-a}\ dx$$

$$= \frac{1}{b-a}\int_a^b x\ dx$$

$$= \frac{1}{b-a}\left[\frac{x^2}{2}\right]_a^b$$

$$= \frac{1}{2(b-a)}\left[b^2 - a^2\right]$$

$$= \frac{(b+a)(b-a)}{2(b-a)}$$

$$\therefore \quad E(x) = \frac{a+b}{2}$$

Now, $r-$th central moment

$$\mu_r = E\left[x - E(x)\right]^r$$

$$= E\left[x - \frac{a+b}{2}\right]^r$$

$$= \int_a^b \left(x - \frac{a+b}{2}\right)^r f(x)\, dx$$

$$= \int_a^b \left(x - \frac{a+b}{2}\right)^r . \frac{1}{b-a}\, dx$$

$$= \frac{1}{b-a} \int_a^b \left(x - \frac{a+b}{2}\right)^r dx$$

$$= \frac{1}{b-a} \left[\frac{\left(x - \frac{a+b}{2}\right)^{r+1}}{r+1}\right]_a^b$$

$$= \frac{1}{(b-a)(r+1)} \left[\left(b - \frac{a+b}{2}\right)^{r+1} - \left(a - \frac{a+b}{2}\right)^{r+1}\right]$$

$$= \frac{1}{(b-a)(r+1)} \left[\left(\frac{2b-a-b}{2}\right)^{r+1} - \left(\frac{2a-a-b}{2}\right)^{r+1}\right]$$

$$\therefore \quad \mu_r = \frac{1}{(b-a)(r+1)} \left[\left(\frac{b-a}{2}\right)^{r+1} - \left(\frac{a-b}{2}\right)^{r+1}\right] \text{..............(i)}$$

Putting $r=1$ in (i), we get

$$\mu_1 = \frac{1}{2(b-a)} \left[\left(\frac{b-a}{2}\right)^2 - \left(\frac{a-b}{2}\right)^2\right]$$

$$= \frac{1}{2(b-a)} \left[\left(\frac{b-a}{2}\right)^2 - \left(\frac{b-a}{2}\right)^2\right]$$

$$\therefore \quad \mu_1 = 0$$

Again, putting $r = 2$ in (i), we get

$$\mu_2 = \frac{1}{3(b-a)} \left[\left(\frac{b-a}{2}\right)^3 - \left(\frac{a-b}{2}\right)^3\right]$$

$$= \frac{1}{3(b-a)}\left[\left(\frac{b-a}{2}\right)^3 + \left(\frac{b-a}{2}\right)^3\right]$$

$$= \frac{2}{3(b-a)} \times \left(\frac{b-a}{2}\right)^3$$

$$= \frac{2}{3(b-a)} \times \frac{(b-a)^3}{8}$$

$$= \frac{1}{3} \times \frac{(b-a)^2}{4}$$

$\therefore$ Variance, $\mu_2 = \frac{(a-b)^2}{12}$

Further, putting $r = 3$ in (i), we get

$$\mu_3 = \frac{1}{4(b-a)}\left[\left(\frac{b-a}{2}\right)^4 - \left(\frac{a-b}{2}\right)^4\right]$$

$$= \frac{1}{4(b-a)}\left[\left(\frac{b-a}{2}\right)^4 - \left(\frac{b-a}{2}\right)^4\right]$$

$\therefore$ $\mu_3 = 0$

Again, putting $r = 4$ in (i), we get

$$\mu_4 = \frac{1}{5(b-a)}\left[\left(\frac{b-a}{2}\right)^5 - \left(\frac{a-b}{2}\right)^5\right]$$

$$= \frac{1}{5(b-a)}\left[\left(\frac{b-a}{2}\right)^5 + \left(\frac{b-a}{2}\right)^5\right]$$

$$= \frac{2}{5(b-a)} \times \left(\frac{b-a}{2}\right)^5$$

$$= \frac{2}{5(b-a)} \times \frac{(b-a)^5}{32}$$

$$= \frac{1}{5} \times \frac{(b-a)^4}{16}$$

$\therefore$ $\mu_4 = \frac{(a-b)^4}{80}$

Now, $\beta_1 = \frac{\mu_3^{\ 2}}{\mu_2^{\ 3}} = \frac{0}{\mu_2^{\ 3}} = 0$

$$\beta_2 = \frac{\mu_4}{\mu_2^{\ 2}} = \frac{(a-b)^4}{80} \times \frac{144}{(a-b)^4} = \frac{9}{5}$$

Since $\beta_1 = 0$ and $\beta_2 = \frac{9}{5} < 3$, so the distribution is symmetric and platykurtic.

Q. Find the moment generating function of continuous uniform distribution and hence find mean and variance of the distribution (NU-15)

Or, Find the moment generating function (m.g.f) of continuous uniform distribution and hence find mean, variance, β_1 and β_2.

Ans: We know the pdf of uniform distribution with parameters a and b is

$$f(x) = \frac{1}{b-a}; \qquad a \le x \le b$$

Now, by definition of m.g.f, we can write

$$M_x(t) = E\left(e^{tx}\right)$$

$$= \int_a^b e^{tx}\ f(x)\ dx$$

$$= \int_a^b e^{tx}.\ \frac{1}{b-a} dx$$

$$= \frac{1}{b-a} \int_a^b e^{tx}\ dx$$

$$= \frac{1}{b-a} \left[\frac{e^{tx}}{t}\right]_a^b$$

$$= \frac{e^{bt} - e^{at}}{t\,(b-a)}$$

$$= \frac{1}{t(b-a)} \left[1 + bt + \frac{b^2t^2}{2!} + \frac{b^3t^3}{3!} + \frac{b^4t^4}{4!} + \frac{b^5t^5}{5!} + \cdots - 1 - at - \frac{a^2t^2}{2!} - \frac{a^3t^3}{3!} - \frac{a^4t^4}{4!} - \frac{a^5t^5}{5!} - \cdots\right]$$

$$= \frac{1}{t(b-a)} \left[(b-a)t + \left(b^2 - a^2\right)\frac{t^2}{2!} + \left(b^3 - a^3\right)\frac{t^3}{3!} + \left(b^4 - a^4\right)\frac{t^4}{4!} + \left(b^5 - a^5\right)\frac{t^5}{5!} + \cdots\right]$$

$$=1+(a+b)\frac{t}{2!}+\left(b^2+ba+a^2\right)\frac{t^2}{3!}+\frac{b^4-a^4}{b-a}\frac{t^3}{4!}+\frac{b^5-a^5}{b-a}\frac{t^4}{5!}+\cdots$$

$$=1+\frac{a+b}{2}\frac{t}{1!}+\frac{b^2+ba+a^2}{6}.\,2!.\,\frac{t^2}{2!}+\frac{b^4-a^4}{24(b-a)}.\,3!.\,\frac{t^3}{3!}+\frac{b^5-a^5}{120\,(b-a)}.\,4!.\,\frac{t^4}{4!}+\cdots$$

$$=1+\frac{a+b}{2}\frac{t}{1!}+\frac{b^2+ba+a^2}{3}\frac{t^2}{2!}+\frac{b^4-a^4}{4(b-a)}\frac{t^3}{3!}+\frac{b^5-a^5}{5(b-a)}\frac{t^4}{4!}+\cdots$$

Now, $\mu_r' =$ Coefficient of $\frac{t^r}{r!}$ in $M_x(t)$

Putting $r=1, 2, 3, 4$ we get respectively,

$$\mu_1' = \frac{a+b}{2}$$

$$\mu_2' = \frac{a^2+ab+b^2}{3}$$

$$\mu_3' = \frac{b^4-a^4}{4(b-a)} = \frac{(b^2)^2-(a^2)^2}{4(b-a)} = \frac{(b^2+a^2)(b^2-a^2)}{4(b-a)} = \frac{(b^2+a^2)(b+a)(b-a)}{4(b-a)} = \frac{(a+b)(a^2+b^2)}{4}$$

$$\mu_4' = \frac{b^5-a^5}{5(b-a)}$$

Now, Mean, $\mu_1' = \frac{a+b}{2}$

Variance, $\mu_2 = \mu_2' - \mu_1'^2$

$$= \frac{a^2+ab+b^2}{3} - \frac{(a+b)^2}{4}$$

$$= \frac{a^2+ab+b^2}{3} - \frac{a^2+2ab+b^2}{4}$$

$$= \frac{4a^2+4ab+4b^2-3a^2-6ab-3b^2}{12}$$

$$= \frac{a^2-2ab+b^2}{12}$$

$$= \frac{(a-b)^2}{12}$$

$$\mu_3 = \mu_3' - 3\,\mu_2'\,\mu_1' + 2\,\mu_1'^3$$

$$= \frac{(a+b)(a^2+b^2)}{4} - 3\,\frac{a^2+ab+b^2}{3}.\frac{a+b}{2} + 2.\,\frac{(a+b)^3}{8}$$

$$= \frac{a^3+ab^2+a^2b+b^3}{4} - \frac{a^3+a^2b+ab^2+a^2b+ab^2+b^3}{2} + \frac{a^3+3a^2b+3ab^2+b^3}{4}$$

$$= \frac{1}{4}\left[a^3+ab^2+a^2b+b^3-2a^3-4a^2b-4ab^2-2b^3+a^3+3a^2b+3ab^2+b^3\right]$$

$$= \frac{1}{4}\times 0$$

$$= 0$$

$$\mu_4 = \mu_4' - 4\,\mu_3'\,\mu_1' + 6\,\mu_2'\,\mu_1'^2 - 3\,\mu_1'^4$$

$$= \frac{b^5-a^5}{5(b-a)} - 4\frac{(a+b)(a^2+b^2)}{4}.\frac{a+b}{2} + 6\frac{a^2+ab+b^2}{3}.\frac{(a+b)^2}{4} - 3.\frac{(a+b)^4}{16}$$

$$= \frac{b^5-a^5}{5(b-a)} - \frac{(a+b)^2(a^2+b^2)}{2} + \frac{(a+b)^2(a^2+ab+b^2)}{2} - \frac{3(a+b)^4}{16}$$

$$= \frac{16(b^5-a^5)-40(b-a)(a+b)(a+b)(a^2+b^2)+40(b-a)(a+b)^2(a^2+ab+b^2)-15(b-a)(a+b)^4}{80(b-a)}$$

$$= \frac{1}{80(b-a)}\left[16b^5-16a^5-40(b^2-a^2)(b^2+a^2)(a+b)+40(b^2-a^2)(a+b)(a^2+ab+b^2)-15(b^2-a^2)(a+b)^3\right]$$

$$= \frac{1}{80(b-a)}\left[16b^5-16a^5-40(b^4-a^4)(a+b)+40(b^2-a^2)(a^3+a^2b+ab^2+a^2b+ab^2+b^3)-15(b^2-a^2)(a+b)^3\right]$$

$$= \frac{1}{80(b-a)}(b-a)^5 \text{(on simplification)}$$

$$= \frac{(b-a)^4}{80}$$

So, $\beta_1 = \dfrac{\mu_3^{\,2}}{\mu_2^{\,3}} = \dfrac{0}{\mu_2^{\,3}} = 0$

$$\beta_2 = \frac{\mu_4}{\mu_2^{\,2}} = \frac{(b-a)^4}{80}\times\frac{144}{(b-a)^4} = \frac{9}{5}$$

Since $\beta_1 = 0$, so the distribution is symmetric and $\beta_2 = \dfrac{9}{5} < 3$, so the distribution is platykurtic.

NB: $z = tx$

$$\therefore \frac{dz}{dx} = t$$

$$\Rightarrow \frac{dz}{t} = dx$$

When $x = a$, then $z = at$

$x = b$, then $z = bt$

$$\therefore \frac{1}{b-a}\int_a^b e^{tx}\,dx = \frac{1}{b-a}\int_{at}^{bt} e^z.\frac{dz}{t} = \frac{1}{t(b-a)}\left[e^z\right]_{at}^{bt} = \frac{e^{bt}-e^{at}}{t\,(b-a)}$$

Q. Find the mean deviation about mean of continuous uniform distribution (NU-11)

Ans: We know the pdf of uniform distribution with parameters a and b is

$$f(x) = \frac{1}{b-a}; \quad a \le x \le b$$

We know, $E(x) = \dfrac{a+b}{2}$

Now, Mean deviation about mean is

$$MD = E\,|x - E(x)|$$

$$= E\left|x - \frac{a+b}{2}\right|$$

$$= \int_a^b \left|x - \frac{a+b}{2}\right| f(x)\,dx$$

$$= \int_a^b \left|x - \frac{a+b}{2}\right|.\frac{1}{b-a}\,dx$$

$$\therefore \quad MD = \frac{1}{b-a}\int_a^b \left|x - \frac{a+b}{2}\right| dx \text{.............(i)}$$

Let $z = x - \dfrac{a+b}{2}$

$$\therefore \ \frac{dz}{dx} = 1$$

$$\Rightarrow dz = dx$$

If $x = a$, then $z = a - \frac{a+b}{2} = \frac{2a-a-b}{2} = \frac{a-b}{2} = -\frac{b-a}{2}$

If $x = b$, then $z = b - \frac{a+b}{2} = \frac{2b-a-b}{2} = \frac{b-a}{2}$

Now (i) can be written as

$$MD = \frac{1}{b-a}\int_{-\frac{b-a}{2}}^{\frac{b-a}{2}} |z| \, dz$$

$$= \frac{2}{b-a}\int_{0}^{\frac{b-a}{2}} z \, dz$$

$$= \frac{2}{b-a}\left[\frac{z^2}{2}\right]_0^{\frac{b-a}{2}}$$

$$= \frac{1}{b-a}\left[\left(\frac{b-a}{2}\right)^2 - 0\right]$$

$$= \frac{1}{b-a} \times \frac{(b-a)^2}{4}$$

$$\therefore \ MD = \frac{b-a}{4}$$

NB: (i) $\int_{-3}^{3} |x| \, dx = \int_{-3}^{0} |x| \, dx + \int_{0}^{3} |x| \, dx = \left|\left[\frac{x^2}{2}\right]_{-3}^{0}\right| + \left|\left[\frac{x^2}{2}\right]_{0}^{3}\right| = \frac{1}{2}\left|0^2 - (-3)^2\right| + \frac{1}{2}\left|3^2 - 0^2\right| = \frac{9}{2} + \frac{9}{2} = 2.\frac{9}{2} = 9$

(ii) $\int_{-3}^{3} |x| \, dx = 2\int_{0}^{3} x \, dx = 2\left[\frac{x^2}{2}\right]_0^3 = \left[3^2 - 0\right] = 9$

Q. Show that for a rectangular distribution $f(x;\, a) = \frac{1}{2a}; \quad -a < x < a$ **the mgf about origin is** $\frac{1}{at}\sinh\,(at)$**. Also show that moments of even order are given by** $\mu_{2r} = \frac{a^{2r}}{2r+1}$**.**

Q. Show that all odd order central moments of continuous uniform distribution is zero (NU-11)

Solution: The given probability density function of rectangular distribution is

$$f(x;\; a) = \frac{1}{2a}; \qquad -a < x < a$$

The m.g.f about origin is

$$M_x(t) = E\left(e^{tx}\right)$$

$$= \int_{-a}^{a} e^{tx}\; f(x)\, dx$$

$$= \int_{-a}^{a} e^{tx}.\; \frac{1}{2a} dx$$

$$= \frac{1}{2a}\int_{-a}^{a} e^{tx}\; dx$$

$$= \frac{1}{2a}\left[\frac{e^{tx}}{t}\right]_{-a}^{a}$$

$$= \frac{1}{2at}\left[e^{at} - e^{-at}\right]$$

$$= \frac{1}{at}.\; \frac{e^{at} - e^{-at}}{2}$$

$$= \frac{\sinh\,(at)}{at} \qquad \left[\because\; \sinh\,(at) = \frac{e^{at} - e^{-at}}{2}\right]$$

$$= \frac{1}{at}\left[at + \frac{(at)^3}{3!} + \frac{(at)^5}{5!} + \cdots\right]$$

$$= 1 + \frac{(at)^2}{3!} + \frac{(at)^4}{5!} + \cdots$$

$$=1+\frac{a^2t^2}{3!}+\frac{a^4t^4}{5!}+\cdots$$

$$\therefore \quad M_x(t)=1+\frac{a^2}{(2+1)!}\times 2!\times\frac{t^2}{2!}+\frac{a^4}{(4+1)!}\times 4!\times\frac{t^4}{4!}+\cdots$$

$$=1+0\times\frac{t}{1!}+\frac{a^2}{(2+1)!}\times 2!\times\frac{t^2}{2!}+0\times\frac{t^3}{3!}+\frac{a^4}{(4+1)!}\times 4!\times\frac{t^4}{4!}+\cdots$$

Now, $\mu_r{}' =$ coefficient of $\frac{t^r}{r!}$ in $M_x(t)$

Putting $r=1,\ 2, 3, 4$ we get respectively,

$$\mu_1{}'=0,\ \ \mu_2{}'=\frac{a^2}{(2+1)!}\times 2!,\ \ \ \mu_3'=0,\ \ \ \mu_4{}'=\frac{a^4}{(4+1)!}\times 4!$$

All moments of odd order about origin vanish i.e., $\mu'_{2r+1}=0$ ....(i)

In particular, $\mu_1'=0$

Thus, all raw moments are central moments for the distribution. Hence

$$\mu'_{2r}=\mu_{2r} \text{ and } \mu'_{2r+1}=\mu_{2r+1}$$

So, from (i) we can write, $\mu'_{2r+1}=\mu_{2r+1}=0$ i.e., all odd order central moments of continuous uniform distribution is zero.

The moments for even order:

Since, $\mu'_{2r}=\mu_{2r}$, so

$$\mu_2{}'=\mu_2=\frac{a^2}{(2+1)!}\times 2!$$

$$\mu_4{}'=\mu_4=\frac{a^4}{(4+1)!}\times 4!$$

Similarly, $\mu_{2r}=\frac{a^{2r}\,(2r)!}{(2r+1)!}=\frac{a^{2r}\,(2r)!}{(2r+1)\,(2r)!}=\frac{a^{2r}}{2r+1}$

NB:

(i) $f(x;\,a)=\frac{1}{2a};\ \ -a<x<a$ **;**

Now, the r-th raw moment about origin is

$$\mu_r' = E\left(x^r\right)$$

$$= \int_{-a}^{a} x^r \, f(x)\, dx$$

$$= \int_{-a}^{a} x^r . \frac{1}{2a} dx$$

$$= \frac{1}{2a} \int_{-a}^{a} x^r \, dx$$

$$= \frac{1}{2a} \left[\frac{x^{r+1}}{r+1}\right]_{-a}^{a}$$

$$= \frac{1}{2a\,(r+1)} \left[x^{r+1}\right]_{-a}^{a}$$

$$\therefore \quad \mu_r' = \frac{1}{2a(r+1)}\left[a^{r+1} - (-a)^{r+1}\right] \text{................(i)}$$

If $r = 1$ in (i), then

$$\mu_1' = \frac{1}{4a}\left[a^2 - (-a)^2\right] = \frac{1}{4a}\left[a^2 - a^2\right] = 0$$

If $r = 3$ in (i), then

$$\mu_3' = \frac{1}{8a}\left[a^4 - (-a)^4\right] = \frac{1}{8a}\left[a^4 - a^4\right] = 0$$

$\therefore \; \mu_1' = \mu_3' = \cdots = 0$ i.e., all moments of odd order about origin vanish.

In particular, if $\mu_1' = 0$, then

$$\mu_2 = \mu_2' - \mu_1'^2 = \mu_2' - 0 = \mu_2'$$

$$\mu_3 = \mu_3' - 3\,\mu_2'\,\mu_1' + 2\,\mu_1'^3 = \mu_3' - 3\,\mu_2'.0 + 2.0 = \mu_3'$$

$$\mu_4 = \mu_4' - 4\,\mu_3'\,\mu_1' + 6\,\mu_2'\,\mu_1'^2 - 3\,\mu_1'^4 = \mu_4' - 4\,\mu_3'.0 + 6\,\mu_2'.0 - 3.0 = \mu_4'$$

So, $\mu_{2r} = \mu_{2r}'$ i.e., all raw moments are central moments.

(ii) $M_x(t) = 1 + \frac{a^2}{(2+1)!} \times 2! \times \frac{t^2}{2!} + \frac{a^4}{(4+1)!} \times 4! \times \frac{t^4}{4!} + \cdots$

Now, $\mu_2 =$ Coefficient of $\frac{t^2}{2!}$ in $M_x(t) = \frac{a^2}{(2+1)!} \times 2!$

$\mu_4 =$ Coefficient of $\frac{t^4}{4!}$ in $M_x(t) = \frac{a^4}{(4+1)!} \times 4!$

$\therefore\ \mu_{2r} =$ Coefficient of $\frac{t^{2r}}{(2r)!}$ in $M_x(t) = \frac{a^{2r}}{(2r+1)!} \times (2r)!$

Short questions and answers:

(1) Find the mean of continuous uniform distribution (NU-11, 14)

Ans: We know the pdf of uniform distribution with parameter a and b is

$$f(x) = \frac{1}{b-a}; \quad a \le x \le b$$

By definition,

Mean, $E(x) = \int_a^b x\, f(x)\, dx$

$$= \int_a^b x.\ \frac{1}{b-a} dx$$

$$= \frac{1}{b-a} \int_a^b x\ dx$$

$$= \frac{1}{b-a} \left[\frac{x^2}{2}\right]_a^b$$

$$= \frac{1}{2(b-a)} \left[b^2 - a^2\right]$$

$$= \frac{(b+a)(b-a)}{2(b-a)}$$

$$\therefore \text{ Mean, } E(x) = \frac{a+b}{2}$$

(2) What is the β_1 and β_2 of continuous uniform distribution? (NU-11)

Ans: For continuous uniform distribution $\beta_1 = 0$ and $\beta_2 = \frac{9}{5}$ which indicate the distribution is symmetric and platykurtic.

Mathematical Problems:

(1) Subway trains on a certain line run every half hour between mid-night and six in the morning. What is the probability that a man entering the station at a random time during this period will have to wait at least twenty minutes?

Ans: Let the random variable X denote the waiting time (in minutes) for the next train. Under the assumption that a man arrives at the station at random, X is distributed uniformly on (0,30) with probability density function

$$f\left(x; 30)\right) = \frac{1}{30}; \quad 0 < x < 30$$

The probability that he has to wait at least 20 minutes is

$$P[x \geq 20] = \int_{20}^{30} f(x)\, dx$$

$$= \int_{20}^{30} \frac{1}{30}\, dx$$

$$= \frac{1}{30} \int_{20}^{30} dx$$

$$= \frac{1}{30} [x]_{20}^{30}$$

$$= \frac{1}{30}\left[30-20\right]$$

$$= \frac{1}{3}$$

(2) If X has a uniform distribution in $[0, 1]$ with pdf $f(x)=1;\ \ 0<x<1$. Find the distribution of $-2\log X$.

Or, If variate x follows continuous uniform distribution over the range $[0, 1]$, then find the distribution of $Y=-2\log x$ (**NU-14)**

Solution: Here $X \sim U\left[0,\ 1\right]$

$$\therefore \quad f\left(x\right)) = \frac{1}{1-0} = 1$$

Let $Y=-2\log X$

The distribution function of Y is

$$F(y) = P\left[Y \le y\right]$$

$$= P\left[-2\log X \le y\right]$$

$$= P\left[2\log X \ge -y\right]$$

$$= P\left[\log X \ge -\frac{y}{2}\right]$$

$$= P\left[X \ge e^{-\frac{y}{2}}\right]$$

$$= 1 - P\left[X < e^{-\frac{y}{2}}\right]$$

$$= 1 - \int_0^{e^{-\frac{y}{2}}} f(x)\,dx$$

$$= 1 - \int_0^{e^{-\frac{y}{2}}} 1\,dx$$

$$= 1 - \left[x\right]_0^{e^{-\frac{y}{2}}}$$

$$\therefore \quad F(y) = 1 - e^{-\frac{y}{2}}$$

So, $f(y) = \frac{d}{dy} F(y) = \frac{d}{dy}\left(1 - e^{-\frac{y}{2}}\right) = -e^{-\frac{y}{2}}\left(-\frac{1}{2}\right) = \frac{1}{2} e^{-\frac{y}{2}}$

Since X ranges from 0 to 1, so $Y = -2 \log X$ ranges from 0 to ∞.

$$\therefore \quad f(y) = \frac{1}{2} e^{-\frac{y}{2}}; \qquad 0 < y < \infty$$

which is the pdf of chi-square distribution with 2 df.

NB: (i) $\log(0) = \infty$

(ii) $\log(1) = 0$

(iii) The pdf of χ^2 distribution with n d.f is

$$f(\chi^2) = \frac{1}{2^{\frac{n}{2}} \Gamma\left(\frac{n}{2}\right)} e^{-\frac{1}{2}\chi^2} (\chi^2)^{\frac{n}{2}-1}; \qquad 0 < \chi^2 < \infty$$

So, if $y = \chi^2$ and $n = 2$, then

$$f(y) = \frac{1}{2^{\frac{2}{2}} \Gamma\left(\frac{2}{2}\right)} e^{-\frac{1}{2}y} (y)^{\frac{2}{2}-1}$$

$$\therefore \quad f(y) = \frac{1}{2} e^{-\frac{y}{2}}; \qquad 0 < y < \infty$$

which is the pdf of chi-square distribution with 2 df.

Beta distribution of first kind

There are two types of beta distribution. They are known as first and second kind of Beta distribution. The form of the distribution depends on the form of the beta functions.

Q. Define beta distribution of first kind.

Or, What is beta distribution of first kind? (NU-12, 14, 16)

Ans: A continuous random variable X is said to have a beta distribution of the first kind if its pdf is given by

$$f(x;\ l, m) = \frac{x^{l-1}\ (1-x)^{m-1}}{\beta(l,\ m)};\quad 0 < x < 1,\quad l > 0,\quad m > 0$$

where l and m are the two parameters of the distribution.

This is the standard form of the beta distribution.

NB: The pdf of beta distribution of the first kind with parameters l and m is

$$f(x;\ l, m) = \frac{x^{l-1}\ (1-x)^{m-1}}{\beta(l,\ m)} \text{..........(i)}$$

(a) If $l = m = 1$, then (i) reduces to uniform distribution over the range $(0,1)$

(b) If $\mathrm{m} = 1$, then (i) turns into standard power function distribution with pdf

$$f(x;\ l) = l\ \ x^{l-1};\quad 0 < x < 1$$

(c) $\int_0^1 f(x;\ l, m)\, dx = 1$

$$\Rightarrow \int_0^1 \frac{x^{l-1}\ (1-x)^{m-1}}{\beta(l,\ m)}\, dx = 1$$

$$\Rightarrow \frac{1}{\beta(l,\ m)} \int_0^1 x^{l-1}\ (1-x)^{m-1}\, dx = 1$$

$$\therefore \int_0^1 x^{l-1}\ (1-x)^{m-1}\, dx = \beta(l,\ m)$$

Remarks: (i) $\Gamma(\alpha+1) = \alpha\ \Gamma(\alpha)$

(ii) $\Gamma\alpha = (\alpha-1)\,!$

(iii) $\Gamma(\alpha+1) = \alpha! = \alpha\,(\alpha-1)! = \alpha\,\Gamma\alpha$

(iv) $\beta(l,\ m) = \dfrac{\Gamma l\ \Gamma m}{\Gamma(l+m)}$

(vi) Harmonic mean, $\mathrm{HM} = \dfrac{1}{E\left(\dfrac{1}{x}\right)}$

(vii) The r-th moment about origin, usually denoted by μ_r' is defined as

$$\mu_r' = E\left[x^r\right] = \begin{cases} \sum x^r\ f(x); & \text{if x is discrete} \\ \int_{-\infty}^{\infty} x^r\ f(x)\ dx; & \text{if x is continuous} \end{cases}$$

Q. Find the r-th raw moment and hence find mean and variance of the beta distribution of first kind.

Or, Find the mean and variance of beta distribution of first kind **(NU-12)**

Or, Find the mean and variance of beta distribution (**NU-15**)

Ans: Let x is beta variate of the first kind with parameters l and m, so the pdf is

$$f(x;\ l, m) = \frac{x^{l-1}\ (1-x)^{m-1}}{\beta(l,\ m)};\quad 0<x<1,\quad l>0,\quad m>0$$

Now, by definition r-th raw moment about origin is

$$\mu_r' = E(x^r) = \int_0^1 x^r\ f(x)\ dx$$

$$= \int_0^1 x^r .\ \frac{x^{l-1}\ (1-x)^{m-1}}{\beta(l,\ m)}\ dx$$

$$= \frac{1}{\beta(l, m)} \int_0^1 x^{l+r-1}\ (1-x)^{m-1}\ dx$$

$$= \frac{1}{\beta(l, m)}.\ \beta(l+r,\ m) \quad \left[\because\ \int_0^1 x^{l-1}\ (1-x)^{m-1}\ dx = \beta(l, m)\right]$$

$$= \frac{\Gamma(l+m)}{\Gamma l\ \Gamma m}.\ \frac{\Gamma(l+r)\ \Gamma m}{\Gamma(l+r+m)}$$

$$\therefore \quad \mu_r' = \frac{\Gamma(l+m)\ \Gamma(l+r)}{\Gamma(l)\ \Gamma(l+r+m)} \text{ ....(i)}$$

Now, putting $r = 1$ in (i), we get

$$\mu_1' = \frac{\Gamma(l+m)\ \Gamma(l+1)}{\Gamma(l)\ \Gamma(l+1+m)}$$

$$= \frac{\Gamma(l+m).\ l\ \Gamma(l)}{\Gamma(l)\ (l+m)\ \Gamma(l+m)}$$

$$\therefore \text{ Mean, } \mu_1' = \frac{l}{l+m}$$

Again, putting $r = 2$ in (i), we get

$$\mu_2' = \frac{\Gamma(l+m)\ \Gamma(l+2)}{\Gamma(l)\ \Gamma(l+2+m)}$$

$$= \frac{\Gamma(l+m)\ (l+1)\ l\ \Gamma(l)}{\Gamma(l)\ (l+m+1)\ (l+m)\ \Gamma(l+m)}$$

$$\therefore \quad \mu_2' = \frac{l\ (l+1)}{(l+m)\ (l+m+1)}$$

So, Variance, $\mu_2 = \mu_2' - \mu_1'^2$

$$= \frac{l\ (l+1)}{(l+m)\ (l+m+1)} - \frac{l^2}{(l+m)^2}$$

$$= \frac{(l^2+l)\ (l+m) - l^2\ (l+m+1)}{(l+m)^2\ (l+m+1)}$$

$$= \frac{l^3 + l^2 m + l^2 + l m - l^3 - l^2 m - l^2}{(l+m)^2\ (l+m+1)}$$

$$\therefore \text{ Variance, } \mu_2 = \frac{l\ m}{(l+m)^2\ (l+m+1)}$$

Q. Find the Harmonic mean of beta distribution of 1st kind (NU-13, 16)

Ans: We know the pdf of beta distribution of first kind with parameters l and m is

$$f(x)=\frac{x^{l-1}\ (1-x)^{m-1}}{\beta(l,\ m)};\ \ 0<x<1$$

By definition of Harmonic mean, we can write

$$\text{HM}=\frac{1}{\text{E}\left(\frac{1}{\text{x}}\right)}\ \ldots\ldots\ldots\text{(i)}$$

$$\text{Now, E}\left(\frac{1}{\text{x}}\right)=\int_0^1 \frac{1}{\text{x}}\ \text{f(x) dx}$$

$$=\int_0^1 \frac{1}{x}.\ \frac{x^{l-1}\ (1-x)^{m-1}}{\beta(l,\ m)}\ dx$$

$$=\frac{1}{\beta(l,\ m)}\int_0^1 x^{l-2}\ (1-x)^{m-1}\,dx$$

$$=\frac{1}{\beta(l,\ m)}\int_0^1 x^{(l-1)-1}\ (1-x)^{m-1}\,dx$$

$$=\frac{1}{\beta(l,\ m)}.\ \ \beta(l-1,\ m)$$

$$=\frac{\Gamma(l+m)}{\Gamma(l)\ \Gamma(m)}.\ \frac{\Gamma(l-1)\ \Gamma(m)}{\Gamma(l-1+m)}$$

$$=\frac{(l+m-1)\ \Gamma(l+m-1)}{(l-1)\ \Gamma(l-1)}.\ \ \frac{\Gamma(l-1)}{\Gamma(l+m-1)}$$

$$\therefore\ \ E\left(\frac{1}{x}\right)=\frac{l+m-1}{l-1}$$

So (i) can be written as

$$HM=\frac{1}{E\left(\frac{1}{x}\right)}=\frac{1}{\frac{l+m-1}{l-1}}=\frac{l-1}{l+m-1}$$

Q. Find the Mode of the beta distribution of first kind (NU-12, 14)

Ans: Let x is a beta variate of first kind with parameters l and m, so the pdf is

$$f(x)=\frac{1}{\beta(l,\ m)}\ x^{l-1}\ (1-x)^{m-1};\quad 0<x<1 ....(i)$$

The mode of the distribution will be the solution of $f'(x)=0$, provided $f''(x)<0$.

Taking log on both sides of (i), we get

$$\log\ f(x)=\log\left\{\frac{1}{\beta(l,\ m)}\right\}+(l-1)\log x+(m-1)\log(1-x) ....(ii)$$

Now, differentiating (ii) w.r.t x, we have

$$\frac{1}{f(x)}\ f'(x)=(l-1).\ \frac{1}{x}+(m-1).\ \frac{1}{1-x}.\ (-1)$$

$$f'(x)=\left(\frac{l-1}{x}-\frac{m-1}{1-x}\right)\ f(x)$$

Using $f'(x)=0$, we get

$$\left(\frac{l-1}{x}-\frac{m-1}{1-x}\right)\ f(x)=0$$

$$\Rightarrow \frac{l-1}{x}-\frac{m-1}{1-x}=0 \quad [\because\ f(x)\neq 0]$$

$$\Rightarrow \frac{(l-1)(1-x)-x(m-1)}{x(1-x)}=0$$

$$\Rightarrow l-l\ x-1+x-m\ x+x=0$$

$$\Rightarrow -l\ x-m\ x+2x=-\ l+1$$

$$\Rightarrow -x\ (l+m-2)=-\ (l-1)$$

$$\therefore\ x=\frac{l-1}{l+m-2}$$

It can be easily seen that $f''(x)<0$ at $x=\dfrac{l-1}{l+m-2}$

So, Mode of the distribution is $\dfrac{l-1}{l+m-2}$.

Theorem: Transformation of Beta variate of first kind to Beta variate of second kind.

Or, How beta distribution of first kind is transferred to beta distribution of second kind **(NU-12, 14)**

Or, How will you find the beta distribution of second kind from beta distribution of first kind? **(NU-15)**

Proof: If x is a beta variate of first kind with parameter l and m, so

$$f(x;\ l, m) = \frac{1}{\beta(l, m)}\ x^{l-1}\ (1-x)^{m-1};\quad 0 < x < 1,\quad l > 0,\quad m > 0$$

Let, $y = \frac{1-x}{x}$

$\Rightarrow x\ y = 1 - x$

$\Rightarrow x + x\ y = 1$

$\Rightarrow x\ (1+y) = 1$

$\Rightarrow x = \frac{1}{1+y}$

$\therefore\ \frac{dx}{dy} = -\frac{1}{(1+y)^2}$

So, $|J| = \frac{1}{(1+y)^2}$

If $x = 0$, then $y = \infty$

$x = 1$, then $y = 0$.

Now, the pdf of y is given by

$$f(y) = f(x)\ |J|$$

$$= \frac{1}{\beta(l,\ m)} \left(\frac{1}{1+y}\right)^{l-1} \left(1 - \frac{1}{1+y}\right)^{m-1} \cdot\ \frac{1}{(1+y)^2}$$

$$= \frac{1}{\beta(l,\ m)}\ \frac{1}{(1+y)^{l-1}}\ \frac{(1+y-1)^{m-1}}{(1+y)^{m-1}} \cdot\ \frac{1}{(1+y)^2}$$

$$= \frac{1}{\beta(l,\ m)} \quad \frac{y^{m-1}}{(1+y)^{l-1+m-1+2}}$$

$$= \frac{1}{\beta(m,\ l)} \quad \frac{y^{m-1}}{(1+y)^{m+l}}$$

$$\therefore \quad f(y) = \beta_2(m,\ l)$$

That is $y = \frac{1-x}{x}$ is a Beta variate of second kind with parameter m and l.

NB: (i) $\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$

(ii) The pdf of second kind of beta distribution is

$$f(x;\ l,\ m) = \frac{x^{l-1}}{\beta(l,\ m)\ (1+x)^{l+m}}; \quad x > 0,\ l > 0,\ m > 0$$

Where l and m are the two parameters of the distribution.

Short questions and answers:

(1) Define beta distribution of first/ 1st kind (NU-11, 13)

Or, What is beta distribution of first kind? (NU-12, 14)

Ans: A continuous random variable x is said to have a beta distribution of the first kind if its pdf is given by

$$f(x;\ l,\ m) = \frac{x^{l-1}\ (1-x)^{m-1}}{\beta(l,\ m)}; \quad 0 < x < 1, \quad l > 0,\ m > 0$$

where l and m are the two parameters of the distribution.

This is the standard form of the beta distribution.

(2) If $X \sim \beta_1(l,\ m)$ then what is the mean of variate x? (NU-14)

Ans: If $X \sim \beta_1(l,\ m)$ then mean of variate x is

$$\text{Mean } = \frac{l}{l+m}$$

NB: Various Measures-

(i) The r-th moment about origin, usually denoted by μ'_r is defined as

$$\mu'_r = E[x^r] = \begin{cases} \sum x^r\ f(x); & \text{if x is discrete} \\ \int_{-\infty}^{\infty} x^r\ f(x)\ dx; & \text{if x is continuous} \end{cases}$$

(ii) The r-th moment about point 'a', usually denoted by $\mu'_r(a)$ is defined as

$$\mu'_r(a) = E[x-a]^r = \begin{cases} \sum (x-a)^r\ f(x); & \text{if x is discrete} \\ \int_{-\infty}^{\infty} (x-a)^r\ f(x)\ dx; & \text{if x is continuous} \end{cases}$$

(iii) The r-th central or corrected moment about mean μ, usually denoted by μ_r is defined as

$$\mu_r = E[x-\mu]^r = \begin{cases} \sum (x-\mu)^r\ f(x); & \text{if x is discrete} \\ \int_{-\infty}^{\infty} (x-\mu)^r\ f(x)\ dx; & \text{if x is continuous} \end{cases}$$

(iv) **Median**: Median is the point which divides the total area two equal parts. Thus, if M is the median, then

$$\int_a^M f(x)\ dx = \int_M^b f(x)\ dx = \frac{1}{2}$$

Thus solving $\int_a^M f(x)\ dx = \frac{1}{2}$ or $\int_M^b f(x)\ dx = \frac{1}{2}$ for M, we get the value of median.

(v) **Mean Deviation**: Mean deviation about mean is given by

$$MD = \int_a^b |x - \text{mean}|\ f(x)\ dx$$

(vi) **Mode**: Mode is the value of x for which $f(x)$ is maximum. Thus, Mode is given by $f'(x) = 0$ and $f''(x) < 0$

Beta distribution of second kind

Definition: A continuous random variable X is said to have a beta distribution of second kind if its pdf is given by

$$f(x;\ l,\ m) = \frac{x^{l-1}}{\beta(l,\ m)\ (1+x)^{l+m}}; \quad x > 0, \quad l > 0, \quad m > 0$$

where l and m are the two parameters of the distribution.

NB: $\int_0^\infty f(x;\ l,\ m)\ dx = 1$

$$\Rightarrow \int_0^\infty \frac{x^{l-1}}{\beta\,(l,\ m)\ (1+x)^{l+m}}\ dx = 1$$

$$\Rightarrow \frac{1}{\beta(l,\ m)} \int_0^\infty \frac{x^{l-1}}{(1+x)^{l+m}}\ dx = 1$$

$$\therefore \int_0^\infty \frac{x^{l-1}}{(1+x)^{l+m}}\ dx = \beta(l,\ m)$$

Q. Find the r-th raw moment and hence find and variance of the beta distribution of second kind.

Or, Find the r-th moment of beta distribution of 2^{nd} kind and hence find mean and variance (NU-13, 16)

Ans: Let x is a beta variate of second kind with parameter l and m. So, the pdf of x is

$$f(x) = \frac{x^{l-1}}{\beta(l,\ m)\ (1+x)^{l+m}}; \quad x > 0, \quad l > 0, \quad m > 0$$

So, r-th raw moment about origin is

$$\mu_r' = E(x^r) = \int_0^\infty x^r\ f(x)\ dx$$

$$= \int_0^\infty x^r \cdot \frac{x^{l-1}}{\beta(l,\ m)\ (1+x)^{l+m}}\ dx$$

$$= \frac{1}{\beta(l,\ m)} \int_0^\infty \frac{x^{l+r-1}}{(1+x)^{l+r+m-r}}\, dx$$

$$= \frac{1}{\beta(l, m)}. \quad \beta(l+r,\ \ m-r)$$

$$= \frac{\Gamma(l+m)}{\Gamma l \ \Gamma m}. \quad \frac{\Gamma(l+r)\ \Gamma(m-r)}{\Gamma(l+r+m-r)}$$

$$\therefore \ \ \mu_r' = \frac{\Gamma(l+r)\ \ \Gamma(m-r)}{\Gamma l \ \ \Gamma m} \text{ ....(i)}$$

Now, putting $r=1$ in (i), we get

$$\mu_1' = \frac{\Gamma(l+1)\ \ \Gamma(m-1)}{\Gamma l \ \ \Gamma m}$$

$$= \frac{l.\ \Gamma l \ \ \Gamma(m-1)}{\Gamma l.\ (m-1)\ \Gamma(m-1)}$$

$\therefore$ Mean, $\mu_1' = \dfrac{l}{m-1}$

Again, putting $r = 2$ in (i), we get

$$\mu_2' = \frac{\Gamma(l+2)\ \ \Gamma(m-2)}{\Gamma l \ \ \Gamma m}$$

$$= \frac{(l+1)\ l\ \Gamma l \ \ \Gamma(m-2)}{\Gamma l\ (m-1)\ (m-2)\ \Gamma(m-2)}$$

$$= \frac{l\ (l+1)}{(m-1)\ (m-2)}$$

So, Variance, $\mu_2 = \mu_2' - \mu_1'^2$

$$= \frac{l\,(l+1)}{(m-1)\,(m-2)} - \frac{l^2}{(m-1)^2}$$

$$= \frac{(l^2+l)(m-1)-l^2(m-2)}{(m-1)^2(m-2)}$$

$$= \frac{l^2 m - l^2 + l m - l - l^2 m + 2 l^2}{(m-1)^2(m-2)}$$

$$= \frac{l^2 + l\, m - l}{(m-1)^2(m-2)}$$

$$\therefore \text{ Variance, } \mu_2 = \frac{l\,(l+m-1)}{(m-1)^2(m-2)}$$

Q. Find the Harmonic mean of beta distribution of 2nd kind (NU-11, 14, 16)

Ans: We know the pdf of beta distribution of second kind with parameters l and m is

$$f(x) = \frac{x^{l-1}}{\beta(l,\ m)\ (1+x)^{l+m}}; \qquad 0 < x < \infty$$

By definition of Harmonic mean, we can write

$$\text{HM} = \frac{1}{E\left(\frac{1}{x}\right)} \ \ldots\ldots\ldots.(i)$$

$$\text{Now, } E\left(\frac{1}{x}\right) = \int_0^\infty \frac{1}{x} f(x)\, dx$$

$$= \int_0^\infty \frac{1}{x}.\ \frac{x^{l-1}}{\beta(l,\ m)\ (1+x)^{l+m}}\ dx$$

$$= \frac{1}{\beta(l,\ m)} \int_0^\infty \frac{x^{l-2}}{(1+x)^{l+m}}\ dx$$

$$= \frac{1}{\beta(l,\ m)} \int_0^\infty \frac{x^{(l-1)-1}}{(1+x)^{l-1+m+1}}\ dx$$

$$= \frac{1}{\beta(l,\ m)}.\quad \beta(l-1,\ m+1)$$

$$= \frac{1}{\beta(l,\ m)}. \quad \beta(l-1,\ m+1)$$

$$= \frac{\overline{|l+m}}{\overline{|l}\ \overline{|m}}. \ \frac{\overline{|l-1}\ \overline{|m+1}}{\overline{|l-1+m+1}}$$

$$= \frac{\overline{|l-1}\ \overline{|m+1}}{\overline{|l}\ \overline{|m}}$$

$$\therefore\ E\left(\frac{1}{x}\right) = \frac{m}{l-1}$$

So, (i) can be written as

$$HM = \frac{1}{E\left(\frac{1}{x}\right)} = \frac{1}{\frac{m}{l-1}} = \frac{l-1}{m}$$

Q. Find the Mode of the Beta distribution of second kind (NU-13)

Ans: We know the pdf of beta distribution of second kind with parameters l and m is

$$f(x) = \frac{1}{\beta(l,\ m)} \ \frac{x^{l-1}}{(1+x)^{l+m}}; \quad x > 0 \text{ ....(i)}$$

The mode of the distribution will be the solution of $f'(x) = 0$, provided $f''(x) < 0$.

Taking log on both sides of (i), we get

$$\log\ f(x) = \log\left\{\frac{1}{\beta(l,\ m)}\right\} + (l-1)\log x - (l+m)\ \log\ (1+x) \text{ ....(ii)}$$

Now, differentiating (ii) w.r.t x, we have

$$\frac{1}{f(x)}\ f'(x) = (l-1).\ \frac{1}{x} - (l+m).\ \frac{1}{1+x}$$

$$\Rightarrow f'(x) = \left(\frac{l-1}{x} - \frac{l+m}{1+x}\right)\ f(x)$$

Using $f'(x) = 0$, we get

$$\left(\frac{l-1}{x} - \frac{l+m}{1+x}\right)\ f(x) = 0$$

$$\Rightarrow \frac{l-1}{x} - \frac{l+m}{1+x} = 0 \qquad [\because\ f(x) \neq 0]$$

$$\Rightarrow \frac{(l-1)(1+x)-x(l+m)}{x(1+x)}=0$$

$$\Rightarrow l+lx-1-x-lx-mx=0$$

$$\Rightarrow l-1-x-mx=0$$

$$\Rightarrow -mx-x=-l+1$$

$$\Rightarrow -x(m+1)=-(l-1)$$

$$\therefore \quad x=\frac{l-1}{m+1}$$

It can be easily seen that $f''(x)<0$ at $x=\frac{l-1}{m+1}$

So, Mode of the distribution is $\frac{l-1}{m+1}$.

Theorem: Transformation of beta variate of second kind to beta variate of first kind.

Or, How will you find the beta distribution of first kind from beta distribution of second kind? (NU-15)

Or, If x is a beta variate of second kind with parameter l and m, then $y=\frac{1}{1+x}$ is a beta variate of first kind with parameter m and l i.e., if $x\sim\beta_2(l,\ m)$ then $y=\frac{1}{1+x}\sim\beta_1(l,\ m)$.

Ans: Since $x\sim\beta_2(l,\ m)$, so

$$f(x)=\frac{1}{\beta(l,\ m)}\ \frac{x^{l-1}}{(1+x)^{l+m}};\qquad x>0,\quad l>0,\quad m>0$$

Let $y=\frac{1}{1+x}$

$$\Rightarrow 1+x=\frac{1}{y}$$

$$\Rightarrow x=\frac{1}{y}-1$$

$$\Rightarrow x = \frac{1-y}{y}$$

$$\therefore \ \frac{dx}{dy} = \frac{y(-1)-(1-y).1}{y^2} = \frac{-y-1+y}{y^2} = -\frac{1}{y^2}$$

$$\text{So, } |J| = \frac{1}{y^2}$$

When, $x=0$ then $y=1$ and when $x=\infty$, then $y=0$.

The pdf of y is given by

$$f(y) = f(x)\ |J|$$

$$= \frac{1}{\beta(l,\ m)} \ \frac{\left(\frac{1-y}{y}\right)^{l-1}}{\left(\frac{1}{y}\right)^{l+m}} . \ \frac{1}{y^2}$$

$$= \frac{1}{\beta(l,m)}\ (1-y)^{l-1}\ y^{-l+1}\ y^{l+m}\ y^{-2}$$

$$= \frac{1}{\beta(l,m)}\ (1-y)^{l-1}\ y^{-l+1+l+m-2}$$

$$= \frac{1}{\beta(m,\ l)}\ y^{m-1}\ (1-y)^{l-1}$$

$$= \beta_1(m,\ l)$$

That is $y = \frac{1}{1+x}$ is a Beta variate of first kind with parameter m and l.

NB: (i) $\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\ \frac{du}{dx} - u\ \frac{dv}{dx}}{v^2}$

(ii) The pdf of first kind of beta distribution is

$$f(x;\ l,\ m) = \frac{x^{l-1}\ (1-x)^{m-1}}{\beta(l,\ m)}; \qquad 0 < x < 1, \quad l > 0, \quad m > 0$$

Theorem: If $x \sim \beta_2(l,\ m)$**, then** $y = \dfrac{1}{x} \sim \beta_2(m,\ l)$**.**

Proof: Since $x \sim \beta_2(l, m)$

$$f(x) = \frac{1}{\beta(l,\ m)}\ \frac{x^{l-1}}{(1+x)^{l+m}}; \qquad x > 0, \quad l > 0, \quad m > 0$$

Let $\mathrm{y} = \dfrac{1}{\mathrm{x}}$

$$\Rightarrow \mathrm{x} = \frac{1}{\mathrm{y}}$$

$$\therefore\ \frac{\mathrm{dx}}{\mathrm{dy}} = -\frac{1}{\mathrm{y}^2}$$

So, $|\mathrm{J}| = \dfrac{1}{\mathrm{y}^2}$

If $\mathrm{x} = 0$ then $\mathrm{y} = \infty$ and if $\mathrm{x} = \infty$, then $\mathrm{y} = 0$.

The pdf of y is given by

$$\mathrm{f}(\mathrm{y}) = \mathrm{f}(\mathrm{x})\ |\mathrm{J}|$$

$$= \frac{1}{\beta(l,\ m)}\ \frac{\left(\frac{1}{y}\right)^{l-1}}{\left(1+\frac{1}{y}\right)^{l+m}} \cdot \frac{1}{y^2}$$

$$= \frac{1}{\beta(l,\ m)}\ \frac{y^{l+m}}{y^{l-1}\ (1+y)^{l+m}} \cdot \frac{1}{y^2}$$

$$= \frac{1}{\beta(l,\ m)}\ \frac{y^{l+m-l+1-2}}{(1+y)^{l+m}}$$

$$= \frac{1}{\beta(m,\ l)}\ \frac{y^{m-1}}{(1+y)^{l+m}}$$

$$= \beta_2(m,\ l)$$

That is $\mathrm{y} = \dfrac{1}{\mathrm{x}}$ is a Beta variate of second kind with parameter m and l.

Q. Show that $\beta(m, n) = \dfrac{\Gamma m \ \Gamma n}{\Gamma m+n}$ **(NU-15)**

Proof: We know, $\Gamma m = \int_0^\infty e^{-x} \ x^{m-1} \ dx$

Let, $x = \lambda y$

$\therefore \ dx = \lambda . dy$

When $x = 0$ then $y = 0$ and when $x = \infty$ then $y = \infty$

So, $\Gamma m = \int_0^\infty e^{-\lambda y} \ (\lambda y)^{m-1} . \ \lambda \ dy$

$\Rightarrow \Gamma m = \int_0^\infty e^{-\lambda y} \ \lambda^m \ y^{m-1} \ dy$............(i)

Again, $\Gamma n = \int_0^\infty e^{-x} \ x^{n-1} \ dx \qquad \left[\because \ \Gamma m = \int_0^\infty e^{-x} \ x^{m-1} \ dx\right]$

Let, $x = \lambda$

$\therefore \ dx = d\lambda$

When $x = 0$ then $\lambda = 0$ and when $x = \infty$ then $\lambda = \infty$

So, $\Gamma n = \int_0^\infty e^{-\lambda} \ \lambda^{n-1} \ d\lambda$............(ii)

Now multiplying (i) and (ii), we get

$$\Gamma m \ \Gamma n = \int_0^\infty \int_0^\infty e^{-(1+y)\lambda} \ \lambda^{m+n-1} \ y^{m-1} \ d\lambda \ dy$$

$$= \int_0^\infty \left[\int_0^\infty e^{-(1+y)\lambda} \ \lambda^{m+n-1} \ d\lambda\right] \ y^{m-1} \ dy$$

$$= \int_0^\infty \frac{\Gamma m+n}{(1+y)^{m+n}} \ y^{m-1} \ dy \qquad \left[\because \ \int_0^\infty e^{-\beta x} \ x^{\alpha-1} \ dx = \frac{\Gamma \alpha}{\beta^\alpha}\right]$$

$$= \Gamma m+n \int_0^\infty \frac{y^{m-1}}{(1+y)^{m+n}} \ dy$$

$$\Rightarrow \Gamma m \ \Gamma n = \Gamma m+n \ \ \beta(m, n)$$

$$\therefore \ \beta(m, n) = \frac{\Gamma m \ \Gamma n}{\Gamma m+n}$$

NB: The pdf of gamma distribution is

$$f(x;\ \alpha,\ \beta)=\frac{\beta^{\alpha}\ e^{-\beta x}\ x^{\alpha-1}}{\lceil\alpha};\qquad x>0,\quad \alpha>0,\quad \beta>0\ \ \text{.............(i)}$$

and $\int_0^{\infty} f(x;\ \alpha,\ \beta)\ dx=1$

$$\Rightarrow\int_0^{\infty}\frac{\beta^{\alpha}\ e^{-\beta x}\ x^{\alpha-1}}{\lceil\alpha}\ dx=1$$

$$\Rightarrow\int_0^{\infty} e^{-\beta x}\ x^{\alpha-1}\ dx=\frac{\lceil\alpha}{\beta^{\alpha}}$$

(i) If $\beta=1$, then (i) reduces to the form

$$f(x;\ \alpha)=\frac{e^{-x}\ x^{\alpha-1}}{\lceil\alpha};\qquad x>0,\quad \alpha>0$$

which is called standard gamma distribution with parameter α

(ii) $\int_0^{\infty} f(x;\ \alpha)\ dx=1$

$$\Rightarrow\int_0^{\infty}\frac{e^{-x}\ x^{\alpha-1}}{\lceil\alpha}\ dx=1$$

$$\Rightarrow\lceil\alpha=\int_0^{\infty} e^{-x}\ x^{\alpha-1}\ dx$$

So, $\lceil m=\int_0^{\infty} e^{-x}\ x^{m-1}\ dx$

Similarly, $\lceil n=\int_0^{\infty} e^{-x}\ x^{n-1}\ dx$

P-762, Advanced Calculus-I, Titas Math Series.

(iii) The pdf of 2^{nd} kind of Beta distribution is given by

$$f(x;\ l,\ m)=\frac{x^{l-1}}{\beta(l,\ m)\ (1+x)^{l+m}};\qquad x>0,\quad l>0,\quad m>0$$

and $\int_0^\infty f(x;\ l, m)\, dx = 1$

$$\Rightarrow \int_0^\infty \frac{x^{l-1}}{\beta(l,\ m)\ (1+x)^{l+m}}\, dx = 1$$

$$\Rightarrow \frac{1}{\beta(l,\ m)} \int_0^\infty \frac{x^{l-1}}{(1+x)^{l+m}}\, dx = 1$$

$$\therefore \int_0^\infty \frac{x^{l-1}}{(1+x)^{l+m}}\, dx = \beta(l, m)$$

Q. Find out the value of $\beta\left(\frac{1}{2}, \frac{1}{2}\right)$. Hence find out the value of $\Gamma\left(\frac{1}{2}\right)$ (NU-2020)

Ans: We know, $\beta(l,\ m) = \int_0^1 x^{l-1}\ (1-x)^{m-1}\ dx$

So, $\beta\left(\frac{1}{2}, \frac{1}{2}\right) = \int_0^1 x^{\frac{1}{2}-1}\ (1-x)^{\frac{1}{2}-1}\, dx$

$$= \int_0^1 \frac{1}{\sqrt{x}}\ \frac{1}{\sqrt{1-x}}\, dx$$

$$= \int_0^1 \frac{1}{\sqrt{x-x^2}}\, dx$$

$$= \int_0^1 \frac{1}{\sqrt{\left(\frac{1}{2}\right)^2 - \left(x-\frac{1}{2}\right)^2}}\, dx$$

$$= \left[\mathrm{Sin}^{-1} \frac{x-\frac{1}{2}}{\frac{1}{2}}\right]_0^1$$

$$= \mathrm{Sin}^{-1} \frac{1-\frac{1}{2}}{\frac{1}{2}} - \mathrm{Sin}^{-1} \frac{-\frac{1}{2}}{\frac{1}{2}}$$

$$= \mathrm{Sin}^{-1}(1) - \mathrm{Sin}^{-1}(-1)$$

$$= \text{Sin}^{-1}(1) + \text{Sin}^{-1}(1)$$

$$= \frac{\pi}{2} + \frac{\pi}{2}$$

$$\therefore\ \beta\left(\frac{1}{2}, \frac{1}{2}\right) = \frac{\pi}{2}$$

Again, $\beta\left(\frac{1}{2}, \frac{1}{2}\right) = \frac{\pi}{2}$

$$\Rightarrow \frac{\Gamma\frac{1}{2}\ \Gamma\frac{1}{2}}{\Gamma\left(\frac{1}{2}+\frac{1}{2}\right)} = \frac{\pi}{2}$$

$$\Rightarrow \left(\Gamma\frac{1}{2}\right)^2 = \frac{\pi}{2} \qquad \left[\because\ \Gamma 1 = 1\right]$$

$$\therefore\ \Gamma\frac{1}{2} = \sqrt{\frac{\pi}{2}}$$

NB: (i) The pdf of beta distribution of the first kind with parameters l and m is

$$f(x;\ l, m) = \frac{x^{l-1}\ (1-x)^{m-1}}{\beta(l,\ m)};\quad 0 < x < 1,\quad l > 0,\quad m > 0$$

Now, $\int_0^1 f(x;\ l, m)\ dx = 1$

$$\Rightarrow \int_0^1 \frac{x^{l-1}\ (1-x)^{m-1}}{\beta(l,\ m)}\ dx = 1$$

$$\Rightarrow \frac{1}{\beta(l,\ m)} \int_0^1 x^{l-1}\ (1-x)^{m-1}\ dx = 1$$

$$\therefore\ \int_0^1 x^{l-1}\ (1-x)^{m-1}\ dx = \beta(l,\ m)$$

(ii) $\int \frac{dx}{\sqrt{a^2 - x^2}} = \text{Sin}^{-1}\left(\frac{x}{a}\right)$

(iii) $\text{Sin}(-\theta) = -\theta$, (P-252, Intermediate book, Akkharpatra prokashani)

(iv) $\text{Sin}^{-1}(-1) = \dfrac{1}{\text{Sin}(-1)} = \dfrac{1}{-\text{Sin}(1)} = -\text{Sin}^{-1}(1)$

Short questions and answers:

(1) Define beta distribution of second kind (NU-11, 14)

Ans: A continuous random variable x is said to have a beta distribution of second kind if its pdf is given by

$$f(x;\ l,\ m) = \frac{x^{l-1}}{\beta(l,\ m)\ (1+x)^{l+m}}; \quad x>0,\ \ l>0,\ \ m>0$$

Where, l and m are the two parameters of the distribution.

(2) If $f(x) = \dfrac{x^{m-1}}{\beta(m,\ n)\ (1+x)^{m+n}};\ 0 < x < \infty$ **then what is the mean of variable x? (NU-12)**

Ans: The mean of variable x is $\dfrac{l}{m-1}$.

(3) Write down the p.d.f of beta distribution of 2nd kind (NU-15)

Ans: The pdf of beta distribution of 2nd kind is

$$f(x;\ l,\ m) = \frac{x^{l-1}}{\beta(l,\ m)\ (1+x)^{l+m}}; \quad x>0,\ \ l>0,\ \ m>0$$

Where, l and m are the two parameters of the distribution.

Gamma Distribution

Q. What is gamma distribution? (NU-14)

Or, Define gamma distribution **(NU-16)**

Ans: A continuous random variable X is said to have a gamma distribution with parameters α and β if its probability density function is given by

$$f(x;\ \alpha,\ \beta) = \frac{\beta^{\alpha}\ e^{-\beta x}\ x^{\alpha-1}}{\overline{\lfloor}\alpha};\qquad x > 0,\quad \alpha > 0,\quad \beta > 0$$

NB: The pdf of gamma distribution is

$$f(x;\ \alpha,\ \beta) = \frac{\beta^{\alpha}\ e^{-\beta x}\ x^{\alpha-1}}{\overline{\lfloor}\alpha};\qquad x > 0,\quad \alpha > 0,\quad \beta > 0\ \ ....(i)$$

(i) If $\beta = 1$, then (i) reduces to the form

$$f(x;\ \alpha) = \frac{e^{-x}\ x^{\alpha-1}}{\overline{\lfloor}\alpha};\qquad x > 0,\quad \alpha > 0$$

which is called standard gamma distribution with parameter α

(ii) If $\alpha = 1$, then (i) reduces to the form

$$f(x;\ \beta) = \beta\ e^{-\beta x};\qquad x > 0,\quad \beta > 0$$

which is the pdf of exponential distribution with parameter β

(iii) $\int_0^{\infty} f(x;\ \alpha,\ \beta)\, dx = 1$

$$\Rightarrow \int_0^{\infty} \frac{\beta^{\alpha}\ e^{-\beta x}\ x^{\alpha-1}}{\overline{\lfloor}\alpha}\, dx = 1$$

$$\Rightarrow \frac{\beta^{\alpha}}{\overline{\lfloor}\alpha} \int_0^{\infty} e^{-\beta x}\ x^{\alpha-1}\, dx = 1$$

$$\therefore \int_0^{\infty} e^{-\beta x}\ x^{\alpha-1}\, dx = \frac{\overline{\lfloor}\alpha}{\beta^{\alpha}}$$

(iv) $\int_0^{\infty} f(x;\ \alpha, \beta)\, dx = 1$

$$\Rightarrow \int_0^\infty \frac{\beta^\alpha\ e^{-\beta x}\ x^{\alpha-1}}{\Gamma\alpha}\, dx = 1$$

$$\therefore \int_0^\infty \beta^\alpha\ e^{-\beta x}\ x^{\alpha-1}\, dx = \Gamma\alpha = (\alpha-1)!$$

Q. Find the mgf of gamma distribution and hence find mean, variance, β_1 and β_2.

Or, Find β_1 and β_2 of gamma distribution **(NU-16)**

Ans: Let x is a gamma variate with parameters α and β, so the pdf of x is

$$f(x;\ \alpha,\ \beta) = \frac{\beta^\alpha\ e^{-\beta x}\ x^{\alpha-1}}{\Gamma\alpha}; \qquad x>0,\quad \alpha>0,\quad \beta>0$$

Now, the mgf of x is

$$M_x(t) = E\left(e^{tx}\right) = \int_0^\infty e^{tx}\ f(x)\, dx$$

$$= \int_0^\infty e^{tx}.\ \frac{\beta^\alpha\ e^{-\beta x}\ x^{\alpha-1}}{\Gamma\alpha}\, dx$$

$$= \frac{\beta^\alpha}{\Gamma\alpha} \int_0^\infty e^{-\beta x + tx}\ x^{\alpha-1}\, dx$$

$$= \frac{\beta^\alpha}{\Gamma\alpha} \int_0^\infty e^{-(\beta-t)x}\ x^{\alpha-1}\, dx$$

$$= \frac{\beta^\alpha}{\Gamma\alpha}.\ \frac{\Gamma\alpha}{(\beta-t)^\alpha} \qquad \left[\because \int_0^\infty e^{-\beta x}\ x^{\alpha-1}\, dx = \frac{\Gamma\alpha}{\beta^\alpha}\right]$$

$$= \frac{\beta^\alpha}{(\beta-t)^\alpha}$$

$$= \frac{\beta^\alpha}{\left\{\beta\left(1-\frac{t}{\beta}\right)\right\}^\alpha}$$

$$= \frac{\beta^\alpha}{\beta^\alpha\left(1-\frac{t}{\beta}\right)^\alpha}$$

$$=\frac{1}{\left(1-\frac{t}{\beta}\right)^{\alpha}}$$

$$\therefore\ \ M_x(t)=\left(1-\frac{t}{\beta}\right)^{-\alpha}$$

Now, cumulant generating function,

$$K_x(t)=\log M_x(t)$$

$$=\log\left(1-\frac{t}{\beta}\right)^{-\alpha}$$

$$=-\alpha\ \log\left(1-\frac{t}{\beta}\right)$$

$$=\alpha\left[\frac{t}{\beta}+\frac{t^2}{2\,\beta^2}+\frac{t^3}{3\,\beta^3}+\frac{t^4}{4\,\beta^4}+\cdots\right]$$

$$=\frac{\alpha}{\beta}\frac{t}{1!}+\frac{\alpha}{\beta^2}.\frac{t^2}{2!}+\frac{2\alpha}{\beta^3}.\frac{t^3}{3!}+\frac{6\alpha}{\beta^4}.\frac{t^4}{4!}+\cdots$$

We know, $K_r=\text{Coefficient of }\frac{t^r}{r!}\text{ in }\ K_x(t)$

Now, putting $r=1,\ 2,\ 3,\ 4$ we get respectively,

$$K_1=\text{Coefficient of }\frac{t}{1!}\text{ in }\ K_x(t)=\frac{\alpha}{\beta}$$

$$K_2=\text{Coefficient of }\frac{t^2}{2!}\text{ in }\ K_x(t)=\frac{\alpha}{\beta^2}$$

$$K_3=\text{Coefficient of }\frac{t^3}{3!}\text{ in }\ K_x(t)=\frac{2\,\alpha}{\beta^3}$$

$$K_4=\text{Coefficient of }\frac{t^4}{4!}\text{ in }\ K_x(t)=\frac{6\,\alpha}{\beta^4}$$

So, Mean, $\mu_1'=K_1=\frac{\alpha}{\beta}$

Variance, $\mu_2=K_2=\frac{\alpha}{\beta^2}$

$$\mu_3 = K_3 = \frac{2\,\alpha}{\beta^3}$$

and $$\mu_4 = K_4 + 3\,K_2{}^2 = \frac{6\,\alpha}{\beta^4} + 3.\ \frac{\alpha^2}{\beta^4} = \frac{3\,\alpha^2 + 6\,\alpha}{\beta^4}$$

$$\beta_1 = \frac{\mu_3{}^2}{\mu_2{}^3} = \frac{4\,\alpha^2}{\beta^6} \times \frac{\beta^6}{\alpha^3} = \frac{4}{\alpha}$$

and $$\beta_2 = \frac{\mu_4}{\mu_2{}^2} = \frac{3\,\alpha^2 + 6\,\alpha}{\beta^4} \times \frac{\beta^4}{\alpha^2} = 3 + \frac{6}{\alpha}$$

Since, $\beta_1 = \frac{4}{\alpha} > 0$ so the distribution is positively skewed and since $\beta_2 = 3 + \frac{6}{\alpha} > 3$, so the distribution is leptokurtic.

NB: (i) If $\alpha \to \infty$, $\beta_1 \to 0$ and $\beta_2 \to 3$, then gamma distribution tends to normal distribution as α becomes very large.

(ii) $$\log(1 - x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \cdots$$

(iii) The pdf of gamma distribution with parameters α and β is

$$f\left(x;\ \alpha,\ \beta\right) = \frac{\beta^{\alpha}\ e^{-\beta x}\ x^{\alpha - 1}}{\lceil\overline{\alpha}};\qquad x > 0,\quad \alpha > 0,\quad \beta > 0$$

The mgf is, $M_x(t) = \left(1 - \frac{t}{\beta}\right)^{-\alpha}$ i.e., $x \sim G\left(\alpha,\ \beta\right)$

Now, $$M_{\bar{x}}(t) = M_{\left(\frac{x_1 + x_2 + \cdots + x_n}{n}\right)}(t)$$

$$= M_{(x_1 + x_2 + \cdots + x_n)}\left(\frac{t}{n}\right)$$

$$= M_{x_1}\left(\frac{t}{n}\right).\ \ M_{x_2}\left(\frac{t}{n}\right) \cdots M_{x_n}\left(\frac{t}{n}\right) \qquad \left[\because\ x_1,\ x_2, \cdots, x_n \text{ are independent}\right]$$

$$= \left(1 - \frac{t}{n\,\beta}\right)^{-\alpha}.\ \left(1 - \frac{t}{n\,\beta}\right)^{-\alpha} \cdots \left(1 - \frac{t}{n\,\beta}\right)^{-\alpha}$$

$$= \left\{ \left(1 - \frac{t}{n\beta}\right)^{-\alpha} \right\}^{n}$$

$$\therefore \quad M_{\bar{x}}(t) = \left(1 - \frac{t}{n\beta}\right)^{-n\alpha} \quad \text{i.e.,} \quad \bar{x} \sim G(n\,\alpha,\ n\,\beta)$$

Q. Find the mgf of standard gamma distribution and hence find mean, variance, β_1 and β_2.

Ans: We know, the pdf of standard gamma distribution with parameter α is

$$f(x;\ \alpha) = \frac{e^{-x}\ x^{\alpha-1}}{\Gamma\alpha}; \qquad x > 0, \quad \alpha > 0$$

Now, the mgf of x is

$$M_x(t) = E\left(e^{tx}\right) = \int_0^\infty e^{tx}\ f(x)\ dx$$

$$= \int_0^\infty e^{tx}.\ \frac{e^{-x}\ x^{\alpha-1}}{\Gamma\alpha}\ dx$$

$$= \frac{1}{\Gamma\alpha} \int_0^\infty e^{-x+tx}\ x^{\alpha-1}\ dx$$

$$= \frac{1}{\Gamma\alpha} \int_0^\infty e^{-(1-t)x}\ x^{\alpha-1}\ dx$$

$$= \frac{1}{\Gamma\alpha}.\ \frac{\Gamma\alpha}{(1-t)^{\alpha}} \qquad \left[\because\ \int_0^\infty e^{-\beta x}\ x^{\alpha-1} dx = \frac{\Gamma\alpha}{\beta^{\alpha}}\right]$$

$$\therefore\ M_x(t) = (1-t)^{-\alpha}$$

Now, cumulant generating function,

$$K_x(t) = \log\ M_x(t)$$

$$= \log(1-t)^{-\alpha}$$

$$= -\alpha\ \log(1-t)$$

$$= \alpha\ \left[t + \frac{t^2}{2} + \frac{t^3}{3} + \frac{t^4}{4} + \cdots\right]$$

$$= \alpha \ \frac{t}{1!} + \alpha. \ \frac{t^2}{2!} + 2\alpha. \ \frac{t^3}{3!} + 6\alpha. \ \frac{t^4}{4!} + \cdots$$

We know, $K_r =$ Coefficient of $\frac{t^r}{r!}$ in $K_x(t)$

Now, putting $r = 1, 2, 3, 4$ we get respectively,

$$K_1 = \text{Coefficient of } \frac{t}{1!} \text{ in } K_x(t) = \alpha$$

$$K_2 = \text{Coefficient of } \frac{t^2}{2!} \text{ in } K_x(t) = \alpha$$

$$K_3 = \text{Coefficient of } \frac{t^3}{3!} \text{ in } K_x(t) = 2\,\alpha$$

$$K_4 = \text{Coefficient of } \frac{t^4}{4!} \text{ in } K_x(t) = 6\,\alpha$$

So, Mean, $\mu_1' = K_1 = \alpha$

Variance, $\mu_2 = K_2 = \alpha$

$$\mu_3 = K_3 = 2\alpha$$

and $\mu_4 = K_4 + 3\,K_2^{\,2} = 6\,\alpha + 3\,\alpha^2$

$$\beta_1 = \frac{\mu_3^{\,2}}{\mu_2^{\,3}} = \frac{4\,\alpha^2}{\alpha^3} = \frac{4}{\alpha}$$

and $\beta_2 = \frac{\mu_4}{\mu_2^{\,2}} = \frac{6\,\alpha + 3\,\alpha^2}{\alpha^2} = 3 + \frac{6}{\alpha}$

Since, $\beta_1 = \frac{4}{\alpha} > 0$ so the distribution is positively skewed and since $\beta_2 = 3 + \frac{6}{\alpha} > 3$, so the distribution is leptokurtic.

Q. Find the harmonic mean and mode of gamma distribution (NU-15)

Ans: Harmonic mean of gamma distribution-

We know, the pdf of standard gamma distribution with parameter α is

$$f\left(x;\ \alpha\right) = \frac{e^{-x}\ x^{\alpha-1}}{\Gamma\alpha}; \qquad x > 0, \quad \alpha > 0$$

Suppose HM is the harmonic mean of the standard gamma distribution. By definition

$$\frac{1}{HM} = E\left(\frac{1}{x}\right) = \int_0^\infty \frac{1}{x} \quad f(x;\ \alpha)\,dx$$

$$= \int_0^\infty \frac{1}{x} \quad \frac{e^{-x} \ x^{\alpha-1}}{\lceil\alpha} \ dx$$

$$= \frac{1}{\lceil\alpha} \int_0^\infty e^{-x} \ x^{(\alpha-1)-1} \ dx$$

$$= \frac{1}{\lceil\alpha} \ \lceil\alpha-1 \qquad \left[\because \int_0^\infty e^{-x} \ x^{\alpha-1} \ dx = \lceil\alpha\right]$$

$$= \frac{1}{(\alpha-1)\lceil\alpha-1} \ \lceil\alpha-1 \qquad \left[\because \ \lceil\alpha = (\alpha-1) \ \lceil\alpha-1\right]$$

$$\Rightarrow \frac{1}{HM} = \frac{1}{\alpha-1}$$

$$\therefore \ HM = \alpha - 1$$

Mode of gamma distribution-

We know, the pdf of standard gamma distribution with parameter α is

$$f(x;\ \alpha) = \frac{e^{-x} \ x^{\alpha-1}}{\lceil\alpha}; \qquad x > 0, \quad \alpha > 0$$

$$\Rightarrow \log \ f(x;\ \alpha) = -\log\lceil\alpha - x + (\alpha-1)\log x \ \text{...........(i)}$$

Now differentiating (i) w.r.t x and equating to zero i.e.,

$$\frac{d\log f(x;\alpha)}{dx} = 0$$

$$\Rightarrow 0 - 1 + \frac{\alpha-1}{x} = 0$$

$$\Rightarrow \frac{\alpha-1}{x} = 1$$

$$\therefore \ x = \alpha - 1$$

which is the mode of gamma distribution when $\alpha > 1$.

NB: P-516, 518, M.K. Roy (New book)

Q. Show that for parameter $n \to \infty$, gamma distribution tends to normal distribution (NU-11)

Or, Show that for parameter $m \to \infty$, gamma distribution tends to normal distribution **(NU-13)**

Or, The standard gamma distribution tends to standard normal distribution as the value of the parameter m tends to infinity.

Ans: We know the pdf of standard gamma distribution with parameter m is

$$f(x;\ m) = \frac{e^{-x}\ x^{m-1}}{\overline{|m}}; \qquad x > 0, \quad m > 0$$

Moment generating function

$$M_x(t) = (1-t)^{-m}$$

We also know that $E(x) = V(x) = m$

Let us define the standardized variable, $z = \dfrac{x-m}{\sqrt{m}}$

The mgf of z is

$$M_z(t) = E\left(e^{tz}\right)$$

$$= E\left(e^{t\,\frac{x-m}{\sqrt{m}}}\right)$$

$$= E\left(e^{\frac{xt}{\sqrt{m}} - \sqrt{m}\,t}\right)$$

$$= E\left(e^{\frac{xt}{\sqrt{m}}}.\quad e^{-\sqrt{m}\,t}\right)$$

$$= e^{-\sqrt{m}\,t}\quad E\left(e^{\frac{t}{\sqrt{m}}\,x}\right)$$

$$= e^{-\sqrt{m}\,t}\quad M_x\left(\frac{t}{\sqrt{m}}\right) \qquad \left[\because\ M_x(t) = E\left(e^{tx}\right)\right]$$

$$\Rightarrow M_z(t) = e^{-\sqrt{m}\,t}\left(1-\frac{t}{\sqrt{m}}\right)^{-m} \qquad \left[\because\ M_x(t) = (1-t)^{-m}\right]$$

$$\therefore \quad K_z(t) = \log M_z(t) = -\sqrt{m}\ t - m\ \log\left(1-\frac{t}{\sqrt{m}}\right)$$

$$= -\sqrt{m}\ t - m\left(-\frac{t}{\sqrt{m}} - \frac{t^2}{2m} - \frac{t^3}{3\,m^{\frac{3}{2}}} - \cdots\right)$$

$$= -\sqrt{m}\ t + m\left(\frac{t}{\sqrt{m}} + \frac{t^2}{2m} + \frac{t^3}{3m\ \sqrt{m}} + \cdots\right)$$

$$= -\sqrt{m}\ t + \sqrt{m}\ t + \frac{t^2}{2} + \frac{t^3}{3\ \sqrt{m}} + \cdots$$

$$\Rightarrow K_z(t) = \frac{t^2}{2} + \frac{t^3}{3\ \sqrt{m}} + \cdots$$

$$\Rightarrow \lim_{m\to\infty} K_z(t) = \frac{t^2}{2}$$

$\therefore \lim_{m\to\infty} M_z(t) = e^{\frac{t^2}{2}}$; which is the mgf of standard normal variate.

Hence gamma variate can be approximated by standard normal variate if $m \to \infty$.

Q. If x is normal variate then show that $u = \frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2$ is a gamma variate with parameter $\frac{1}{2}$ (NU-13)

Ans: We are given $u = \frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2$; where $x \sim N(\mu, \sigma^2)$

So, we have, $f(x) = \frac{1}{\sigma\ \sqrt{2\pi}}\ e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$; $\quad -\infty < x < \infty$

Let, $u = \frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2$

$$\Rightarrow 2u = \left(\frac{x-\mu}{\sigma}\right)^2$$

$$\Rightarrow \sqrt{2u} = \frac{x-\mu}{\sigma}$$

$$\Rightarrow x = \mu + \sigma\ \sqrt{2u}$$

$$\Rightarrow x = \mu + \sigma\sqrt{2}\ u^{\frac{1}{2}}$$

Now, $\frac{dx}{du} = \sigma\sqrt{2}.\ \frac{1}{2}\ u^{\frac{1}{2}-1} = \frac{\sigma}{\sqrt{2}}\ u^{-\frac{1}{2}}$

$$\therefore \ |J| = \frac{\sigma}{\sqrt{2}} \ u^{-\frac{1}{2}}$$

So, the pdf of u is

$$f(u) = f(x). \ |J|$$

$$= \frac{1}{\sigma\sqrt{2\pi}} \ e^{-u}. \ \frac{\sigma}{\sqrt{2}} \ u^{-\frac{1}{2}}$$

$$= \frac{1}{2\sqrt{\pi}} \ e^{-u}. \ u^{-\frac{1}{2}}$$

$$\Rightarrow f(u) = \frac{1}{2\sqrt{\pi}} \ e^{-u}. \ u^{-\frac{1}{2}}$$

$$= \frac{1}{\Gamma\left(\frac{1}{2}\right)} e^{-u}. \ u^{\frac{1}{2}-1}; \ 0 < u < \infty \qquad \left[\because \sqrt{\pi} = \Gamma\left(\frac{1}{2}\right)\right]$$

The factor $\frac{1}{2}$ disappearing from the fact that total probability is unity.

So, $f(u) = \frac{1}{\Gamma\left(\frac{1}{2}\right)} e^{-u}. \ u^{\frac{1}{2}-1}; \quad 0 < u < \infty$

which is the pdf of standard form of gamma variate with parameter $\frac{1}{2}$.

Hence, $u = \frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2$ is a $\gamma\left(\frac{1}{2}\right)$ variate.

NB: We know the pdf of standard gamma distribution with parameter m is

$$f(x; \ m) = \frac{e^{-x} \ x^{m-1}}{\Gamma(m)}; \quad x > 0, \quad m > 0$$

Q. Find mean and variance of gamma distribution (NU-11, 13, 16)

Ans: The pdf of gamma distribution with parameters α and β is

$$f(x; \ \alpha, \ \beta) = \frac{\beta^{\alpha} \ e^{-\beta x} \ x^{\alpha-1}}{\Gamma(\alpha)}; \quad x > 0, \quad \alpha > 0, \quad \beta > 0$$

Now, Mean, $E(x) = \int_0^\infty x \quad f(x;\, \alpha,\, \beta)\, dx$

$$= \int_0^\infty x \quad \frac{\beta^\alpha \; e^{-\beta x} \; x^{\alpha-1}}{\Gamma(\alpha)}\, dx$$

$$= \frac{\beta^\alpha}{\Gamma(\alpha)} \int_0^\infty e^{-\beta x} \; x^{\alpha+1-1}\, dx$$

$$= \frac{\beta^\alpha}{\Gamma(\alpha)} \cdot \frac{\Gamma(\alpha+1)}{\beta^{\alpha+1}} \qquad \left[\because \int_0^\infty e^{-\beta x} \; x^{\alpha-1}\, dx = \frac{\Gamma(\alpha)}{\beta^\alpha} \right]$$

$$= \frac{1}{\Gamma(\alpha)} \cdot \frac{\alpha \; \Gamma(\alpha)}{\beta}$$

$$\therefore \quad E(x) = \frac{\alpha}{\beta}$$

Again, $E(x^2) = \int_0^\infty x^2 \; f(x)\, dx$

$$= \int_0^\infty x^2 \quad \frac{\beta^\alpha \; e^{-\beta x} \; x^{\alpha-1}}{\Gamma(\alpha)}\, dx$$

$$= \frac{\beta^\alpha}{\Gamma(\alpha)} \int_0^\infty e^{-\beta x} \; x^{(\alpha+2)-1}\, dx$$

$$= \frac{\beta^\alpha}{\Gamma(\alpha)} \cdot \frac{\Gamma(\alpha+2)}{\beta^{\alpha+2}} \qquad \left[\because \int_0^\infty e^{-\beta x} \; x^{\alpha-1}\, dx = \frac{\Gamma(\alpha)}{\beta^\alpha} \right]$$

$$= \frac{1}{\Gamma(\alpha)} \cdot \frac{(\alpha+1)\, \alpha \; \Gamma(\alpha)}{\beta^2}$$

$$\therefore \quad E(x^2) = \frac{\alpha\, (\alpha+1)}{\beta^2} = \frac{\alpha^2 + \alpha}{\beta^2}$$

Now, Variance $V(x) = E(x^2) - \{E(x)\}^2$

$$= \frac{\alpha^2 + \alpha}{\beta^2} - \frac{\alpha^2}{\beta^2}$$

$$= \frac{\alpha^2 + \alpha - \alpha^2}{\beta^2}$$

$$= \frac{\alpha}{\beta^2}$$

NB: (i) $\Gamma(\alpha+1) = \alpha\ \Gamma(\alpha)$

(ii) $\Gamma(\alpha) = (\alpha - 1)!$

(iii) $\Gamma(\alpha+1) = \alpha! = \alpha\ (\alpha-1)! = \alpha\ \Gamma(\alpha)$

Theorem: State and prove the additive property of gamma distribution (NU-12)

Statement: If $x_1, x_2, \ldots, x_k$ be k independent gamma variates with parameters $\alpha_1, \alpha_2, \ldots, \alpha_k$ then $\sum_{i=1}^{k} x_i = x_1 + x_2 + \cdots + x_k$ is also a gamma variate with parameter $\sum_{i=1}^{k} \alpha_i = \alpha_1 + \alpha_2 + \cdots + \alpha_k$.

Proof: We know the mgf of x_i is

$$M_{x_i}(t) = (1-t)^{-\alpha_i} \text{ ; for all } i = 1, 2, \ldots, k$$

So, $M_{x_1}(t) = (1-t)^{-\alpha_1}$

$M_{x_2}(t) = (1-t)^{-\alpha_2}$

....

$M_{x_k}(t) = (1-t)^{-\alpha_k}$

Now, the mgf of $\sum_{i=1}^{k} x_i$ is

$$M_{\sum_{i=1}^{k} x_i}(t) = E\left(e^{t \sum_{i=1}^{k} x_i}\right)$$

$$= E\left[e^{t\ (x_1 + x_2 + \cdots + x_k)} \right]$$

$$= E\left[e^{t x_1 + t x_2 + \cdots + t x_k} \right]$$

$$= E\left[e^{t x_1}.\ \ e^{t x_2} \cdots e^{t x_k}\right]$$

$$= E\,[e^{t x_1}].\ \ E\,[e^{t x_2}] \cdots E\,[e^{t x_k}] \quad [\because\ x_1, x_2, ..., x_k \text{ are independent }]$$

$$= M_{x_1}(t).\, M_{x_2}(t) \cdots M_{x_k}(t)$$

$$= (1-t)^{-\alpha_1}.(1-t)^{-\alpha_2} \cdots (1-t)^{-\alpha_k}$$

$$= (1-t)^{-(\alpha_1+\alpha_2+...+\alpha_k)}$$

$$\therefore\ M_{\sum_{i=1}^{k} x_i}(t) = (1-t)^{-\sum_{i=1}^{k}\alpha_i}$$

Which is the mgf of gamma variate with parameter $\sum_{i=1}^{k}\alpha_i$.

Hence by uniqueness property of mgf, we have $x_1 + x_2 + \cdots + x_k$ is also a gamma variate with parameter $\alpha_1 + \alpha_2 + \cdots + \alpha_k$.

Q. If $x \sim G\,(l)$ **and** $y \sim G\,(m)$ **and they are independent, then**

a) Find the density of $U = \dfrac{x}{x+y}$

b) Find the density of $U = \dfrac{x}{y}$

Or, If x and y are two independent gamma variate then find the distribution of $z = \dfrac{x}{y}$ **(NU-11)**

Or, Let x and y be two independent gamma variates with parameters l and m respectively, then find the distribution of $z = \dfrac{x}{y}$ **(NU-13)**

Or, Let X and Y be two independent gamma variates with parameters l and m respectively, then find the distribution of $Z = \dfrac{X}{Y}$ **(NU-15)**

Solution: Since $x \sim G\,(l)$ and $y \sim G\,(m)$, so the pdf of x and y are respectively

$$f_1(x) = \frac{e^{-x}\ x^{l-1}}{\Gamma l}; \qquad x > 0, \quad l > 0$$

and $$f_2(y) = \frac{e^{-y}\ y^{m-1}}{\Gamma m}; \qquad y > 0, \quad m > 0$$

Since x and y are independent, so the joint pdf of x and y is

$$f(x,y) = f_1(x).\ f_2(y)$$

$$= \frac{e^{-x}\ x^{l-1}}{\lceil l}.\ \frac{e^{-y}\ y^{m-1}}{\lceil m}$$

$$= \frac{e^{-(x+y)}\ x^{l-1}\ y^{m-1}}{\lceil l\ \lceil m}$$

Given $U = \frac{x}{x+y}$ and let $V = x+y$

$\Rightarrow U = \frac{x}{V}$ $\qquad \Rightarrow V = U\,V + y$

$\Rightarrow x = U\,V$ $\qquad \Rightarrow y = V(1-U)$

Now, Jacobian of the transformation is

$$J = \begin{vmatrix} \frac{\partial x}{\partial U} & \frac{\partial x}{\partial V} \\ \frac{\partial y}{\partial U} & \frac{\partial y}{\partial V} \end{vmatrix} = \begin{vmatrix} V & U \\ -V & 1\text{-}U \end{vmatrix} = V - U\ V + U\ V = V$$

$\therefore\ |J| = V$

Since x and y range from 0 to ∞, so the range of U from 0 to 1 and range of V from 0 to ∞.

Hence the joint pdf of U and V is

$$f(U,V) = f(x,\ y)\ \ |J|$$

$$= \frac{e^{-V}\ (U\ V)^{l-1}\ \ \{V\ (1-U)\}^{m-1}}{\lceil l\ \ \lceil m}.\ \ V$$

$$= \frac{U^{l-1}\ (1-U)^{m-1}\ e^{-V}\ \ V^{l-1+m-1+1}}{\lceil l\ \ \lceil m}$$

$$= \frac{U^{l-1}\ \ (1-U)^{m-1}}{\lceil l\ \ \lceil m} e^{-V}\ \ V^{l+m-1}; \qquad 0 < U < 1, \quad 0 < V < \infty$$

Now, the marginal pdf of $U = \frac{x}{x+y}$ is

$$f(U) = \int_0^\infty f(U\,V)\ dV$$

$$= \frac{U^{l-1}\,(1-U)^{m-1}}{\Gamma l \;\; \Gamma m} \int_0^{\infty} e^{-V}\; V^{l+m-1}\, dV$$

$$= \frac{U^{l-1}\,(1-U)^{m-1}}{\Gamma l \;\; \Gamma m}.\;\; \Gamma(l+m) \qquad \left[\because \int_0^{\infty} e^{-\beta x}\; x^{\alpha-1}\, dx = \frac{\Gamma \alpha}{\beta^{\alpha}}\right]$$

$$\therefore \;\; \therefore \;\; f(U) = \frac{U^{l-1}\,(1-U)^{m-1}}{\beta(l,\;m)}; \qquad 0 < U < 1$$

which is the pdf of Beta distribution of first kind with parameter l and m.

Hence $U = \dfrac{x}{x+y} \sim \beta_1(l,\; m)$

(b) Since $x \sim G\,(l)$ and $\mathrm{y} \sim \mathrm{G}\ (\mathrm{m})$, so the pdf of x and y are respectively

$$f_1(x) = \frac{e^{-x}\; x^{l-1}}{\Gamma l}; \qquad x > 0,\;\; l > 0$$

and $$f_2(y) = \frac{e^{-y}\; y^{m-1}}{\Gamma m}; \qquad y > 0,\;\; m > 0$$

Since x and y are independent, so the joint pdf of x and y is

$$\mathrm{f}(\mathrm{x},\mathrm{y}) = \mathrm{f}_1(\mathrm{x}).\; \mathrm{f}_2(\mathrm{y})$$

$$= \frac{e^{-x}\; x^{l-1}}{\Gamma l}.\; \frac{e^{-y}\; y^{m-1}}{\Gamma m}$$

$$= \frac{e^{-(x+y)}\;\; x^{l-1}\;\; y^{m-1}}{\Gamma l \;\; \Gamma m}$$

Given $\mathrm{U} = \dfrac{\mathrm{x}}{\mathrm{y}}$ and let $\mathrm{V} = \mathrm{y}$

$\Rightarrow \mathrm{U} = \dfrac{\mathrm{x}}{\mathrm{V}}$ $\qquad \Rightarrow \mathrm{y} = \mathrm{V}$

$\Rightarrow \mathrm{x} = \mathrm{UV}$

Now, Jacobian of the transformation is

$$\mathrm{J} = \begin{vmatrix} \dfrac{\partial \mathrm{x}}{\partial \mathrm{U}} & \dfrac{\partial \mathrm{x}}{\partial \mathrm{V}} \\[2ex] \dfrac{\partial \mathrm{y}}{\partial \mathrm{U}} & \dfrac{\partial \mathrm{y}}{\partial \mathrm{V}} \end{vmatrix} = \begin{vmatrix} \mathrm{V} & \mathrm{U} \\ 0 & 1 \end{vmatrix} = \mathrm{V}$$

$\therefore \; |\mathrm{J}| = \mathrm{V}$

Since x and y range from 0 to ∞, so both U and V ranges from 0 to ∞.

Hence the joint pdf of U and V is

$$f(U, V) = f(x, y)\ |J|$$

$$= \frac{e^{-(x+y)}\ x^{l-1}\ y^{m-1}}{\Gamma l\ \Gamma m} .\ V$$

$$= \frac{e^{-(UV+V)}\ (UV)^{l-1}\ V^{m-1}}{\Gamma l\ \Gamma m} .\ V \qquad [\because\ x = UV \ \ and \ \ y = V]$$

$$= \frac{e^{-V(U+1)}\ U^{l-1}\ V^{l-1+m-1+1}}{\Gamma l\ \Gamma m}$$

$$= \frac{e^{-\ V(U+1)}\ U^{l-1}\ V^{l+m-1}}{\Gamma l\ \Gamma m}; \qquad 0 < U < \infty, \quad 0 < V < \infty$$

Hence the marginal pdf of $U = \frac{x}{y}$ is

$$f(U) = \int_0^\infty f(UV)\ dV$$

$$= \frac{U^{l-1}}{\Gamma l\ \Gamma m} \int_0^\infty e^{-\ (U+1)\ V}\ V^{l+m-1}\ dV$$

$$= \frac{U^{l-1}}{\Gamma l\ \Gamma m} . \frac{\Gamma(l+m)}{(U+1)^{l+m}} \qquad \left[\because\ \int_0^\infty e^{-\beta x}\ x^{\alpha-1}\ dx = \frac{\Gamma \alpha}{\beta^\alpha}\right]$$

$$\therefore \quad f(U) = \frac{U^{l-1}}{\beta(l,\ m)\ \ (1+U)^{l+m}}; \qquad 0 < U < \infty$$

which is the pdf of Beta distribution of 2nd kind with parameters l and m.

Hence $U = \frac{x}{y} \sim \beta_2(l,\ m)$

NB: (i) The pdf of beta distribution of first kind with parameters l and m is

$$f(x;\ \ l,\ \ m) = \frac{x^{l-1}\ \ (1-x)^{m-1}}{\beta(l,\ m)}; \qquad 0 < x < 1, \quad l > 0, \quad m > 0$$

(ii) The pdf of beta distribution of second kind with parameters l and m is

$$f(x)=\frac{x^{l-1}}{\beta(l,\ m)\ (1+x)^{l+m}};\qquad x>0,\quad l>0,\quad m>0$$

(iii) $\beta(l,\ m)=\dfrac{\Gamma l.\ \Gamma m}{\Gamma l+m}$

Q. If x and y are independent gamma variate with parameter l and m respectively and if $u=x+y$ and $v=\dfrac{x}{x+y}$ are independent, then show that u is a gamma variate with parameter $l+m$ and v is a beta variate of first kind with parameter l and m.

Proof: Since $x\sim G(l)$, so the pdf of x is

$$f(x)=\frac{e^{-x}\ x^{l-1}}{\Gamma l};\qquad x>0,\quad l>0$$

Again, since $y\sim G(m)$, so the pdf of y is

$$f(y)=\frac{e^{-y}\ y^{m-1}}{\Gamma m};\qquad y>0,\quad m>0$$

Since x and y are independent, so the joint pdf of x and y is

$$f(x,y)=f(x)\ f(y)$$

$$=\frac{e^{-x}\ x^{l-1}}{\Gamma l}.\quad\frac{e^{-y}\ y^{m-1}}{\Gamma m}$$

$$\Rightarrow f(x,\ y)=\frac{1}{\Gamma l\ \Gamma m}\ e^{-(x+y)}\ x^{l-1}\ y^{m-1};\qquad 0<x<\infty,\quad 0<y<\infty$$

Now, we have

$$u=x+y\quad\text{and}\qquad v=\frac{x}{x+y}$$

$$\Rightarrow u=u\,v+y\qquad\Rightarrow v=\frac{x}{u}$$

$$\Rightarrow y=u-u\,v\qquad\Rightarrow x=u\,v$$

$$\Rightarrow y=u\ (1-v)$$

The Jacobian of the transformation J is given by

$$J = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} v & u \\ 1-v & -u \end{vmatrix} = -uv - u + uv = -u$$

$\therefore \ |J| = u$

As the ranges of x and y from 0 to ∞, so the range of u from 0 to ∞ and the range of v from 0 to 1.

The joint pdf of u and v is

$$f(u,v) = f(x,y)\ |J|$$

$$= \frac{1}{\lceil l \ \lceil m}\ e^{-u}\ (u\ v)^{l-1}\ \{u\ (1-v)\}^{m-1}.\ u$$

$$= \frac{1}{\lceil l \ \lceil m}\ e^{-u}\ u^{l-1+m-1+1}\ v^{l-1}\ (1-v)^{m-1}$$

$$= \frac{e^{-u}\ u^{l+m-1}}{\lceil l+m}.\ \frac{\lceil l+m}{\lceil l \ \lceil m}\ v^{l-1}\ (1-v)^{m-1}$$

$$= \frac{e^{-u}\ u^{l+m-1}}{\lceil l+m}.\ \frac{v^{l-1}\ (1-v)^{m-1}}{\beta(l,\ m)}$$

$\therefore \ f(u,v) = f_1(u).\ f_2(v)$

Where, $f_1(u) = \dfrac{e^{-u}\ u^{l+m-1}}{\lceil l+m}$; $u > 0$ and $l+m > 0$ is the pdf of standard gamma variate with parameter $l+m$ and $f_2(v) = \dfrac{v^{l-1}\ (1-v)^{m-1}}{\beta(l,\ m)}$ is the pdf of beta variate of first kind with parameter l and m.

So, we can conclude that u and v are independently distributed where $u \sim G\ (l+m)$ and $v \sim \beta_1(l,\ m)$.

Remarks:

(i) A continuous random variable x is said to have a gamma distribution with parameters α and β if its probability density function is given by

$$f(x;\ \alpha,\ \beta) = \frac{\beta^{\alpha}\ e^{-\beta x}\ x^{\alpha-1}}{\lceil\alpha};\qquad x>0,\ \ \alpha>0,\ \ \beta>0$$

If $\beta = 1$, then (i) reduces to the form

$$f(x;\ \alpha) = \frac{e^{-x}\ x^{\alpha-1}}{\lceil\alpha};\qquad x>0,\ \ \alpha>0$$

which is called standard gamma distribution with parameter α

(ii) We have, $u = x + y$ and $\quad v = \dfrac{x}{x+y} = \dfrac{\frac{x}{x}}{\frac{x}{x}+\frac{y}{x}} = \dfrac{1}{1+\frac{y}{x}}$

As the ranges of x and y from 0 to ∞, so the range of u from 0 to ∞ and the range of v from 0 to 1.

(iii) $\dfrac{1}{\infty} = 0$, $\dfrac{\infty}{\infty} = \infty$, $\infty + \infty = \infty$

Q. If x and y are two independent gamma variates with parameters m and n respectively, then show that (i) the variates $u = x + y$ and $v = \dfrac{x}{x+y}$ are independent; (ii) u is a $\gamma(m+n)$ variate and v is a $\beta_1(m,\ n)$ variate (NU-12, 14)

Or, If x and y are two independent gamma variates with parameters m and n respectively, show that variates $u = x + y$ and $v = \dfrac{x}{y}$ are independent **(NU-16)**

Proof: Since $x \sim G\ (m)$, so the pdf of x is

$$f(x) = \frac{e^{-x}\ x^{m-1}}{\lceil m};\qquad x>0,\ \ m>0$$

Since $y \sim G\ (n)$, so the pdf of y is

$$f(y) = \frac{e^{-y}\ y^{n-1}}{\lceil n};\qquad y>0,\ \ n>0$$

Since x and y are independent, so the joint pdf of x and y is

$$f(x,\ y) = f(x)\ f(y)$$

$$= \frac{e^{-x}\ x^{m-1}}{\lceil m}.\ \ \frac{e^{-y}\ y^{n-1}}{\lceil n}$$

$$\Rightarrow f(x,\ y) = \frac{1}{\overline{|m}\ \ \overline{|n}}\ e^{-(x+y)}\ \ x^{m-1}\ y^{n-1}; \quad 0 < x < \infty, \quad 0 < y < \infty$$

Now, we have

$$u = x + y \quad \text{and} \quad v = \frac{x}{x+y}$$

$$\Rightarrow u = u\,v + y \qquad \Rightarrow v = \frac{x}{u}$$

$$\Rightarrow y = u - u\,v \qquad \Rightarrow x = u\,v$$

$$\Rightarrow y = u\,(1 - v)$$

The Jacobian of the transformation J is given by

$$J = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\[2ex] \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} v & u \\ 1-v & -u \end{vmatrix} = -u\,v - u + u\,v = -u$$

$$\therefore\ |J| = u$$

As the ranges of x and y from 0 to ∞, so the range of u from 0 to ∞ and the range of v from 0 to 1.

The joint pdf of u and v is

$$f(u,\ v) = f(x, y)\ |J|$$

$$= \frac{e^{-u}\ (u\,v)^{m-1}\ \{u\,(1-v)\}^{n-1}}{\overline{|m}\ \ \overline{|n}}.\ \ u$$

$$= \frac{1}{\overline{|m}\ \overline{|n}}\ e^{-u}\ \ u^{m-1+n-1+1}\ \ v^{m-1}\ (1-v)^{n-1}$$

$$= \frac{e^{-u}\ \ u^{m+n-1}}{\overline{|m+n}}.\ \ \frac{\Gamma(m+n)}{\overline{|m}\ \ \overline{|n}}\ v^{m-1}\ \ (1-v)^{n-1}$$

$$= \frac{e^{-u}\ u^{m+n-1}}{\Gamma(m+n)} . \frac{v^{m-1}\ (1-v)^{n-1}}{\beta(m,\ n)}$$

$\therefore\ f(u,\ v) = f(u).\ f(v)$; which indicates u and v are independent.

Where, $f(u)=\frac{e^{-u}\ u^{m+n-1}}{\Gamma(m+n)}$; $u>0$ and $m+n>0$ is the pdf of standard gamma variate with parameter $m+n$ and $f(v)=\frac{v^{m-1}\ (1-v)^{n-1}}{\beta(m,\ n)}$ is the pdf of beta variate of first kind with parameter m and n.

So, we can conclude that u and v are independently distributed where $u\sim\gamma\ (m+n)$ and $v\sim\beta_1\ (m,\ n)$.

Theorem: If x and y are independent gamma variate with parameter l and m respectively and if $u=x+y$ and $v=\frac{x}{y}$ are independent, then show that $u\sim G\ (l+m)$ and $v\sim\beta_2\ (l,\ m)$.

Proof: Since $x\sim G\ (l)$, so $f(x)=\frac{e^{-x}\ x^{l-1}}{\Gamma l}$; $\quad x>0,\ \ l>0$

and since $y\sim G\ (m)$, so $f(y)=\frac{e^{-y}\ y^{m-1}}{\Gamma m}$; $\quad y>0,\quad m>0$

Since x and y are independent, so the joint pdf of x and y is

$$f(x,\ y)=f(x).\ f(y)$$

$$=\frac{e^{-x}\ x^{l-1}}{\Gamma l}.\quad\frac{e^{-y}\ y^{m-1}}{\Gamma m}$$

$$=\frac{1}{\Gamma l\ \ \Gamma m}.\ e^{-(x+y)}\ \ x^{l-1}\ \ y^{m-1};\qquad 0<x<\infty,\quad 0<y<\infty$$

Now, we have

$$u=x+y\qquad\text{and}\qquad v=\frac{x}{y}$$

$$\Rightarrow u=v\,y+y\qquad\qquad\Rightarrow x=v\,y$$

$$\Rightarrow u=y\ (1+v)\qquad\qquad\Rightarrow x=v.\frac{u}{1+v}=u\left(1-\frac{1}{1+v}\right)$$

$$\Rightarrow y=\frac{u}{1+v}$$

The Jacobian of the transformation 'J' is given by

$$J = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \frac{v}{1+v} & \frac{u}{(1+v)^2} \\ \frac{1}{1+v} & -\frac{u}{(1+v)^2} \end{vmatrix} = -\frac{uv}{(1+v)^3} - \frac{u}{(1+v)^3} = \frac{-uv-u}{(1+v)^3} = \frac{-u(1+v)}{(1+v)^3} = \frac{-u}{(1+v)^2}$$

$$\therefore \ |J| = \frac{u}{(1+v)^2}$$

If x and y range from 0 to ∞, so u and v range also from 0 to ∞.

The joint pdf of u and v is

$$f(u, v) = f(x, y)\ |J|$$

$$= \frac{1}{\Gamma l \ \Gamma m} \cdot \ e^{-(x+y)} \ x^{l-1} \ y^{m-1} \cdot \frac{u}{(1+v)^2}$$

$$= \frac{1}{\Gamma l \ \Gamma m} \cdot \ e^{-u} \left(\frac{uv}{1+v}\right)^{l-1} \left(\frac{u}{1+v}\right)^{m-1} \cdot \frac{u}{(1+v)^2}$$

$$= \frac{1}{\Gamma l \ \Gamma m} \cdot \ \frac{e^{-u} \ u^{l-1+m-1+1} \ v^{l-1}}{(1+v)^{l-1+m-1+2}}$$

$$= \frac{1}{\Gamma l \ \Gamma m} \cdot \ \frac{e^{-u} \ u^{l+m-1} \ v^{l-1}}{(1+v)^{l+m}}$$

$$= \frac{e^{-u} \ u^{l+m-1}}{\Gamma (l+m)} \cdot \frac{\Gamma(l+m)}{\Gamma l \ \Gamma m} \cdot \frac{v^{l-1}}{(1+v)^{l+m}}$$

$$= \frac{e^{-u} \ u^{l+m-1}}{\Gamma (l+m)} \cdot \ \frac{v^{l-1}}{\beta(l, m) \ (1+v)^{l+m}}$$

$$= f_1(u). \ f_2(v)$$

where, $f_1(u) = \frac{e^{-u} \ u^{l+m-1}}{\Gamma(l+m)}$; $u > 0, \ l+m > 0$ is the pdf of standard gamma variate with parameter $(l+m)$ and $f_2(v) = \frac{v^{l-1}}{\beta(l, m) \ (1+v)^{l+m}}$ is the pdf of beta variate of second kind with parameter l and m.

Therefore, we can conclude that u and v are independently distributed where $u \sim G\ (l+m)$ and $v \sim \beta_2\ (l, m)$.

Short questions and answers:

(1) Define gamma distribution (NU-11)

Ans: A continuous random variable x is said to have a gamma distribution with parameters α and β if its probability density function is given by

$$f(x;\ \alpha,\ \beta) = \frac{\beta^{\alpha}\ e^{-\beta x}\ x^{\alpha-1}}{\Gamma\alpha};\quad x > 0,\quad \alpha > 0,\quad \beta > 0$$

(2) When gamma distribution reduces to normal distribution (NU-11)

Or, Under what condition gamma distribution tends to normal distribution **(NU-13)**

Ans: Gamma distribution reduces to normal distribution as α becomes very large i.e., $\alpha \to \infty$.

(3) What is the mean and variance of gamma distribution? (NU-12)

Ans: The mean and variance of gamma distribution are $\frac{\alpha}{\beta}$ and $\frac{\alpha}{\beta^2}$ respectively.

(4) If $X \sim G(4, 5)$ then, what is $f(x)$? (NU-14)

Ans: We know the pdf of gamma distribution with parameters α and β is

$$f(x;\ \alpha,\ \beta) = \frac{\beta^{\alpha}\ e^{-\beta x}\ x^{\alpha-1}}{\Gamma\alpha};\quad x > 0,\quad \alpha > 0,\quad \beta > 0$$

So, if $X \sim G(4, 5)$ then

$$f(x;\ 4,\ 5) = \frac{5^{4}\ e^{-5x}\ x^{4-1}}{\Gamma 4}$$

$$= \frac{625\ e^{-5x}\ x^{3}}{6};\quad x > 0$$

Cauchy Distribution

This distribution was discovered by Cauchy (1787-1857).

Q. Define Cauchy distribution (NU-14)

Ans: A continuous random variable y is said to have a generalized Cauchy distribution with parameters μ and λ if its probability density function is given by

$$f(y;\,\mu,\,\lambda)=\frac{1}{\pi\,\lambda}\,\frac{1}{\left[1+\left(\frac{y-\mu}{\lambda}\right)^2\right]};\quad -\infty<y<\infty$$

Where $\lambda>0$ and $\pi=3.143$.

NB:

(i) It is often quoted as a distribution for which moments do not exist.

(ii) Mean and Variance of the distribution do not exist.

(iii) If $\lambda=1$, Cauchy distribution reduces to the form as

$$f(y;\,\mu)=\frac{1}{\pi}\,\frac{1}{\left[1+(y-\mu)^2\right]};\quad -\infty<y<\infty$$

(iv) The pdf of Cauchy distribution is

$$f(y;\,\mu,\,\lambda)=\frac{1}{\pi\,\lambda}\,\frac{1}{\left[1+\left(\frac{y-\mu}{\lambda}\right)^2\right]};\quad -\infty<y<\infty$$

Let, $x=\frac{y-\mu}{\lambda}$

$$\Rightarrow y-\mu=\lambda x$$

$$\Rightarrow y=\mu+\lambda x$$

Now, $\frac{dy}{dx}=\lambda$ i.e., $|J|=\lambda$

So, the pdf of x is

$$f(x)=f(y).\,|J|$$

$$=\frac{1}{\pi\,\lambda}\,\frac{1}{(1+x^2)}\times\lambda$$

$\therefore\ f(x)=\frac{1}{\pi\,(1+x^2)};\ -\infty<x<\infty$; which is the pdf of standard Cauchy distribution.

Theorem: If x is a Cauchy variate with parameter μ and λ, then median and mode of the distribution is μ.

Or, Find the median and mode of Cauchy distribution (NU-11, 13)

Or, Find median of Cauchy distribution (NU-16)

Proof: The pdf of Cauchy distribution with parameter μ and λ is

$$f(x) = \frac{1}{\pi\lambda}\frac{1}{\left[1+\left(\frac{x-\mu}{\lambda}\right)^2\right]}$$

$$= \frac{1}{\pi\lambda}\frac{1}{\left[\frac{\lambda^2+(x-\mu)^2}{\lambda^2}\right]}$$

$$= \frac{1}{\pi\lambda}\cdot\frac{\lambda^2}{\lambda^2+(x-\mu)^2}$$

$$\Rightarrow f(x) = \frac{\lambda}{\pi}\cdot\frac{1}{\lambda^2+(x-\mu)^2} \quad \text{................(i)}$$

Let $\hat{\mu}$ is the median of the distribution, then

$$\int_{-\infty}^{\hat{\mu}} f(x)\,dx = \frac{1}{2}$$

$$\Rightarrow \int_{-\infty}^{\hat{\mu}} \frac{\lambda}{\pi}\cdot\frac{1}{\lambda^2+(x-\mu)^2}\,dx = \frac{1}{2}$$

$$\Rightarrow \frac{\lambda}{\pi}\left[\frac{1}{\lambda}\tan^{-1}\left(\frac{x-\mu}{\lambda}\right)\right]_{-\infty}^{\hat{\mu}} = \frac{1}{2}$$

$$\Rightarrow \frac{\lambda}{\pi}\left[\frac{1}{\lambda}\tan^{-1}\left(\frac{x-\mu}{\lambda}\right)\right]_{-\infty}^{\hat{\mu}} = \frac{1}{2}$$

$$\Rightarrow \frac{\lambda}{\pi}\cdot\frac{1}{\lambda}\left[\tan^{-1}\left(\frac{\hat{\mu}-\mu}{\lambda}\right)-\tan^{-1}\left(\frac{-\infty-\mu}{\lambda}\right)\right] = \frac{1}{2}$$

$$\Rightarrow \tan^{-1}\left(\frac{\hat{\mu}-\mu}{\lambda}\right)-\tan^{-1}(-\infty) = \frac{\pi}{2}$$

$$\Rightarrow \tan^{-1}\left(\frac{\hat{\mu}-\mu}{\lambda}\right)+\tan^{-1}(\infty) = \frac{\pi}{2}$$

$$\Rightarrow \tan^{-1}\left(\frac{\hat{\mu}-\mu}{\lambda}\right)+\tan^{-1}\ \tan\frac{\pi}{2}=\frac{\pi}{2}$$

$$\Rightarrow \tan^{-1}\left(\frac{\hat{\mu}-\mu}{\lambda}\right)+\frac{\pi}{2}=\frac{\pi}{2}$$

$$\Rightarrow \tan^{-1}\left(\frac{\hat{\mu}-\mu}{\lambda}\right)=0$$

$$\Rightarrow \frac{\hat{\mu}-\mu}{\lambda}=0$$

$$\therefore\ \hat{\mu}=\mu$$

So, μ is the median of Cauchy distribution.

The mode of the distribution will be the solution of $f'(x)=0$, provided $f''(x)<0$.

Taking log on both sides of (i), we get

$$\log f(x)=\log\left(\frac{\lambda}{\pi}\right)+\log\left\{\frac{1}{\lambda^2+(x-\mu)^2}\right\}$$

$$=C+\log\left\{\lambda^2+(x-\mu)^2\right\}^{-1}$$

$$\Rightarrow \log f(x)=C-\log\left\{\lambda^2+(x-\mu)^2\right\} \text{..........(ii)}$$

Now, differentiating (ii) w.r.t x, we have

$$\frac{1}{f(x)}\ f'(x)=-\frac{2(x-\mu)}{\lambda^2+(x-\mu)^2}$$

$$\Rightarrow f'(x)=-\frac{2(x-\mu)}{\lambda^2+(x-\mu)^2}\ f(x)$$

Using $f'(x)=0$, we get

$$-\frac{2(x-\mu)}{\lambda^2+(x-\mu)^2}\ f(x)=0$$

$$\Rightarrow -\frac{2(x-\mu)}{\lambda^2+(x-\mu)^2}=0 \qquad [\because f(x)\neq 0]$$

$$\Rightarrow -2\,(x-\mu)=0$$

$$\therefore\ \ x=\mu$$

It can be easily seen that $f''(x)<0$ at $x=\mu$

Hence, μ is the mode of Cauchy distribution.

$\therefore$ Median $=$ Mode $= \mu$

NB: (i) $\tan 0^0 = 0$, $\tan 90^0 = \tan\frac{\pi}{2} = \infty$, $\tan(-\theta) = -\tan\theta$

(ii) $\frac{d}{dx}(\tan x) = \sec^2 x$

(iii) $\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}$

(iv) $1+\tan^2 x = \sec^2 x$

(v) Find the value of $\int \frac{dx}{a^2+x^2}$

Solution: Let, $x = a\tan\theta$

$$\therefore \frac{dx}{d\theta} = a\sec^2\theta$$

$$\Rightarrow dx = a\sec^2\theta\, d\theta$$

and $a^2+x^2 = a^2 + a^2\tan^2\theta = a^2(1+\tan^2\theta) = a^2\sec^2\theta$

and since, $x = a\tan\theta$

$$\Rightarrow \tan\theta = \frac{x}{a}$$

$$\Rightarrow \theta = \tan^{-1}\frac{x}{a}$$

$$\therefore \int\frac{dx}{a^2+x^2} = \int\frac{a\sec^2\theta\, d\theta}{a^2\sec^2\theta} = \int\frac{1}{a}\,d\theta = \frac{1}{a}\int d\theta = \frac{1}{a}\theta = \frac{1}{a}\tan^{-1}\frac{x}{a}$$

Similarly, $\int\frac{1}{\lambda^2+(x-\mu)^2}\,dx = \frac{1}{\lambda}\tan^{-1}\left(\frac{x-\mu}{\lambda}\right)$

Q. If X is a Cauchy variate, show that X^2 **is** $\beta_2\left(\frac{1}{2},\frac{1}{2}\right)$ **variate (NU-11, 13, 16)**

Ans: The pdf standard Cauchy distribution is

$$f(x) = \frac{1}{\pi(1+x^2)}; \quad -\infty < x < \infty$$

Let $u = x^2$

$$\Rightarrow x = \sqrt{u}$$

$$\Rightarrow \frac{dx}{du} = \frac{1}{2} u^{-\frac{1}{2}} = \frac{1}{2\sqrt{u}}$$

$$\therefore \quad |J| = \frac{1}{2\sqrt{u}}$$

So, the distribution of u is

$$f(u) = f(x)\,|J|$$

$$= \frac{2}{\pi\,(1+u)} \cdot \frac{1}{2\sqrt{u}}; \quad 0 < u < \infty$$

$$= \frac{1}{\pi} \frac{u^{-\frac{1}{2}}}{1+u}$$

$$= \frac{1}{\beta\left(\frac{1}{2}, \frac{1}{2}\right)} \frac{u^{\frac{1}{2}-1}}{(1+u)^{\frac{1}{2}+\frac{1}{2}}} \sim \beta_2\left(\frac{1}{2}, \frac{1}{2}\right)$$

$$\therefore \quad f(x^2) = \frac{1}{\beta\left(\frac{1}{2}, \frac{1}{2}\right)} \frac{(x^2)^{\frac{1}{2}-1}}{(1+x^2)^{\frac{1}{2}+\frac{1}{2}}}$$

Hence if x is a Cauchy variate, then x^2 is a $\beta_2\left(\frac{1}{2}, \frac{1}{2}\right)$ variate.

NB: (i) $\Gamma\frac{1}{2} = \sqrt{\pi}$

(ii) $\beta\left(\frac{1}{2}, \frac{1}{2}\right) = \frac{\Gamma\frac{1}{2}\,\Gamma\frac{1}{2}}{\Gamma\left(\frac{1}{2}+\frac{1}{2}\right)} = \frac{\sqrt{\pi}.\ \sqrt{\pi}}{\Gamma 1} = \pi$

(iii) The pdf of second kind of beta with parameters l and m is

$$f(x; l, m) = \frac{x^{l-1}}{\beta(l, m)\,(1+x)^{l+m}}; \quad x > 0,\ l > 0,\ m > 0$$

Q. Find the characteristic function of this distribution (NU-14)

Ans: If x is a standard Cauchy variate, then the pdf of x is

$$f(x) = \frac{1}{\pi(1+x^2)}; \quad -\infty < x < \infty$$

So, the characteristic function of x is

$$\phi_x(t) = E\left(e^{itx}\right) = \int_{-\infty}^{\infty} e^{itx}\ f(x)\ dx = \int_{-\infty}^{\infty} e^{itx}\ \frac{1}{\pi\left(1+x^2\right)}\ dx = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{e^{itx}}{1+x^2}\ dx \text{ ....(1)}$$

To evaluate (i), let is consider the standard Laplace double exponential distribution

$$f(z) = \frac{1}{2} e^{-|z|}; \quad -\infty < z < \infty$$

Then the characteristic function of z is

$$\phi_z(t) = E\left(e^{itz}\right) = \frac{1}{1+t^2}$$

Since $\phi_z(t)$ is absolutely integrable in $(-\infty, \infty)$, we have by inversion theorem

$$f(z) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-itz}\ \phi_z(t)\ dt$$

$$\Rightarrow \frac{1}{2} e^{-|z|} = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-itz} \cdot \frac{1}{1+t^2}\ dt$$

$$\Rightarrow e^{-|z|} = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{e^{itz}}{1+t^2}\ dt; \text{ [changing t to -t]}$$

On interchanging t and z, we get

$$e^{-|t|} = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{e^{itz}}{1+z^2}\ dz \text{ ....(ii)}$$

Comparing (i) and (ii), we get $\phi_x(t) = e^{-|t|}$

which is the characteristic function of Cauchy distribution.

NB:

(i) The standard form of double exponential distribution is $f(x) = \frac{1}{2} e^{-|x|}$. It is also known as Poisson's first law of error. P- 474, M. K. Roy

(ii) The characteristic function of standard Laplace double exponential distribution is

$$\phi_x(t) = \frac{1}{1+t^2} \quad \text{(P- 475, M. K. Roy)}$$

(iii) **Inversion theorem**:

$$f(x)=\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-itx}\ \phi(t)\ dt\ ;$$ P- 6.87, S. C. Gupta and P- 301, M. K. Roy

(iv) $-\infty<t<\infty$

$$\Rightarrow -(-\infty)>-t>-\infty$$

$$\Rightarrow \infty>-t>-\infty$$

$$\Rightarrow -\infty<-t<\infty$$

Short questions and answers:

(1) Define Cauchy distribution (NU-11, 14)

Ans: A continuous random variable y is said to have a generalized Cauchy distribution with parameters μ and λ if its probability density function is given by

$$f(y;\ \mu,\ \lambda)=\frac{1}{\pi\lambda}\frac{1}{\left[1+\left(\frac{y-\mu}{\lambda}\right)^2\right]};\quad -\infty<y<\infty$$

Where $\lambda>0$ and $\pi=3.143$.

(2) What is the characteristic function of Cauchy distribution? (NU-13)

Ans: The characteristic function of Cauchy distribution is $\phi_x(t)=e^{-|t|}$.

Maxwell Distribution

Q. Define Maxwell distribution.

Ans: A continuous random variable x is said to have a Maxwell distribution if its pdf is defined as

$$f(x;\lambda)=\frac{\sqrt{2}}{\lambda^3\sqrt{\pi}}\,x^2\,e^{-\frac{x^2}{2\lambda^2}};\quad 0\le x\le\infty$$

$\lambda>0$ is the only parameter of the distribution.

Q. Find the r-th raw moment and hence find mean and variance of Maxwell distribution.

Or, Find the mean and variance of Maxwell distribution (NU-11, 14)

Ans: The pdf of Maxwell distribution with parameter λ is

$$f(x;\lambda)=\frac{\sqrt{2}}{\lambda^3\sqrt{\pi}}\,x^2\,e^{-\frac{x^2}{2\lambda^2}};\quad 0\le x\le\infty$$

By definition,

$$\mu_r'=E(x^r)$$

$$=\int_0^\infty x^r\,f(x)\,dx$$

$$=\int_0^\infty x^r\,\frac{\sqrt{2}}{\lambda^3\sqrt{\pi}}\,x^2\,e^{-\frac{x^2}{2\lambda^2}}\,dx$$

$$\Rightarrow \mu_r'=\frac{\sqrt{2}}{\lambda^3\sqrt{\pi}}\int_0^\infty x^{r+2}\,e^{-\frac{x^2}{2\lambda^2}}\,dx \quad \text{.........(i)}$$

Let, $t=\dfrac{x^2}{2\lambda^2}$

$\Rightarrow x^2=t\,2\,\lambda^2$

$\Rightarrow x=\sqrt{t}.\,\lambda\sqrt{2}$

$\therefore\ \dfrac{dx}{dt}=\dfrac{1}{2\sqrt{t}}\,\lambda\,\sqrt{2}$

$\Rightarrow dx=\dfrac{\lambda}{\sqrt{2t}}\,dt$

If $x=0$, then $t=0$

and if $x=\infty$, then $t=\infty$

So (i) can be written as

$$\mu_r' = \frac{\sqrt{2}}{\lambda^3 \sqrt{\pi}} \int_0^{\infty} \left(\sqrt{t}.\, \lambda\, \sqrt{2}\right)^{r+2} e^{-t}.\, \frac{\lambda}{\sqrt{2t}}\, dt$$

$$= \frac{\lambda^{r+3} \left(\sqrt{2}\right)^{r+2} \sqrt{2}}{\sqrt{2}\, \lambda^3\, \sqrt{\pi}} \int_0^{\infty} \left(\sqrt{t}\right)^{r+2} e^{-t}.\, \frac{1}{\sqrt{t}}\, dt$$

$$= \frac{\lambda^r\, 2^{\frac{r}{2}+1}}{\sqrt{\pi}} \int_0^{\infty} e^{-t}\, t^{\frac{r}{2}+1-\frac{1}{2}}\, dt$$

$$= \frac{\lambda^r\, 2^{\frac{r}{2}+1}}{\sqrt{\pi}} \int_0^{\infty} e^{-t}\, t^{\frac{r+1}{2}}\, dt$$

$$= \frac{\lambda^r\, 2^{\frac{r}{2}+1}}{\sqrt{\pi}} \int_0^{\infty} e^{-t}\, t^{\frac{r+3}{2}-1}\, dt$$

$$\therefore \quad \mu_r' = \frac{\lambda^r\, 2^{\frac{r}{2}+1}}{\sqrt{\pi}} \Gamma\left(\frac{r}{2}+\frac{3}{2}\right)$$

Now, putting $r=1$ in (i), we get

$$\mu_1' = \frac{\lambda\, 2^{\frac{1}{2}+1}}{\sqrt{\pi}} \Gamma 2 = \frac{2\sqrt{2}}{\sqrt{\pi}}\, \lambda$$

$\therefore$ Mean, $\mu_1' = \dfrac{2\lambda\sqrt{2}}{\sqrt{\pi}}$

Again, putting $r=2$ in (i), we get

$$\mu_2' = \frac{\lambda^2\, 2^{\frac{2}{2}+1}}{\sqrt{\pi}} \Gamma\left(\frac{2}{2}+\frac{3}{2}\right) = \frac{\lambda^2\, 4}{\Gamma\frac{1}{2}} \Gamma\left(\frac{3}{2}+1\right) = \frac{\lambda^2\, 4}{\Gamma\frac{1}{2}}\, \frac{3}{2}.\, \frac{1}{2}\, \Gamma\frac{1}{2} = 3\lambda^2$$

$$\therefore \text{ Variance, } \mu_2 = \mu_2' - \mu_1'^2 = 3\lambda^2 - \frac{4.\, \lambda^2.\, 2}{\pi} = \lambda^2\left(3 - \frac{8}{\pi}\right) = \lambda^2\left(3 - \frac{8}{3.1429}\right) = 0.454\,\lambda^2$$

NB: The pdf of gamma distribution is

$$f(x;\, \alpha,\, \beta) = \frac{\beta^{\alpha}\ e^{-\beta x}\ x^{\alpha-1}}{\Gamma\alpha};\quad x > 0,\ \alpha > 0,\ \beta > 0 \text{ ....(i)}$$

(i) If $\beta = 1$, then (i) reduces to the form

$$f(x;\ \alpha) = \frac{e^{-x}\ x^{\alpha-1}}{\Gamma\alpha};\quad x > 0,\ \alpha > 0$$

which is called standard gamma distribution with parameter α

(ii) $\int_0^{\infty} f(x)\ dx = 1$

$$\Rightarrow \int_0^{\infty} \frac{e^{-x}\ x^{\alpha-1}}{\Gamma\alpha}\ dx = 1$$

$$\therefore \int_0^{\infty} e^{-x}\ x^{\alpha-1}\ dx = \Gamma\alpha$$

(iii) $\Gamma 2 = (2-1)! = 1$

Q. Find the mode of Maxwell distribution.

Ans: The pdf of Maxwell distribution with parameter λ is

$$f(x;\, \lambda) = \frac{\sqrt{2}}{\lambda^{3}\ \sqrt{\pi}}\ x^{2}\ e^{-\frac{x^{2}}{2\lambda^{2}}};\quad 0 \le x \le \infty \text{ ....(i)}$$

The mode of the distribution will be the solution of $f'(x) = 0$, provided $f''(x) < 0$.

Taking log on both sides of (i), we get

$$\log f(x) = C + 2\log x - \frac{x^{2}}{2\lambda^{2}} \text{ ....(ii)}$$

Now differentiating (ii) w.r.t x, we have

$$\frac{1}{f(x)}\ f'(x) = \frac{2}{x} - \frac{2x}{2\lambda^{2}}$$

$$f'(x) = \left(\frac{2}{x} - \frac{x}{\lambda^{2}}\right) f(x)$$

Using $f'(x) = 0$, we get

$$\left(\frac{2}{x} - \frac{x}{\lambda^2}\right) f(x) = 0$$

$$\Rightarrow \left(\frac{2}{x} - \frac{x}{\lambda^2}\right) = 0 \quad [\because f(x) \neq 0]$$

$$\Rightarrow 2\lambda^2 - x^2 = 0$$

$$\Rightarrow x^2 = 2\lambda^2$$

$$\therefore \quad x = \lambda\sqrt{2}$$

It can be easily seen that $f''(x) < 0$ at $x = \lambda\sqrt{2}$. So, Mode of the distribution is $\lambda\sqrt{2}$.

Short questions and answers:

(1) Define Maxwell distribution (NU-11)

Ans: A continuous random variable x is said to have a Maxwell distribution if its pdf is defined as

$$f(x;\lambda) = \frac{\sqrt{2}}{\lambda^3\sqrt{\pi}}\, x^2\, e^{-\frac{x^2}{2\lambda^2}}; \quad 0 \leq x \leq \infty$$

$\lambda > 0$ is the only parameter of the distribution.

(2) Write down the probability density function of Maxwell distribution (NU-14)

Ans: The pdf of Maxwell distribution with parameter λ is

$$f(x;\lambda) = \frac{\sqrt{2}}{\lambda^3\sqrt{\pi}}\, x^2\, e^{-\frac{x^2}{2\lambda^2}}; \quad 0 \leq x \leq \infty$$

Weibull Distribution

The distribution is named is after Waloddi Weibull, a Swedish physicist, who used it in 1939 to represent the distribution of the breaking strength of materials.

Q. Define Weibull distribution (NU-11, 14)

Ans: A continuous random variable x is said to have a Weibull distribution if its pdf is given by

$$f(x;\, a,\, b) = a\ b\ x^{b-1}\ e^{-ax^{b}};\quad 0 < x < \infty$$

where, $a > 0$ and $b > 0$.

NB:

1. If $b = 1$, the Weibull distribution reduces to exponential distribution.

2. The moment generating function of Weibull distribution is $a^{-\frac{t}{b}}\ \Gamma\left(1+\frac{t}{b}\right)$

3. The r-th raw moment of Weibull distribution is $a^{-\frac{r}{b}}\ \Gamma\left(1+\frac{r}{b}\right)$

4. The mean of Weibull distribution is $\left(\frac{1}{a}\right)^{\frac{1}{b}}\ \Gamma\left(1+\frac{1}{b}\right)$

5. The variance of Weibull distribution is $\left(\frac{1}{a}\right)^{\frac{2}{b}}\left\{\Gamma\left(1+\frac{2}{b}\right)-\Gamma^{2}\left(1+\frac{1}{b}\right)\right\}$

Bivariate Normal Distribution

(দ্বিচলক পরিমিত বিন্যাস)

Q. Define Bivariate Normal distribution (NU-15)

Ans: Suppose X and Y are two normally correlated variables with correlation coefficient ρ and $E(X)=\mu_x$, $V(X)=\sigma_x{}^2$, $E(Y)=\mu_y$, $V(Y)=\sigma_y{}^2$. Then the joint probability density function of X and Y is said to have a bivariate normal distribution if its pdf is defined as

$$f(x,y)=\frac{1}{2\pi\sigma_x\sigma_y\sqrt{1-\rho^2}}e^{-\frac{1}{2(1-\rho^2)}\left[\left(\frac{x-\mu_x}{\sigma_x}\right)^2-2\rho\left(\frac{x-\mu_x}{\sigma_x}\right)\left(\frac{y-\mu_y}{\sigma_y}\right)+\left(\frac{y-\mu_y}{\sigma_y}\right)^2\right]}$$

$$-\infty<x<\infty,\ -\infty<y<\infty,\ \sigma_x>0,\ \sigma_y>0$$
$$-\infty<\mu_x<\infty,\ -\infty<\mu_y<\infty,\ -1<\rho<1$$

Here, μ_x, μ_y, σ_x, σ_y and ρ are the five parameters of the distribution.

Q. Derive the pdf of bivariate normal distribution.

Ans: The probability density function of a normal distribution with mean μ and variance σ^2 is

$$f\left(x;\mu,\sigma^2\right)=\frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{1}{2}\frac{(x-\mu)^2}{\sigma^2}}\ ;\quad -\infty<x<\infty$$

$$=k\,e^{-\frac{1}{2}(x-\mu)\left(\sigma^2\right)^{-1}(x-\mu)}\ ;\ \text{Where } k=\frac{1}{\sigma\sqrt{2\pi}}\ \ldots\ldots\ldots\text{(i)}$$

In case of bivariate normal distribution, X is replaced by a vector $X=\begin{pmatrix}x\\y\end{pmatrix}$, μ is replaced by a vector $\mu=\begin{pmatrix}\mu_x\\\mu_y\end{pmatrix}$ and $\left(\sigma^2\right)^{-1}$ is replaced by positive definite symmetric matrix

$$A=\begin{pmatrix}a_{11} & a_{12}\\ a_{21} & a_{22}\end{pmatrix}_{2\times2}$$

which is the variance-covariance matrix and k is a positive constant. The square $\left(\frac{X-\mu}{\sigma}\right)^2=\left(X-\mu\right)'\left(\sigma^2\right)^{-1}\left(X-\mu\right)$ is replaced by the quadratic form $(X-\mu)'A(X-\mu)$. Thus, the density function of a bivariate normal distribution is

$$f(X) = f(x,\ y) = k\ e^{-\frac{1}{2}(X-\mu)' A\,(X-\mu)} \quad \text{.........(i)}$$

Since A is a positive definite symmetric matrix, there exists a non-singular matrix C such that $C'AC = I_2$ where,

$$C = \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix} \text{ and } I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Now let us make a transformation

$$X - \mu = CY$$

$$\Rightarrow X = \mu + CY$$

$$\Rightarrow \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \mu_x \\ \mu_y \end{pmatrix} + \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} \mu_x \\ \mu_y \end{pmatrix} + \begin{pmatrix} c_{11}\ y_1\ +\ c_{12}\ y_2 \\ c_{21}\ y_1\ +\ c_{22}\ y_2 \end{pmatrix}$$

The Jacobian of the transformation is

$$J = \begin{vmatrix} \dfrac{\partial x}{\partial y_1} & \dfrac{\partial x}{\partial y_2} \\ \dfrac{\partial y}{\partial y_1} & \dfrac{\partial y}{\partial y_2} \end{vmatrix} = \begin{vmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{vmatrix}$$

$$\therefore\ |J| = \text{mod}\,|C|$$

Where "$\text{mod}\,|C|$" indicates the absolute value of the determinant of C.

Again, $(X-\mu)' A(X-\mu) = (CY)' A(CY)$

$$= Y'C'ACY$$

$$= Y'\ Y \quad [\because\ C'AC = I_2]$$

So (i) can be written as, $f(X) = k\,e^{-\frac{1}{2}Y'\,Y}$

Since $C'AC = I_2$

$$\Rightarrow |C'\,A\,C| = |I_2|$$

$$\Rightarrow |C'|\ |A|\ |C| = 1$$

$$\Rightarrow |C|\ |A|\ |C| = 1$$

$$\Rightarrow |C|^2 = \frac{1}{|A|}$$

$$\Rightarrow \text{mod}|C| = \frac{1}{|A|^{\frac{1}{2}}}$$

So, $|J| = \text{mod}|C| = \dfrac{1}{|A|^{\frac{1}{2}}}$ $\quad \left[\because\ |J| = \text{mod}|C|\right]$

Again, $\quad C'AC = I_2$

$$\Rightarrow (C')^{-1} C'ACC^{-1} = (C')^{-1} I_2 C^{-1}$$

$$\Rightarrow A = (CC')^{-1}$$

$\therefore \quad A = V^{-1};$ Where $\quad V = CC'$

So, $|J| = \dfrac{1}{|A|^{\frac{1}{2}}} = \dfrac{1}{|V|^{-\frac{1}{2}}} = |V|^{\frac{1}{2}}$

Now, $f(Y) = f(X).\,|J| = k\, e^{-\frac{1}{2}Y'Y}.\,|V|^{\frac{1}{2}}$

Now we have to find the value of k such that

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(Y)\, dY = 1$$

$$\Rightarrow \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} k\,|V|^{\frac{1}{2}}\, e^{-\frac{1}{2}Y'Y} dY = 1$$

$$\Rightarrow k\,|V|^{\frac{1}{2}} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-\frac{1}{2}\sum_{i=1}^{2} y_i^2} dy_1.\, dy_2 = 1$$

$$\Rightarrow k\,|V|^{\frac{1}{2}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}y_1^2} dy_1 \int_{-\infty}^{\infty} e^{-\frac{1}{2}y_2^2} dy_2 = 1$$

$$\Rightarrow k\,|V|^{\frac{1}{2}} \prod_{i=1}^{2} \int_{-\infty}^{\infty} e^{-\frac{1}{2}y_i^2} dy_i = 1$$

$$\Rightarrow k\,|V|^{\frac{1}{2}} \left(\sqrt{2\pi}\right)^2 = 1$$

$$\therefore \quad k = \frac{1}{2\pi\ |V|^{\frac{1}{2}}}$$

Now putting $A = V^{-1}$ and $k = \dfrac{1}{2\pi \ |V|^{\frac{1}{2}}}$ in (i), we get

$$f(x,\ y) = \frac{1}{2\pi \ |V|^{\frac{1}{2}}} e^{-\frac{1}{2}(X-\mu)' V^{-1} (X-\mu)} \quad \ldots\ldots\text{(ii)}$$

Suppose, $V = E\left[X - E(X)\right]\left[X - E(X)\right]'$

$$= E\begin{pmatrix} x-\mu_x \\ y-\mu_y \end{pmatrix}\begin{pmatrix} x-\mu_x & y-\mu_y \end{pmatrix}$$

$$= E\begin{bmatrix} (x-\mu_x)^2 & (x-\mu_x)(y-\mu_y) \\ (x-\mu_x)(y-\mu_y) & (y-\mu_y)^2 \end{bmatrix}$$

$$= \begin{bmatrix} E(x-\mu_x)^2 & E(x-\mu_x)(y-\mu_y) \\ E(x-\mu_x)(y-\mu_y) & E(y-\mu_y)^2 \end{bmatrix}$$

$$\Rightarrow V = \begin{bmatrix} \sigma_x^{\ 2} & \rho\ \sigma_x\ \sigma_y \\ \rho\ \sigma_x\ \sigma_y & \sigma_y^{\ 2} \end{bmatrix}$$

$$\Rightarrow |V| = \sigma_x^{\ 2}\ \sigma_y^{\ 2} - \rho^2\ \sigma_x^{\ 2}\ \sigma_y^{\ 2} = \sigma_x^{\ 2}\ \sigma_y^{\ 2}\left(1 - \rho^2\right)$$

So, $V^{-1} = \dfrac{1}{|V|}\begin{pmatrix} \sigma_y^{\ 2} & -\rho\ \sigma_x\ \sigma_y \\ -\rho\ \sigma_x\ \sigma_y & \sigma_x^{\ 2} \end{pmatrix}$

$$= \frac{1}{\sigma_x^{\ 2}\ \sigma_y^{\ 2}\ (1-\rho^2)}\begin{pmatrix} \sigma_y^{\ 2} & -\rho\ \sigma_x\ \sigma_y \\ -\rho\ \sigma_x\ \sigma_y & \sigma_x^{\ 2} \end{pmatrix}$$

$$= \frac{1}{1-\rho^2}\begin{pmatrix} \dfrac{1}{\sigma_x^{\ 2}} & -\dfrac{\rho}{\sigma_x\ \sigma_y} \\ -\dfrac{\rho}{\sigma_x\,\sigma_y} & \dfrac{1}{\sigma_y^{\ 2}} \end{pmatrix}$$

Now (ii) can be written as

$$f(x,\ y) = \frac{1}{2\pi \ |V|^{\frac{1}{2}}} e^{-\frac{1}{2}(X-\mu)' V^{-1} (X-\mu)}$$

$$= \frac{1}{2\pi \sqrt{\sigma_x^{\,2}\ \sigma_y^{\,2}\left(1-\rho^2\right)}}\, e^{-\frac{1}{2}\left(x-\mu_x \quad y-\mu_y\right)\frac{1}{1-\rho^2}\begin{pmatrix} \frac{1}{\sigma_x^{\,2}} & -\frac{\rho}{\sigma_x\sigma_y} \\ -\frac{\rho}{\sigma_x\sigma_y} & \frac{1}{\sigma_y^{\,2}} \end{pmatrix}\begin{pmatrix} x-\mu_x \\ y-\mu_y \end{pmatrix}}$$

$$= \frac{1}{2\pi\ \sigma_x\ \sigma_y\ \sqrt{1-\rho^2}}\, e^{-\frac{1}{2\left(1-\rho^2\right)}\left[\frac{x-\mu_x}{\sigma_x^{\,2}} - \frac{\rho\left(y-\mu_y\right)}{\sigma_x\sigma_y} \qquad -\frac{\rho\left(x-\mu_x\right)}{\sigma_x\ \sigma_y} + \frac{y-\mu_y}{\sigma_y^{\,2}}\right]\begin{pmatrix} x-\mu_x \\ y-\mu_y \end{pmatrix}}$$

$$= \frac{1}{2\pi\ \sigma_x\ \sigma_y\ \sqrt{1-\rho^2}}\, e^{-\frac{1}{2\left(1-\rho^2\right)}\left[\frac{\left(x-\mu_x\right)^2}{\sigma_x^{\,2}} - \frac{\rho\left(x-\mu_x\right)\left(y-\mu_y\right)}{\sigma_x\ \sigma_y} - \frac{\rho\left(x-\mu_x\right)\left(y-\mu_y\right)}{\sigma_x\sigma_y} + \frac{\left(y-\mu_y\right)^2}{\sigma_y^{\,2}}\right]}$$

$$\therefore \quad f(x,\ y) = \frac{1}{2\pi\ \sigma_x\ \sigma_y\ \sqrt{1-\rho^2}}\, e^{-\frac{1}{2\left(1-\rho^2\right)}\left[\left(\frac{x-\mu_x}{\sigma_x}\right)^2 - 2\rho\left(\frac{x-\mu_x}{\sigma_x}\right)\left(\frac{y-\mu_y}{\sigma_y}\right) + \left(\frac{y-\mu_y}{\sigma_y}\right)^2\right]}$$

This is the pdf of a bivariate normal distribution with parameters $\mu_x, \mu_y, \sigma_x, \sigma_y$ and ρ .

NB: (i) If $x \sim N(0, 1)$, then $f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}; \quad -\infty < x < \infty$

So, $\int_{-\infty}^{\infty} f(x)\, dx = 1$

$$\Rightarrow \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}\, dx = 1$$

$$\therefore \int_{-\infty}^{\infty} e^{-\frac{1}{2}x^2}\, dx = \sqrt{2\pi}$$

(ii) $\rho = \frac{Cov(x,\ y)}{\sigma_x\sigma_y}$

$$= \frac{E\{x-E(x)\}\ \{y-E(y)\}}{\sigma_x\ \sigma_y}$$

$$\Rightarrow \rho = \frac{E\left(x-\mu_x\right)\ \left(y-\mu_y\right)}{\sigma_x\ \sigma_y}$$

$$\therefore\ E\left(x-\mu_x\right)\ \left(y-\mu_y\right) = \rho\ \sigma_x\ \sigma_y$$

(iii) $X = A + CY$

$$\Rightarrow E(X) = A + C\ E(Y) = A \quad [\because Y \sim NID(0, I)]$$

$$\therefore\ A = E(X) = \begin{bmatrix} E(x) \\ E(y) \end{bmatrix} = \begin{pmatrix} \mu_x \\ \mu_y \end{pmatrix}$$

Alternative Method:

The probability density function of a normal distribution with mean μ and variance σ^2 is

$$f(x;\ \mu,\ \sigma^2) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\frac{(x-\mu)^2}{\sigma^2}} \ ; \quad -\infty < x < \infty$$

$$= \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}(x-\mu)\ (\sigma^2)^{-1}\ (x-\mu)}$$

$$= k\ \ e^{-\frac{1}{2}(x-\mu)\,B\,(x-\mu)} \text{..........(i)}$$

Where B is a positive quantity and k is a constant such that the integration over the whole range of X is one.

Now we shall show that bivariate normal distribution is a generalization of univariate normal distribution. Suppose the pdf of a bivariate normal distribution can be written in a similar form of (i) as

$$f(X) = k\ e^{-\frac{1}{2}(X-A)'\ B\,(X-A)}$$

where, $X = \begin{pmatrix} x \\ y \end{pmatrix}$ and B is a positive definite matrix.

We shall find k, A and B in such a way that

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(X)\ dX = 1$$

$$\Rightarrow \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} k\ e^{-\frac{1}{2}(X-A)'\ B\,(X-A)}\ dX = 1 \text{...........(ii)}$$

Since B is a positive definite matrix, we can find always a non-singular matrix C such that

$$C'\ B\ C = I$$

Let us make a transformation

$$X - A = CY$$

$$\Rightarrow X = A + CY$$

So, $\dfrac{dX}{dY} = |C|$

$$\Rightarrow dX = |C|\ dY$$

Again, $(X-A)'\ B\ (X-A) = (CY)'\ B\ (CY)$

$$= Y'\ C'\ B\ C\ Y$$

$$= Y'Y \quad (\because C'BC = I)$$

$$= {y_1}^2 + {y_2}^2$$

Now (ii) can be written as

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} k\ e^{-\frac{1}{2}\left({y_1}^2+{y_2}^2\right)}\ |C|\ dY = 1$$

$$\Rightarrow k\ |C| \int_{-\infty}^{\infty} e^{-\frac{1}{2}{y_1}^2}\ dy_1 \int_{-\infty}^{\infty} e^{-\frac{1}{2}{y_2}^2}\ dy_2 = 1$$

$$\Rightarrow k\ |C|\ \sqrt{2\pi}.\ \sqrt{2\pi} = 1$$

$$\Rightarrow k = \frac{1}{2\pi\ |C|} \text{ ....(iii)}$$

Again, $X = A + CY$

$$\Rightarrow E(X) = A + C\ E(Y) = A \quad [\because Y \sim NID(0, I)]$$

$$\Rightarrow A = E(X) = \begin{bmatrix} E(x) \\ E(y) \end{bmatrix} = \begin{pmatrix} \mu_x \\ \mu_y \end{pmatrix}$$

Suppose, $\Sigma = E[X - E(X)]\,[X - E(X)]'$

$$= E[A + CY - A]\,[A + CY - A]'$$

$$= E[C\ Y Y' C']$$

$$= C\ E[YY']\ C'$$

$$\Rightarrow \Sigma = C\ C' \quad [\because E(YY') = I_p]$$

$$\Rightarrow |\Sigma| = |C\ C'| = |C|.\ |C'| = |C|.\ |C| = |C|^2$$

$$\Rightarrow |C| = |\Sigma|^{\frac{1}{2}}$$

Now (iii) can be written as, $k = \dfrac{1}{2\pi\ |\Sigma|^{\frac{1}{2}}}$ ....(iv)

Again, $C'\ B\ C = I$

$$\Rightarrow (C')^{-1}\ C'\ B\ C\ C^{-1} = (C')^{-1}\ C^{-1}$$

$$\Rightarrow B = (C\ C')^{-1}$$

$$\therefore\ B = \Sigma^{-1} \quad [\because\ \Sigma = C\ C']$$

Now, $\Sigma = E[X - E(X)]\ [X - E(X)]'$

$$= E\begin{pmatrix} x-\mu_x \\ y-\mu_y \end{pmatrix}\begin{pmatrix} x-\mu_x & y-\mu_y \end{pmatrix}$$

$$= E\begin{bmatrix} (x-\mu_x)^2 & (x-\mu_x)(y-\mu_y) \\ (x-\mu_x)(y-\mu_y) & (y-\mu_y)^2 \end{bmatrix}$$

$$= \begin{bmatrix} E(x-\mu_x)^2 & E(x-\mu_x)(y-\mu_y) \\ E(x-\mu_x)(y-\mu_y) & E(y-\mu_y)^2 \end{bmatrix}$$

$$\Rightarrow \Sigma = \begin{bmatrix} \sigma_x^{\ 2} & \rho\ \sigma_x\ \sigma_y \\ \rho\ \sigma_x\ \sigma_y & \sigma_y^{\ 2} \end{bmatrix}$$

$$\Rightarrow |\Sigma| = \sigma_x^{\ 2}\ \sigma_y^{\ 2} - \rho^2\ \sigma_x^{\ 2}\ \sigma_y^{\ 2} = \sigma_x^{\ 2}\ \sigma_y^{\ 2}\ (1-\rho^2)$$

Now (iv) can be written as, $k = \dfrac{1}{2\pi\sqrt{\sigma_x^{\ 2}\ \sigma_y^{\ 2}\ (1-\rho^2)}} = \dfrac{1}{2\pi\ \sigma_x\ \sigma_y\ \sqrt{1-\rho^2}}$

So, $\Sigma^{-1} = \dfrac{1}{|\Sigma|}\begin{pmatrix} \sigma_y^{\ 2} & -\rho\ \sigma_x\ \sigma_y \\ -\rho\ \sigma_x\ \sigma_y & \sigma_x^{\ 2} \end{pmatrix}$

$$= \frac{1}{\sigma_x^{\ 2}\ \sigma_y^{\ 2}\ (1-\rho^2)}\begin{pmatrix} \sigma_y^{\ 2} & -\rho\ \sigma_x\ \sigma_y \\ -\rho\ \sigma_x\ \sigma_y & \sigma_x^{\ 2} \end{pmatrix}$$

$$= \frac{1}{1-\rho^2}\begin{pmatrix} \dfrac{1}{\sigma_x^{\ 2}} & -\dfrac{\rho}{\sigma_x\ \sigma_y} \\ -\dfrac{\rho}{\sigma_x\ \sigma_y} & \dfrac{1}{\sigma_y^{\ 2}} \end{pmatrix}$$

Now putting the values of k, $(X-A)$ and B in (ii), we get

$$f(X)=k\ e^{-\frac{1}{2}(X-A)'\,B\,(X-A)}$$

$$=\frac{1}{2\pi\ \ \sigma_x\ \sigma_y\ \sqrt{1-\rho^2}}\,e^{-\frac{1}{2}\left(x-\mu_x\quad y-\mu_y\right)\Sigma^{-1}\begin{pmatrix}x-\mu_x\\ y-\mu_y\end{pmatrix}}$$

$$=\frac{1}{2\pi\ \sigma_x\ \sigma_y\ \sqrt{1-\rho^2}}\,e^{-\frac{1}{2}\left(x-\mu_x\quad y-\mu_y\right)\frac{1}{1-\rho^2}\begin{pmatrix}\frac{1}{\sigma_x^{\,2}} & -\frac{\rho}{\sigma_x\sigma_y}\\ -\frac{\rho}{\sigma_x\sigma_y} & \frac{1}{\sigma_y^{\,2}}\end{pmatrix}\begin{pmatrix}x-\mu_x\\ y-\mu_y\end{pmatrix}}$$

$$=\frac{1}{2\pi\ \sigma_x\ \sigma_y\ \sqrt{1-\rho^2}}\,e^{-\frac{1}{2\left(1-\rho^2\right)}\left[\frac{x-\mu_x}{\sigma_x^{\,2}}-\frac{\rho\left(y-\mu_y\right)}{\sigma_x\sigma_y}\qquad -\frac{\rho\left(x-\mu_x\right)}{\sigma_x\ \sigma_y}+\frac{y-\mu_y}{\sigma_y^{\,2}}\right]\begin{pmatrix}x-\mu_x\\ y-\mu_y\end{pmatrix}}$$

$$=\frac{1}{2\pi\ \sigma_x\ \sigma_y\ \sqrt{1-\rho^2}}\,e^{-\frac{1}{2\left(1-\rho^2\right)}\left[\frac{\left(x-\mu_x\right)^2}{\sigma_x^{\,2}}-\frac{\rho\left(x-\mu_x\right)\left(y-\mu_y\right)}{\sigma_x\ \sigma_y}-\frac{\rho\left(x-\mu_x\right)\left(y-\mu_y\right)}{\sigma_x\sigma_y}+\frac{\left(y-\mu_y\right)^2}{\sigma_y^{\,2}}\right]}$$

$$\therefore\quad f(x,\ y)=\frac{1}{2\pi\ \sigma_x\ \sigma_y\ \sqrt{1-\rho^2}}\,e^{-\frac{1}{2\left(1-\rho^2\right)}\left[\left(\frac{x-\mu_x}{\sigma_x}\right)^2-2\rho\left(\frac{x-\mu_x}{\sigma_x}\right)\left(\frac{y-\mu_y}{\sigma_y}\right)+\left(\frac{y-\mu_y}{\sigma_y}\right)^2\right]}$$

This is the pdf of a bivariate normal distribution with parameters μ_x, μ_y, σ_x, σ_y and ρ.

NB: (i) Let, $X=\mu+CY$

$$\Rightarrow\begin{pmatrix}x\\ y\end{pmatrix}=\begin{pmatrix}\mu_x\\ \mu_y\end{pmatrix}+\begin{pmatrix}c_{11} & c_{12}\\ c_{21} & c_{22}\end{pmatrix}\begin{pmatrix}y_1\\ y_2\end{pmatrix}=\begin{pmatrix}\mu_x\\ \mu_y\end{pmatrix}+\begin{pmatrix}c_{11}\ y_1\ +\ c_{12}\ y_2\\ c_{21}\ y_1\ +\ c_{22}\ y_2\end{pmatrix}$$

The Jacobian of the transformation is

$$J=\begin{vmatrix}\frac{\partial x}{\partial y_1} & \frac{\partial x}{\partial y_2}\\ \frac{\partial y}{\partial y_1} & \frac{\partial y}{\partial y_2}\end{vmatrix}=\begin{vmatrix}c_{11} & c_{12}\\ c_{21} & c_{22}\end{vmatrix}$$

$$\therefore\quad |J|=\text{mod}\,|C|$$

Where "$\text{mod}\,|C|$" indicates the absolute value of the determinant of C.

(ii) If $x\sim N(0,1)$, then $f(x)=\dfrac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}x^2};\quad -\infty<x<\infty$

So, $\int_{-\infty}^{\infty} f(x)\, dx = 1$

$$\Rightarrow \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}\, dx = 1$$

$$\therefore \int_{-\infty}^{\infty} e^{-\frac{1}{2}x^2}\, dx = \sqrt{2\pi}$$

(iii) $V(Y) = E[Y - E(Y)][Y - E(Y)]'$

$$\Rightarrow I = E(YY') \quad [\because Y \sim NID(0, I)]$$

$$\therefore E(YY') = I$$

(iv) $\rho = \dfrac{Cov(x,\ y)}{\sigma_x \sigma_y} = \dfrac{E\{x - E(x)\}\ \{y - E(y)\}}{\sigma_x\ \sigma_y} = \dfrac{E(x - \mu_x)\ (y - \mu_y)}{\sigma_x\ \sigma_y}$

$$\therefore E(x - \mu_x)\ (y - \mu_y) = \rho\ \sigma_x\ \sigma_y$$

Q. Find out the marginal distributions of bi-variate normal distribution (NU-15)

Ans: We know the pdf of bivariate normal distribution is

$$f(x,\ y) = \frac{1}{2\pi\ \sigma_x\ \sigma_y\ \sqrt{1-\rho^2}} e^{-\frac{1}{2(1-\rho^2)}\left[\left(\frac{x-\mu_x}{\sigma_x}\right)^2 - 2\rho\left(\frac{x-\mu_x}{\sigma_x}\right)\left(\frac{y-\mu_y}{\sigma_y}\right) + \left(\frac{y-\mu_y}{\sigma_y}\right)^2\right]}$$

$$-\infty < x < \infty,\ -\infty < y < \infty,\ \sigma_x > 0,\ \sigma_y > 0$$

$$-\infty < \mu_x < \infty,\ -\infty < \mu_y < \infty,\ -1 < \rho < 1$$

By definition, the marginal distribution of X is given by

$$g(x) = \int_{-\infty}^{\infty} f(x,\ y)\ dy$$

$$\Rightarrow g(x) = \frac{1}{2\pi\ \sigma_x\ \sigma_y\ \sqrt{1-\rho^2}} \int_{-\infty}^{\infty} e^{-\frac{1}{2(1-\rho^2)}\left[\left(\frac{x-\mu_x}{\sigma_x}\right)^2 - 2\rho\left(\frac{x-\mu_x}{\sigma_x}\right)\left(\frac{y-\mu_y}{\sigma_y}\right) + \left(\frac{y-\mu_y}{\sigma_y}\right)^2\right]} dy \text{ ....(i)}$$

Let, $v = \dfrac{y - \mu_y}{\sigma_y}$

$$\Rightarrow y - \mu_y = v\ \sigma_y$$

$$\Rightarrow y = \mu_y + v\ \sigma_y$$

Now, $\frac{dy}{dv} = \sigma_y$

$\therefore \; dy = \sigma_y \; dv$

So (i) can be written as

$$g(x) = \frac{1}{2\pi \; \sigma_x \; \sigma_y \; \sqrt{1-\rho^2}} \int_{-\infty}^{\infty} e^{-\frac{1}{2(1-\rho^2)}\left[\left(\frac{x-\mu_x}{\sigma_x}\right)^2 - 2\rho\left(\frac{x-\mu_x}{\sigma_x}\right)v \; + v^2\right]} \sigma_y \; dv$$

$$= \frac{1}{2\pi \; \sigma_x \; \sqrt{1-\rho^2}} \int_{-\infty}^{\infty} e^{-\frac{1}{2(1-\rho^2)}\left[\left(\frac{x-\mu_x}{\sigma_x}\right)^2 + \left\{\rho\left(\frac{x-\mu_x}{\sigma_x}\right)\right\}^2 - 2\rho\left(\frac{x-\mu_x}{\sigma_x}\right)v \; + v^2 - \left\{\rho\left(\frac{x-\mu_x}{\sigma_x}\right)\right\}^2\right]} dv$$

$$= \frac{1}{2\pi \; \sigma_x \; \sqrt{1-\rho^2}} \int_{-\infty}^{\infty} e^{-\frac{1}{2(1-\rho^2)}\left[\left(\frac{x-\mu_x}{\sigma_x}\right)^2 - \rho^2\left(\frac{x-\mu_x}{\sigma_x}\right)^2 + \left\{\rho\left(\frac{x-\mu_x}{\sigma_x}\right) - v\right\}^2\right]} dv$$

$$= \frac{1}{2\pi \; \sigma_x \; \sqrt{1-\rho^2}} \int_{-\infty}^{\infty} e^{-\frac{1}{2(1-\rho^2)}\left[\left(\frac{x-\mu_x}{\sigma_x}\right)^2 (1-\rho^2) + \left\{v - \rho\left(\frac{x-\mu_x}{\sigma_x}\right)\right\}^2\right]} dv$$

$$= \frac{1}{2\pi \; \sigma_x \sqrt{1-\rho^2}} \int_{-\infty}^{\infty} e^{-\frac{1}{2(1-\rho^2)}\left(\frac{x-\mu_x}{\sigma_x}\right)^2 (1-\rho^2)} \, e^{-\frac{1}{2(1-\rho^2)}\left\{v-\rho\left(\frac{x-\mu_x}{\sigma_x}\right)\right\}^2} \; dv$$

$$= \frac{1}{2\pi \; \sigma_x \sqrt{1-\rho^2}} e^{-\frac{1}{2}\left(\frac{x-\mu_x}{\sigma_x}\right)^2} \int_{-\infty}^{\infty} e^{-\frac{1}{2}\left\{\frac{v-\rho\left(\frac{x-\mu_x}{\sigma_x}\right)}{\sqrt{1-\rho^2}}\right\}^2} \; dv \text{ ....(ii)}$$

Again, let $w = \frac{v-\rho\left(\frac{x-\mu_x}{\sigma_x}\right)}{\sqrt{1-\rho^2}}$

$$\Rightarrow v = \rho\left(\frac{x - \mu_x}{\sigma_x}\right) + w\sqrt{1-\rho^2}$$

Now, $\frac{dv}{dw} = w\sqrt{1-\rho^2}$

$$\Rightarrow dv = \sqrt{1-\rho^2} \; dw$$

So (ii) can be written as

$$g(x) = \frac{1}{2\pi \; \sigma_x \; \sqrt{1-\rho^2}} e^{-\frac{1}{2}\left(\frac{x-\mu_x}{\sigma_x}\right)^2} \int_{-\infty}^{\infty} e^{-\frac{1}{2}w^2} \sqrt{1-\rho^2} \; dw$$

$$= \frac{1}{2\pi\ \sigma_x} e^{-\frac{1}{2}\left(\frac{x-\mu_x}{\sigma_x}\right)^2} \int_{-\infty}^{\infty} e^{-\frac{1}{2}w^2}\ dw$$

$$= \frac{1}{\sigma_x\ \sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu_x}{\sigma_x}\right)^2} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}w^2}\ dw$$

$$\therefore\ \ g(x) = \frac{1}{\sigma_x\ \sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu_x}{\sigma_x}\right)^2} \qquad \left[\because\ \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}w^2}\ dw\right]$$

Hence the marginal distribution of X is a univariate normal with mean μ_x and variance $\sigma_x^{\ 2}$.

Similarly, it can be shown that the marginal distribution of Y is

$$h(y) = \frac{1}{\sigma_y\ \sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{y-\mu_y}{\sigma_y}\right)^2}$$

Q. **Let** $(X,\ Y) \sim BVN\left(\mu_1,\ \mu_2,\ \sigma_1^{\ 2},\ \sigma_2^{\ 2},\ \rho\right)$**, then show that X and Y are independent if and only if** $\rho = 0$ **(NU-15)**

Ans: We know the pdf of bivariate normal distribution is

$$f(x,\ y) = \frac{1}{2\pi\ \sigma_x\ \sigma_y\ \sqrt{1-\rho^2}} e^{-\frac{1}{2(1-\rho^2)}\left[\left(\frac{x-\mu_x}{\sigma_x}\right)^2 - 2\rho\left(\frac{x-\mu_x}{\sigma_x}\right)\left(\frac{y-\mu_y}{\sigma_y}\right) + \left(\frac{y-\mu_y}{\sigma_y}\right)^2\right]} \quad .....(i)$$

We also know that the marginal distributions of X and Y are

$$g(x) = \frac{1}{\sigma_x\ \sqrt{2\pi}}\ e^{-\frac{1}{2}\left(\frac{x-\mu_x}{\sigma_x}\right)^2} \ \text{ and }\ h(y) = \frac{1}{\sigma_y\ \sqrt{2\pi}}\ e^{-\frac{1}{2}\left(\frac{y-\mu_y}{\sigma_y}\right)^2}$$

If $\rho = 0$, then (i) can be written as

$$f(x,\ y) = \frac{1}{2\pi\ \sigma_x\ \sigma_y} e^{-\frac{1}{2}\left[\left(\frac{x-\mu_x}{\sigma_x}\right)^2 + \left(\frac{y-\mu_y}{\sigma_y}\right)^2\right]}$$

$$= \frac{1}{\sigma_x\ \sqrt{2\pi}\ \sigma_y\ \sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu_x}{\sigma_x}\right)^2}\ \ e^{-\frac{1}{2}\left(\frac{y-\mu_y}{\sigma_y}\right)^2}$$

$$= \frac{1}{\sigma_x \sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu_x}{\sigma_x}\right)^2} \frac{1}{\sigma_y \sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{y-\mu_y}{\sigma_y}\right)^2}$$

$$\therefore \ f(x,\ y) = g(x).\ h(y)$$

So, if $\rho = 0$, then X and Y are independent.

Q. Find the moment generating function of bi-variate normal distribution (NU-15)

Ans: We know the pdf of bivariate normal distribution is

$$f(x,y) = \frac{1}{2\pi\ \sigma_x\ \sigma_y\ \sqrt{1-\rho^2}} e^{-\frac{1}{2(1-\rho^2)}\left[\left(\frac{x-\mu_x}{\sigma_x}\right)^2 - 2\rho\left(\frac{x-\mu_x}{\sigma_x}\right)\left(\frac{y-\mu_y}{\sigma_y}\right) + \left(\frac{y-\mu_y}{\sigma_y}\right)^2\right]}$$

Now moment generating function of a bivariate normal distribution about their origin is

$$M_{x,\,y}(t_1,\ t_2) = E\left[e^{t_1 x + t_2 y}\right]$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{t_1 x + t_2 y}\ f(x,\ y)\ dx\ dy$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{t_1 x + t_2 y} \frac{1}{2\pi\ \sigma_x\ \sigma_y\ \sqrt{1-\rho^2}} e^{-\frac{1}{2(1-\rho^2)}\left[\left(\frac{x-\mu_x}{\sigma_x}\right)^2 - 2\rho\left(\frac{x-\mu_x}{\sigma_x}\right)\left(\frac{y-\mu_y}{\sigma_y}\right) + \left(\frac{y-\mu_y}{\sigma_y}\right)^2\right]} dx\ dy$$

Let, $u = \frac{x-\mu_x}{\sigma_x}$ and $v = \frac{y-\mu_y}{\sigma_y}$

$\Rightarrow x = \mu_x + u\,\sigma_x$ $\qquad \Rightarrow y = \mu_y + v\,\sigma_y$

Now, $J = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \sigma_x & 0 \\ 0 & \sigma_y \end{vmatrix} = \sigma_x\ \sigma_y$

So, $dx.\ dy = |J|.\ du.\ dv = \sigma_x\ \sigma_y.du.\ dv$

$$M_{x,\,y}(t_1, t_2) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{1}{2\pi\ \sigma_x\ \sigma_y\ \sqrt{1-\rho^2}}\ e^{t_1(\mu_x + u\,\sigma_x) + t_2(\mu_y + v\,\sigma_y)}\ e^{-\frac{1}{2(1-\rho^2)}(u^2 - 2\rho uv + v^2)}\ \sigma_x\ \sigma_y\ du\ dv$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{1}{2\pi\ \sqrt{1-\rho^2}} e^{t_1\,\mu_x + t_1\,u\,\sigma_x + t_2\,\mu_y + t_2\,v\,\sigma_y} e^{-\frac{1}{2(1-\rho^2)}(u^2 - 2\rho uv + v^2)}\ du\ dv$$

$$= \frac{e^{t_1\mu_x + t_2\mu_y}}{2\pi\sqrt{1-\rho^2}} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{t_1\sigma_x u + t_2\sigma_y v} e^{-\frac{1}{2(1-\rho^2)}(u^2 - 2\rho uv + v^2)} \, du \, dv$$

$$= \frac{e^{t_1\mu_x + t_2\mu_y}}{2\pi\sqrt{1-\rho^2}} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{t_1\sigma_x u + t_2\sigma_y v - \frac{1}{2(1-\rho^2)}(u^2 - 2\rho uv + v^2)} \, du \, dv$$

$$= \frac{e^{t_1\mu_x + t_2\mu_y}}{2\pi\sqrt{1-\rho^2}} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{\frac{2(1-\rho^2)(t_1\sigma_x u + t_2\sigma_y v) - (u^2 - 2\rho uv + v^2)}{2(1-\rho^2)}} \, du \, dv$$

$$= \frac{e^{t_1\mu_x + t_2\mu_y}}{2\pi\sqrt{1-\rho^2}} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-\frac{1}{2(1-\rho^2)}\left\{(u^2 - 2\rho uv + v^2) - 2(1-\rho^2)(t_1\sigma_x u + t_2\sigma_y v)\right\}} \, du \, dv$$

$$= \frac{e^{t_1\mu_x + t_2\mu_y}}{2\pi\sqrt{1-\rho^2}} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-\frac{1}{2(1-\rho^2)}\left[\left\{u - \rho v - (1-\rho^2)t_1\sigma_x\right\}^2 + (1-\rho^2)(v - \rho t_1\sigma_x - t_2\sigma_y)^2 - (1-\rho^2)(t_1^2\sigma_x^2 + 2\rho\sigma_x\sigma_y t_1 t_2 + t_2^2\sigma_y^2)\right]} \, du \, dv$$

(on simplification)

$$= \frac{e^{t_1\mu_x + t_2\mu_y}}{2\pi\sqrt{1-\rho^2}} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-\frac{1}{2}\left\{\frac{u - \rho v - (1-\rho^2)t_1\sigma_x}{\sqrt{1-\rho^2}}\right\}^2 - \frac{1}{2}(v - \rho t_1\sigma_x - t_2\sigma_y)^2 + \frac{1}{2}(t_1^2\sigma_x^2 + 2\rho\sigma_x\sigma_y t_1 t_2 + t_2^2\sigma_y^2)} \, du \, dv$$

Let, $w = \frac{u - \rho v - (1-\rho^2)t_1\sigma_x}{\sqrt{1-\rho^2}}$ and $z = v - \rho t_1\sigma_x - t_2\sigma_y$

$\Rightarrow u = \rho v + (1-\rho^2)t_1\sigma_x + w\sqrt{1-\rho^2}$ $\Rightarrow v = z + \rho t_1\sigma_x + t_2\sigma_y$

Now, $J = \begin{vmatrix} \frac{\partial u}{\partial w} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial w} & \frac{\partial v}{\partial z} \end{vmatrix} = \begin{vmatrix} \sqrt{1-\rho^2} & 0 \\ 0 & 1 \end{vmatrix} = \sqrt{1-\rho^2}$

So, $du.\,dv = |J|.\,dw.\,dz = \sqrt{1-\rho^2}.dw.\,dz$

Therefore,

$$M_{x,y}(t_1, t_2) = \frac{e^{t_1\mu_x + t_2\mu_y}}{2\pi\sqrt{1-\rho^2}} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-\frac{1}{2}w^2 - \frac{1}{2}z^2 + \frac{1}{2}(t_1^2\sigma_x^2 + 2\rho\sigma_x\sigma_y t_1 t_2 + t_2^2\sigma_y^2)} \sqrt{1-\rho^2} \, dw \, dz$$

$$= e^{t_1\mu_x + t_2\mu_y + \frac{1}{2}(t_1^2\sigma_x^2 + 2\rho\sigma_x\sigma_y t_1 t_2 + t_2^2\sigma_y^2)} \frac{1}{2\pi} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-\frac{1}{2}w^2 - \frac{1}{2}z^2} \, dw \, dz$$

$$= e^{t_1 \mu_x + t_2 \mu_y + \frac{1}{2}\left(t_1^2\sigma_x^2 + 2\rho\sigma_x\sigma_y t_1 t_2 + t_2^2\sigma_y^2\right)} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}w^2} \, dw \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}z^2} \, dz$$

$$= e^{t_1 \mu_x + t_2 \mu_y + \frac{1}{2}\left(t_1^2\sigma_x^2 + 2\rho \sigma_x \sigma_y t_1 t_2 + t_2^2 \sigma_y^2\right)}$$

This is the mgf of a bivariate normal distribution.

NB: (i) If $x \sim N\left(\mu, \sigma^2\right)$, then $f\left(x; \mu, \sigma^2\right) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}; \quad -\infty < x < \infty$

(ii) If $x \sim N(0, 1)$, then $f(x; 0, 1) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}; \quad -\infty < x < \infty$

(iii) Let $u = \frac{x-a}{c}$

$$\Rightarrow x = a + cu$$

Now, $\frac{dx}{du} = c = |J|$

$$\therefore \; dx = |J| \, du$$

(iv) $\left(u^2 - 2\rho uv + v^2\right) - 2\left(1-\rho^2\right)\left(t_1 \sigma_x u + t_2 \sigma_y v\right)$

$$= \left(u^2 - 2\rho uv + v^2\right) - \left(2-2\rho^2\right)\left(t_1 \sigma_x u + t_2 \sigma_y v\right)$$

$$= u^2 - 2\rho uv + v^2 - \left(2t_1\sigma_x u + 2t_2\sigma_y v - 2\rho^2 t_1\sigma_x u - 2\rho^2 t_2\sigma_y v\right)$$

$$= u^2 - 2\rho uv + v^2 - 2t_1 \sigma_x u - 2t_2 \sigma_y v + 2\rho^2 t_1 \sigma_x u + 2\rho^2 t_2 \sigma_y v$$

$$= u^2 + \rho^2 v^2 + t_1^2\sigma_x^2 - 2\rho^2 t_1^2\sigma_x^2 + \rho^4 t_1^2\sigma_x^2 - 2\rho uv + 2\rho v t_1\sigma_x - 2\rho^3 v t_1\sigma_x - 2t_1\sigma_x u + 2\rho^2 t_1\sigma_x u$$
$$+ v^2 - \rho^2 v^2 + \rho^2 t_1^2\sigma_x^2 - \rho^4 t_1^2\sigma_x^2 + t_2^2\sigma_y^2 - \rho^2 t_2^2\sigma_y^2 - 2\rho v\sigma_x t_1 + 2\rho^3 v\sigma_x t_1 + 2\rho\sigma_x\sigma_y t_1 t_2 - 2\rho^3\sigma_x\sigma_y t_1 t_2$$
$$- 2v\sigma_y t_2 + 2\rho^2 v\sigma_y t_2 - t_1^2\sigma_x^2 - 2\rho\sigma_x\sigma_y t_1 t_2 - t_2^2\sigma_y^2 + \rho^2\sigma_x^2 t_1^2 + 2\rho^3\sigma_x\sigma_y t_1 t_2 + \rho^2\sigma_y^2 t_2^2$$

$$= u^2 + \rho^2 v^2 + \left(1-\rho^2\right)^2 t_1^2\sigma_x^2 - 2\rho uv + 2\rho v\left(t_1\sigma_x - \rho^2 t_1\sigma_x\right) - 2t_1\sigma_x u + 2\rho^2 t_1\sigma_x u + \left(1-\rho^2\right)v^2$$
$$+ \left(1-\rho^2\right)\rho^2 t_1^2\sigma_x^2 + \left(1-\rho^2\right)t_2^2\sigma_y^2 - 2\rho v\sigma_x t_1\left(1-\rho^2\right) + 2\rho\sigma_x\sigma_y t_1 t_2\left(1-\rho^2\right) - 2v\sigma_y t_2\left(1-\rho^2\right)$$
$$- t_1^2\sigma_x^2 - 2\rho\sigma_x\sigma_y t_1 t_2 - t_2^2\sigma_y^2 + \rho^2\sigma_x^2 t_1^2 + 2\rho^3\sigma_x\sigma_y t_1 t_2 + \rho^2\sigma_y^2 t_2^2$$

$$= u^2 + (-\rho v)^2 + \left\{-(1-\rho^2)t_1\sigma_x\right\}^2 + 2u.(-\rho v) + 2(-\rho v).\left\{-(1-\rho^2)t_1\sigma_x\right\} + 2\left\{-(1-\rho^2)t_1\sigma_x\right\}.u$$

$$+(1-\rho^2)\left[v^2 + (-\rho t_1\sigma_x)^2 + (-t_2\sigma_y)^2 + 2v(-\rho t_1\sigma_x) + 2(-\rho t_1\sigma_x)(-t_2\sigma_y) + 2v(-t_2\sigma_y)\right]$$

$$-\left(t_1^2\sigma_x^2 + 2\rho\sigma_x\sigma_y t_1 t_2 + t_2^2\sigma_y^2 - \rho^2 t_1^2\sigma_x^2 - 2\rho^3\sigma_x\sigma_y t_1 t_2 - \rho^2 t_2^2\sigma_y^2\right)$$

$$= \left\{u-\rho v-(1-\rho^2)t_1\sigma_x\right\}^2 + (1-\rho^2)\left(v-\rho t_1\sigma_x - t_2\sigma_y\right)^2 - (1-\rho^2)\left(t_1^2\sigma_x^2 + 2\rho\sigma_x\sigma_y t_1 t_2 + t_2^2\sigma_y^2\right)$$

Q. Find the conditional probability density function of x for given y of bi-variate normal distribution (NU-14)

Ans: We know the pdf of bivariate normal distribution is

$$f(x,y) = \frac{1}{2\pi\,\sigma_x\,\sigma_y\,\sqrt{1-\rho^2}}\, e^{-\frac{1}{2(1-\rho^2)}\left[\left(\frac{x-\mu_x}{\sigma_x}\right)^2 - 2\rho\left(\frac{x-\mu_x}{\sigma_x}\right)\left(\frac{y-\mu_y}{\sigma_y}\right) + \left(\frac{y-\mu_y}{\sigma_y}\right)^2\right]}$$

$$-\infty < x < \infty,\ -\infty < y < \infty,\ \sigma_x > 0,\ \sigma_y > 0$$
$$-\infty < \mu_x < \infty,\ -\infty < \mu_y < \infty,\ -1 < \rho < 1$$

We also know that the marginal distribution of Y is

$$h(y) = \frac{1}{\sigma_y\,\sqrt{2\pi}}\, e^{-\frac{1}{2}\left(\frac{y-\mu_y}{\sigma_y}\right)^2}$$

Now the conditional probability density function of x for given y is

$$f(x|y) = \frac{f(x,\,y)}{h(y)}$$

$$= \frac{\dfrac{1}{2\pi\,\sigma_x\,\sigma_y\,\sqrt{1-\rho^2}}\, e^{-\frac{1}{2(1-\rho^2)}\left[\left(\frac{x-\mu_x}{\sigma_x}\right)^2 - 2\rho\left(\frac{x-\mu_x}{\sigma_x}\right)\left(\frac{y-\mu_y}{\sigma_y}\right) + \left(\frac{y-\mu_y}{\sigma_y}\right)^2\right]}}{\dfrac{1}{\sigma_y\sqrt{2\pi}}\, e^{-\frac{1}{2}\left(\frac{y-\mu_y}{\sigma_y}\right)^2}}$$

$$= \frac{\dfrac{1}{\sqrt{2\pi}\,\sigma_x\,\sqrt{1-\rho^2}}\, e^{-\frac{1}{2(1-\rho^2)}\left[\left(\frac{x-\mu_x}{\sigma_x}\right)^2 - 2\left(\frac{x-\mu_x}{\sigma_x}\right)\rho\left(\frac{y-\mu_y}{\sigma_y}\right) + \left\{\rho\left(\frac{y-\mu_y}{\sigma_y}\right)\right\}^2 + \left(\frac{y-\mu_y}{\sigma_y}\right)^2 - \left\{\rho\left(\frac{y-\mu_y}{\sigma_y}\right)\right\}^2\right]}}{e^{-\frac{1}{2}\left(\frac{y-\mu_y}{\sigma_y}\right)^2}}$$

$$=\frac{\frac{1}{\sqrt{2\pi}\ \sigma_x\ \sqrt{1-\rho^2}}\, e^{-\frac{1}{2\left(1-\rho^2\right)}\left[\left\{\frac{x-\mu_x}{\sigma_x}-\rho\left(\frac{y-\mu_y}{\sigma_y}\right)\right\}^2+\left(\frac{y-\mu_y}{\sigma_y}\right)^2\left(1-\rho^2\right)\right]}}{e^{-\frac{1}{2}\left(\frac{y-\mu_y}{\sigma_y}\right)^2}}$$

$$=\frac{\frac{1}{\sqrt{2\pi}\ \sigma_x\ \sqrt{1-\rho^2}}\, e^{-\frac{1}{2\left(1-\rho^2\right)}\left\{\frac{x-\mu_x}{\sigma_x}\sigma_x-\rho\frac{y-\mu_y}{\sigma_y}\sigma_x\right\}^2\frac{1}{\sigma_x^2}}\, e^{-\frac{1}{2\left(1-\rho^2\right)}\left(\frac{y-\mu_y}{\sigma_y}\right)^2\left(1-\rho^2\right)}}{e^{-\frac{1}{2}\left(\frac{y-\mu_y}{\sigma_y}\right)^2}}$$

$$=\frac{\frac{1}{\sigma_x\sqrt{1-\rho^2}\sqrt{2\pi}}\, e^{-\frac{1}{2\left(1-\rho^2\right)\sigma_x^2}\left\{x-\mu_x-\rho\frac{\sigma_x}{\sigma_y}\left(y-\mu_y\right)\right\}^2}\, e^{-\frac{1}{2}\left(\frac{y-\mu_y}{\sigma_y}\right)^2}}{e^{-\frac{1}{2}\left(\frac{y-\mu_y}{\sigma_y}\right)^2}}$$

$$=\frac{1}{\sqrt{\sigma_x^2\left(1-\rho^2\right)}\ \sqrt{2\pi}}\, e^{-\frac{1}{2\left(1-\rho^2\right)\sigma_x^2}\left[x-\left\{\mu_x+\rho\frac{\sigma_x}{\sigma_y}\left(y-\mu_y\right)\right\}\right]^2}$$

$$=\frac{1}{\sqrt{\sigma_x^2\left(1-\rho^2\right)}\sqrt{2\pi}}\, e^{-\frac{1}{2}\left[\frac{x-\left\{\mu_x+\rho\frac{\sigma_x}{\sigma_y}\left(y-\mu_y\right)\right\}}{\sqrt{\sigma_x^2\left(1-\rho^2\right)}}\right]^2}$$

This is the conditional probability density function of x for given y.

NB: Obviously, it is a univariate normal with mean $\mu_x+\rho\frac{\sigma_x}{\sigma_y}\left(y-\mu_y\right)$ and variance $\sigma_x^2\left(1-\rho^2\right)$.

Q. If x and y have bi-variate normal distribution with mean zero and standard deviation are σ_x and σ_y respectively, then show that $u=\frac{x}{\sigma_x}+\frac{y}{\sigma_y}$ and $v=\frac{x}{\sigma_x}-\frac{y}{\sigma_y}$ are independently distributed (NU-14)

Ans: We know the pdf of bivariate normal distribution is

$$f(x,\ y)=\frac{1}{2\pi\ \sigma_x\ \sigma_y\ \sqrt{1-\rho^2}}\, e^{-\frac{1}{2\left(1-\rho^2\right)}\left[\left(\frac{x-\mu_x}{\sigma_x}\right)^2-2\rho\left(\frac{x-\mu_x}{\sigma_x}\right)\left(\frac{y-\mu_y}{\sigma_y}\right)+\left(\frac{y-\mu_y}{\sigma_y}\right)^2\right]}$$

Since $x\sim N\left(0,\ \sigma_x^2\right)$ and $y\sim N\left(0,\ \sigma_y^2\right)$, so

$$f(x,\ y) = \frac{1}{2\pi\ \sigma_x\ \ \sigma_y\sqrt{1-\rho^2}}\, e^{-\frac{1}{2\left(1-\rho^2\right)}\left\{\frac{x^2}{\sigma_x{}^2}-2\rho\frac{xy}{\sigma_x\sigma_y}+\frac{y^2}{\sigma_y{}^2}\right\}}$$

Now, $u = \frac{x}{\sigma_x} + \frac{y}{\sigma_y}$ and $v = \frac{x}{\sigma_x} - \frac{y}{\sigma_y}$

So, $u + v = \frac{2x}{\sigma_x}$ and $u - v = \frac{2y}{\sigma_y}$

$\Rightarrow x = \frac{\sigma_x}{2}(u+v)$ $\Rightarrow y = \frac{\sigma_y}{2}(u-v)$

The Jacobian of the transformation is

$$J = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \frac{\sigma_x}{2} & \frac{\sigma_x}{2} \\ \frac{\sigma_y}{2} & -\frac{\sigma_y}{2} \end{vmatrix} = -\frac{\sigma_x\sigma_y}{4} - \frac{\sigma_x\sigma_y}{4} = -2.\frac{\sigma_x\sigma_y}{4} = -\frac{\sigma_x\sigma_y}{2}$$

$$\therefore\ \ |J| = \frac{\sigma_x\sigma_y}{2}$$

The joint probability density function of U and V is given by

$$f(u,v) = f(x,y).\ |J|$$

$$= \frac{1}{2\pi\sigma_x\sigma_y\sqrt{1-\rho^2}}\, e^{-\frac{1}{2\left(1-\rho^2\right)}\left[\frac{\left\{\frac{\sigma_x}{2}(u+v)\right\}^2}{\sigma_x{}^2}-2\rho\frac{\frac{\sigma_x}{2}(u+v).\frac{\sigma_y}{2}(u-v)}{\sigma_x\sigma_y}+\frac{\left\{\frac{\sigma_y}{2}(u-v)\right\}^2}{\sigma_y{}^2}\right]}.\frac{\sigma_x\sigma_y}{2}$$

$$= \frac{1}{4\pi\sqrt{1-\rho^2}}\, e^{-\frac{1}{2\left(1-\rho^2\right)}\left\{\frac{u^2+2uv+v^2}{4}-2\rho\frac{u^2-v^2}{4}+\frac{u^2-2uv+v^2}{4}\right\}}$$

$$= \frac{1}{4\pi\sqrt{1-\rho^2}}\, e^{-\frac{1}{2\left(1-\rho^2\right)}\left\{\frac{u^2+2uv+v^2+u^2-2uv+v^2}{4}-2\rho\frac{u^2-v^2}{4}\right\}}$$

$$= \frac{1}{4\pi\sqrt{1-\rho^2}}\, e^{-\frac{1}{2\left(1-\rho^2\right)}\left\{\frac{u^2+v^2}{2}-\rho\frac{u^2-v^2}{2}\right\}}$$

$$= \frac{1}{4\pi\sqrt{1-\rho^2}}\, e^{-\frac{1}{2\left(1-\rho^2\right)}\left\{\frac{u^2+v^2-\rho u^2+\rho v^2}{2}\right\}}$$

$$= \frac{1}{2.2\pi\sqrt{(1+\rho)(1-\rho)}}\, e^{-\frac{1}{4\left(1-\rho^2\right)}\left\{(1-\rho)u^2+(1+\rho)\,v^2\right\}}$$

$$= \frac{1}{\sqrt{2}.\sqrt{2}.\sqrt{2\pi}.\sqrt{2\pi}\ \sqrt{1+\rho}\ \sqrt{1-\rho}}\ e^{-\frac{u^2}{4(1+\rho)}}.e^{-\frac{v^2}{4(1-\rho)}}$$

$$= \frac{1}{\sqrt{2(1+\rho)}\sqrt{2\pi}}e^{-\frac{1}{2}\left\{\frac{u}{\sqrt{2(1+\rho)}}\right\}^2}.\frac{1}{\sqrt{2(1-\rho)}\sqrt{2\pi}}e^{-\frac{1}{2}\left\{\frac{v}{\sqrt{2(1-\rho)}}\right\}^2}$$

$$= f(u).f(v)$$

Where, $f(u) = \frac{1}{\sqrt{2(1+\rho)}\sqrt{2\pi}}e^{-\frac{1}{2}\left\{\frac{u}{\sqrt{2(1+\rho)}}\right\}^2}$ and $f(v) = \frac{1}{\sqrt{2(1-\rho)}\sqrt{2\pi}}e^{-\frac{1}{2}\left\{\frac{v}{\sqrt{2(1-\rho)}}\right\}^2}$

Hence u and v are independently normally distributed as

$$u \sim N\left[0,\ 2(1+\rho)\right] \text{ and } v \sim N\left[0,\ 2(1-\rho)\right]$$

NB: We know the pdf of normal distribution is

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$$

If mean is zero i.e., $\mu = 0$, then

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{1}{2}\left(\frac{x}{\sigma}\right)^2}$$

i.e., $x \sim N(0, \sigma^2)$

Similarly, $f(u) = \frac{1}{\sqrt{2(1+\rho)}\ \sqrt{2\pi}}e^{-\frac{1}{2}\left\{\frac{u}{\sqrt{2(1+\rho)}}\right\}^2}$

i.e., $u \sim N\left[0,\ 2(1+\rho)\right]$

and $f(v) = \frac{1}{\sqrt{2(1-\rho)}\ \sqrt{2\pi}}e^{-\frac{1}{2}\left\{\frac{v}{\sqrt{2(1-\rho)}}\right\}^2}$

i.e., $v \sim N\left[0,\ 2(1-\rho)\right]$

P-643, M. K. Roy (New book)

পরিসংখ্যান (তত্ত্বীয়)-২০১৫

(Probability Distribution)

বিষয় কোডঃ 223601

সময়-৪ ঘণ্টা

পূর্ণমান-৮০

[দ্রষ্টব্যঃ- প্রত্যেক বিভাগ হতে ধারাবাহিকভাবে প্রশ্নের উত্তর দিতে হবে।]

ক-বিভাগ

(যে কোন দশটি সংক্ষিপ্ত প্রশ্নের উত্তর দাও)

মান- ১ x ১০ = ১০

১। (ক) বার্ণোলী বিন্যাসের সংজ্ঞা দাও।
[Define Bernoulli distribution.]

(খ) সুষম বিন্যাসের সংজ্ঞা দাও।
[Define uniform distribution.]

(গ) দেখাও যে, দ্বিপদী বিন্যাসের গড় > ভেদাংক।
[Show that, mean > variance of binomial distribution.]

(ঘ) পরিমিত বিন্যাসের ক্ষেত্রে ৫ম কেন্দ্রীয় পরিঘাতের মান কত?
[What is the value of 5^{th} central moment of normal distribution?]

(ঙ) জ্যামিতিক বিন্যাসের "স্মৃতি ভ্রম" ধর্মটি কি?
[What is the "lack of memory" property of geometric distribution?]

(চ) পরিমিত বিন্যাসের ক্ষেত্রে $\overline{X}, Me$ ও Mo এর মধ্যে সম্পর্ক কি?
[What is the relation among $\overline{X}, Me$ and Mo of normal distribution?]

(ছ) পরা-জ্যামিতিক বিন্যাসের সংজ্ঞা দাও।
[Define hyper-geometric distribution.]

(জ) প্রথম প্রকার বিটা বিন্যাসের পরিঘাত উৎপাদকী ফাংশনটি লিখ।
[Write down the moment generating function of beta distribution of first kind.]

(ঝ) পরিমিত রেখা কি?
[What is normal curve?]

(ঞ) দ্বিতীয় প্রকার বিটা বিন্যাসের সম্ভাবনা অপেক্ষকটি লিখ।
[Write down the p.d.f of beta distribution of 2^{nd} kind.]

(ট) আয়তাকার বিন্যাসের মধ্যমা বের কর।
[Find the median of rectangular distribution.]

(ঠ) দ্বি-চলক পৈঁসু বিন্যাসের সংজ্ঞা দাও।
[Define bi-variate Poission distribution.]

খ বিভাগ

(যে কোন পাঁচটি প্রশ্নের উত্তর দাও)

মান- ৪x ৫ = ২০

২। দ্বিপদী বিন্যাসের পরিঘাত উৎপাদনকারী অপেক্ষকটি উদ্ভাবন কর এবং এখান থেকে বিন্যাসটির ভেদাংক নির্ণয় কর।
[Derive the m.g.f of binomial distribution, hence find out variance of the distribution.]

৩। অবিচ্ছিন্ন আয়তাকার বিন্যাসের পরিঘাত উৎপাদকী ফাংশন বের কর এবং সেখান থেকে বিন্যাসটির গড় ও ভেদাংক বের কর।
[Find the moment generating function of continuous uniform distribution and hence find mean and variance of the distribution.]

৪। পরাজ্যামিতিক বিন্যাসের গড় ও ভেদাংক নির্ণয় কর।
[Find out the mean and variance of hyper-geometric distribution.]

৫। দেখাও যে, $\beta(m,n)=\frac{\Gamma m\Gamma n}{\Gamma(m+n)}$

[Show that $\beta(m,n)=\frac{\Gamma m\Gamma n}{\Gamma(m+n)}$]

৬। দেখাও যে, পৈঁসু বিন্যাসের ক্ষেত্রে, $\mu_{r+1}=rm\mu_{r-1}+m\frac{\partial \mu_r}{\partial m}$

[Show that for Poission distribution $\mu_{r+1}=rm\mu_{r-1}+m\frac{\partial \mu_r}{\partial m}$.]

৭। মনে কর, X ও Y দুটি স্বাধীন গামা চলক যাদের পরামিতি l এবং m, তাহলে $Z=\frac{X}{Y}$ এর বিন্যাস নির্ণয় কর।

[Let X and Y be two independent gamma variates with parameters l and m respectively, then find the distribution of $Z=\frac{X}{Y}$.]

৮। যদি $f(x)=be^{-b(x-a)}$; $a\le x\le\infty$, $b>0$ সম্ভাবনা ঘনত্ব অপেক্ষক বিশিষ্ট x একটি অবিচ্ছিন্ন দৈব চলক হয়, তবে দেখাও যে, $a=m-\sigma$ যেখানে m ও σ যথাক্রমে বিন্যাসটির গড় ও পরিমিত ব্যবধান।

[If x is a continuous random variable with pdf $f(x)=be^{-b(x-a)}$; $a\le x\le\infty$, $b>0$, then show that $a=m-\sigma$, where m and σ are mean and standard deviation of the distribution respecitvely .]

৯। দ্বিপদী পরিমিত বিন্যাসের প্রান্তীয় বিন্যাসগুলো নির্ণয় কর।

[Find out the marginal distributions of bi-varitate normal distribution.]

গ বিভাগ

(যে কোন পাঁচটি প্রশ্নের উত্তর দাও)

মান- ১০ x ৫ = ৫০

১০। (ক) দ্বিপদী বিন্যাসের সম্ভাবনা অপেক্ষক উদ্ভাবন কর।

(খ) দ্বিপদী বিন্যাসের ক্ষেত্রে দেখাও যে, $\mu_{r+1}=pq\left(nr\mu_{r-1}+\frac{d\mu_{r-1}}{dp}\right)$ যেখানে প্রতীকসমূহ সচরাচর অর্থ বহন করে। এখান থেকে μ_2 , μ_3 এবং μ_4 এর মান বের কর।

[(a) Derive the probability function of binomial distribution.

(b) For the binomial distribution show that $\mu_{r+1}=pq\left(nr\mu_{r-1}+\frac{d\mu_{r-1}}{dp}\right)$, where the symbols have their usual meanings. Hence find μ_2 , μ_3 and μ_4 .]

১১। (ক) পৈঁসু বিন্যাসের কুমুল্যান্ট উৎপাদনকারী ফাংশন বের কর এবং সেখান থেকে β_1 ও β_2 বের কর ও মন্তব্য কর।

(খ) যদি X একটি m পরামানবিশিষ্ট পৈঁসু চলক হয়, তবে দেখাও যে, $Z=\frac{X-m}{\sqrt{m}}$ একটি আদর্শ পরিমিত চলক, যখন $m\to\infty$ ।

[(a) Find the cumulant generating function of Poission distribution and hence find β_1 and β_2 and comment.

(b) If X is a Poission variate with parameter m then show that $Z=\frac{X-m}{\sqrt{m}}$ is a standard normal variate, when $m\to\infty$.]

১২। (ক) ঋণাত্মক দ্বিপদী বিন্যাসের যোগবোধক ধর্মটি বিবৃতিসহ প্রমাণ কর। দেখাও যে, ঋণাত্মক দ্বিপদী বিন্যাসের মোট সম্ভাবনা এক।

(খ) ঋণাত্মক দ্বিপদী বিন্যাসের পরিঘাত উৎপাদকী ফাংশন বের কর এবং সেখান থেকে বিন্যাসটির গড় ও ভেদাংক বের কর।

[(a) State and prove the additive property of negative binomial distribution. Show that the total probability of negative binomial distribution is one.

(b) Find the moment generating function of negative binomial distribution and hence find the mean and variance of the distribution.]

১৩। (ক) পরা-জ্যামিতিক বিন্যাসের সাথে দ্বিপদী বিন্যাসের সম্পর্ক স্থাপন কর।

(খ) জ্যামির্তিক বিন্যাসের পরিঘাত উৎপাদকী ফাংশন বের কর এবং সেখান থেকে বা অন্যভাবে β_1 ও β_2 বের কর এবং মন্তব্য কর।

[(a) Establish the relation between hyper-geometric distribution and binomial distribution

(b) Find the moment generating function of geometric distribution and hence or otherwise find β_1 and β_2 and comment.]

১৪। (ক) পৈঁসু বিন্যাস হতে পরিমিত বিন্যাস উদ্ভাবন কর।

(খ) দেখাও যে, পরিমিত বিন্যাসের গড়, মধ্যমা ও প্রচুরক সমান।

[(a) Derive normal distribution from Poission distribution.

(b) Show that mean, median and mode of normal distribution coincides.]

১৫। (ক) গামা বিন্যাসের তরঙ্গ গড় ও প্রচুরক বের কর।

(খ) অক্ষাভিসারী বিন্যাসের মধ্যমা এবং গড় হতে গড় ব্যবধান বের কর।

[(a) Find the harmonic mean and mode of gamma distribution.

(b) Find the median and mean deviation from mean of exponential distribution.]

১৬। (ক) বিটা বিন্যাসের গড় ও ভেদাংক বের কর।

(খ) তুমি কিভাবে প্রথম প্রকার বিটা বিন্যাস হতে দ্বিতীয় প্রকার বিটা বিন্যাস এবং দ্বিতীয় প্রকার বিটা বিন্যাস হতে প্রথম প্রকার বিটা বিন্যাস বের করবে?

[(a) Find the mean and variance of beta distribution.

(b) How will you find the beta distribution of second kind from beta distribution of first kind and beta distribution of first kind from beta distribution of second kind?]

১৭। (ক) দ্বি-চলক পরিমিত বিন্যাসের সংজ্ঞা দাও। বিন্যাসটির পরিঘাত উৎপাদকী ফাংশন বের কর।

(খ) মনে কর, $(X,Y) \sim BVN(\mu_1, \mu_2, \sigma_1^2, \sigma_2^2, \rho)$, তাহলে দেখাও যে, X ও Y স্বাধীন হবে যদি ও কেবল যদি $\rho = 0$ হয়।

[(a) Define bi-variate normal distribution. Find the moment generating function of this distribution.

(b) Let $(X,Y) \sim BVN(\mu_1, \mu_2, \sigma_1^2, \sigma_2^2, \rho)$, then show that X and Y are independent if and only if $\rho = 0$.]

পরিসংখ্যান (তত্ত্বীয়)-২০১৬

[২০১৩-২০১৪ সালের সিলেবাস অনুযায়ী]

বিষয় কোডঃ 223601

(Probability Distribution)

সময়-৪ ঘণ্টা

পূর্ণমান-৮০

[দ্রষ্টব্যঃ- প্রত্যেক বিভাগ হতে ধারাবাহিকভাবে প্রশ্নের উত্তর দিতে হবে।]

ক-বিভাগ

(যে কোন দশটি সংক্ষিপ্ত প্রশ্নের উত্তর দাও)

মান- ১ x ১০ = ১০

১। (ক) ঋণাত্মক দ্বিপদী বিন্যাসের গড় ও ভেদাংক কত?

[What is the mean and variance of negative binomial distribution?]

(খ) জ্যামিতিক বিন্যাসের সংজ্ঞা দাও।

[Define Geometric distribution.]

(গ) পরাজ্যামিতিক বিন্যাসের পরামিতিগুলো কি কি?

[What are the parameters of Hyper Geometric distribution?]

(ঘ) কি কি শর্তে গামা বিন্যাস পরিমিত বিন্যাসে পরিণত হয়?

[Under what conditions gamma distribution tends to normal distribution?]

(ঙ) প্রথম প্রকার বেটা বিন্যাসের সম্ভাবনা অপেক্ষকটি লিখ।

[Write the pdf of beta distribution of 1^{st} kind.]

(চ) পরিমিত বিন্যাসের সম্ভাবনা ঘনত্ব অপেক্ষকটি লিখ।

[Write down the probability density function of normal distribution.]

(ছ) দ্বিচলক পরিমিত বিন্যাসের গড় ও ভেদাংক কত?

[What is the mean and variance of bivariate normal distribution?]

(জ) যদি $f(x) = \theta e^{-\theta x}$, যেখানে $0 < x < \infty$ হয়, তবে x চলকের গড় ও ভেদাংক কত?

[If $f(x) = \theta e^{-\theta x}$,where $0 < x < \infty$, then what is the mean and variance of x variable?]

(ঝ) অবিচ্ছিন্ন আয়তাকার বিন্যাসের গড় কত, যার ব্যাপ্তি a থেকে b পর্যন্ত?

[What is the mean of a continuous uniform distribution within the interval a to b?]

(ঞ) আদর্শায়িত গামা বিন্যাসের সম্ভাবনা অপেক্ষকটি লিখ।

[Write the pdf of standard gamma distribution.]

(ট) পরাজ্যামিতিক বিন্যাসের গড় ও ভেদাংক কত?

[What is the mean and variance of Hyper Geometric distribution.]

(ঠ) কৌশি বিন্যাসের তিনটি বৈশিষ্ট্য লিখ।

[Write the three properties of Cauchy distribution.]

খ বিভাগ

(যে কোন পাঁচটি প্রশ্নের উত্তর দাও)

মান- ৪x ৫ = ২০

২। পৈঁসু বিন্যাসের পরিঘাত উৎপাদনকারী ফাংশনটি উদ্ভাবন কর এবং এখান থেকে গড় ও ভেদাংক বের কর।

[Derive the mgf of Poission distribution, hence find mean and variance.]

৩। ঋণাত্মক দ্বিপদী বিন্যাসের সম্ভাবনা অপেক্ষকটি উদ্ভাবন কর।

[Derive the probability function of negative binomial distribution.]

৪। পরাজ্যামিতিক বিন্যাসের সম্ভাবনা ফাংশনটি উদ্ভাবন কর।

[Derive the probability function of Hyper Geometric distribution.]

৫। অক্ষাভিসারী বিন্যাসের গড় ও ভেদাংক নির্ণয় কর।

[Find mean and variance of exponential distribution.]

৬। দ্বিতীয় প্রকার বিটা বিন্যাসের তরঙ্গ গড় নির্ণয় কর।

[Find the harmonic mean of beta distribution of 2^{nd} kind.]

৭। গামা বিন্যাসের β_1 ও β_2 নির্ণয় কর।

[Find β_1 and β_2 of gamma distribution.]

৮। জ্যামিতিক বিন্যাসের গড় ও ভেদাংক নির্ণয় কর।

[Find mean and variance of Geometric distribution.]

৯। পরিমিত বিন্যাসের সংজ্ঞা দাও। পরিমিত বিন্যাসের গড়, মধ্যমা ও প্রচুরকের মান কত? পরিমিত বিন্যাসের ব্যবহারগুলো লিখ

[Define normal distribution. What are the values of mean, median and mode of normal distribution? Write the uses of normal distributoin.]

গ বিভাগ

(যে কোন পাঁচটি প্রশ্নের উত্তর দাও)

মান- ১০ x ৫ = ৫০

১০। (ক) পৈঁসু বিন্যাসের সম্ভাবনা অপেক্ষক উদ্ভাবন কর।
(খ) পৈঁসু বিন্যাসের প্রচুরক নির্ণয় কর।
[(a) Derive the probability function of Poission distribution.
(b) Find mode of Poission distribution.]

১১। (ক) জ্যামিতিক বিন্যাসের সম্ভাবনা অপেক্ষক উদ্ভাবন কর।
(খ) জ্যামিতিক বিন্যাসের যোজন ধর্মটি বিবৃতিসহ প্রমাণ কর।
[(a) Derive probability function of Geometric distribution.
(b) State and prove the additive property of Geometric distribution.]

১২। (ক) ঋণাত্মক দ্বিপদী বিন্যাসের প্রচুরক নির্ণয় কর।
(খ) এ বিন্যাসের পরিঘাতের ক্ষেত্রে পৌনঃপুনিক সূত্রটি উদ্ভাবন কর।
[(a) Find the mode of negative binomial distribution.
(b) Derive the recursion formula of negative binomial distribution for moments.]

১৩। (ক) বহুপদী বিন্যাসের সংজ্ঞা দাও। বহুপদী বিন্যাসের সম্ভাবনা অপেক্ষক উদ্ভাবন কর।
(খ) বহুপদী বিন্যাসের গড় ও ভেদাংক বের কর।
[(a) Define multinomial distribution. Derive probability function of this distribution.
(b) Find mean and variance of multinomial distribution.]

১৪। (ক) প্রথম প্রকার বিটা বিন্যাস কি? এই বিন্যাসের তরঙ্গ গড় নির্ণয় কর।
(খ) দ্বিতীয় প্রকার বিটা বিন্যাসের r-তম পরিঘাত নির্ণয় কর এবং এখান থেকে গড় ও ভেদাংক নির্ণয় কর।

[(a) What is beta distribution of first kind? Find the harmonic mean of this distribution.

(b) Find the r-th moment of beta distribution of 2^{nd} kind and hence find mean and variance.]

১৫। (ক) কৌশি বিন্যাসের মধ্যমা নির্ণয় কর।

(খ) যদি x একটি কৌশি চলক হয় তবে দেখাও যে, x^2 একটি $\beta_2\left(\frac{1}{2},\frac{1}{2}\right)$ চলক হবে।

[(a) Find median of Cauchy distribution.

(b) If x is a Cauchy variate, show that x^2 is $\beta_2\left(\frac{1}{2},\frac{1}{2}\right)$ variate.]

১৬। দ্বিচলক পরিমিত বিন্যাস কি? দ্বিচলক পরিমিত বিন্যাসের প্রান্তীয় বিন্যাস $f(x)$ এবং শর্তাধীন বিন্যাস $f\left(\frac{y}{x}\right)$ বের কর।

[What is bi-variate normal distribution? Find marginal distribution of $f(x)$ and condition distribution of $f\left(\frac{y}{x}\right)$ of bi-variate normal distribution.]

১৭। (ক) গামা বিন্যাসের সংজ্ঞা দাও। ইহার গড় ও ভেদাংক বের কর।

(খ) x ও y যথাক্রমে m ও n পরামানবিশিষ্ট স্বাধীন গামা চলক হলে, দেখাও যে, $u = x + y$ এবং $v = \frac{x}{y}$ স্বাধীন হবে।

[(a) Define gamma distribution. Find its mean and variance.

(b) If x and y are two independent gamma variates with parameters m and n respectively, show that variates $u = x + y$ and $v = \frac{x}{y}$ are independent.]

পরিসংখ্যান (তত্ত্বীয়)-২০১৭

বিষয় কোড: 223601

(Probability Distribution)

সময়—৪ ঘণ্টা

পূর্ণমান—৮০

[দ্রষ্টব্য:—একই বিভাগের বিভিন্ন প্রশ্নের উত্তর ধারাবাহিকভাবে লিখতে হবে।]

ক বিভাগ

(যে কোন দশটি প্রশ্নের উত্তর দাও)

মান—১×১০=১০ ১

১। (ক) 'বার্ণোলী চেষ্টা' কি?

[What is a 'Bernouli trial'?]

(খ) পৈঁসু বিন্যাসের গড় ও ভেদাংক কত?

[What is the mean and varriance of Poisson distreibution?]

(গ) বিচ্ছিন্ন সুষম বিন্যাসের সংজ্ঞা দাও।

[Define discrete uniform distribution.]

(ঘ) কি কি শর্তে ঋণাত্মক দ্বিপদী বিন্যাস পৈঁসু বিন্যাসের রূপান্তরিত হয়?

[Under what conditions negative binomial distribution becomes to Poisson distribution.]

(ঙ) ঋণাত্মক দ্বিপদী বিন্যাসের সংজ্ঞা দাও।

[Define negative binomial distribution.]

(চ) পরিমিত বিন্যাসের গড় ও ভেদাংক কত?

[What is the mean and variance of normal distribution?]

(ছ) দ্বিতীয় প্রকারের বিটা বিন্যাসের সম্ভাবনা ঘনত্ব অপেক্ষকটি লিখ।

[Write down the probability density function of beta distribution of 2^{nd} kind.]

(জ) দ্বি-চলক পরিমিত বিন্যাসের পরিমিতিগুলো কি কি?

[Wht are the parameters of bi-variate normal distribution?]

(ঝ) অক্ষাভিসরী বিন্যাসের সংজ্ঞা দাও।

[Define exponential distribution?]

(ঞ) আদর্শ পরিমিত বিন্যাসের লেখচিত্র অংকনপূর্বক বৈশিষ্ট্যগুলো দেখাও।

[Sketch a standard normal probability curve and show its area properties.]

(ট) কৌশি বিন্যাসের সম্ভাবনা ঘনত্ব অপেক্ষকটি লিখ।

[Write down the probability density function of Cauchy distribution.]

(ঠ) পরা-জ্যামিতিক বিন্যাসের একটি ব্যবহার লিখ।

[Write down any one use of hyper-geometric distribution.]

খ বিভাগ

(যে কোন পাঁচটি প্রশ্নের উত্তর দাও)

মান—৪×৫=২০

২। বার্ণোলী বিন্যাসের গড় ও ভেদাংক নির্ণয় কর।

[Find the mean and variance of Bernouli distribution.]

৩। দ্বিপদী বিন্যাসের পরিঘাত উৎপাদক ফাংশনটি উদ্ভাবন কর এবং এখান থেকে বিন্যাসটির গড় নির্ণয় কর।

[Derive the moment generating function of binomial distribution and hence find mean of the distribution.]

৪। পরা-জ্যামিতিক বিন্যাসের গড় ও ভেদাংক নির্ণয় কর।

[Find out the mean and variance of hyper-geometric distribution.]

৫। পৈঁসু বিন্যাসের ধর্ম ও ব্যবহার লিখ।

[Write down the properties and uses of Poisson distribution.]

৬। অবিচ্ছিন্ন আয়তাকার বিন্যাসের গড় ও ভেদাংক বের কর।

[Find out the mean and variance of continuous uniform distribution.]

৭। ঋণাত্মক দ্বিপদী বিন্যাসের পৌনঃপুনিক সূত্রটি উদ্ভাবন কর।

[Derive the recursion formula of negative binomial distribution.]

৮। পরিমিত বিন্যাসের গড় কেন্দ্রিক গড় ব্যবধান নির্ণয় কর।

[Find mean deviation about mean of normal distribution.]

৯। প্রচলিত প্রতীকে প্রমাণ কর যে, $\beta(m,n)=\dfrac{\lceil m \lceil n}{\lceil m+n}$ ।

[In usual notations, prove that, $\beta(m,n)=\dfrac{\lceil m\, \lceil n}{\lceil m+n}$]

গ বিভাগ

(যে কোন পাঁচটি প্রশ্নের উত্তর দাও)

মান—১০×৫ = ৫০

১৩। (ক) দ্বিপদী বিন্যাসের কুমুল্যান্ট উৎপাদনী অপেক্ষক নির্ণয় কর। এখান থেকে বিন্যাসটির β_1 ও β_2 নির্ণয় কর।

(খ) দ্বিপদী বিন্যাসের ধর্ম ও ব্যবহার আলোচনা কর।

[(a) Find the cumulant generating function of binomial distribution. Hence find β_1 and β_2 of the distribution.

(b) Discuss the uses and properties of binomial distribution.]

১১। (ক) দেখাও যে, যদি N -এর মান '∞' হয় তবে পরা-জ্যামিতিক বিন্যাস দ্বিপদী বিন্যাসে রূপান্তরিত হয়।

(খ) পরা-জ্যামিতিক বিন্যাসের সংজ্ঞা দাও। এ বিন্যাসের ধর্মগুলো লিখ ।

[(a) Show that, if N tends to '∞' then hyper-geometric distribution tends to binomial distribution.

(b) Define hyper-geometric distribution. Write down the properties of this distribution.]

১২। (ক) পৈঁসু বিন্যাস হতে পরিমিত বিন্যাস উদ্ভাবন কর।

(খ) প্রমাণ কর যে, পরিমিত বিন্যাসের জোড় কেন্দ্রীয় পরিঘাতের মান

$$\mu_{2r}=1\times 3\times 5\times\cdots\times(2r-1)\sigma^{2r};r=1,2,3\cdots$$

[(a) Derive normal distribution from Poisson distribution.

(b) Prove that, even order central moment of a normal distribution is:

$$\mu_{2r}=1\times 3\times 5\times\cdots\times(2r-1)\sigma^{2r};r=1,2,3\cdots$$

১৩। (ক) সমরূপ বিন্যাসের সংজ্ঞা দাও। দেখাও যে, এটি একটি অনতি সূঁচালো বিন্যাস।

(খ) যদি $X_i \sim U(0,1);i=1,2,3,\cdots,n$ পরস্পর স্বাধীন হয় তবে $V=2\sum_{i=1}^{jn}\log_e X_i$ এর সম্ভাবনা বিন্যাস নির্ণয় কর।

[(a) Define uniform distribution. Show that, it is a platykurtic distribution.

(b) If $X_i \sim U(0,1);i=1,2,3,\cdots,n$ are independent then find the distribution of $V=2\sum_{i=1}^{jn}\log_e X_i$.]

১৪। (ক) দেখাও যে, পরামিত $m\to\infty$ এর জন্য গামা বিন্যাস পরিমিত বিন্যাসে রূপান্তরিত হয়।

(খ) যদি x একটি পরিমিত চলক হয় তবে দেখাও যে, $u=\dfrac{1}{2}\left(\dfrac{x-\mu}{\sigma}\right)^2$ একটি '$\dfrac{1}{2}$' পরামানবিশিষ্ঠ গামা চলক।

[(a) Show that for parameter $m\to\infty$, gamma distribution tends to normal distribution.

(b) If x is a normal variate then show that $u=\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2$ is a gamma variate with parameter $'\frac{1}{2}'$.]

১৫। (ক) প্রথম প্রকার বিটা বিন্যাসের প্রচুরক বের কর।

(খ) যদি x ও y যথাক্রমে m এবং n পরামানবিশিষ্ট দু'টি স্বাধীন গামা চলক হয় তবে দেখাও যে,

(i) $u=x+y$ এবং $v=\frac{x}{x+y}$ চলকদ্বয় স্বাধীন হবে;

(ii) u একটি $\gamma(m+n)$ চলক এবং v একটি $\beta_1(m,n)$ চলক হবে।

[(a) Find the mode of beta distribution of first kind.

(b) If x and y are two independent gamma variates with parameter m and n respectively, then show that

(i) $u=x+y$ and $v=\frac{x}{x+y}$ are independent variates;

(ii) u is a $\gamma(m+n)$ variate and v is a $\beta_1(m,n)$ variate.]

১৬। (ক) কৌশি বিন্যাসের সংজ্ঞা দাও। যদি $x_1, x_2, \cdots, x_n$ পরিমিত কৌশি বিন্যাস হতে গৃহীত n সংখ্যক দৈব পর্যবেক্ষণ হয় তবে $\overline{X}=\frac{1}{n}\sum_{i=1}^{n}x_i$ এর বিন্যাস নির্ণয় কর।

(খ) অক্ষাভিসারী বিন্যাসের সংজ্ঞা দাও। θ পরামিতিবিশিষ্ট একটি অক্ষাভিসারী বিন্যাসের গড় কেন্দ্রীক গড় ব্যবধান এবং মধ্যমা নির্ণয় কর।

[(a) Define Cauchy distribution. If are randomly taken observations from standard Cauchy distribution then find the distribution of $\overline{X}=\frac{1}{n}\sum_{i=1}^{n}x_i$.

(b) Define exponential distribution. Find the median and mean deviation about mean of an exponential distribution with parameter θ.]

১৭। (ক) দ্বি-চলক পরিমিত বিন্যাসের সংজ্ঞা দাও। ইহার mgf নির্ণয় কর। এখান থেকে বিন্যাসটির গড় বের কর।

(খ) যদি x ও y দু'টি আদর্শ পরিমিত চলক হয় যাদের সংশ্লেষাংক ρ, তবে প্রমাণ কর যে, $cor(x^2, y^2)=\rho^2$.

[(a) Define bi-variate normal distribution. Find it mgf. Hence find mean of this distribution.

(b) If x and y are two standard normal variates with corelation co-efficient, then prove that, $cor(x^2, y^2)=\rho^2$.]

পরিসংখ্যান-২০১৮

বিষয় কোড ঃ 223601

(Probability Distribution)

সময় - ৪ ঘন্টা

পূর্ণমান - ৮০

[দ্রষ্টব্য ঃ – একই বিভাগের প্রশ্নের উত্তর ধারাবাহিকভাবে লিখতে হবে।]

ক বিভাগ

(যে কোন দশটি প্রশ্নের উত্তর দাও)

মান – ১ × ১০ = ১০

১। (ক) বার্নৌলী বিন্যাস কি?

[What is Bernoulli distribution?]

(খ) দ্বিপদী বিন্যাসের গড় ও ভেদাংক যথাক্রমে ২০ ও ৪ হলে চেষ্টার সংখ্যা কত?

[If the mean and variance of a binomial distribution are 20 and 4 respectively then find the number of trials.]

(গ) জ্যামিতিক বিন্যাস সংজ্ঞায়িত কর।

[Define Geometric distribution.]

(ঘ) ঋণাত্মক দ্বিপদী বিন্যাসের পরিঘাত উৎপাদক অপেক্ষকটি লিখ।

[Write down the moment generating function of negative binomial distribution.]

(ঙ) পরাজ্যামিতিক বিন্যাসের ভেদাংক কত?

[What is the variance of Hyper Geometric distribution?]

(চ) একটি অক্ষাভিসারী বিন্যাসের গড় ৪ হলে উহার সম্ভাবনা ঘনত্ব অপেক্ষকটি লিখ।

[Write down the probability density function of an exponential distribution with mean 4.]

(ছ) অবিচ্ছিন্ন আয়তাকার বিন্যাসের E(x) বের কর যখন $2 \le x \le 7$।

[Find E(x) of a continuous uniform distribution when $2 \le x \le 7$.]

(জ) প্রথম প্রকার বিটা বিন্যাসের সম্ভাবনা ঘনত্ব অপেক্ষকটি লিখ।

[Write the probability density function of beta distributio of 1st kind.]

(ঝ) পরিমিত বিন্যাসের গড় ব্যবধান ও পরিমিত ব্যবধানের সম্পর্কটি লিখ।

[Write down the relationship between the standard deviation and mean deviation of normal distribution.]

(ঞ) Weibull বিন্যাসের সংজ্ঞা দাও।

[Define Weibull distribution.]

(ট) Maxwell বিন্যাসের সম্ভাবনা ঘনত্ব অপেক্ষকটি লিখ।

[Write down the probability density function of Maxwell distribution.]

(ঠ) দ্বিচলক পরিমিত বিন্যাসের পরামিতি কতটি?

[How many parameters have a bi-variate normal distribution?]

খ বিভাগ

(যে কোনো পাঁচটি প্রশ্নের উত্তর দাও)

মান – ৪ × ৫ = ২০

২। একটি দ্বিপদী চলক x-এর জন্য $p(x = 0) = 2p(x = 1) = 9p(x = 2)$ হলে সম্ভাবনা অপেক্ষকটি নির্ণয় কর।

[For a binomial variate x, if $p(x = 0) = 2p(x = 1) = 9p(x = 2)$ then find the probability function.]

৩। পৈসঁু বিন্যাসের সম্ভাবনা অপেক্ষকটি উদ্ভাবন কর।

[Drive the probability function of Poisson distribution.]

৪। ঋণাত্মক দ্বিপদী বিন্যাসের গড় ও ভেদাংক নির্ণয় কর।

[Find the mean and variance of Negative binomial distribution.]

৫। জ্যামিতিক বিন্যাসের পরিঘাত উৎপাদকী অপেক্ষক বের কর। অতঃপর গড় ও ভেদাংক নির্ণয় কর।

[Determine the probability generating function of Geometric distribution. Hence find the mean and variance.]

৬। পরিমিত বিন্যাস কি? এর ধর্মাবলি উল্লেখ কর।

[What is Normal distribution? Mention its properties.]

৭। গামা চলক কি? গামা বিন্যাসের নিয়ামক অপেক্ষকটি নিরূপণ কর।

[What id Gamma Variate? Determine the characteristic function of Gamma distribution.]

৮। দ্বিতীয় প্রকারের বিটা বিন্যাস কি? এর তরঙ্গ গড় নির্ণয় কর।

[What is beta distribution of 2nd kind? Find its harmonic mean.]

৯। কৌশি বিন্যাসের নিয়ামক অপেক্ষক নির্ণয় কর।

[Find out the characteristic function of Cauchy distribution.]

গ বিভাগ

(যে কোনো পাঁচটি প্রশ্নের উত্তর দাও)

মান – ১০ × ৫ = ৫০

১০। (ক) দ্বিপদী বিন্যাসের সম্ভাবনা অপেক্ষটি উদ্ভাবন কর।

(খ) দ্বিপদী বিন্যাসের ক্ষেত্রে প্রচলিত প্রতীকে দেখাও যে, $\mu_{r+1} = pq\left[\frac{d}{dp}\mu_r + nr\,\mu_{r-1}\right]$। অতঃপর μ_2 এবং μ_3 নির্ণয় কর।

[(a) Drive the probability function of binomial distribution.

(b) For the binomial distribution in usual notation show that, $\mu_{r+1} = pq\left[\frac{d}{dp}\mu_r + nr\,\mu_{r-1}\right]$. Hence find μ_2 and μ_3.]

১১। (ক) পৈসোঁ বিন্যাসের কুমুল্যান্ট উৎপাদকী অপেক্ষক নির্ণয় কর। অতঃপর β_1 ও β_2 বের কর।

(খ) পৈসোঁ বিন্যাসের যোগবোধক ধর্মটি বিবৃতিসহ প্রমাণ কর।

[(a) Find the cumulant generating function of Poission distribution. Hence find β_1 & β_2.

(b) State and prove the additive property of Poission distribution.]

১২। (ক) জ্যামিতিক বিন্যাসের সম্ভাবনা অপেক্ষকটি উদ্ভাবন কর।

(খ) জ্যামিতিক বিন্যাসের 'স্মৃতিভ্রম' ধর্মটি বিবৃতিসহ প্রমাণ কর।

[(a) Derive the probability function of Geometric distribution.

(b) State and prove the "Lack of Memory" property Geometric distribution]

১৩। (ক) পরাজ্যামিতিক বিন্যাসের সম্ভাবনা অপেক্ষকটি উদ্ভাবন কর।

(খ) বহুপদী বিন্যাসের পরিঘাত উৎপাদকী অপেক্ষক নিরূপণ কর। অতঃপর এ বিন্যাসের গড় ও ভেদাংক নির্ণয় কর।

[(a) Derive the probability function of Hyper Geometric distribution.

(b) Determine the moment generating function of multinomial distribution. Hence find the mean & variance of this distribution.]

১৪। (ক) পরিমিত বিন্যাসের মধ্যমা ও প্রচুরক নির্ণয় কর।

(খ) দেখাও যে, পরিমিত বিন্যাসের ক্ষেত্রে $\beta_1 = 0$ এবং $\beta_2 = 3$।

[(a) Find the median and mode of normal distribution.

(b) Show that in case of normal distribution $\beta_1 = 0$ and $\beta_2 = 3$.]

১৫। (ক) আয়তাকার বিন্যাস কি? অবিচ্ছিন্ন আয়তাকার বিন্যাসের ক্ষেত্রে গড় হতে গড় ব্যবধান নির্ণয় কর।

(খ) অক্ষাভিসারী বিন্যাসের কুমুল্যান্ট উৎপাদকী অপেক্ষক বের কর এবং অতঃপর দেখাও যে, অক্ষাভিসারী বিন্যাসটি হলো অতি সূচাঁলো বিন্যাস।

[(a) What is Rectangular distribution? Find the mean deviation from mean in case of continuous rectangular distribution.

(b) Find out the cumulant generating function of exponential distribution and hence show that exponential distribution is platykurtic distribution.]

১৬। (ক) বিটা বিন্যাস কি? কিভাবে প্রথম প্রকার বিটা বিন্যাস হতে ২য় প্রকার বিটা বিন্যাস এবং ২য় প্রকার বিটা বিন্যাস হতে প্রথম প্রকার বিটা বিন্যাস পাওয়া যায়?

(খ) যদি x ও y দুইটি স্বাধীন গামা চলক হয় তবে দেখাও যে, (x + y) গামা বিন্যাস মেনে চলে কিন্তু $\left(\frac{x}{y}\right)$ ২য় প্রকার বিটা বিন্যাস মেনে চলে।

[(a) What is Beta distribution? How will you get Beta distribution of 2nd kind from Beta distribution of 1st kind and Beta distribution of 1st kind from Beta distribution of 2nd kind?

(b) If x and y are two independent Gamma variates, then show that (x + y) follows Gamma distribution, but $\left(\frac{x}{y}\right)$ follows Beta distribution of 2nd kind.]

১৭। (ক) দ্বিচলক পরিমিত বিন্যাসের সম্ভাবনা ঘনত্ব অপেক্ষকটি উপস্থাপন কর এবং এ বিন্যাসের ধর্মাবলি বিবৃত কর।

(খ) যদি $(x_1, x_2) \sim$ BVN $(\mu_1, \mu_2, \sigma_1^2, \sigma_2^2, \rho)$ হয় তবে x_1 দৈবচলকের প্রান্তীয় বিন্যাসের সম্ভাবনা ঘনত্ব অপেক্ষক নির্ণয় কর।

[(a) Represent the probability density function of bivariate normal distribution and state the properties of this distribution.

(b) If $(x_1, x_2) \sim$ BVN $(\mu_1, \mu_2, \sigma_1^2, \sigma_2^2, \rho)$ then determine the p.d.f of the marginal distribution of the random variable x_1.]

পরিসংখ্যান (তত্ত্বীয়)-২০১৯

বিষয় কোড ঃ 223601

Probability Distribution

সময় - ৪ ঘন্টা

পূর্ণমান - ৮০

[**দ্রষ্টব্য ঃ –** বিভিন্ন বিভাগের প্রশ্নের উত্তর ধারাবাহিকভাবে লিখতে হবে।]

ক বিভাগ

(যে কোন **দশটি** প্রশ্নের উত্তর দাও)

মান- ১ x ১০ = ১০

১। (ক) বার্ণোলী চেষ্টা কী?
[What is a Bernoulli trial?]
(খ) পৈঁসু বিন্যাসের গড় ৫ হলে সম্ভাবনা অপেক্ষকটি লিখ।
[Suppose mean of a Poisson distribution be 5, then write down the probability function.]
(গ) ঋণাত্মক দ্বিপদী বিন্যাসের সম্ভাবনা অপেক্ষকটি লিখ।
[Write down the probability function of negation binomial distribution.]
(ঘ) জ্যামিতিক বিন্যাসের পরিঘাত উৎপাদকী অপেক্ষকটি লিখ।
[Write down the moment generation function of geometric distribution?]
(ঙ) কখন পরাজ্যামিতিক বিন্যাস দ্বিপদী বিন্যাসে রূপান্তরিত হয়?
[When hyper geometric distribution converted to a binomial distribution?]
(চ) বহুপদিক বিন্যাসে সম্ভাবনা অপেক্ষকটি লিখ।
[Write down the probability function of multinomial distribution.]
(ছ) পরিমিতি বিন্যাসের ৪র্থ কেন্দ্রীয় পরিঘাতের মান কত?
[What is the value of the 4th central moment of normal distribution?]
(জ) অক্ষাভিসারী বিন্যাস সংজ্ঞায়িত কর।
[Define exponential distribution.]
(ঝ) ২য় প্রকার বিটা বিন্যাসের সম্ভাবনা ঘনত্ব অপেক্ষকটি লিখ।
[Write down the p.d.f. of beta distribution of 2nd kind.]
(ঞ) দ্বি-পরামিতিক গামা বিন্যাসের সম্ভাবনা ঘনত্ব ফাংশনটি লিখ।
[Write down the probability density function of a bi-parametric gamma distribution.]
(ট) দ্বি-চলক পরিমিত বিন্যাসের সম্ভাবনা ঘনত্ব অপেক্ষকটি লিখ।
[Write down the probability density function of bi-variate normal distribution.]
(ঠ) সাধারণ কৌশি বিন্যাসের গড় কত?
[What is the mean of general Cauchy distribution?]

খ বিভাগ

(যে কোন **পাঁচটি** প্রশ্নের উত্তর দাও)

মান- ৪ x ৫ = ২০

২। দ্বি-পদী বিন্যাসের পরিঘাত উৎপাদনকারী অপেক্ষক বের কর এবং এখান থেকে উক্ত বিন্যাসের গড় ও ভেদাংক বের কর।
[Find the moment generating function of bi-nominal distribution and hence find its mean and variance.]

৩। দেখাও যে, ঋণাত্মক দ্বিপদী বিন্যাসের সীমাস্থ রূপ হলো পৈঁসু বিন্যাস।
[Show that, Poisson distribution is a limiting form of the negative binomial distribution.]

৪। জ্যামিতিক বিন্যাসের ক্ষেত্রে দেখাও যে, $P(x \geq r+j \mid x \geq j) = P(x \geq r) = q^r$.
[In case of geometric distribution show that, $P(x \geq r+j \mid x \geq j) = P(x \geq r) = q^r$.]

৫। পরা-জ্যামিতিক বিন্যাস কি? দ্বিপদী বিন্যাসের সাথে এর সম্পর্ক প্রতিষ্ঠা কর।
[What is hyper-geometric distribution? Establish its relation with binomial distribution.]

৬। যদি দৈব চলক $x \sim N(\mu, \sigma^2)$ হয় তবে, দেখাও যে, $u = \frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2$ হবে " $\frac{1}{2}$ " পরামিতি বিশিষ্ট গামা চলক।

[If random variable $x \sim N(\mu, \sigma^2)$, then show that $u = \frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2$ be a gamma variate with parameter " $\frac{1}{2}$ " .]

৭। অবিচ্ছিন্ন আয়তাকার বিন্যাসের গড় ও ভেদাংক নির্ণয় কর।
[Find the mean and variance of continuous uniform distribution.]

৮। অক্ষাভিসারী বিন্যাসের মধ্যমা ও তরঙ্গ গড় নির্ণয় কর।
[Find median and harmonic mean of exponential distribution.]

৯। দ্বি-চলক পরিমিত বিন্যাসের পরিঘাত উৎপাদকী অপেক্ষক নির্ণয় কর।
[Find the moment generating function of bi-variate normal distribution.]

গ বিভাগ

(যে কোনো **পাঁচটি** প্রশ্নের উত্তর দাও)

মান- ১০ x ৫ = ৫০

১০। (ক) পরিঘাতের ক্ষেত্রে পৈঁসু বিন্যাসের পৌনঃপুনিক সূত্রটি প্রতিষ্ঠা কর।

(খ) পরা-জ্যামিতিক বিন্যাসের গড় ও ভেদাংকের মান বের কর।

[(a) Establish the recurrence relation for the moments of Poisson distribution.

(b) Determine the mean and variance of hyper-geometric distribution.]

১১। (ক) ঋণাত্মক দ্বিপদী বিন্যাসের সম্ভাবনা অপেক্ষকটি উদ্ভাবন কর।

(খ) ঋণাত্মক দ্বিপদী বিন্যাসের কুমুল্যান্ট উৎপাদকী অপেক্ষক নির্ণয় কর এবং এখান থেকে β_1 ও β_2 নির্ণয় কর।

[(a) Derive the probability function of negative binomial distribution.

(b) Determine the cumulant generating function of negative binomial distribution and hence find β_1 and β_2 .]

১২। (ক) অক্ষাভিসারী বিন্যাসের গড় ও ভেদাংক নির্ণয় কর।

(খ) অবিচ্ছিন্ন আয়তাকার বিন্যাসের r -তম কেন্দ্রীয় পরিঘাত নির্ণয় কর। অতঃপর β_1 ও β_2 বের কর।

[(a) Find the mean and variance of exponential distribution.

(b) Determine r-th central moment of the continuous rectangular distribution. Hence find β_1 and β_2 .]

১৩। (ক) পরিমিত বিন্যাসের গড় কেন্দ্রিক গড় ব্যবধান নির্ণয় কর।

(খ) প্রমাণ কর যে, পরিমিত বিন্যাসের জোড় স্থানীয় কেন্দ্রীয় পরিঘাত হলো-

$$\mu_{2r} = (2r-1)(2r-3) \ldots \ldots 5.3.1.\sigma^{2r}; r = 1, 2, 3, \ldots \ldots$$

[(a) Find the mean deviation about mean for normal distribution.

(b) Prove that the even order central moment of normal distribution is

$$\mu_{2r} = (2r-1)(2r-3) \ldots \ldots 5.3.1.\sigma^{2r}; r = 1, 2, 3, \ldots \ldots]$$

১৪। (ক) ১ম প্রকার বিটা বিন্যাস বলতে কী বুঝ? এর তরঙ্গ গড় নির্ণয় কর।

(খ) ২য় প্রকার বিটা বিন্যাসের r -তম পরিঘাত নির্ণয় কর এবং এখান থেকে বিন্যাসের গড় ও ভেদাঙ্ক নির্ণয় কর।

[(a) What do you mean by beta distribution of 1st kind? Find its harmonic mean.

(b) Determine the r-th moment of beta distribution of 2nd kind and hence find the mean and variance of the distribution.]

১৫। (ক) সমরূপ বিন্যাসের সংজ্ঞা দাও। দেখাও যে, সমরূপ বিন্যাস একটি অনতিসূঁচালো বিন্যাস।

(খ) যদি $X_i \sim U(0,1), i = 1,\ 2, 3, \dots \dots, n$ পরস্পর স্বাধীন হয়, তবে, $W = -2\sum_{i=1}^{n} \log_e X_i$ - এর সম্ভাবনা বিন্যাস নির্ণয় কর।

[(a) Define uniform distribution. Show that, uniform distribution is platykurtic.

(b) If $X_i \sim U(0,1), i = 1,\ 2, 3, \dots \dots, n$ are independent, then find the probability distribution of $W = -2\sum_{i=1}^{n} \log_e X_i$.]

১৬। (ক) কৌশি বিন্যাস কি? এর মধ্যমা নির্ণয় কর।

(খ) যদি দৈবচলক x আদর্শ কৌশি বিন্যাস অনুসরণ করে, তবে দেখাও যে, $x^2 \sim \beta_2\left(\frac{1}{2}, \frac{1}{2}\right)$.

[(a) What is Cauchy distribution? Find its median.

(b) If the random variable x follows standard Cauchy distribution, then show that $x^2 \sim \beta_2\left(\frac{1}{2}, \frac{1}{2}\right)$.]

১৭। (ক) যদি x ও y দুটি আদর্শ পরিমিত চলক হয় যাদের সংশ্লেষাংক ρ তবে প্রমাণ কর যে $\text{cor}\,(x^2, y^2) = \rho^2$.

(খ) x ও y এর দ্বিচলক পরিমিত বিন্যাসের ক্ষেত্রে শর্তাধীন সম্ভাবনা ঘনত্ব অপেক্ষক $f\left(y|x\right)$ নির্ণয় কর।

[(a) If x and y are two standard normal variate with correlation co-efficient ρ then prove that $\text{cor}\left(x^2, y^2\right) = \rho^2$.

(b) In case of bi-variate normal distribution of x and y determine the conditional probability density function $f\left(y|x\right)$.]

পরিসংখ্যান (তত্ত্বীয়)-২০২০

বিষয় কোড ঃ 223601

Probability Distribution

সময় - ৪ ঘন্টা

পূর্ণমান - ৮০

[**দ্রষ্টব্য ঃ –** প্রতিটি বিভাগের বিভিন্ন প্রশ্নের উত্তর ধারাবাহিকভাবে লিখতে হবে ।]

ক বিভাগ

(যে কোন **দশটি** প্রশ্নের উত্তর দাও)

মান- ১ x ১০ = ১০

১। (ক) বার্ণোলি বিন্যাস কী?

[What is Bernoulli distribution?]

(খ) কখন দ্বিপদী বিন্যাস অতিসূচাল হয়?

[When binomial distribution is leptokurtic?]

(গ) পৈঁসো বিন্যাসের পরিঘাতের পুনঃপৌনিক সূত্রটি লেখ।

[Write down the recurrence relation for moments of Poisson distribution.]

(ঘ) কী শর্তে ঋণাত্মক দ্বিপদী বিন্যাস জ্যামিতিক বিন্যাসে পরিণত হয়?

[Under what conditions negative binomial distribution reduces to geometric distribution?]

(ঙ) ঋণাত্মক দ্বিপদী বিন্যাসের একটি বাস্তব উদাহরণ দাও।

[Give a real-life example of negative binomial distribution.]

(চ) জ্যামিতিক বিন্যাসের যোগবোধক ধর্মটি বর্ণনা কর।

[State the additive property of geometric distribution.]

(ছ) জ্যামিতিক বিন্যাস বলতে কী বুঝ?

[What do you mean by geometric distribution?]

(জ) পরাজ্যামিতিক বিন্যাসের ভেদাংক কত?

[What is the variance of hypergeometric distribution?]

(ঝ) পরিমিত বিন্যাসের জোড় কেন্দ্রীয় পরিঘাতের মান কত?

[What is the value of ever order central moments of normal distribution?]

(ঞ) অক্ষাভিসারী বিন্যাসের সম্ভাবনা ঘনত্ব অপেক্ষকটি লেখ।

[Write down the probability density function of exponential distribution?]

(ট) Weibul বিন্যাস সংজ্ঞায়িত কর।

[Define Weibul distribution.]

(ঠ) দ্বিচলক বিশিষ্ট পরিমিত বিন্যাসের পরামানসমূহ উল্লেখ কর।

[Mention the parameters of bi-variate distribution.]

খ বিভাগ

(যে কোন **পাঁচটি** প্রশ্নের উত্তর দাও)

মান- ৪ x ৫ = ২০

২। বার্ণোলি বিন্যাসের ১ম চারটি কেন্দ্রীয় পরিঘাতের মান নির্ণয় কর।

[Find out the first four central moments of Bernoulli distribution.]

৩। দেখাও যে গামা বিন্যাস হলো পরিমিত বিন্যাসের একটি বিশেষ রূপ।

[Show that gamma distribution is a special case of normal distribution.]

৪। ঋণাত্মক দ্বিপদী বিন্যাসের সংজ্ঞা দাও। এ বিন্যাসের ব্যবহার আলোচনা কর।

[Define negative binomial distribution. Discuss the uses of this distribution.]

৫। জ্যামিতিক বিন্যাসের স্মৃতিভ্রম ধর্মটি প্রমাণ কর।

[Prove the lack of memory property of geometric distribution.]

৬। ১ম প্রকারের বিটা বিন্যাসের সংজ্ঞা দাও। এ বিন্যাসের গড় নির্ণয় কর।

[Define beta distribution of 1^{st} kind. Find out the mean of this distribution.]

৭। পরিমিত বিন্যাসের বৈশিষ্ট্য আলোচনা কর।

[Discuss the characteristics of normal distribution.]

৮। আয়তাকার বিন্যাসের ক্ষেত্রে প্রচলিত প্রতীকে দেখাও যে, $MD = \frac{\sqrt{3}}{2} SD$.

[In case of rectangular distribution, in usual notation show that $MD = \frac{\sqrt{3}}{2} SD$.]

৯। দেখাও যে, দুটি প্রমিত পরিমিত চলকের অনুপাত প্রমিত বিন্যাসকে মেনে চলে।

[Show that the ratio of two independent standard normal variates follow standard Cauchy distribution.]

গ বিভাগ

(যে কোনো **পাঁচটি** প্রশ্নের উত্তর দাও)

মান- ১০ X ৫ = ৫০

১০। (ক) প্রয়োজনীয় অনুমিতি উল্লেখপূর্বক দ্বিপদী বিন্যাসের সম্ভাবনা অপেক্ষক উদ্ভাবন কর।

(খ) দ্বিপদী বিন্যাসের পরিঘাতের পুনঃপৌনিক সূত্রটি উদ্ভাবন কর। এখান থেকে বিন্যাসটির β_1 নির্ণয় কর।

[(a) Derive the probability function of binomial distribution mentioning the necessary assumptions.

(b) Derive the recurrence relation for moments of binomial distribution. Hence find out β_1 of the distribution.]

১১। (ক) পৈঁসু চলক কী? পৈঁসু চলকের ব্যবহার লেখ।

(খ) পৈঁসু বিন্যাসের কুমুল্যান্ট উৎপাদনকারী অপেক্ষক নির্ণয় কর এবং এখান থেকে গড় ও ভেদাংক নির্ণয় কর।

[(a) What is Poisson variate? Write down the uses of Poisson variate.

(b) Determine the cumulant generating function of Poisson distribution and hence find its mean and variance.]

১২। (ক) বহুপদী বিন্যাস কাকে বলে? বহুপদী বিন্যাসের সম্ভাবনা অপেক্ষক উদ্ভাবনা কর।

(খ) যদি দৈব চলক $x_1, x_2, \cdots, x_k$ এর বিন্যাস n ও $p_i \left(i=1, 2, \cdots, k\right)$ পরামিতি বিশিষ্ট বহুপদী বিন্যাস হয় তবে দেখাও যে,

(i) x_i এর প্রান্তীয় বিন্যাস n ও $p_i \left(i=1, 2, \cdots, k\right)$ পরামিতি বিশিষ্ট দ্বিপদী বিন্যাস এবং

(ii) $Cov\left(x_i, x_j\right) = -n\, p_i\, p_j;\ \forall\ i \neq j = 1, 2, \cdots, k$.

[(a) What is multinomial distribution? Derive the probability function of multinomial distribution.

(b) If the distribution of random variables $x_1, x_2, \cdots, x_k$ is a multinomial distribution with parameters n and $p_i \left(i=1, 2, \cdots, k\right)$ then show that

(i) The marginal distribution of x_i is binomial with parameters n and $p_i \left(i=1, 2, \cdots, k\right)$, (ii) $Cov\left(x_i, x_j\right) = -n\, p_i\, p_j;\ \forall\ i \neq j = 1, 2, \cdots, k$]

১৩। (ক) জ্যামিতিক বিন্যাসের পরিঘাত উৎপাদনকারী অপেক্ষক বের কর এবং এখান থেকে উক্ত বিন্যাসের গড় ও ভেদাংক নির্ণয় কর।

(খ) পরাজ্যামিতিক বিন্যাসের মডেল ব্যবহার করে একটি লেকের মাছের সংখ্যা কীভাবে পরিমাপ করা যায়, ব্যাখ্যা কর।

[(a) Find the moment generating function of geometric distribution and hence find its mean and variance.

(b) Explain how you will use hyper geometric model to estimate the number of fish in a lake.]

১৪। (ক) অক্ষাভিসারী বিন্যাস কী? ইহার মধ্যমা এবং গড় কেন্দ্রিক গড় ব্যবধান নির্ণয় কর।

(খ) যদি দুটি স্বাধীন দৈব চলক X ও Y এর সাধারণ অক্ষাভিসারী pdf-

$$f\left(x\right) = \begin{cases} e^{-x} & ;\ x \geq 0 \\ 0 & ;\ \text{Otherwise} \end{cases}$$

থাকে তবে $\left(Y - X\right)$ এর pdf নির্ণয় কর এবং মন্তব্য কর।

[(a) What is an exponential distribution? Find its median and mean deviation about mean.

(b) If X and Y are two independent random variables with a common exponential pdf-

$$f(x)=\begin{cases} e^{-x} & ; \ x \geq 0 \\ 0 & ; \ \text{Otherwise} \end{cases}$$

Then find the pdf of $(Y-X)$ and comment.]

১৫। (ক) দ্বিতীয় প্রকার বিটা বিন্যাসের প্রচুরক নির্ণয় কর। প্রচলিত প্রতীকে প্রমাণ কর যে, $\beta(m, n)=\frac{\overline{|m}\,\overline{|n}}{\overline{|m+n}}$.

(খ) $\beta\left(\frac{1}{2}, \frac{1}{2}\right)$ এর মান বের কর। এখান থেকে $\overline{\left|\frac{1}{2}\right.}$ এর মান বের কর।

[(a) Find out the mode of beta distribution of 2nd kind. In usual notation prove that, $\beta(m, n)=\frac{\overline{|m}\,\overline{|n}}{\overline{|m+n}}$.

(b) Find out the value of $\beta\left(\frac{1}{2}, \frac{1}{2}\right)$. Hence find out the value of $\overline{\left|\frac{1}{2}\right.}$.]

১৬। (ক) সাধারণ গামা বিন্যাসের সংজ্ঞা দাও। ইহার পরিঘাত উৎপাদনী অপেক্ষক নির্ণয় কর। এখান থেকে বা অন্যভাবে বিন্যাসটির ভেদাংক, β_1 ও β_2 নির্ণয় কর।

(খ) দেখাও যে, পরিমিত বিন্যাসের গড়, মধ্যমা ও প্রচুরক ইহার পরামিতি μ -এর সমান।

[(a) Define generalized gamma distribution. Find its moment generating function. Hence or otherwise find its variance, β_1 and β_2.

(b) Show that, the mean, median and mode of normal distribution is equal to its parameter μ.]

১৭। (ক) দ্বি-চলক পরিমিত বিন্যাস কী? ইহার প্রান্তীয় বিন্যাস নির্ণয় কর।

(খ) দুটি দৈব চলক X ও Y-এর গড় শূন্য, ভেদাংক যথাক্রমে ${\sigma_1}^2$ ও ${\sigma_2}^2$ এবং সংশ্লেষাংক ρ. চলক দুটি পরিমিত বিন্যাসে বিন্যস্ত হলে দেখাও যে, $U=\frac{X}{\sigma_1}+\frac{Y}{\sigma_2}$ এবং $V=\frac{X}{\sigma_1}-\frac{Y}{\sigma_2}$ পরস্পর স্বাধীন পরিমিত চলক যাদের ভেদাংক যথাক্রমে $2(1+\rho)$ এবং $2(1-\rho)$.

[(a) What is bi-variate normal distribution? Find its marginal distribution.

(b) Two random variables X and Y have zero mean, variances ${\sigma_1}^2$ and ${\sigma_2}^2$ respectively with correlation co-efficient ρ. If the two variables are normally distributed then show that, $U=\frac{X}{\sigma_1}+\frac{Y}{\sigma_2}$ and $V=\frac{X}{\sigma_1}-\frac{Y}{\sigma_2}$ are independent normal variableswith variances $2(1+\rho)$ and $2(1-\rho)$.]