Bernoulli Distribution

The name Bernoulli trial was given in honor of Swiss mathematician James Bernoulli, who wrote in Latin, about the year 1700.

Q. What is Bernoulli trial? (NU-11)

Ans: When in an experiment, trials are independent and each trial contains only two possible outcomes-success and failure and the probabilities of these two outcomes remain same from trial to trial, then each trial of that experiment is called Bernoulli trial.

Example: Suppose a fair coin is tossed once and to get head is success and to get tail is failure. Let 'p' be the probability of success and 'q' be the probability of failure so that $p+q=1$ where $p=P(S)=\frac{1}{2}$ and $q=P(F)=\frac{1}{2}$. So, each trial in coin tossing experiment is Bernoulli trial.

Q. What is Bernoulli variate?

Ans: A discrete random variable X is called a Bernoulli variate if it takes the value '1' when the Bernoulli trial is success and the value '0' when the same Bernoulli trial is a failure.

Example: Let a fair coin is tossed once, then the sample space is, $S=\{H,\ T\}$

Let to get head is success and x indicates the number of successes, then $x=0, 1$. Here x is a Bernoulli variate.

Q. What is Bernoulli Distribution?

Or, Define Bernoulli distribution **(NU-15)**

Ans: A discrete random variable X is said to have a Bernoulli distribution if its probability function is given by

$$f(x;p)=p^x\ q^{1-x};\ \text{for}\ \ x=0,1$$

where p is the parameter of the distribution and $p+q=1$.

NB: $f\left(X=x;p\right)=f(x;p)=p^x\ q^{1-x};\ \text{for}\ \ x=0,1$

Q. Determine the mean and variance of Bernoulli distribution (NU-10)

Ans: We know the probability function of Bernoulli distribution is

$$f(x;\, p) = p^x\ q^{1-x};\quad x = 0, 1$$

Now according to the definition,

$$E(x) = \sum_{x=0}^{1} x\ \ f(x;\, p)$$

$$= \sum_{x=0}^{1} x\ p^x\ q^{1-x}$$

$$= 0.\ p^0\ q^{1-0} + 1.\ p^1\ q^{1-1}$$

$$\Rightarrow E(x) = 0 + p = p$$

$\therefore$ Mean, $E(x) = p$

and Variance, $V(x) = E(x^2) - \{E(x)\}^2$ ....(i)

Now, $\quad E(x^2) = \sum_{x=0}^{1} x^2\ f(x;\, p)$

$$= \sum_{x=0}^{1} x^2\ p^x\ q^{1-x}$$

$$= 0^2.\ p^0\ q^{1-0} + 1^2.\ p^1\ q^{1-1}$$

$$\Rightarrow E(x^2) = 0 + p = p$$

Putting the values of $E(x^2)$ and $E(x)$ in (i), we get

$$V(x) = p - p^2 = p\,(1-p) = pq \quad [\because\ p + q = 1]$$

$\therefore$ Variance, $V(x) = pq$

Q. Find out the first four central moments of Bernoulli distribution (NU-2020)

Or, Find out the first four central moments of Bernoulli distribution and find β_1 and β_2.

Ans: We know the probability function of Bernoulli distribution is

$$f(x;\, p) = p^x\ q^{1-x};\ \text{for}\ \ x = 0, 1$$

The r-th moment of the distribution about origin is

$$\mu_r' = E(x^r)$$

$$= \sum_{x=0}^{1} x^r \ f(x; p)$$

$$= \sum_{x=0}^{1} x^r \ p^x \ q^{1-x}$$

$$= \sum_{x=0}^{1} x^r \ p^x \ (1-p)^{1-x}$$

$$= 0.\ p^0 \ (1-p)^{1-0} + 1.\ p^1 \ (1-p)^{1-1}$$

$$\therefore \ \mu_r' = p \ ....(i)$$

Putting $r = 1, 2, 3, 4$ in (i), we get respectively

$$\mu_1' = p, \ \ \mu_2' = p, \ \ \mu_3' = p \ \text{ and } \ \mu_4' = p$$

We know, $\mu_1 = 0$

Now, $\mu_2 = \mu_2' - \mu_1'^2$

$$= p - p^2$$

$$= p(1-p)$$

$$\therefore \ \mu_2 = pq$$

Again, $\mu_3 = \mu_3' - 3\,\mu_2'\,\mu_1' + 2\,\mu_1'^3$

$$= p - 3\,p^2 + 2\,p^3$$

$$= p - p^2 - 2p^2 + 2\,p^3$$

$$= p(1-p) - 2p^2(1-p)$$

$$= (1-p)\,(p - 2p^2)$$

$$= q.\ p\,(1-2p)$$

$$= q.\ p\,(1-p-p)$$

$$\therefore \ \mu_3 = pq\,(q-p)$$

Further, $\mu_4 = \mu_4' - 4\,\mu_3'\,\mu_1' + 6\,\mu_2'\,\mu_1'^2 - 3\,\mu_1'^4$

$$= p - 4\,p^2 + 6\,p^3 - 3\,p^4$$

$$= p - p^2 - 3p^2 + 3\,p^3 + 3\,p^3 - 3\,p^4$$

$$= p(1-p) - 3p^2(1-p) + 3\,p^3(1-p)$$

$$= (1-p)\left(p - 3p^2 + 3\,p^3\right)$$

$$= q\,p\left(1 - 3p + 3\,p^2\right)$$

$$= pq\,\{1 - 3p(1-p)\}$$

$$\therefore\ \mu_4 = pq\,(1 - 3pq)$$

Hence, $\beta_1 = \dfrac{\mu_3^{\,2}}{\mu_2^{\,3}} = \dfrac{p^2 q^2 (q-p)^2}{p^3 q^3} = \dfrac{(q-p)^2}{pq}$

and $\beta_2 = \dfrac{\mu_4}{\mu_2^{\,2}} = \dfrac{pq\,(1-3pq)}{p^2 q^2} = \dfrac{1-3pq}{pq} = 3 + \dfrac{1-6pq}{pq}$

NB: Various Measures-

(i) The **r-th central or corrected moment about mean** μ, usually denoted by μ_r and is defined as

$$\mu_r = E\left[x - E(x)\right]^r = E\left[x - \mu\right]^r = \begin{cases} \sum (x-\mu)^r f(x); & \text{if x is discrete} \\ \int_{-\infty}^{\infty} (x-\mu)^r f(x)\,dx; & \text{if x is continuous} \end{cases}$$

(ii) The **r-th moment about point 'a'**, usually denoted by $\mu_r'(a)$ is defined as

$$\mu_r'(a) = E\left[x - a\right]^r = \begin{cases} \sum (x-a)^r f(x); & \text{if x is discrete} \\ \int_{-\infty}^{\infty} (x-a)^r f(x)\,dx; & \text{if x is continuous} \end{cases}$$

(iii) The **r-th moment about origin**, usually denoted by μ_r' is defined as

$$\mu_r' = E\left[x^r\right] = \begin{cases} \sum x^r\, f(x); & \text{if x is discrete} \\ \int_{-\infty}^{\infty} x^r\, f(x)\,dx; & \text{if x is continuous} \end{cases}$$

(iv) **Median**: Median is the point which divides the total area into two equal parts. Thus, if M is the median, then

$$\int_a^M f(x)\,dx = \int_M^b f(x)\,dx = \frac{1}{2}$$

Thus solving $\int_a^M f(x)\,dx = \frac{1}{2}$ or, $\int_M^b f(x)\,dx = \frac{1}{2}$ for M, we get the value of median.

(v) **Mean Deviation**: Mean deviation about mean is given by

$$MD = E\left[\,|x - \text{mean}|\,\right] = \begin{cases} \sum |x - \text{mean}|\, f(x); & \text{if x is discrete} \\ \int_a^b |x - \text{mean}|\, f(x)\,dx; & \text{if x is continuous} \end{cases}$$

(vi) **Mode**: Mode is the value of x for which $f(x)$ is maximum. Thus, Mode is given by $f'(x) = 0$ and $f''(x) < 0$.

Q. Find the moment generating function of Bernoulli distribution.

Ans: We know the probability function of Bernoulli distribution is

$$f(x; p) = p^x\, q^{1-x}; \text{ for } x = 0, 1$$

Now the moment generating function of the distribution is

$$\begin{aligned} M_x(t) &= E\left(e^{tx}\right) \\ &= \sum_{x=0}^{1} e^{tx}\, f(x; p) \\ &= \sum_{x=0}^{1} e^{tx}\, p^x\, q^{1-x} \\ &= \sum_{x=0}^{1} e^{tx}\, p^x\, (1-p)^{1-x} \\ &= e^0\, p^0\, (1-p)^{1-0} + e^t\, p^1\, (1-p)^{1-1} \\ &= 1 - p + p e^t \end{aligned}$$

$$\therefore\ M_x(t) = q + p e^t$$

Short questions and answers:

(1) Write the formula of Bernoulli distribution and also write mean (NU-10)

Ans: The probability function Bernoulli distribution is

$$f(x; p) = p^{x}\ q^{1-x};\ \ \text{for}\ \ x = 0, 1$$

and the mean of Bernoulli distribution is p.

(2) What is the mean and variance of Bernoulli distribution?

Ans: The mean and variance of Bernoulli distribution is p and pq respectively.

Binomial Distribution

The binomial distribution was first derived by Swiss mathematician James Bernoulli (1654-1705) in the year 1700 and was first published posthumously (মরণোত্তর, মৃত্যুর পরে) in 1713, eight years after his death by his nephew Nicholas Bernoulli.

Q. What is Binomial Experiment?

Ans: When an experiment has two possible outcomes-success and failure and the experiment is repeated n times independently and the probability of success 'p' and the probability of failure 'q' of any given trial remains constant from trial to trial, then the experiment is known as binomial experiment.

Example: Suppose a fair coin is tossed thrice and to get head is success and to get tail is failure. Let 'p' be the probability of success and 'q' be the probability of failure so that $p+q=1$ where $p=P(S)=\frac{1}{2}$ and $q=P(F)=\frac{1}{2}$. This is a binomial experiment.

Q. What is binomial variate?

Ans: A discrete random variable X is called a Binomial variate if it indicates the number of successes in n Bernoulli trials and taking a set of values $\{0, 1, 2, \cdots, n\}$.

Example: Let a fair coin is tossed twice, then the sample space is

$$S=\{HH, HT, TH, TT\}$$

Let to get head is success and x indicates the number of successes, then $x=0, 1, 2$. Here x is a binomial variate.

Q. What is binomial distribution? (NU-11, 12)

Ans: A discrete random variable X is said to have a binomial distribution if its probability function is defined by

$$f(x;\, n,\, p) = {}^{n}c_{x}\; p^{x}\; q^{n-x};\quad x = 0, 1, 2, \cdots, n$$

Here, $n=$ number of trials

$p=$ Probability of success

$q=1-p=$ Probability of failure

The n and p are called the parameters of the binomial distribution and n is a positive integer.

NB:

(i) If the number of trials in binomial distribution is one i.e., $n=1$, then it becomes Bernoulli distribution.

(ii) If the number of trials greater than 30 i.e., $n>30$ then binomial distribution is not applicable.

(iii) In n trials, the binomial variate x takes the values $0, 1, 2, \cdots, n$, which are integer and discrete, so binomial distribution is a discrete probability distribution.

(iv) $f(X=x;\, n,\, p) = f(x;\, n,\, p) = {}^{n}c_{x}\, p^{x}\, q^{n-x};\quad x = 0, 1, 2, \cdots, n$

Q. Derive the binomial distribution (NU-10, 11, 12)

Or, Derive the probability function of binomial distribution **(NU-15)**

Ans: The following conditions must be satisfied for the binomial distribution:

(i) There should be a fixed number of trials.

(ii) The trials are independent.

(iii) There are only two outcomes for each trial.

(iv) The probability of success and the probability of failure remain same or constant from trial to trial.

Derivation of Binomial Distribution:

Let us suppose that a sequence of n Bernoulli trials results in x successes and $(n-x)$ failures. If we denote a success by S and a failure by F, then one possible outcome of the sequence is of the following form:

$$S\ S\ F\ S\ F\ F\ F\ S \cdots F\ S\ F$$

$$= \underbrace{S\ S \cdots S}_{\text{x times}}\ \ \underbrace{F\ F \cdots F}_{(\text{n-x})\text{ times}}$$

The probability of success and the probability of failure are denoted by p and q respectively. Since the trials are independent, so the probability of this particular sequence is

$$P(S\ S \cdots S\ F\ F \cdots F) = \underbrace{P(S)\ P(S) \cdots P(S)}_{\text{x times}}\ \ \underbrace{P(F)\ P(F) \cdots P(F)}_{(\text{n-x})\text{ times}}$$

$$= \underbrace{p.\ p \ldots p}_{\text{x times}}.\ \underbrace{q.\ q \ldots q}_{(\text{n-x})\text{ times}}$$

$$= p^{x}\ q^{n-x}$$

This is one possible way of x successes and $(n-x)$ failures in n trials. But x successes in n trials can occur in different positions in n_{c_x} ways and the probability of each of these ways is $p^x\ q^{n-x}$.

Since all the ways (or, arrangements) are all mutually exclusive, so the probability of x successes in n trials is given by the addition theorem of probability by the expression-

$$f(x;\,n,\,p) = \underbrace{p^x\ q^{n-x} + p^x\ q^{n-x} + \cdots + p^x\ q^{n-x}}_{{}^n c_x \text{ ways}}$$

$$\Rightarrow f(x;\,n,\,p) = {}^n c_x\ p^x\ q^{n-x};\quad x = 0, 1, 2, \cdots, n$$

which is the probability function of binomial distribution with parameters n and p.

NB: Tossing a coin thrice, we can get one success ${}^3c_1$ ways which are as follows:

Sample points	:	SFF	FSF	FFS
Probability of success	:	$p^1\,q^2$	$p^1\,q^2$	$p^1\,q^2$

$$f(x) = p^1\ q^2 + p^1\ q^2 + p^1\ q^2$$

$$= 3\ p^1\ q^2$$

$$\therefore\ f(x) = {}^3c_1\ p^1\ q^{3-1}$$

Theorem: Prove that the sum of all probabilities of binomial distribution is one.

Or, Show that $\sum\limits_{x=0}^{n} {}^n c_x p^x q^{n-x} = 1$

Proof: Let x is a binomial variate with parameters n and p.

$\therefore$ Probability function, $f(x;\,n,\,p) = {}^n c_x\ p^x\ q^{n-x};\ \ x = 0, 1, 2, \cdots, n$

Now, $\sum\limits_{x=0}^{n} f(x;\,n,\,p) = \sum\limits_{x=0}^{n} {}^n c_x\ p^x\ q^{n-x}$

$$= {}^n c_o\ p^0\ q^{n-0} + {}^n c_1\ p^1\ q^{n-1} + {}^n c_2\ p^2\ q^{n-2} + \cdots + {}^n c_n\ p^n\ q^{n-n}$$

$$= q^n + {}^n c_1\ q^{n-1}\ p^1 + {}^n c_2\ q^{n-2}\ p^2 + \cdots + p^n$$

$$= (q+p)^n$$

$$= (1)^n\quad [\because\ q + p = 1]$$

$$\therefore \sum_{x=0}^{n} f(x;n,p) = 1$$

So, the summation of total probability for binomial distribution is one (Proved)

NB: (i) $(q+p)^n = {}^nc_0\ q^{n-0}\ p^0 + {}^nc_1\ q^{n-1}\ p^1 + {}^nc_2\ q^{n-2}\ p^2 + \cdots + {}^nc_n\ q^{n-n}\ p^n$

$$= q^n + {}^nc_1\ q^{n-1}\ p^1 + {}^nc_2\ q^{n-2}\ p^2 + \cdots + p^n$$

(ii) $(q+p)^{n-1} = {}^{n-1}c_0\ q^{n-1-0}\ p^0 + {}^{n-1}c_1\ q^{n-1-1}\ p^1 + {}^{n-1}c_2\ q^{n-1-2}\ p^2 + \cdots + {}^{n-1}c_{n-1}\ q^{n-1-(n-1)}\ p^{n-1}$

$$= q^{n-1} + {}^{n-1}c_1\ q^{n-1-1}\ p^1 + {}^{n-1}c_2\ q^{n-1-2}\ p^2 + \cdots + p^{n-1}$$

Q. Find mean and variance of binomial distribution (NU-10)

Ans: Let x is a binomial variate with parameters n and p.

So, Probability function, $f(x;n,p) = \binom{n}{x} p^x\ q^{n-x};\quad x = 0, 1, 2, \cdots, n$

Now, Mean $E(x) = \sum_{x=0}^{n} x\ f(x;n,p)$

$$= \sum_{x=0}^{n} x. \binom{n}{x} p^x\ q^{n-x}$$

$$= \sum_{x=1}^{n} x. \frac{n!}{x!(n-x)!} p^x\ q^{n-x}$$

$$= \sum_{x=1}^{n} x. \frac{n(n-1)!}{x(x-1)!(n-x)!} p\ p^{x-1}\ q^{n-x}$$

$$= np \sum_{x=1}^{n} \frac{(n-1)!}{(x-1)!(n-x)!} p^{x-1}\ q^{n-x}$$

$$= np \sum_{x=1}^{n} \binom{n-1}{x-1} p^{x-1}\ q^{n-1-(x-1)}$$

$$= np\ (q+p)^{n-1}$$

$$= np$$

$\therefore$ Mean, $E(x) = np$

and Variance, $V(x) = E(x^2) - \{E(x)\}^2$

$$= E\{x(x-1)+x\}-\{E(x)\}^2$$

$$= E\{x(x-1)\}+E(x)-\{E(x)\}^2$$

$$= E\{x\ (x-1)\}+n\,p-n^2\ p^2 \quad [\because E(x)=n\,p] \quad \text{............(i)}$$

Here, $E\{x(x-1)\}=\sum_{x=0}^{n} x\ (x-1)\ f(x;\ n,\ p)$

$$= \sum_{x=0}^{n} x\ (x-1).\ \binom{n}{x} p^x\ q^{n-x}$$

$$= \sum_{x=2}^{n} x\ (x-1).\ \frac{n!}{x!\,(n-x)!}\ p^x\ q^{n-x}$$

$$= \sum_{x=2}^{n} x\ (x-1).\ \frac{n\,(n-1)\,(n-2)!}{x\,(x-1)\,(x-2)!\ (n-x)!}\ p^2\ p^{x-2}\ q^{n-x}$$

$$= n\,(n-1)\,p^2\ \sum_{x=2}^{n} \frac{(n-2)!}{(x-2)!\,(n-x)!}\ p^{x-2}\ q^{n-x}$$

$$= n\,(n-1)\,p^2\ \sum_{x=2}^{n} \binom{n-2}{x-2} p^{x-2}\ q^{n-2-(x-2)}$$

$$= n(n-1)\,p^2\,(q+p)^{n-2}$$

$$= n\,(n-1)\,p^2 \quad [\because\ q+p=1]$$

$$= n^2\ p^2-n\ p^2$$

Now putting the value of $E\{x(x-1)\}$ in (i), we get

$$V(x)=n^2\ p^2-n\ p^2+n\ p-n^2\ p^2$$

$$= n\,p-n\,p^2$$

$$= n\,p\ (1-p)$$

$$= n\,p\,q \quad [\because\ q+p=1]$$

$\therefore$ Variance, $V(x)=n\,p\,q$

NB: We know, $f(x;\ n,\ p)=\binom{n}{x} p^x\ q^{n-x};\quad x=0,1,\ 2,\cdots,n$

Now, $\sum_{x=0}^{n} f(x;\, n,\, p) = \sum_{x=0}^{n} \binom{n}{x} p^x\ q^{n-x} = (q+p)^n = 1$

Similarly, $\sum_{x=1}^{n} \binom{n-1}{x-1} p^{x-1}\ q^{n-1-(x-1)} = (q+p)^{n-1} = 1$

and $\sum_{x=2}^{n} \binom{n-2}{x-2} p^{x-2}\ q^{n-2-(x-2)} = (q+p)^{n-2} = 1$

Alternative Method: To find Mean and Variance

Let x is a binomial variate with parameters n and p.

So, Probability function, $f(x;\, n,\, p) = n_{c_x}\ p^x\ q^{n-x};\quad x = 0, 1, 2, \cdots, n$

$$
\begin{aligned}
\text{Mean } E(x) &= \sum_{x=0}^{n} x\ f(x;\, n,\, p) \\
&= \sum_{x=0}^{n} x.\ {}^nc_x\ p^x\ q^{n-x} \\
&= 0.\ {}^nc_0\ p^0\ q^{n-0} + 1.\ {}^nc_1\ p^1\ q^{n-1} + 2.\ {}^nc_2\ p^2\ q^{n-2} + \cdots + n.\ {}^nc_n\ p^n\ q^{n-n} \\
&= 0 + n\ p\ q^{n-1} + 2.\frac{n!}{2!\,(n-2)!}\ p^2\ q^{n-2} + \cdots + n\ p^n \\
&= n\ p\ q^{n-1} + 2.\frac{n\ (n-1)\ (n-2)!}{2\ (n-2)!}\ p^2\ q^{n-2} + \cdots + n\ p^n \\
&= n\ p\ q^{n-1} + n\ (n-1)\ p^2\ q^{n-2} + \cdots + n\ p.\ p^{n-1} \\
&= n\ p\ \left[q^{n-1} + (n-1)\ p\ q^{n-2} + \cdots + p^{n-1}\right] \\
&= n\,p\ \left[q^{n-1} + {}^{n-1}c_1\ q^{(n-1)-1}.\ p^1 + \cdots + p^{n-1}\right] \\
&= n\,p\ (q+p)^{n-1} \\
&= n\ p\ (1)^{n-1} \quad [\because\ q+p=1] \\
&= n\ p
\end{aligned}
$$

$\therefore$ Mean, $E(x) = n\ p$

and Variance, $V(x) = E(x^2) - \{E(x)\}^2$

$$= E\{x\ (x-1) + x\} - \{E(x)\}^2$$

$$= E\{x\ (x-1)\} + E(x) - \{E(x)\}^2$$

$$= E\{\ x(x-1)\} + n\,p - n^2\ p^2 \qquad [\because\ E(x) = n\,p] \qquad ....(i)$$

Here, $E\{x\ (x-1)\} = \sum_{x=0}^{n} x\ (x-1)\ f(x;\ n,\ p)$

$$= \sum_{x=0}^{n} x\ (x-1).\ {}^nc_x\ p^x\ q^{n-x}$$

$$= 0\ (0-1)\ {}^nc_0\ p^0\ q^{n-0} + 1\ (1-1)\ {}^nc_1\ p^1\ q^{n-1} + 2\ (2-1)\ {}^nc_2\ p^2\ q^{n-2} +$$
$$3\ (3-1)\ {}^nc_3\ p^3\ q^{n-3} + \cdots + n\ (n-1)\ {}^nc_n\ p^n\ q^{n-n}$$

$$= 0 + 0 + 2.\ \frac{n!}{2!\ (n-2)!}\ p^2\ q^{n-2} + 6.\ \frac{n!}{3!\ (n-3)!}\ p^3\ q^{n-3} + \cdots + n\ (n-1)\ p^n$$

$$= n\ (n-1)\ p^2\ q^{n-2} + n\ (n-1)\ (n-2)\ p^3\ q^{n-3} + \cdots + n\ (n-1)\ p^n$$

$$= n\ (n-1)\ p^2\ \left[q^{n-2} + (n-2)\ p\,q^{n-3} + \cdots + p^{n-2}\right]$$

$$= n\ (n-1)\ p^2\ \left[q^{n-2} + {}^{n-2}c_1\ q^{(n-2)-1}\ p + \cdots + p^{n-2}\right]$$

$$= n\ (n-1)\ p^2\ (q+p)^{n-2}$$

$$= n\ (n-1)\ p^2\ (1)^{n-2} \qquad [\because\ q + p = 1]$$

$$\Rightarrow E\{x\ (x-1)\} = n^2\,p^2 - n\,p^2$$

Now putting the value of $E\{x\ (x-1)\}$ in (i), we get

$$V(x) = n^2\,p^2 - n\,p^2 + n\,p - n^2\,p^2$$
$$= n\,p - n\,p^2$$
$$= n\,p\ (1-p)$$
$$= n\,p\,q \qquad [\because\ q + p = 1]$$

$\therefore$ Variance, $V(x) = n\,p\,q$

Theorem: Prove that for binomial distribution Mean > Variance **(NU-10, 12)**

Or, Show that, mean > variance of binomial distribution **(NU-15)**

Or, Prove that the mean is greater than the variance of binomial distribution.

Proof: Let x is a binomial variate with parameters n and p, where $p + q = 1$.

We know for binomial distribution,

Mean, $E(x) = n\,p$

and Variance $V(x) = n\,p\,q$

$$= n\,p\,(1-p) \quad [\because\ q = 1-p]$$

$$= n\,p - n\,p^2$$

$$\Rightarrow V(x) = E(x) - n\,p^2 \quad [\because\ E(x) = n\,p]$$

$$\Rightarrow E(x) - n\,p^2 = V(x)$$

$$\Rightarrow E(x) = V(x) + n\,p^2$$

But in case of binomial distribution $n > 0,\ p > 0$, so $np^2 > 0$

$\therefore\ E(x) = V(x) +$ positive value

Hence, $E(x) > V(x)$

So, mean is greater than variance of binomial distribution **(Proved)**

Q. Derivation of recursion formula of binomial Distribution.

Ans: Let x is a binomial variate with parameters n and p.

$\therefore$ Probability function, $f(x) = {}^{n}c_{x}\ p^{x}\ q^{n-x};\quad x = 0, 1, 2, \cdots, n$

Now, $f(x+1) = {}^{n}c_{x+1}\ p^{x+1}\ q^{n-x-1}$

$$\therefore\ \frac{f(x+1)}{f(x)} = \frac{{}^{n}c_{x+1}\ p^{x+1}\ q^{n-x-1}}{{}^{n}c_{x}\ p^{x}\ q^{n-x}}$$

$$= \frac{\dfrac{n!}{(x+1)!\ (n-x-1)!}}{\dfrac{n!}{x!\ (n-x)!}}.\ p\,q^{-1}$$

$$= \frac{n!}{(x+1)!\ (n-x-1)!} \times \frac{x!\,(n-x)!}{n!}.\ p\,q^{-1}$$

$$= \frac{x!\ (n-x)!}{(x+1)!\ (n-x-1)!}.\ p\,q^{-1}$$

$$= \frac{x!\ (n-x)\ (n-x-1)!}{(x+1)\ x!\ (n-x-1)!}.\ p\,q^{-1}$$

$$= \frac{n-x}{x+1}.\ \frac{p}{q}$$

$$\Rightarrow \frac{f(x+1)}{f(x)} = \frac{n-x}{x+1} \cdot \frac{p}{q}$$

$$\therefore \quad f(x+1) = \frac{n-x}{x+1} \cdot \frac{p}{q} \; f(x)$$

which is the required recurrence relation of binomial distribution.

Q. Establish the recurrence relation for the moments of binomial distribution.

Or, For the binomial distribution show that $\mu_{r+1} = p\,q\left(n\,r\,\mu_{r-1} + \frac{d\mu_r}{dp}\right)$, where the symbols have their usual meanings. Hence find μ_2 , μ_3 and μ_4 **(NU-15)**

Proof: Let x is a binomial variate with probability function,

$$f(x;\; n, p) = n_{c_x} \; p^x \; q^{n-x}; \quad x = 0, 1, 2, \cdots, n \quad \text{where, } p+q=1$$

Now, r-th central moment

$$\mu_r = E\left[x - E(x)\right]^r$$

$$= E\left(x - n\,p\right)^r$$

$$\Rightarrow \mu_r = \sum_{x=0}^{n} \left(x - n\,p\right)^r \; f(x) \text{.............(i)}$$

$$= \sum_{x=0}^{n} \left(x - n\,p\right)^r \; n_{c_x} \; p^x \; q^{n-x}$$

$$= \sum_{x=0}^{n} n_{c_x} \; p^x \; q^{n-x} \left(x - n\,p\right)^r$$

$$\Rightarrow \mu_r = \sum_{x=0}^{n} n_{c_x} \; p^x \; (1-p)^{n-x} \; \left(x - n\,p\right)^r \quad \left[\because\; q = 1-p\right]$$

Now differentiating μ_r with respect to p, we get,

$$\frac{d\mu_r}{dp} = \sum_{x=0}^{n} n_{c_x} \left[p^x \; (1-p)^{n-x} .\, r\left(x - n\,p\right)^{r-1} . (-n) + \left(x - n\,p\right)^r \left\{p^x . (n-x)\; (1-p)^{n-x-1} . (-1) + (1-p)^{n-x} .\, x\,p^{x-1}\right\}\right]$$

$$= \sum_{x=0}^{n} n_{c_x} \left[-n\,r.\; p^x \, q^{n-x} \left(x - n\,p\right)^{r-1} + \left(x - n\,p\right)^r \left\{x\,p^{x-1} \; q^{n-x} - p^x \; (n-x)\; q^{n-x-1}\right\}\right]$$

$$= -n\,r \sum_{x=0}^{n} n_{c_x} \; p^x \; q^{n-x} \left(x - n\,p\right)^{r-1} + \sum_{x=0}^{n} n_{c_x} \left(x - n\,p\right)^r \left\{x\,p^x .\; p^{-1} \; q^{n-x} - p^x \; (n-x)\; q^{n-x} .\; q^{-1}\right\}$$

$$= -nr\sum_{x=0}^{n}(x-np)^{r-1}\ f(x) + \sum_{x=0}^{n} n_{c_x}\ (x-np)^r\ p^x\ q^{n-x}\left\{\frac{x}{p}-\frac{n-x}{q}\right\}$$

$$= -nr\ \mu_{r-1} + \sum_{x=0}^{n}(x-np)^r\ f(x)\ \left\{\frac{qx-np+px}{pq}\right\}\quad \text{[according to (i)]}$$

$$= -nr\ \mu_{r-1} + \sum_{x=0}^{n}(x-n\ p)^r\ f(x)\ \frac{x-np}{pq}$$

$$= -\ n\ r\mu_{r-1} + \frac{1}{pq}\sum_{x=0}^{n}(x-np)^{r+1}\ f(x)$$

$$\Rightarrow \frac{d\mu_r}{dp} = -\ nr\mu_{r-1} + \frac{1}{pq}\ \mu_{r+1}\quad \text{[according to (i)]}$$

$$\Rightarrow \frac{1}{pq}\ \mu_{r+1} = nr\mu_{r-1} + \frac{d\mu_r}{dp}$$

So, $\mu_{r+1} = pq\left(nr\ \mu_{r-1} + \dfrac{d\mu_r}{dp}\right)$...............(ii)

which is the recurrence relation for moments of Binomial distribution.

Now putting $r = 1,\ 2,\ 3$ in (ii), we get respectively

$$\mu_2 = pq\ \left(n\mu_0 + \frac{d\mu_1}{dp}\right) = npq\quad [\because\ \mu_0 = 1,\quad \mu_1 = 0]$$

$$\mu_3 = pq\left(2n\mu_1 + \frac{d\mu_2}{dp}\right) = pq\ \left\{0 + \frac{d\,(npq)}{dp}\right\}\qquad [\because\ \mu_1 = 0]$$

$$= npq\ \ \frac{d(pq)}{dp}$$

$$= npq\ \ \frac{d\{\,p(1-p)\}}{dp}$$

$$= npq\frac{d\left(p-p^2\right)}{dp} = npq(1-2p) = npq(p+q-2p) = npq\ (q-p)$$

$$\mu_4 = p\ q\ \left(3n\ \mu_2 + \frac{d\mu_3}{dp}\right)$$

$$= pq\ \left\{3n.\ npq + \frac{d\{npq\ (1-2p)\}}{dp}\right\}$$

$$= pq\left\{3n^2 pq + n\,\frac{d\{p(1-p)(1-2p)\}}{dp}\right\}$$

$$= pq\left\{3n^2 pq + n\,\frac{d\left\{p\left(1-3p+2p^2\right)\right\}}{dp}\right\}$$

$$= pq\left\{3n^2 pq + n\,\frac{d\left(p-3p^2+2p^3\right)}{dp}\right\}$$

$$= pq\left\{3n^2 pq + n\left(1-6p+6p^2\right)\right\}$$

$$= npq\left\{3npq + 1 - 6p(1-p)\right\}$$

$$= npq\left\{1 + 3npq - 6pq\right\}$$

$$= npq\left[1 + 3(n-2)pq\right]$$

NB: (i) The probability function of binomial distribution is,

$$f(x) = {}^n c_x \; p^x \; q^{n-x}; \qquad x = 0, 1, 2, \cdots, n$$

From above theorem, we have

$$\mu_r = \sum_{x=0}^{n}\left(x - np\right)^r \; f(x)$$

So, $\mu_0 = \sum_{x=0}^{n}\left(x-np\right)^0 \; f(x) = \sum_{x=0}^{n} f(x) = \sum_{x=0}^{n} {}^n c_x \; p^x \; q^{n-x} = 1$

(ii) $(x-a)(x-b) = x^2 - (a+b)x + ab$

Q. Determine the moment generating function of binomial distribution and hence determine β_1 and β_2 (NU-11).

Or, Derive the m.g.f of binomial distribution, hence find out variance of the distribution **(NU-15)**

Ans: Let x is a binomial variate with parameters n and p. So, probability function is

$$f(x;\; n, p) = {}^n c_x \; p^x \; q^{n-x}; \qquad x = 0, 1, 2, \cdots, n$$

Now, the m.g.f of x is

$$M_x(t) = E(e^{tx})$$

$$= \sum_{x=0}^{n} e^{tx} \; f(x;\; n, p)$$

$$= \sum_{x=0}^{n} e^{tx}.\ {}^{n}c_x\ p^x\ q^{n-x}$$

$$= \sum_{x=0}^{n} {}^{n}c_x\ q^{n-x}\ (p\ e^t)^x$$

$$= \left(q + p\ e^t\right)^n$$

$$\therefore\quad M_x(t) = \left(q + p\ e^t\right)^n$$

We know, the cumulant generating function (c.g.f),

$$K_x(t) = \log\ M_x(t)$$

$$= \log\ \left(q + p\ e^t\right)^n$$

$$= n\ \log\ \left(q + p\ e^t\right)$$

$$\Rightarrow K_x(t) = n\ \log\left[q + p\ \left(1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \frac{t^4}{4!} + \cdots\right)\right]$$

$$\Rightarrow K_x(t) = n\ \log\left[q + p + p\ \left(t + \frac{t^2}{2!} + \frac{t^3}{3!} + \frac{t^4}{4!} + \cdots\right)\right]$$

$$\Rightarrow K_x(t) = n\ \log\left[1 + p\left(t + \frac{t^2}{2!} + \frac{t^3}{3!} + \frac{t^4}{4!} + \cdots\right)\right]$$

$$\Rightarrow K_x(t) = n\left[p\left(t + \frac{t^2}{2!} + \frac{t^3}{3!} + \frac{t^4}{4!} + \cdots\right) - \frac{p^2}{2}\left(t + \frac{t^2}{2!} + \frac{t^3}{3!} + \cdots\right)^2 + \frac{p^3}{3}\left(t + \frac{t^2}{2!} + \cdots\right)^3 - \frac{p^4}{4}\left(t + \frac{t^2}{2!} + \cdots\right)^4 + \cdots\right]$$

$$\Rightarrow K_x(t) = n\left[p\left(t + \frac{t^2}{2!} + \frac{t^3}{3!} + \frac{t^4}{4!} + \cdots\right) - \frac{p^2}{2}\left(t^2 + \frac{t^4}{4} + 2.t.\frac{t^2}{2!} + 2.t.\frac{t^3}{3!} + \cdots\right) + \frac{p^3}{3}\left(t^3 + 3.t^2.\frac{t^2}{2!} + \cdots\right) - \frac{p^4}{4}\left(t^4 + \cdots\right) + \cdots\right]$$

$$\Rightarrow K_x(t) = n\left[\left(pt + p\frac{t^2}{2!} + p\frac{t^3}{3!} + p\frac{t^4}{4!} + \cdots\right) - \left(\frac{p^2}{2}t^2 + \frac{p^2}{8}t^4 + \frac{p^2}{2}t^3 + \frac{p^2}{6}t^4 + \cdots\right) + \left(\frac{p^3}{3}t^3 + \frac{p^3}{2}t^4 + \cdots\right) - \left(\frac{p^4}{4}t^4 + \cdots\right) + \cdots\right]$$

$$\Rightarrow K_x(t) = n\left[pt + \left(p\frac{t^2}{2!} - \frac{p^2}{2}t^2\right) + \left(p\frac{t^3}{3!} - \frac{p^2}{2}t^3 + \frac{p^3}{3}t^3\right) + \left(p\frac{t^4}{4!} - \frac{p^2}{8}t^4 - \frac{p^2}{6}t^4 + \frac{p^3}{2}t^4 - \frac{p^4}{4}t^4\right) + \cdots\right]$$

$$\Rightarrow K_x(t) = n\,p\,t + n\ \left(p - p^2\right)\frac{t^2}{2!} + n\ \left(p - 3p^2 + 2p^3\right)\frac{t^3}{3!} + n\ \left(p - 3\ p^2 - 4\ p^2 + 12\ p^3 - 6\ p^4\right)\frac{t^4}{4!} + \cdots$$

Now, if r-th cumulant is denoted by K_r, then

$$K_r = \text{the coefficient of } \frac{t^r}{r!} \text{ in } K_x(t)$$

Mean, $K_1 =$ the coefficient of t in $K_x(t) = np$

Variance, $K_2 = \mu_2 =$ the coefficient of $\frac{t^2}{2!}$ in $K_x(t) = n\ (p - p^2) = n\,p\ (1-p) = n\,p\,q$

Similarly, $K_3 = \mu_3 = n\,(p - 3\,p^2 + 2\,p^3)$

$$= n\,p\,\,(2\,p^2 - 3\,p + 1)$$

$$= n\,p\,\,(2\,p^2 - 2\,p - p + 1)$$

$$= n\,p\,\,\{2\,p\,(p-1) - 1\,(p-1)\}$$

$$= n\,p\,\,(p-1)\,(2\,p - 1)$$

$$= n\,p\,(-q)\,(2\,p - p - q)$$

$$= n\,p\,q\,(q - p)$$

and $\quad K_4 = n\left(p - 3\,p^2 - 4\,p^2 + 12\,p^3 - 6\,p^4\right)$

$$= n\left(p - 7\,p^2 + 12\,p^3 - 6\,p^4\right)$$

$$= n\,p\left(1 - 7\,p + 12\,p^2 - 6\,p^3\right)$$

$$= n\,p\left[1 - p - 6\,p + 6\,p^2 + 6\,p^2 - 6\,p^3\right]$$

$$= n\,p\,\left[1\,(1-p) - 6\,p\,(1-p) + 6\,p^2\,(1-p)\right]$$

$$= n\,p\,(1-p)\,\left(1 - 6\,p + 6\,p^2\right)$$

$$= n\,p\,q\,[1 - 6\,p\,(1-p)] \qquad [\because\ p + q = 1 \Rightarrow q = 1 - p]$$

$$= n\,p\,q\,(1 - 6\,p\,q)$$

We know, $\mu_4 = K_4 + 3\,K_2^{\,2} = n\,p\,q\,(1 - 6\,p\,q) + 3\,(n\,p\,q)^2 = n\,p\,q\,\left[(1 - 6\,p\,q) + 3\,n\,p\,q\right]$

Now, $\beta_1 = \dfrac{\mu_3^{\,2}}{\mu_2^{\,3}} = \dfrac{\{n\,p\,q\,(q-p)\}^2}{(n\,p\,q)^3} = \dfrac{(q-p)^2}{n\,p\,q}$

and $\beta_2 = \dfrac{\mu_4}{\mu_2^{\,2}} = \dfrac{n\,p\,q\,\left[3\,n\,p\,q + (1 - 6\,p\,q)\right]}{(n\,p\,q)^2} = 3 + \dfrac{1 - 6\,p\,q}{n\,p\,q}$

NB:

(i) $e^x = 1 + x + \dfrac{x^2}{2!} + \dfrac{x^3}{3!} + \cdots$

(ii) $\log\,(1 + x) = x - \dfrac{x^2}{2} + \dfrac{x^3}{3} - \dfrac{x^4}{4} + \cdots$

(iii) $(a + b)^4 = a^4 + 4\,a^3\,b + 6\,a^2\,b^2 + 4\,a\,b^3 + b^4$

(iv) $\sum_{x=0}^{n} {}^{n}c_x\ q^{n-x}\ p^x = {}^{n}c_0\ q^{n-0}\ p^0 + {}^{n}c_1\ q^{n-1}\ p^1 + {}^{n}c_2\ q^{n-2}\ p^2 + \cdots + {}^{n}c_n\ q^{n-n}\ p^n = (q+p)^n$

Similarly, $\sum_{x=0}^{n} {}^{n}c_x\ q^{n-x}\ (p\ e^t)^x = (q + p\ e^t)^n$

(v) **Expression of first four cumulants in terms of moments-**

We know, $M_x(t) = E\left[e^{tx}\right]$

$$= E\left[1 + \frac{tx}{1!} + \frac{(tx)^2}{2!} + \frac{(tx)^3}{3!} + \frac{(tx)^4}{4!} + \cdots\right]$$

$$= E(1) + \frac{t}{1!}E(x) + \frac{t^2}{2!}E(x^2) + \frac{t^3}{3!}E(x^3) + \frac{t^4}{4!}E(x^4) + \cdots$$

$$= 1 + \frac{t}{1!}\ \mu_1' + \frac{t^2}{2!}\ \mu_2' + \frac{t^3}{3!}\ \mu_3' + \frac{t^4}{4!}\ \mu_4' + \cdots \qquad \left[\because \mu_r' = E(x^r)\right]$$

We know, $K_x(t) = \log_e M_x(t)$

$$\Rightarrow K_x(t) = \log_e\left(1 + \mu_1'\ t + \mu_2'\ \frac{t^2}{2!} + \mu_3'\ \frac{t^3}{3!} + \mu_4'\ \frac{t^4}{4!} + \cdots\right)$$

$$= \left(\mu_1' t + \mu_2'\ \frac{t^2}{2!} + \mu_3'\ \frac{t^3}{3!} + \mu_4'\ \frac{t^4}{4!} + \cdots\right) - \frac{1}{2}\left(\mu_1'\ t + \mu_2'\ \frac{t^2}{2!} + \mu_3'\ \frac{t^3}{3!} + \cdots\right)^2$$
$$+ \frac{1}{3}\left(\mu_1' t + \mu_2' \frac{t^2}{2!} + \cdots\right)^3 - \frac{1}{4}\left(\mu_1'\ t + \mu_2'\ \frac{t^2}{2!} + \cdots\right)^4 + \cdots$$

$$= \left(\mu_1'\ t + \mu_2'\ \frac{t^2}{2!} + \mu_3'\ \frac{t^3}{3!} + \mu_4'\ \frac{t^4}{4!} + \cdots\right) - \frac{1}{2}\left(\mu_1'^2\ t^2 + \mu_2'^2\ \frac{t^4}{4} + 2\mu_1'\ t.\ \mu_2'\ \frac{t^2}{2} + 2\mu_1'\ t.\ \mu_3'\ \frac{t^3}{6} + \cdots\right)$$
$$+ \frac{1}{3}\left(\mu_1'^3\ t^3 + 3\mu_1'^2\ t^2.\ \mu_2'\ \frac{t^2}{2} + \cdots\right) - \frac{1}{4}\left(\mu_1'^4\ t^4 + \cdots\right) + \cdots$$

$$= \mu_1'\ t + \mu_2'\ \frac{t^2}{2!} + \mu_3'\ \frac{t^3}{3!} + \mu_4'\ \frac{t^4}{4!} - \mu_1'^2\ \frac{t^2}{2} - \mu_2'^2\ \frac{t^4}{8} - \mu_2'\ \mu_1'\ \frac{t^3}{2} - \mu_3'\ \mu_1'\ \frac{t^4}{6} + \mu_1'^3\ \frac{t^3}{3} + \mu_2'\ \mu_1'^2\ \frac{t^4}{2} - \mu_1'^4\ \frac{t^4}{4} + \cdots$$

$$= \mu_1' t + \left(\mu_2' \frac{t^2}{2!} - \mu_1'^2 \frac{t^2}{2}\right) + \left(\mu_3' \frac{t^3}{3!} - \mu_2' \mu_1' \frac{t^3}{2} + \mu_1'^3 \frac{t^3}{3}\right) + \left(\mu_4' \frac{t^4}{4!} - \mu_2'^2 \frac{t^4}{8} - \mu_3' \mu_1' \frac{t^4}{6} + \mu_2' \mu_1'^2 \frac{t^4}{2} - \mu_1'^4 \frac{t^4}{4}\right) + \cdots$$

$$= \mu_1' t + \left(\mu_2' - \mu_1'^2\right)\frac{t^2}{2!} + \left(\mu_3' - 3\mu_2'\ \mu_1' + 2\mu_1'^3\right)\frac{t^3}{3!} + \left(\mu_4' - 3\ \mu_2'^2 - 4\ \mu_3'\ \mu_1' + 12\ \mu_2'\ \mu_1'^2 - 6\ \mu_1'^4\right)\frac{t^4}{4!} + \cdots$$

Now, $k_r =$ coefficient of $\frac{t^r}{r!}$ in $k_x(t)$ (where $r = 1, 2, 3, 4, \cdots$)

$k_1 = \mu_1' =$ mean

$k_2 = \mu_2' - \mu_1'^2 = \mu_2$ (variance)

$k_3 = \mu_3' - 3\,\mu_2'\,\mu_1' + 2\mu_1'^3 = \mu_3$ (third central moment)

$$k_4 = \mu_4' - 3\mu_2'^2 - 4\,\mu_3'\,\mu_1' + 12\mu_2'\,\mu_1'^2 - 6\,\mu_1'^4$$

$$= \mu_4' - 4\,\mu_3'\,\mu_1' + 6\mu_2'\,\mu_1'^2 - 3\,\mu_1'^4 - 3\,\mu_2'^2 + 6\,\mu_2'\,\mu_1'^2 - 3\mu_1'^4$$

$$= \mu_4 - 3\left\{\mu_2'^2 - 2\,\mu_2'\,\mu_1'^2 + \left(\mu_1'^2\right)^2\right\}$$

$$= \mu_4 - 3\left(\mu_2' - \mu_1'^2\right)^2$$

$\therefore\ k_4 = \mu_4 - 3\,\mu_2^2 \quad [\because\ \mu_2 = \mu_2' - \mu_1'^2]$

$\Rightarrow k_4 = \mu_4 - 3\,k_2^2 \quad [\because\ \mu_2 = k_2]$

$\therefore\ \mu_4 = k_4 + 3\,k_2^2$

Alternative Method:

Determine the moment generating function of binomial distribution and hence find mean and variance.

Ans: Let x is a binomial variate with parameters n and p.

So, probability function, $f(x;\ n, p) = {}^n c_x\ p^x\ q^{n-x}; \quad x = 0, 1, 2, \cdots, n$

Now, the m.g.f of x is

$$M_x(t) = E(e^{tx})$$

$$= \sum_{x=0}^{n} e^{tx}\ f(x;\ n, p)$$

$$= \sum_{x=0}^{n} e^{tx}.\ {}^n c_x\ p^x\ q^{n-x}$$

$$= \sum_{x=0}^{n} {}^n c_x\ q^{n-x}\ (p\ e^t)^x$$

$$= \left(q + p\,e^t\right)^n$$

$\therefore\ M_x(t) = \left(q + p\,e^t\right)^n$ ....(i)

Now differentiating (i) with respect of t, we get,

$$\frac{d\,M_x(t)}{d\,t} = n\,(q+pe^t)^{n-1}.\,pe^t$$

$$= npe^t\,(q+p\,e^t)^{n-1}\ldots\ldots\ldots\ldots\ldots.(ii)$$

$$\therefore\ \mu_1' = \left.\frac{d\,M_x(t)}{d\,t}\right]_{t=0} = npe^0\,(q+pe^0)^{n-1} = np(q+p)^{n-1} = np$$

So, Mean $\mu_1' = np$

Again differentiating (ii) with respect of t, we get

$$\frac{d^2\,M_x(t)}{d\,t^2} = np\left[e^t\,(n-1)\,(q+pe^t)^{n-2}\ pe^t + (q+pe^t)^{n-1}\ e^t\right]$$

$$\therefore\ \mu_2' = \left.\frac{d^2\,M_x(t)}{d\,t^2}\right]_{t=0} = np\left[e^0\,(n-1)\,(q+pe^0)^{n-2}\ pe^0 + (q+pe^0)^{n-1}\ e^0\right]$$

$$= np\left[(n-1)\,(q+p)^{n-2}\,p + (q+p)^{n-1}\right]$$

$$= np\left[(n-1)p+1\right]$$

$$\therefore\ \mu_2' = n(n-1)p^2 + np$$

Variance, $\mu_2 = \mu_2' - \mu_1'^2$

$$= n(n-1)p^2 + np - (np)^2$$

$$= n^2p^2 - np^2 + np - n^2p^2$$

$$= np - np^2$$

$$= np\,(1-p)$$

$$= npq \qquad [\because\ p+q=1]$$

Q. Determine the characteristic function of binomial distribution and hence determine β_1 and β_2

Ans: Let x is a binomial variate with parameters n and p. So, probability function is

$$f(x;\ n,p) = {}^nc_x\ p^x\ q^{n-x};\qquad x = 0, 1, 2, \cdots, n$$

Now, the characteristic function (c.f) of x is

$$\phi_x(t) = E(e^{itx})$$

$$= \sum_{x=0}^{n} e^{itx} \quad f(x;\, n,\, p)$$

$$= \sum_{x=0}^{n} e^{itx}.\ {}^{n}c_x\ p^x\ q^{n-x}$$

$$= \sum_{x=0}^{n} {}^{n}c_x\ q^{n-x}\ (pe^{it})^x$$

$$= \left(q + pe^{it}\right)^n$$

$$\therefore\ \ \phi_x(t) = \left(q + pe^{it}\right)^n$$

We know, the cumulant generating function (cgf),

$$K_x(t) = \log\ \phi_x(t)$$

$$= \log\ \left(q + pe^{it}\right)^n$$

$$= n\ \log\left(q + p\ e^{it}\right)$$

$$\Rightarrow K_x(t) = n\ \ \log\left[q + p\ \left\{1 + (it) + \frac{(it)^2}{2!} + \frac{(it)^3}{3!} + \frac{(it)^4}{4!} + \cdots\right\}\right]$$

$$\Rightarrow K_x(t) = n\ \log\ \left[q + p + p\ \left\{(it) + \frac{(it)^2}{2!} + \frac{(it)^3}{3!} + \frac{(it)^4}{4!} + \cdots\right\}\right]$$

$$\Rightarrow K_x(t) = n\ \log\ \left[1 + p\ \left\{(it) + \frac{(it)^2}{2!} + \frac{(it)^3}{3!} + \frac{(it)^4}{4!} + \cdots\right\}\right]$$

$$\Rightarrow K_x(t) = n\left[p\left\{(it) + \frac{(it)^2}{2!} + \frac{(it)^3}{3!} + \frac{(it)^4}{4!} + \cdots\right\} - \frac{p^2}{2}\left\{(it) + \frac{(it)^2}{2!} + \frac{(it)^3}{3!} + \cdots\right\}^2 + \frac{p^3}{3}\left\{(it) + \frac{(it)^2}{2!} + \cdots\right\}^3 - \frac{p^4}{4}\left\{(it) + \frac{(it)^2}{2!} + \cdots\right\}^4\right]$$

$$\Rightarrow K_x(t) = n\left[p\left\{(it) + \frac{(it)^2}{2!} + \frac{(it)^3}{3!} + \frac{(it)^4}{4!} + \cdots\right\} - \frac{p^2}{2}\left\{(it)^2 + \frac{(it)^4}{4} + 2.it.\frac{(it)^2}{2!} + 2.it.\frac{(it)^3}{3!} + \cdots\right\} + \frac{p^3}{3}\left\{(it)^3 + 3.(it)^2.\frac{(it)^2}{2!} + \cdots\right\} - \frac{p^4}{4}\left\{(it)^4 + \cdots\right\} + \cdots\right]$$

$$\Rightarrow K_x(t) = n\left[\left\{p(it) + p\frac{(it)^2}{2!} + p\frac{(it)^3}{3!} + p\frac{(it)^4}{4!} + \cdots\right\} - \left\{\frac{p^2}{2}(it)^2 + \frac{p^2}{8}(it)^4 + \frac{p^2}{2}(it)^3 + \frac{p^2}{6}(it)^4 + \cdots\right\} + \left\{\frac{p^3}{3}(it)^3 + \frac{p^3}{2}(it)^4 + \cdots\right\} - \left\{\frac{p^4}{4}(it)^4 + \cdots\right\} + \cdots\right]$$

$$\Rightarrow K_x(t) = n\left[p(it) + \left\{p\frac{(it)^2}{2!} - \frac{p^2}{2}(it)^2\right\} + \left\{p\frac{(it)^3}{3!} - \frac{p^2}{2}(it)^3 + \frac{p^3}{3}(it)^3\right\} + \left\{p\frac{(it)^4}{4!} - \frac{p^2}{8}(it)^4 - \frac{p^2}{6}(it)^4 + \frac{p^3}{2}(it)^4 - \frac{p^4}{4}(it)^4\right\} + \cdots\right]$$

$$\Rightarrow K_x(t) = np(it) + n\left(p - p^2\right)\frac{(it)^2}{2!} + n\left(p - 3p^2 + 2p^3\right)\frac{(it)^3}{3!} + n\left(p - 3p^2 - 4p^2 + 12p^3 - 6p^4\right)\frac{(it)^4}{4!} + \cdots$$

Now, if r-th cumulant is denoted by K_r, then

$$K_r = \text{ the coefficient of } \frac{(it)^r}{r!} \text{ in } K_x(t)$$

Mean, $K_1 =$ the coefficient of it in $K_x(t) = np$

Variance, $K_2 = \mu_2 =$ the coefficient of $\dfrac{(it)^2}{2!}$ in $K_x(t) = n(p - p^2) = np(1-p) = npq$

Similarly, $K_3 = \mu_3 = n\,(p - 3p^2 + 2p^3)$

$$= np\,(2p^2 - 3p + 1)$$

$$= np\,(2p^2 - 2p - p + 1)$$

$$= np\,\{2\,p(p-1) - 1(p-1)\}$$

$$= np\,(p-1)\,(2p-1)$$

$$= np\,(-q)\,(2p - p - q)$$

$$= npq\,(q-p)$$

and $K_4 = n\left(p - 3\,p^2 - 4\,p^2 + 12\,p^3 - 6\,p^4\right)$

$$= n\left(p - 7\,p^2 + 12\,p^3 - 6\,p^4\right)$$

$$= np\left(1 - 7p + 12p^2 - 6p^3\right)$$

$$= np\left[1 - p - 6p + 6\,p^2 + 6\,p^2 - 6\,p^3\right]$$

$$= np\left[1(1-p) - 6\,p\,(1-p) + 6\,p^2\,(1-p)\right]$$

$$= np\,(1-p)\left(1 - 6p + 6\,p^2\right)$$

$$= npq\,[1 - 6\,p(1-p)] \qquad [\because\ p + q = 1 \Rightarrow q = 1-p]$$

$$= npq\,(1 - 6\,pq)$$

We know, $\mu_4 = K_4 + 3\,K_2^{\,2} = npq\,(1 - 6\,pq) + 3\,(npq)^2 = npq\,[(1 - 6\,pq) + 3\,npq]$

Now, $\beta_1 = \dfrac{\mu_3^{\,2}}{\mu_2^{\,3}} = \dfrac{\{npq\,(q-p)\}^2}{(npq)^3} = \dfrac{(q-p)^2}{npq}$

and $\beta_2 = \dfrac{\mu_4}{\mu_2^{\,2}} = \dfrac{npq\,\left[3\,npq + (1 - 6\,pq)\right]}{(npq)^2} = 3 + \dfrac{1 - 6\,pq}{npq}$

Theorem: State and prove the additive property of binomial distribution.

Statement: If $X_1, X_2, \cdots, X_k$ are k independent binomial variates with parameters $(n_1, p), (n_2, p), \cdots, (n_k, p)$ respectively, then $X_1 + X_2 + \cdots + X_k = \sum_{i=1}^{k} X_i$ is also a binomial variate with parameters $n_1 + n_2 + \cdots + n_k = \sum_{i=1}^{k} n_i$ and p.

Proof: We know, the m.g.f of binomial distribution with parameters n and p is

$$M_X(t) = \left(q + p\, e^t\right)^n \text{ ....(i)}$$

Thus, $M_{X_i}(t) = \left(q + p\, e^t\right)^{n_i}; \quad i = 1, 2, \cdots, k$

So, $M_{X_1}(t) = \left(q + p\, e^t\right)^{n_1}, \quad M_{X_2}(t) = \left(q + p\, e^t\right)^{n_2}, \cdots, M_{X_k}(t) = \left(q + p\, e^t\right)^{n_k}$

Now, the m.g.f of $\sum_{i=1}^{k} X_i$ is

$$M_{\sum_{i=1}^{k} X_i}(t) = E\left(e^{t \sum_{i=1}^{k} X_i}\right)$$

$$= E\left[e^{t\,(X_1 + X_2 + \cdots + X_k)}\right]$$

$$= E\left[e^{t X_1} .\ e^{t X_2} \cdots e^{t X_k}\right]$$

$$= E\left[e^{t X_1}\right] .\ E\left[e^{t X_2}\right] \cdots E\left[e^{t X_k}\right] \quad (\because X_1, X_2, \cdots, X_k \text{ are independent})$$

$$= M_{X_1}(t) .\ M_{X_2}(t) \cdots M_{X_k}(t)$$

$$= \left(q + p\, e^t\right)^{n_1} .\ \left(q + p\, e^t\right)^{n_2} \cdots \left(q + p\, e^t\right)^{n_k}$$

$$= \left(q + p\, e^t\right)^{n_1 + n_2 + \cdots + n_k}$$

$$\therefore \quad M_{\sum_{i=1}^{k} X_i}(t) = \left(q + p\, e^t\right)^{\sum_{i=1}^{k} n_i} \text{ ....(ii)}$$

Now comparing (i) and (ii), we can say that (ii) is also the m.g.f of binomial distribution with parameters $\sum_{i=1}^{k} n_i$ and p.

Since m.g.f uniquely determines a distribution, therefore $\sum_{i=1}^{k} X_i$ is also a binomial variate with parameters $\sum_{i=1}^{k} n_i$ and p.

This completes the proof of the theorem.

NB: If the probability of success is different for different independent random variables i.e., if $X_i \sim b(n_i, p_i);\quad i = 1, 2, \cdots, k$ then additive property is not true. Because in that case $M_{\sum_{i=1}^{k} X_i}(t)$ can not be expressed in the form $\left(q + p e^t\right)^{\sum_{i=1}^{k} n_i}$. Hence in general binomial distribution does not posses the additive property, except $p_1 = p_2 = \cdots = p_k = p$.

Alternative Method:

Statement: If $X_1, X_2, \cdots, X_k$ are k independent binomial variates with parameters $(n_1, p),\ (n_2, p), \cdots, (n_k, p)$ respectively, then $X_1 + X_2 + \cdots + X_k = \sum_{i=1}^{k} X_i$ is also a binomial variate with parameters $n_1 + n_2 + \cdots + n_k = \sum_{i=1}^{k} n_i$ and p.

Proof: We know, the characteristic function of binomial distribution with parameters n and p is

$$\phi_X(t) = \left(q + p\ e^{it}\right)^n \ldots\ldots\ldots\ldots\ldots\text{(i)}$$

Thus $\phi_{X_i}(t) = \left(q + p\ e^{it}\right)^{n_i};\qquad i = 1, 2, \cdots, k$

So, $\phi_{X_1}(t) = \left(q + p\ e^{it}\right)^{n_1},\quad \phi_{X_2}(t) = \left(q + p\ e^{it}\right)^{n_2}, \cdots, \phi_{X_k}(t) = \left(q + p\ e^{it}\right)^{n_k}$

Now, the c.f of $\sum_{i=1}^{k} X_i$ is

$$\phi_{\sum_{i=1}^{k} X_i}(t) = E\left(e^{it\sum_{i=1}^{k} X_i}\right)$$

$$= E\left[e^{it\,(X_1 + X_2 + \cdots + X_k)}\right]$$

$$= E\left[e^{it\,X_1}.\ \ e^{it\,X_2} \cdots e^{it\,X_k}\right]$$

$$= E\left[e^{it\,X_1}\right].\ E\left[e^{it\,X_2}\right] \cdots E\left[e^{it\,X_k}\right] \qquad (\because\ X_1,\ X_2, \cdots, X_k \text{ are independent})$$

$$= \phi_{X_1}(t).\ \phi_{X_2}(t) \cdots \phi_{X_k}(t)$$

$$= \left(q + p\ e^{it}\right)^{n_1}.\ \ \left(q + p\ e^{it}\right)^{n_2} \cdots \left(q + p\ e^{it}\right)^{n_k}$$

$$= \left(q + p\ e^{it}\right)^{n_1 + n_2 + \cdots + n_k}$$

$$\therefore \quad \phi_{\sum_{i=1}^{k} X_i}(t) = \left(q + p\ e^{it}\right)^{\sum_{i=1}^{k} n_i} \quad \text{.............(ii)}$$

Now comparing (i) and (ii), we can say that (ii) is also the characteristic function of binomial distribution with parameters $\sum_{i=1}^{k} n_i$ and p.

Hence by uniqueness property of characteristic function (cf), therefore $\sum_{i=1}^{k} X_i$ is also a binomial variate with parameters $\sum_{i=1}^{k} n_i$ and p.

Q. Determine the cumulant generating function and hence find mean and variance of binomial distribution.

Ans: Let x is a binomial variate with parameters n and p.

We know, the m.g.f of binomial distribution is

$$M_X(t) = \left(q + p\ e^t\right)^n$$

$\therefore$ Cumulant generating function (c.g.f)

$$K_x(t) = \log\ M_x(t)$$

$$= \log\ \left(q + p\ e^t\right)^n$$

$$= n\ \log\left(q + p\ e^t\right)$$

$$\Rightarrow K_x(t) = n\ \log\left[q + p\left(1 + \frac{t}{1!} + \frac{t^2}{2!} + \cdots\right)\right]$$

$$\Rightarrow K_x(t) = n\ \log\left[q + p + p\ \left(t + \frac{t^2}{2!} + \cdots\right)\right]$$

$$\Rightarrow K_x(t) = n\ \log\left[1 + p\left(t + \frac{t^2}{2!} + \cdots\right)\right]$$

$$\Rightarrow K_x(t) = n\left[p\left(t + \frac{t^2}{2!} + \cdots\right) - \frac{p^2}{2}\left(t + \frac{t^2}{2!} + \cdots\right)^2 + \cdots\right]$$

$$\Rightarrow K_x(t) = n\left[p\left(t + \frac{t^2}{2!} + \cdots\right) - \frac{p^2}{2}\left(t^2 + \cdots\right) + \cdots\right]$$

$$\Rightarrow K_x(t) = n\left[p\,t + \left(p\ \frac{t^2}{2!} - \frac{p^2}{2} t^2\right) + \cdots\right]$$

$$\Rightarrow K_x(t) = npt + n\left(p - p^2\right)\frac{t^2}{2!} + \cdots$$

Now, if r-th cumulant is denoted by K_r, then

$$K_r = \text{ the coefficient of } \frac{t^r}{r!} \text{ in } K_x(t)$$

Mean, $K_1 =$ the coefficient of t in $K_x(t) = np$

Variance, $K_2 = \mu_2 =$ the coefficient of $\frac{t^2}{2!}$ in $K_x(t) = n\ (p - p^2) = np\ (1-p) = npq$

Q. What is expected frequency?

Ans: If the probability for a certain value of a random variable is multiplied by total number of frequencies, then this is known as expected frequency. It is usually denoted by E.

Example: Let 5 unbiased coins are thrown 160 times.

Here, number of trials, n = 5

Number of experiment or Total number of frequencies, N = 160

If the number of head is denoted by x, then probability function is

$$f(x) = {}^5c_x \left(\frac{1}{2}\right)^x \left(\frac{1}{2}\right)^{5-x}; \qquad x = 0, 1, 2, \cdots, 5$$

Here, the probability to get '0' head is

$$f(x=0) = {}^5c_0 \left(\frac{1}{2}\right)^0 \left(\frac{1}{2}\right)^{5-0} = \left(\frac{1}{2}\right)^5 = \frac{1}{32}$$

So, the expected frequency to get '0' head is

$$E = N \times P(x=0) = 160 \times \frac{1}{32} = 5$$

Q. Write down the properties of binomial distribution.

Ans: The properties of binomial distribution are as follows-

(i) Binomial distribution is a probability distribution of a discrete random variable.

(ii) The parameters of binomial distribution are n and p.

(iii) The sum of all probabilities of binomial distribution is one.

(iv) The probability of success (p) and the probability of failure remain same from trial to trial.

(v) The mean of binomial distribution is np.

(vi) The variance and standard deviation of binomial distribution are npq and $\sqrt{npq}$ respectively.

(vii) The mean is greater than variance of binomial distribution.

(viii) If x is a binomial variate with probability function $f(x;n,p)$, then its mgf about origin is $M_x(t) = \left(q + pe^t\right)^n$.

(ix) If in case of binomial distribution, the number of trials is very large i.e., $n \to \infty$ and the probability of success is very small i.e., $p \to 0$, then binomial distribution tends to Poisson distribution.

(x) If in case of binomial distribution, the number of trials is very large i.e., $n \to \infty$ and the probability of success and failure is approximately same i.e., $p \cong q$, then binomial distribution tends to normal distribution.

(xi) The sum of two independent binomial variates is also binomial.

Q. Uses of binomial distribution.

Ans: The uses of binomial distribution are as follows-

(i) Binomial distribution is used to derive different distributions like Poisson and normal distribution.

(ii) Binomial distribution can be successfully used if in an experiment each trial consists only two possible outcomes.

(iii) It is used to fit frequency distribution of an experiment.

Short questions and answers:

(1) What is parameter (NU-12)

Ans: The objective of the statistical analysis is to estimate the unknown characteristic, usually known as parameter.

For example: n and p are the two parameters of binomial distribution.

(2) How many parameters are involved in binomial distribution and what are they? (NU-12)

Ans: Two parameters are involved in binomial distribution and they are n and p. Here n indicates the no. of trials and p indicates the probability of success.

(3) Write down the mean and variance of binomial distribution (NU-10)

Ans: The mean and variance of binomial distribution are np and npq respectively.

Here, n indicates the no. of trials, p indicates probability of success and q indicates the probability of failure.

(4) Under what condition binomial distribution turns into Poisson distribution? (NU-10)

Ans: In binomial distribution, if number of trials are very large i.e., $n \to \infty$ and probability of success is very small i.e. $p \to 0$, then binomial distribution turns into Poisson distribution.

(5) Write down the relation between binomial distribution and Poisson distribution.

Ans: The relation between binomial distribution and Poisson is as follows:

$$\underset{\substack{n\to\infty \\ p\to 0}}{\mathrm{Lt}} \text{ Binomial distribution} = \text{Poission distribution}$$

$$\Rightarrow \underset{\substack{n\to\infty \\ p\to 0}}{\mathrm{Lt}} n_{c_x} \, p^x \, q^{n-x} = \frac{e^{-\lambda}\lambda^x}{x!}; \quad x = 0, 1, 2, \cdots, \infty$$

Here, $n =$ the no. of trials, $p =$ probability of success, $q =$ the probability of failure and $\lambda =$ Parameter of Poisson distribution where $\lambda = np$.

(6) For binomial distribution which is correct? (NU-12)

(i) $E(x) < V(x)$; (ii) $E(x) > V(x)$

Ans: $E(x) > V(x)$ is correct because we know that mean is greater than variance of binomial distribution.

(7) The mean and variance of a binomial distribution are 4 and 5 respectively; comment (NU-11)

Ans: The mean and variance of a binomial distribution are 4 and 5 respectively i.e., mean is less than variance which is impossible. Because we know that in case of binomial distribution, mean is greater than variance.

(8) Write down the recursion formula of a binomial distribution (NU-11)

Ans: Let x is a binomial variate with parameter n and p.

So, the recursion formula of a binomial distribution is

$$P(x+1)=\frac{n-x}{x+1}.\frac{p}{q}\ P(x)$$

(9) In a symmetrical binomial distribution if $P(x=2)=P(x=3)$ **, then** $n=?$

Ans: Let x is a binomial variate with parameter n and p.

$\therefore$ Probability function, $P(x)={}^{n}c_{x}\ p^{x}\ q^{n-x};\quad x=0,1,2,\cdots,n$

We know for symmetrical binomial distribution, $p=q=\frac{1}{2}$

Now, $P(x=2)=P(x=3)$

$$\Rightarrow {}^{n}c_{2}\ p^{2}\ q^{n-2}={}^{n}c_{3}\ p^{3}\ q^{n-3}$$

$$\Rightarrow \frac{n\ (n-1)}{2}.\ p^{2}.\ q^{n-2}=\frac{n\ (n-1)\ (n-2)}{6}.\ p^{2}.\ p.\ q^{n-2}.\ q^{-1}$$

$$\Rightarrow 1=\frac{n-2}{3}.\ \frac{p}{q}$$

$$\Rightarrow 1=\frac{n-2}{3}.\ \frac{\frac{1}{2}}{\frac{1}{2}}$$

$$\Rightarrow n-2=3$$

$$\therefore\ n=5$$

10. When binomial distribution is leptokurtic?

Ans: The coefficient of kurtosis of binomial distribution is,

$$\beta_{2}=3+\frac{1-6\,pq}{npq}$$

The distribution is leptokurtic if $pq<\frac{1}{6}$.

NB: i) The distribution is mesokurtic if $pq=\frac{1}{6}$.

[Mesokurtic means, $\beta_{2}=3$

Now, $\beta_{2}=3+\frac{1-6\,pq}{npq}$

$$\Rightarrow 3 = 3 + \frac{1-6\,pq}{npq} \qquad (\because\ \beta_2 = 3)$$

$$\Rightarrow 0 = \frac{1-6\,pq}{npq}$$

$$\Rightarrow 1-6\,pq = 0$$

$$\Rightarrow 6\,pq = 1$$

$$\therefore\ pq = \frac{1}{6}$$

ii) The distribution is platykurtic if $pq > \frac{1}{6}$.

11. When binomial distribution is symmetric?

Ans: In case of binomial distribution, we know

$$\beta_1 = \frac{(q-p)^2}{npq} = \frac{(1-p-p)^2}{npq} = \frac{(1-2p)^2}{npq}$$

So, coefficient of skewness, $\sqrt{\beta_1} = \frac{1-2p}{\sqrt{npq}}$

It is symmetrical if $p = q = \frac{1}{2}$.

NB: (i) Binomial distribution is positively skewed if $p < \frac{1}{2}$.

(ii) Binomial distribution is negatively skewed if $p > \frac{1}{2}$.

Mathematical Problems:

(1) If $P(x=0) = 2\ P(x=1) = 9\ P(x=2)$ **of binomial distribution, then find its mean and variance. Also find the probability function.**

Solution: Let x is a binomial variate with parameters n and p.

$\therefore$ Probability function, $P(x) = f(x) = {}^n c_x\ p^x\ q^{n-x}; \quad x = 0, 1, 2, \cdots, n$

Now, $P(x=0) = 2\ P(x=1)$

$$\Rightarrow {}^n c_0\ p^0\ q^{n-0} = 2.\ {}^n c_1\ p^1\ q^{n-1}$$

$$\Rightarrow 1.\,1.\,q^n = 2.\; n\,p\,q^{n-1}$$

$$\Rightarrow q^n = 2\; n\,p\,q^n.\,q^{-1}$$

$$\Rightarrow 1 = 2np.\frac{1}{q}$$

$$\Rightarrow q = 2\; n\,p.................(i)$$

and $\quad 2\;P(x=1) = 9\;P(x=2)$

$$\Rightarrow 2.\;{}^nc_1\;p^1\;q^{n-1} = 9.\;{}^nc_2\;p^2\;q^{n-2}$$

$$\Rightarrow 2\;n\,p\,q^{n-1} = 9.\;\frac{n!}{2!\,(n-2)!}\;p^2\;q^{n-2}$$

$$\Rightarrow 2\;n\,p\,q^{n-1} = 9.\;\frac{n\;(n-1)}{2}\;p^2\;q^{n-1}.\,q^{-1}$$

$$\Rightarrow 2 = \frac{9\;(n-1)}{2}\;p.\;\frac{1}{q}$$

$$\Rightarrow 4q = 9\;(n-1)\;p$$

$$\Rightarrow 4.\;2\;n\,p = 9\;(n-1)\,p \quad [\text{ from (i) }]$$

$$\Rightarrow 8n = 9\;(n-1)$$

$$\Rightarrow 8n = 9n - 9$$

$$\Rightarrow 8n - 9n = -9$$

$$\Rightarrow -n = -9$$

$$\therefore\;\; n = 9$$

Putting the value of n in (i), we get

$$q = 2\times 9p = 18p = 18\;(1-q) \qquad [\because\; p+q=1]$$

$$\Rightarrow q = 18 - 18\;q$$

$$\Rightarrow q + 18\;q = 18$$

$$\Rightarrow 19\;q = 18$$

$$\Rightarrow q = \frac{18}{19}$$

$$\therefore\;\; p = 1-q = 1-\frac{18}{19} = \frac{19-18}{19} = \frac{1}{19}$$

Mean, $E(x) = n\;p = 9\times\frac{1}{19} = \frac{9}{19}$

and Variance, $V(x) = npq = 9 \times \frac{1}{19} \times \frac{18}{19} = \frac{162}{361}$

Now $n = 9$, $p = \frac{1}{19} = 0.05$ and $q = \frac{18}{19} = 0.94$

So, the probability function of binomial distribution is

$$P(x) = {}^{9}c_{x} \, (0.05)^{x} \, (0.94)^{9-x}; \qquad x = 0, 1, 2, \cdots, 9$$

(2) With the usual notations, find p for a Binomial variate x if $n = 6$ and $9\ P(x = 4) = P(x = 2)$ (NU-11)

Solution: Let x is a binomial variate with parameters n and p.

Given that $n = 6$

$\therefore$ Probability function, $P(x) = f(x) = {}^{n}c_{x}\ p^{x}\ q^{n-x} = {}^{6}c_{x}\ p^{x}\ q^{6-x}; \quad x = 0, 1, 2, \cdots, 6$

Now, $9\ P(x = 4) = P(x = 2)$

$$\Rightarrow 9 \times {}^{6}c_{4}\ p^{4}\ q^{2} = {}^{6}c_{2}\ p^{2}\ q^{4}$$

$$\Rightarrow 9 \times 15 \times p^{2} \times p^{2} \times q^{2} = 15 \times p^{2} \times q^{2} \times q^{2}$$

$$\Rightarrow 135 \times p^{2} = 15 \times q^{2}$$

$$\Rightarrow 9 \times p^{2} = (1-p)^{2} \quad [\because\ q = 1-p]$$

$$\Rightarrow 9\ p^{2} = 1 - 2p + p^{2}$$

$$\Rightarrow 8p^{2} + 2p - 1 = 0$$

$$\Rightarrow 8p^{2} + 4p - 2p - 1 = 0$$

$$\Rightarrow 4p\ (2p+1) - 1\ (2p+1) = 0$$

$$\Rightarrow (2p+1)\ (4p-1) = 0$$

Either, $(2p+1) \neq 0$ Or, $4p - 1 = 0$

$$\therefore \quad p = \frac{1}{4}$$

(3) A family has 6 children where the probability of child being a boy or a girl is equal. Find the probability that 4 of them are boys.

Solution: Let, the child being a boy is success.

$\therefore$ Probability of success, $p = \frac{1}{2} = 0.5$

Probability of failure, $q = 1 - p = 1 - 0.5 = 0.5$

No. of trials (no. of child), $n = 6$

Since $n < 30$ and p is not too small, so the problem is to be solved through binomial distribution.

Suppose x is a binomial variate which indicates the number of success.

Now the probability function for x is

$$f(x) = {}^{n}c_{x}\ p^{x}\ q^{n-x}$$

$$\Rightarrow f(x) = {}^{6}c_{x}\ (0.5)^{x}\ (0.5)^{6-x}; \qquad x = 0, 1, 2, \cdots, 6$$

$$= {}^{6}c_{x}\ (0.5)^{6}$$

$\therefore$ The probability that 4 children are boys,

$$f(x = 4) = {}^{6}c_{4}\ (0.5)^{6} = 0.2344$$

(4) A family has 5 children. The probability of child being a boy or a girl is equal. Find the probability that the family consists of (i) no boy; (ii) at least 3 boys (NU-11)

Solution: Let, the child being a boy is success.

$\therefore$ Probability of success, $p = \frac{1}{2} = 0.5$

Probability of failure, $q = 1 - p = 1 - 0.5 = 0.5$

No. of trials (no. of child), $n = 5$

Since $n < 30$ and p is not too small, so the problem is to be solved through binomial distribution.

Let, x is a binomial variate which indicates the number of successes.

Now the probability function for x is

$$f(x) = {}^{n}c_{x}\ p^{x}\ q^{n-x}$$

$$\Rightarrow f(x) = {}^{5}c_{x}\ (0.5)^{x}\ (0.5)^{5-x}; \qquad x = 0, 1, 2, \cdots, 5$$

$$= {}^{5}c_{x}\ (0.5)^{5}$$

The probability that the family consists of

(i) no boy, i.e., $f(x = 0) = 5_{C_0}\ (0.5)^{5} = (0.5)^{5} = 0.03125$

(ii) at least 3 boys i.e., $f(x \geq 3) = 1 - f(x = 0) - f(x = 1) - f(x = 2)$

$$= 1 - 5_{c_0}\ (0.5)^{5} - 5_{c_1}\ (0.5)^{5} - 5_{c_2}\ (0.5)^{5}$$

$$= 1 - (0.5)^{5} - 5\ (0.5)^{5} - 10\ (0.5)^{5}$$

$$= 1 - 16\,(0.5)^5$$

$$\therefore \quad f(x \geq 3) = 0.5$$

(5) A tub contains 5 seeds. The probability of germination (অঙ্কুরোদগম) **of each seed is $\frac{2}{3}$. Find the probability that there will be 3 plants in the tub. Find the probability that there will be 5 plants in the tub. Find the probability that there will be at least one plant in the tub (NU-10)**

[একটি টবে পাঁচটি বীজ আছে। প্রতিটি বীজ হতে গাছ হওয়ার সম্ভাবনা $\frac{2}{3}$ । উক্ত টবে তিনটি গাছ হওয়ার সম্ভাবনা নির্ণয় কর। পাঁচটি গাছ হওয়ার সম্ভাবনা নির্ণয় কর। কমপক্ষে একটি গাছ হওয়ার সম্ভাবনা নির্ণয় কর]

Solution: Let, germination indicates the success.

$\therefore$ Probability of success, $p = \frac{2}{3}$

Probability of failure, $q = 1 - p = 1 - \frac{2}{3} = \frac{3-2}{3} = \frac{1}{3}$

No, of trials (no. of seeds), $n = 5$

Since $n < 30$ and p is not too small, so the problem is to be solved through binomial distribution.

Let, x is a binomial variate which indicates the number of successes.

Now the probability function for x is,

$$f(x) = {}^n c_x \; p^x \; q^{n-x}$$

$$\Rightarrow f(x) = {}^5 c_x \left(\frac{2}{3}\right)^x \left(\frac{1}{3}\right)^{5-x}; \qquad x = 0, 1, 2, \cdots, 5$$

$\therefore$ The probability that there will be 3 plants

$$f(x = 3) = {}^5 c_3 \left(\frac{2}{3}\right)^3 \left(\frac{1}{3}\right)^{5-3} = 10 \times \frac{8}{27} \times \frac{1}{9} = \frac{80}{243}$$

The probability that there will be 5 plants

$$f(x = 5) = {}^5 c_5 \left(\frac{2}{3}\right)^5 \left(\frac{1}{3}\right)^{5-5} = \left(\frac{2}{3}\right)^5 = \frac{32}{243}$$

and the probability that there will be at least one plant

$$f(x \geq 1) = 1 - f(x = 0)$$

$$= 1 - \left(\frac{1}{3}\right)^5$$

$$=1-\frac{1}{243}=\frac{243-1}{243}=\frac{242}{243}$$

(6) A fair coin is tossed 5 times. Find the probability of getting (i) 2 heads; (ii) at least 3 heads; (iii) at most 2 heads.

Solution: Let, head indicates the number of successes.

$\therefore$ Probability of success, $p=\frac{1}{2}=0.5$

Probability of failure, $q=\frac{1}{2}=0.5$

No. of trials, $n=5$

Since $n<30$ and p is not too small, so the problem is to be solved through binomial distribution.

Let x is a binomial variate which indicates the number of successes.

Now the probability function for x is,

$$f(x)={}^{n}c_{x}\ p^{x}\ q^{n-x}$$

$$\Rightarrow f(x)={}^{5}c_{x}\ (0.5)^{x}\ (0.5)^{5-x}; \qquad x=0, 1, 2, \cdots, 5$$

$$={}^{5}c_{x}\ (0.5)^{5}$$

The probability of getting

(i) 2 heads i.e., $f(x=2)={}^{5}c_{2}\,(0.5)^{5}=10\times 0.03125=0.3125$

(ii) at least 3 heads i.e., $f(x\geq 3)=1-f(x<3)$

$$=1-f(x=0)-f(x=1)-f(x=2)$$

$$=1-{}^{5}c_{0}\,(0.5)^{5}-{}^{5}c_{1}\,(0.5)^{5}-{}^{5}c_{2}\,(0.5)^{5}$$

$$=1-16\,(0.5)^{5}=0.5$$

(iii) at most 2 heads i.e., $f(x\leq 2)=f(x=0)+f(x=1)+f(x=2)$

$$=16\,(0.5)^{5}=0.5$$

Poisson Distribution

Poisson distribution was discovered by the French mathematician and physicist Simeon Denis Poisson (1781-1840), who published it in 1837.

Q. Define Poisson distribution (NU-11)

Or, What is Poisson distribution? (NU-12)

Ans: A discrete random variable X is said to have a Poisson distribution if its probability function is given by

$$f(x;\ \lambda) = \frac{e^{-\lambda}\ \lambda^{x}}{x!}; \qquad x = 0, 1, 2, \cdots, \infty$$

where, λ is the only parameter of Poisson distribution and $\lambda = n\,p$

NB: $f(X = x;\ \lambda) = f(x;\ \lambda) = \dfrac{e^{-\lambda}\ \lambda^{x}}{x!}; \qquad x = 0, 1, 2, \cdots, \infty$

Q. What is Poisson Variate? Give some examples (NU-10)

Ans: A discrete random variable X is said to be Poisson variate if its probability function can be defined as

$$f(x;\ \lambda) = \frac{e^{-\lambda}\ \lambda^{x}}{x!}; \qquad x = 0, 1, 2, \cdots, \infty \text{ and } \lambda > 0$$

Q. Give some practical examples of Poisson variate.

Ans: Some practical examples of Poisson variate are given below-

1. Number of faulty blades in a packet of 100.
2. Number of printing mistakes at each page of a book.
3. Number of air accidents in some unit of time.
4. Number of suicides reported in a particular day.
5. Number of deaths from a disease such as heart attack or cancer or due to snake bite.
6. The number of robbers caught on a given day in a certain city.

Q. Derive of Poisson probability function from Binomial probability function.

Or, Derive Poisson distribution as a limiting case of Binomial distribution (NU-10)

Or, Stating the necessary conditions derive the probability function of Poisson distribution (NU-12)

Or, Derive the Poisson distribution and hence find its mean and variance. Show that they are equal (NU-10)

Or, Derive the probability function of Poisson distribution (**NU-16**)

Ans: Poisson distribution can be derived from the binomial distribution under the following conditions-

(i) the number of trials 'n' is very large i.e., $n \to \infty$

(ii) the probability of success 'p' is very small i.e., $p \to 0$

(iii) the mean of the binomial distribution is $\lambda = np$, a finite and positive constant.

We know the probability function of binomial variate X with parameters n and p is

$$f(x;\, n, p) = {}^{n}c_{x}\; p^{x}\, q^{n-x}; \qquad x = 0, 1, 2, \cdots, n$$

Let, $np = \lambda$

$$\Rightarrow p = \frac{\lambda}{n}$$

$$\therefore \quad q = 1 - p = 1 - \frac{\lambda}{n}$$

Now, the probability function of Poisson distribution is

$$f(x) = \lim_{n\to\infty} {}^{n}c_{x}\; p^{x}\, q^{n-x}$$

$$= \lim_{n\to\infty} \frac{n!}{x!\,(n-x)!} \cdot \left(\frac{\lambda}{n}\right)^{x} \left(1 - \frac{\lambda}{n}\right)^{n-x}$$

$$= \lim_{n\to\infty} \frac{n(n-1)(n-2)\cdots(n-x+1)(n-x)!}{x!\,(n-x)!} \times \frac{\lambda^{x}}{n^{x}} \times \frac{\left(1-\frac{\lambda}{n}\right)^{n}}{\left(1-\frac{\lambda}{n}\right)^{x}}$$

$$= \lim_{n\to\infty} \frac{n.\, n\left(1-\frac{1}{n}\right).\, n\left(1-\frac{2}{n}\right)\cdots n\left(1-\frac{x-1}{n}\right)}{x!} \times \frac{\lambda^{x}}{n^{x}} \times \frac{\left(1-\frac{\lambda}{n}\right)^{n}}{\left(1-\frac{\lambda}{n}\right)^{x}}$$

$$= \lim_{n\to\infty} \frac{n^{x}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\ldots\left(1-\frac{x-1}{n}\right)\lambda^{x}\left(1-\frac{\lambda}{n}\right)^{n}}{x!\, n^{x}\left(1-\frac{\lambda}{n}\right)^{x}}$$

$$= \frac{\lambda^{x}}{x!} \cdot \frac{\lim\limits_{n\to\infty}\left\{\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\ldots\left(1-\frac{x-1}{n}\right)\right\}.\ \lim\limits_{n\to\infty}\left(1-\frac{\lambda}{n}\right)^{n}}{\lim\limits_{n\to\infty}\left(1-\frac{\lambda}{n}\right)^{x}}$$

$$\Rightarrow f(x) = \frac{\lambda^x}{x!} \lim_{n\to\infty}\left(1-\frac{\lambda}{n}\right)^n \text{.............(i)}$$

Now, Suppose $y = \left(1-\frac{\lambda}{n}\right)^n$

$$\Rightarrow \log y = n \log\left(1-\frac{\lambda}{n}\right)$$

$$= n\left(-\frac{\lambda}{n}-\frac{\lambda^2}{2n^2}-\frac{\lambda^3}{3n^3}-\cdots\right)$$

$$= -\lambda-\frac{\lambda^2}{2n}-\frac{\lambda^3}{3n^2}-\cdots$$

$$\Rightarrow \lim_{n\to\infty} \log y = \lim_{n\to\infty}\left(-\lambda-\frac{\lambda^2}{2n}-\frac{\lambda^3}{3n^2}-\cdots\right)$$

$$= -\lambda-0-0-\cdots \qquad \left[\because \lim_{n\to\infty}\frac{1}{n}=0\right]$$

$$\Rightarrow \lim_{n\to\infty} \log\left(1-\frac{\lambda}{n}\right)^n = -\lambda$$

$$\Rightarrow \lim_{n\to\infty}\left(1-\frac{\lambda}{n}\right)^n = e^{-\lambda} \text{........(ii)}$$

Putting the value from (ii) in (i), we get

$$\therefore\ f(x) = \frac{\lambda^x}{x!}e^{-\lambda} = \frac{e^{-\lambda}\lambda^x}{x!};\ x = 0, 1, 2, \cdots, \infty$$

which is the probability function of Poisson distribution with parameter λ.

Q. Find the mean and variance of Poisson Distribution.

Ans: The probability function of Poisson variate X, with parameter λ is

$$f(x;\lambda) = \frac{e^{-\lambda}\lambda^x}{x!};\ x = 0, 1, 2, \cdots, \infty$$

Now by definition, $E(x) = \sum_{x=0}^{\infty} x\ f(x)$

$$= \sum_{x=0}^{\infty} x.\frac{e^{-\lambda}\lambda^x}{x!}$$

$$= e^{-\lambda}\left[0.\frac{\lambda^0}{0!} + \sum_{x=1}^{\infty} x.\frac{\lambda^x}{x!}\right]$$

$$= e^{-\lambda} \sum_{x=1}^{\infty} x.\ \frac{\lambda\ \lambda^{x-1}}{x\ (x-1)!}$$

$$= \lambda\ e^{-\lambda} \sum_{x=1}^{\infty} \frac{\lambda^{x-1}}{(x-1)!}$$

$$= \lambda\ e^{-\lambda}\ e^{\lambda}$$

$$= \lambda$$

$\therefore$ Mean, $E(x) = \lambda$ ....(i)

Again, Variance, $V(x) = E(x^2) - \{E(x)\}^2$

$$= E\{x\ (x-1) + x\} - \{E(x)\}^2$$

$$= E\{x\ (x-1)\} + E(x) - \{E(x)\}^2$$

$$\Rightarrow V(x) = E\{x\ (x-1)\} + \lambda - \lambda^2 \quad [\because\ E(x) = \lambda] \text{.............(ii)}$$

Here, $E\{x\ (x-1)\} = \sum_{x=0}^{\infty} x\ (x-1).\ f(x)$

$$= \sum_{x=0}^{\infty} x\ (x-1).\ \frac{e^{-\lambda}\ \lambda^x}{x!}$$

$$= e^{-\lambda} \sum_{x=2}^{\infty} x\ (x-1).\ \frac{\lambda^x}{x!} \qquad (\because 1^{st} \text{ two term vanishes})$$

$$= e^{-\lambda} \sum_{x=2}^{\infty} x\ (x-1).\ \frac{\lambda^2\ \lambda^{x-2}}{x\ (x-1)\ (x-2)!}$$

$$= \lambda^2\ e^{-\lambda} \sum_{x=2}^{\infty} \frac{\lambda^{x-2}}{(x-2)!}$$

$$= \lambda^2\ e^{-\lambda}\ e^{\lambda}$$

$$= \lambda^2$$

Now, putting the value of $E\{x\ (x-1)\}$ in (ii), we get

$$V(x) = \lambda^2 + \lambda - \lambda^2 = \lambda$$

$\therefore$ Variance, $V(x) = \lambda$ ....(iii)

From (i) and (iii), it is clear that,

$$E(x) = V(x) = \lambda$$

i.e., Mean and variance of Poisson distribution are equal.

NB: $\sum_{x=0}^{\infty} \frac{\lambda^x}{x!} = 1 + \lambda + \frac{\lambda^2}{2} + \cdots = e^{\lambda}$

Similarly, $\sum_{x=1}^{\infty} \frac{\lambda^{x-1}}{(x-1)!} = 1 + \lambda + \frac{\lambda^2}{2} + \cdots = e^{\lambda}$

Alternative Method: To find mean and variance

Ans: The probability function of Poisson variate X, with parameter λ is

$$f(x;\lambda) = \frac{e^{-\lambda}\lambda^x}{x!}; \qquad x = 0, 1, 2, \cdots, \infty$$

Now, by definition

$$
\begin{aligned}
E(x) &= \sum_{x=0}^{\infty} x\ f(x) \\
&= \sum_{x=0}^{\infty} x.\ \frac{e^{-\lambda}\lambda^x}{x!} \\
&= e^{-\lambda} \sum_{x=0}^{\infty} x.\ \frac{\lambda^x}{x!} \\
&= e^{-\lambda}\left(0.\ \frac{\lambda^0}{0!} + 1.\ \frac{\lambda^1}{1!} + 2.\ \frac{\lambda^2}{2!} + 3.\ \frac{\lambda^3}{3!} + \cdots\right) \\
&= e^{-\lambda}\left(\lambda + 2.\ \frac{\lambda^2}{2} + 3.\ \frac{\lambda^3}{6} + \cdots\right) \\
&= e^{-\lambda}\left(\lambda + \lambda^2 + \frac{\lambda^3}{2} + \cdots\right) \\
&= e^{-\lambda}.\ \lambda\left(1 + \lambda + \frac{\lambda^2}{2} + \cdots\right) \\
&= \lambda\ e^{-\lambda}\left(1 + \lambda + \frac{\lambda^2}{2} + \cdots\right) \\
&= \lambda\ e^{-\lambda}\ e^{\lambda} \\
&= \lambda
\end{aligned}
$$

$\therefore$ Mean, $E(x) = \lambda$ ....(i)

Again, Variance, $V(x) = E(x^2) - \{E(x)\}^2$

$$= E\{x\ (x-1) + x\} - \{E(x)\}^2$$

$$= E\{x(x-1)\} + E(x) - \{E(x)\}^2$$

$$\Rightarrow V(x) = E\{x\ (x-1)\} + \lambda - \lambda^2 \ [\because E(x) = \lambda] \text{..........(ii)}$$

Here, $E\{x\ (x-1)\} = \sum_{x=0}^{\infty} x\ (x-1).\ f(x)$

$$= \sum_{x=0}^{\infty} x\ (x-1).\ \frac{e^{-\lambda}\ \lambda^x}{x!}$$

$$= e^{-\lambda} \sum_{x=0}^{\infty} x\ (x-1).\ \frac{\lambda^x}{x!}$$

$$= e^{-\lambda}\left[0\ (0-1).\ \frac{\lambda^0}{0!} + 1\ (1-1).\ \frac{\lambda^1}{1!} + 2\ (2-1).\ \frac{\lambda^2}{2!} + 3\ (3-1).\ \frac{\lambda^3}{3!} + 4\ (4-1).\ \frac{\lambda^4}{4!} + \cdots\right]$$

$$= e^{-\lambda}\left[0 + 0 + 2.\ 1.\ \frac{\lambda^2}{2} + 3.\ 2.\ \frac{\lambda^3}{6} + 4.\ 3.\ \frac{\lambda^4}{24} + \cdots\right]$$

$$= e^{-\lambda}.\ \lambda^2\left[1 + \lambda + \frac{\lambda^2}{2} + \cdots\right]$$

$$= \lambda^2\ e^{-\lambda}\left[1 + \lambda + \frac{\lambda^2}{2} + \cdots\right]$$

$$= \lambda^2\ e^{-\lambda}\ e^{\lambda}$$

$$= \lambda^2$$

Now, putting the value of $E\{x\ (x-1)\}$ in (ii), we get

$$V(x) = \lambda^2 + \lambda - \lambda^2 = \lambda$$

$\therefore$ Variance, $V(x) = \lambda$ ....(iii)

From (i) and (iii), it is clear that, $E(x) = V(x) = \lambda$

i.e., Mean and variance of Poisson distribution are equal.

Theorem: Prove that the sum of all probabilities of a Poisson distribution is one (NU-11)

Proof: The probability function for the Poisson variate X with parameter λ is

$$f(x;\lambda) = \frac{e^{-\lambda}\ \lambda^x}{x!}; \qquad x = 0, 1, 2, \cdots, \infty$$

Now, $\sum_{x=0}^{\infty} f(x) = \sum_{x=0}^{\infty} \frac{e^{-\lambda}\lambda^x}{x!}$

$$= e^{-\lambda} \sum_{x=0}^{\infty} \frac{\lambda^x}{x!}$$

$$= e^{-\lambda}\left(\frac{\lambda^0}{0!}+\frac{\lambda^1}{1!}+\frac{\lambda^2}{2!}+\cdots\right)$$

$$= e^{-\lambda}\left(1+\frac{\lambda}{1!}+\frac{\lambda^2}{2!}+\cdots\right)$$

$$= e^{-\lambda}.\, e^{\lambda}$$

$$= 1$$

$$\therefore \sum_{x=0}^{\infty} f(x) = 1$$

So, the summation of total probability for Poisson distribution is one (Proved)

Q. Calculate the moment generating function of Poisson distribution and hence find its β_1 and β_2 (NU-10)

Or, Find the moment generating function of Poisson distribution **(NU-11)**

Or, Derive the m.g.f of Poisson distribution, hence find mean and variance **(NU-16)**

Or, Find the moment generating function of Poisson distribution and hence find its mean, variance, β_1 and β_2.

Ans: The probability function of Poisson variate X, with parameter λ is

$$f(x;\lambda) = \frac{e^{-\lambda}\lambda^x}{x!}; \qquad x = 0, 1, 2, \cdots, \infty$$

Now, the mgf of x is

$$M_x(t) = E(e^{tx})$$

$$= \sum_{x=0}^{\infty} e^{tx} f(x)$$

$$= \sum_{x=0}^{\infty} e^{tx} \frac{e^{-\lambda}\lambda^x}{x!}$$

$$= e^{-\lambda} \sum_{x=0}^{\infty} \frac{(\lambda e^t)^x}{x!}$$

$$= e^{-\lambda}\, e^{\lambda e^t}$$

$$= e^{\lambda (e^t - 1)}$$

$\therefore$ The mgf of Poisson distribution is

$$M_x(t) = e^{\lambda (e^t - 1)} \text{.............(i)}$$

We know, cumulant generating function (cgf)

$$K_x(t) = \log M_x(t)$$

$$= \log \; e^{\lambda (e^t - 1)}$$

$$= \lambda \; (e^t - 1)$$

$$= \lambda \left(1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \frac{t^4}{4!} + \cdots - 1\right)$$

$$\therefore \quad K_x(t) = \lambda \left(t + \frac{t^2}{2!} + \frac{t^3}{3!} + \frac{t^4}{4!} + \cdots\right)$$

If r-th cumulant is denoted by K_r, then

$$K_r = \text{ the coefficient of } \frac{t^r}{r!} \text{ in } K_x(t)$$

$\therefore$ Mean, $K_1 = \mu_1' =$ the coefficient of t in $K_x(t) = \lambda$

Variance, $K_2 = \mu_2 =$ the coefficient of $\frac{t^2}{2!}$ in $K_x(t) = \lambda$

$$K_3 = \mu_3 = \text{ the coefficient of } \frac{t^3}{3!} \text{ in } K_x(t) = \lambda$$

$$K_4 = \text{ the coefficient of } \frac{t^4}{4!} \text{ in } K_x(t) = \lambda$$

We know, $\mu_4 = K_4 + 3\,K_2{}^2 = \lambda + 3\,\lambda^2$

Now, $\beta_1 = \dfrac{\mu_3{}^2}{\mu_2{}^3} = \dfrac{\lambda^2}{\lambda^3} = \dfrac{1}{\lambda}$

and $\beta_2 = \dfrac{\mu_4}{\mu_2{}^2} = \dfrac{\lambda + 3\,\lambda^2}{\lambda^2} = \dfrac{1}{\lambda} + 3$

NB: Coefficient of skewness, $\sqrt{\beta_1} = \sqrt{\dfrac{1}{\lambda}} = \dfrac{1}{\sqrt{\lambda}}$

and Coefficient of kurtosis, $\beta_2 = \dfrac{1}{\lambda} + 3$

Alternative method:

The probability function of Poisson variate x, with parameter λ is

$$f(x;\ \lambda) = \frac{e^{-\lambda}\ \lambda^x}{x!}; \qquad x = 0, 1, 2, \cdots, \infty$$

Now the m.g.f of x is

$$M_x(t) = E(e^{tx})$$

$$= \sum_{x=0}^{\infty} e^{tx}\ f(x)$$

$$= \sum_{x=0}^{\infty} e^{tx}\ \frac{e^{-\lambda}\ \lambda^x}{x!}$$

$$= e^{-\lambda}\ \sum_{x=0}^{\infty} \frac{(\lambda\ e^t)^x}{x!}$$

$$= e^{-\lambda}\ e^{\lambda\ e^t}$$

$$= e^{\lambda\ (e^t - 1)}$$

$\therefore$ The mgf of Poisson distribution is

$$M_x(t) = e^{\lambda\ (e^t - 1)} \text{.............(i)}$$

Differentiating (i) with respect to t, we get

$$\frac{d M_x(t)}{dt} = e^{\lambda\ (e^t - 1)}.\ \lambda e^t$$

$$= \lambda\ e^{\lambda\ (e^t - 1)}\ e^t \text{.............(ii)}$$

$$\therefore \text{ Mean},\ \mu_1' = \left.\frac{d\ M_x(t)}{dt}\right]_{t=0} = \lambda\ e^{\lambda\ (e^0 - 1)}\ e^0 = \lambda\ e^{\lambda\ (1-1)}.\ 1 = \lambda e^0 = \lambda$$

Again, differentiating (ii) with respect to t, we get

$$\frac{d^2\ M_x(t)}{dt^2} = \lambda\ \left[e^{\lambda\ (e^t - 1)}\ e^t + e^t\ e^{\lambda\ (e^t - 1)}.\ \lambda\ e^t\right]$$

$$= \lambda\ e^{\lambda\ (e^t - 1)}\ e^t + \lambda^2\ e^{\lambda\ (e^t - 1)}\ e^{2t} \text{.............(iii)}$$

$$\therefore \quad \mu_2' = \left.\frac{d^2\ M_x(t)}{d t^2}\right]_{t=0} = \lambda\ e^{\lambda\ (e^0 - 1)}\ e^0 + \lambda^2\ e^{\lambda\ (e^0 - 1)}\ e^{2 \times 0} = \lambda\ e^{\lambda\ (1-1)}.\ 1 = \lambda + \lambda^2$$

$$\therefore \text{ Variance},\ \mu_2 = \mu_2' - \mu_1'^2 = \lambda + \lambda^2 - \lambda^2 = \lambda$$

Further, differentiating (iii) with respect to t, we get

$$\frac{d^3\ M_x(t)}{d\,t^3} = \lambda\left[e^{\lambda\,(e^t-1)}.\ e^t + e^t.\ e^{\lambda\,(e^t-1)}.\ \lambda\ e^t\right] + \lambda^2\left[e^{\lambda\,(e^t-1)}.\ e^{2t}.\ 2 + e^{2t}.\ e^{\lambda(e^t-1)}.\ \lambda\, e^t\right]$$

$$= \lambda\ e^{\lambda\,(e^t-1)}\ e^t + \lambda^2\ e^{\lambda\,(e^t-1)}\ e^{2t} + 2\lambda^2\ e^{\lambda(e^t-1)}\ e^{2t} + \lambda^3\ e^{\lambda\,(e^t-1)}\ e^{3t}$$

$$= \lambda\ e^{\lambda\,(e^t-1)}\ e^t + 3\,\lambda^2\ e^{\lambda\,(e^t-1)}\ e^{2t} + \lambda^3\ e^{\lambda\,(e^t-1)}\ e^{3t} \text{.............(iv)}$$

$$\therefore\quad \mu_3' = \left.\frac{d^3\,M_x(t)}{d\,t^3}\right]_{t=0} = \lambda\ e^{\lambda\,(e^0-1)}\ e^0 + 3\,\lambda^2\ e^{\lambda\,(e^0-1)}\ e^{2\times 0} + \lambda^3\ e^{\lambda\,(e^0-1)}\ e^{3\times 0} = \lambda + 3\,\lambda^2 + \lambda^3$$

$$\therefore\quad \mu_3 = \mu_3' - 3\ \mu_2'\ \mu_1' + 2\ \mu_1'^3$$

$$= \lambda + 3\,\lambda^2 + \lambda^3 - 3\,(\lambda+\lambda^2)\ \lambda + 2\lambda^3 = \lambda + 3\,\lambda^2 + \lambda^3 - 3\,\lambda^2 - 3\,\lambda^3 + 2\,\lambda^3 = \lambda + 3\,\lambda^3 - 3\,\lambda^3 = \lambda$$

Finally differentiating (iv) with respect to t, we get

$$\frac{d^4\,M_x(t)}{d\,t^4} = \lambda\left[e^{\lambda\,(e^t-1)}.\ e^t + e^t.\ e^{\lambda\,(e^t-1)}.\ \lambda e^t\right] + 3\,\lambda^2\left[e^{\lambda\,(e^t-1)}.\ e^{2t}.\ 2 + e^{2t}.\ e^{\lambda\,(e^t-1)}.\ \lambda\, e^t\right] + \lambda^3\left[e^{\lambda\,(e^t-1)}.\ e^{3t}.3 + e^{3t}.\ e^{\lambda\,(e^t-1)}.\ \lambda\, e^t\right]$$

$$\therefore\ \mu_4' = \left.\frac{d^4\,M_x(t)}{d\,t^4}\right]_{t=0} = \lambda\left[1+\lambda\right] + 3\,\lambda^2\left[2+\lambda\right] + \lambda^3\left[3+\lambda\right]$$

$$= \lambda + \lambda^2 + 6\,\lambda^2 + 3\,\lambda^3 + 3\,\lambda^3 + \lambda^4$$

$$= \lambda + 7\,\lambda^2 + 6\,\lambda^3 + \lambda^4$$

Now, $\mu_4 = \mu_4' - 4\ \mu_3'\ \mu_2' + 6\ \mu_2'\ \mu_1'^2 - 3\ \mu_1'^4$

$$= \lambda + 7\,\lambda^2 + 6\,\lambda^3 + \lambda^4 - 4\,(\lambda + 3\,\lambda^2 + \lambda^3)\ \lambda + 6\,(\lambda+\lambda^2)\ \lambda^2 - 3\,\lambda^4$$

$$= \lambda + 7\,\lambda^2 + 6\,\lambda^3 + \lambda^4 - 4\,\lambda^2 - 12\,\lambda^3 - 4\,\lambda^4 + 6\,\lambda^3 + 6\,\lambda^4 - 3\,\lambda^4$$

$$= \lambda + 3\,\lambda^2 + 12\,\lambda^3 - 12\,\lambda^3 + 7\,\lambda^4 - 7\,\lambda^4$$

$$= \lambda + 3\,\lambda^2$$

$$\therefore\ \beta_1 = \frac{\mu_3{}^2}{\mu_2{}^3} = \frac{\lambda^2}{\lambda^3} = \frac{1}{\lambda}$$

and $\beta_2 = \dfrac{\mu_4}{\mu_2{}^2} = \dfrac{\lambda + 3\lambda^2}{\lambda^2} = \dfrac{1}{\lambda} + 3$

NB: Coefficient of skewness, $\sqrt{\beta_1} = \sqrt{\dfrac{1}{\lambda}} = \dfrac{1}{\sqrt{\lambda}}$

and Coefficient of kurtosis, $\beta_2 = \dfrac{1}{\lambda} + 3$

Q. Find the cumulant generating function of Poisson distribution and hence find β_1 **and** β_2 **and comment (NU-15)**

Or, Calculate the cumulant generating function of Poisson distribution and hence find mean, variance, β_1 and β_2 .

Ans: Let x is a Poisson variate with parameter λ.

We know, the m.g.f of Poisson variate is

$$M_x(t) = e^{\lambda (e^t - 1)}$$

$\therefore$ Cumulant generating function (cgf)

$$\begin{aligned} K_x(t) &= \log M_x(t) \\ &= \log e^{\lambda (e^t - 1)} \\ &= \lambda (e^t - 1) \\ &= \lambda \left(1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \frac{t^4}{4!} + \cdots - 1\right) \\ &= \lambda \left(t + \frac{t^2}{2!} + \frac{t^3}{3!} + \frac{t^4}{4!} + \cdots\right) \end{aligned}$$

If r-th cumulant is denoted by K_r, then

$$K_r = \text{ the coefficient of } \frac{t^r}{r!} \text{ in } K_x(t)$$

$\therefore$ Mean, $K_1 = \mu_1' =$ the coefficient of t in $K_x(t) = \lambda$

Variance, $K_2 = \mu_2 =$ the coefficient of $\frac{t^2}{2!}$ in $K_x(t) = \lambda$

$K_3 = \mu_3 =$ the coefficient of $\frac{t^3}{3!}$ in $K_x(t) = \lambda$

$K_4 =$ the coefficient of $\frac{t^4}{4!}$ in $K_x(t) = \lambda$

We know, $\mu_4 = K_4 + 3\, K_2^{\,2} = \lambda + 3\,\lambda^2$

Now, $\beta_1 = \dfrac{\mu_3^{\,2}}{\mu_2^{\,3}} = \dfrac{\lambda^2}{\lambda^3} = \dfrac{1}{\lambda}$

and $\beta_2 = \dfrac{\mu_4}{\mu_2^{\,2}} = \dfrac{\lambda + 3\,\lambda^2}{\lambda^2} = \dfrac{1}{\lambda} + 3$

NB: Coefficient of skewness, $\sqrt{\beta_1} = \sqrt{\frac{1}{\lambda}} = \frac{1}{\sqrt{\lambda}}$

and Coefficient of kurtosis, $\beta_2 = \frac{1}{\lambda} + 3$

Q. Calculate the characteristic function of Poisson distribution and hence find its β_1 and β_2.

Ans: The probability function of Poisson variate X, with parameter λ is

$$f(x;\ \lambda) = \frac{e^{-\lambda}\ \lambda^x}{x!};\ x = 0, 1, 2, \cdots, \infty$$

Now, the characteristic function (c.f) of x is

$$Q_x(t) = E(e^{itx})$$

$$= \sum_{x=0}^{\infty} e^{itx}\ f(x)$$

$$= \sum_{x=0}^{\infty} e^{itx}\ \frac{e^{-\lambda}\ \lambda^x}{x!}$$

$$= e^{-\lambda} \sum_{x=0}^{\infty} \frac{(\lambda\ e^{it})^x}{x!}$$

$$= e^{-\lambda}\ e^{\lambda\ e^{it}}$$

$$\Rightarrow Q_x(t) = e^{\lambda\,(e^{it}-1)}$$

$\therefore$ The c.f of Poisson distribution is, $Q_x(t) = e^{\lambda\,(e^{it}-1)}$..............(i)

We know, cumulant generating function (c.g.f)

$$K_x(t) = \log\ Q_x(t)$$

$$= \log e^{\lambda(e^{it}-1)}$$

$$= \lambda\ (e^{it} - 1)$$

$$= \lambda \left[1 + it + \frac{(it)^2}{2!} + \frac{(it)^3}{3!} + \frac{(it)^4}{4!} + \cdots - 1\right]$$

$$\therefore\quad K_x(t) = \lambda \left[it + \frac{(it)^2}{2!} + \frac{(it)^3}{3!} + \frac{(it)^4}{4!} + \cdots\right]$$

If r-th cumulant is denoted by K_r, then

$$K_r = \text{ the coefficient of } \frac{(it)^r}{r!} \text{ in } K_x(t)$$

$\therefore$ Mean, $K_1 = \mu_1' =$ the coefficient of it in $K_x(t) = \lambda$

Variance, $K_2 = \mu_2 =$ the coefficient of $\dfrac{(it)^2}{2!}$ in $K_x(t) = \lambda$

$K_3 = \mu_3 =$ the coefficient of $\dfrac{(it)^3}{3!}$ in $K_x(t) = \lambda$

$K_4 =$ the coefficient of $\dfrac{(it)^4}{4!}$ in $K_x(t) = \lambda$

We know, $\mu_4 = K_4 + 3\,K_2{}^2 = \lambda + 3\,\lambda^2$

Now, $\beta_1 = \dfrac{\mu_3{}^2}{\mu_2{}^3} = \dfrac{\lambda^2}{\lambda^3} = \dfrac{1}{\lambda}$

and $\beta_2 = \dfrac{\mu_4}{\mu_2{}^2} = \dfrac{\lambda + 3\,\lambda^2}{\lambda^2} = \dfrac{1}{\lambda} + 3$

NB: Coefficient of skewness, $\sqrt{\beta_1} = \sqrt{\dfrac{1}{\lambda}} = \dfrac{1}{\sqrt{\lambda}}$

and Coefficient of kurtosis, $\beta_2 = \dfrac{1}{\lambda} + 3$

Q. Find mode of Poisson distribution (NU-16)

Ans: Let X is a Poisson variate with parameter λ and M is the mode of the distribution. The mode, M of the distribution is clearly that value of X for which the probability of the distribution is maximum.

We know, $f(x) = \dfrac{e^{-\lambda}\,\lambda^x}{x!}$, so $f(x-1) = \dfrac{e^{-\lambda}\,\lambda^{x-1}}{(x-1)!}$

Now, $\dfrac{f(x)}{f(x-1)} = \dfrac{\dfrac{e^{-\lambda}\,\lambda^x}{x!}}{\dfrac{e^{-\lambda}\,\lambda^{x-1}}{(x-1)!}} = \dfrac{e^{-\lambda}\,\lambda^x}{x!} \times \dfrac{(x-1)!}{e^{-\lambda}\,\lambda^{x-1}} = \dfrac{\lambda}{x}$

We discuss the following cases:

Case-I: When λ is not an integer.

Let us suppose that S is the integral (অখন্ড, পূর্ণ) part of λ.

$$\frac{f(1)}{f(0)} = \frac{\lambda}{1} > 1,\ \ \frac{f(2)}{f(1)} = \frac{\lambda}{2} > 1, \cdots, \frac{f(S-1)}{f(S-2)} = \frac{\lambda}{S-1} > 1,\ \ \frac{f(S)}{f(S-1)} = \frac{\lambda}{S} > 1$$

But, $\dfrac{f(S+1)}{f(S)} = \dfrac{\lambda}{S+1} < 1,\ \dfrac{f(S+2)}{f(S+1)} = \dfrac{\lambda}{S+2} < 1, \cdots$

Combining the above expressions into a single expression, we get

$f(0) < f(1) < f(2) < \cdots < f(S-2) < f(S-1) < f(S) > f(S+1) > f(S+2) > \cdots$, which shows that $f(S)$ is the maximum value. Hence in this case the distribution is unimodal and the integral part of λ is the unique modal value.

Case-II: When $\lambda = k$ (say) is an integer. Here we have

$$\frac{f(1)}{f(0)} = \frac{\lambda}{1} > 1,\ \frac{f(2)}{f(1)} = \frac{\lambda}{2} > 1, \cdots, \frac{f(k-1)}{f(k-2)} = \frac{\lambda}{k-1} > 1$$

and $\dfrac{f(k)}{f(k-1)} = \dfrac{\lambda}{k} = 1,\ \dfrac{f(k+1)}{f(k)} = \dfrac{\lambda}{k+1} < 1,\ \dfrac{f(k+2)}{f(k+1)} = \dfrac{\lambda}{k+2} < 1 \cdots$

Combining the above expressions into a single expression, we get

$$f(0) < f(1) < f(2) < \cdots < f(k-2) < f(k-1) = f(k) > f(k+1) > f(k+2) > \cdots$$

In this case we have two maximum values, viz., $f(k-1)$ and $f(k)$ and thus the distribution is bimodal and two modes are at $(k-1)$ and k, i.e., at $(\lambda-1)$ and λ (since $k = \lambda$).

NB: (i) **When λ is not an integer**, then suppose S is the integral (অখন্ড, পূর্ণ) part of λ.

Now let $\lambda = 3.5$ so $S = 3$, then

$$\frac{f(1)}{f(0)} = \frac{\lambda}{1} = \frac{3.5}{1} > 1,\ \frac{f(2)}{f(1)} = \frac{\lambda}{2} = \frac{3.5}{2} > 1, \cdots, \frac{f(S-1)}{f(S-2)} = \frac{\lambda}{S-1} = \frac{3.5}{3-1} > 1,\ \frac{f(S)}{f(S-1)} = \frac{\lambda}{S} = \frac{3.5}{3} > 1$$

But $\dfrac{f(S+1)}{f(S)} = \dfrac{\lambda}{S+1} = \dfrac{3.5}{3+1} < 1,\ \dfrac{f(S+2)}{f(S+1)} = \dfrac{\lambda}{S+2} = \dfrac{3.5}{3+2} < 1, \cdots$

When $\lambda = k$ (say) is an integer. Let $\lambda = k = 4$, then

$$\frac{f(1)}{f(0)} = \frac{\lambda}{1} = \frac{4}{1} > 1,\ \frac{f(2)}{f(1)} = \frac{\lambda}{2} = \frac{4}{2} > 1, \cdots, \frac{f(k-1)}{f(k-2)} = \frac{\lambda}{k-1} = \frac{4}{4-1} > 1$$

and $\dfrac{f(k)}{f(k-1)} = \dfrac{\lambda}{k} = \dfrac{4}{4} = 1,\ \dfrac{f(k+1)}{f(k)} = \dfrac{\lambda}{k+1} = \dfrac{4}{4+1} < 1,\ \dfrac{f(k+2)}{f(k+1)} = \dfrac{\lambda}{k+2} = \dfrac{4}{4+2} < 1 \cdots$

(ii) $\dfrac{f(1)}{f(0)} > 1$ i.e., $f(1) > f(0) \Rightarrow f(0) < f(1)$

Again, $\dfrac{f(2)}{f(1)} > 1$ i.e., $f(2) > f(1) \Rightarrow f(1) < f(2)$

Similarly, $\frac{f(S)}{f(S-1)} > 1$ i.e., $f(S-1) < f(S)$

Combining the above expressions into a single expression, we get

$$f(0) < f(1) < f(2) < \cdots < f(S-2) < f(S-1) < f(S)$$

(iii) যদি কোন পৈঁসু বিন্যাসে $\lambda = 3.5$ হয়, তবে বিন্যাসটির প্রচুরক হবে $S = 3$ । যদি পৈঁসু বিন্যাসে $\lambda = 3$ হয়, তবে বিন্যাসটির দুটি প্রচুরক হবে যথাক্রমে $k-1=2$ এবং $k=3$ ।

(iv) Videlicet শব্দের সংক্ষেপ viz অর্থ হচ্ছে যথা, উদাহরণস্বরূপ।

(v) P-7.45, S.C. Gupta, Fundamental of Mathematical Statistics.

Alternative Method:

Let X is a Poisson variate with parameter λ and M is the mode of the distribution. The mode, M of the distribution is clearly that value of X for which the probability i.e., $\frac{e^{-\lambda}\lambda^{M}}{M!}$ will be greater than the term that precedes it and the term that follows it. That is

$$f(M-1) \le f(M) \ge f(M+1)$$

$$\Rightarrow \frac{e^{-\lambda}\lambda^{M-1}}{(M-1)!} \le \frac{e^{-\lambda}\lambda^{M}}{M!} \ge \frac{e^{-\lambda}\lambda^{M+1}}{(M+1)!}$$

$$\Rightarrow \frac{\lambda^{M}\lambda^{-1}}{(M-1)!} \le \frac{\lambda^{M}}{M(M-1)!} \ge \frac{\lambda^{M}\lambda}{(M+1)M(M-1)!}$$

$$\Rightarrow \frac{\lambda^{-1}}{1} \le \frac{1}{M} \ge \frac{\lambda}{(M+1)M}$$

$$\Rightarrow \frac{M}{\lambda} \le 1 \ge \frac{\lambda}{M+1} \quad \text{(Multiplying by M on both sides)}$$

$$\Rightarrow \frac{M}{\lambda} \le 1 \text{ or, } \quad 1 \ge \frac{\lambda}{M+1}$$

$$\Rightarrow M \le \lambda \text{ or, } \quad M+1 \ge \lambda$$

$$\Rightarrow M \le \lambda \text{ or, } \quad M \ge \lambda - 1$$

$$\Rightarrow M \le \lambda \text{ or, } \quad \lambda - 1 \le M$$

$$\Rightarrow \lambda - 1 \le M \le \lambda$$

If λ is not an integer, then mode is the integral part (পূর্ণসংখ্যা অংশ) of λ. Hence in this case the distribution is unimodal.

If λ is an integer, then $M = \lambda - 1$ and $M = \lambda$. Hence in this case the distribution can be regarded as bimodal.

NB: P- 464, Siddiqur Rahman and **P- 429,** M. K. Roy

Theorem: State and prove additive property of Poisson distribution (NU-11)

Statement: Let $X_1, X_2, \cdots, X_k$ be k independent Poisson variates with parameters $\lambda_1, \lambda_2, \cdots, \lambda_k$. Then $\sum_{i=1}^{k} X_i$ is a Poisson variate with parameter $\sum_{i=1}^{k} \lambda_i$.

Proof: We know the m.g.f of Poisson distribution with parameter λ is

$$M_X(t) = e^{\lambda (e^t - 1)} \text{ ....(i)}$$

Thus, $M_{X_i}(t) = e^{\lambda_i (e^t - 1)};\ i = 1, 2, \cdots, k$

So, $M_{X_1}(t) = e^{\lambda_1 (e^t - 1)},\ M_{X_2}(t) = e^{\lambda_2 (e^t - 1)}, \cdots, M_{X_k}(t) = e^{\lambda_k (e^t - 1)}$

Now, the m.g.f of $\sum_{i=1}^{k} X_i$ is

$$M_{\sum_{i=1}^{k} X_i}(t) = E\left(e^{t \sum_{i=1}^{k} X_i}\right)$$

$$= E\left[e^{t (X_1 + X_2 + \cdots + X_k)}\right]$$

$$= E\left(e^{t X_1} . e^{t X_2} \cdots e^{t X_k}\right)$$

$$= E\left(e^{t X_1}\right) . E\left(e^{t X_2}\right) \cdots E\left(e^{t X_k}\right) \quad (\because X_1, X_2, \cdots, X_k \text{ are independent})$$

$$= M_{X_1}(t) . M_{X_2}(t) \cdots M_{X_k}(t)$$

$$= e^{\lambda_1 (e^t - 1)} . e^{\lambda_2 (e^t - 1)} \cdots e^{\lambda_k (e^t - 1)}$$

$$= e^{(\lambda_1 + \lambda_2 + \cdots + \lambda_k)(e^t - 1)}$$

$$\therefore \quad M_{\sum_{i=1}^{k} X_i}(t) = e^{\sum_{i=1}^{k} \lambda_i (e^t - 1)} \text{ ....(ii)}$$

Now comparing (i) and (ii), we can say that (ii) is also the m.g.f of Poisson distribution with parameter $\sum_{i=1}^{k} \lambda_i$.

Since m.g.f uniquely determines a distribution, therefore $\sum_{i=1}^{k} X_i$ is also a Poisson variate with parameter $\sum_{i=1}^{k} \lambda_i$. This completes the proof of the theorem.

Q. If X is a Poisson variate with parameter m then show that $Z = \frac{X-m}{\sqrt{m}}$ **is a standard normal variate, when** $m \to \infty$ **(NU-15)**

Ans: Let X be a Poisson variate with parameter m. Then the mean and variance of X are m. The standard Poisson variate Z is defined by

$$Z = \frac{X-m}{\sqrt{m}}$$

The moment generating function of Z is

$$M_Z(t) = E\left[e^{tZ}\right]$$

$$= E\left[e^{t\left(\frac{X-m}{\sqrt{m}}\right)}\right]$$

$$= E\left[e^{\frac{tX}{\sqrt{m}} - t\sqrt{m}}\right]$$

$$= e^{-t\sqrt{m}} \; E\left[e^{\frac{tX}{\sqrt{m}}}\right]$$

$$= e^{-t\sqrt{m}} \sum_{X=0}^{\infty} e^{\frac{tX}{\sqrt{m}}} f(X)$$

$$= e^{-t\sqrt{m}} \sum_{X=0}^{\infty} e^{\frac{tX}{\sqrt{m}}} \frac{e^{-m} m^X}{X!} \qquad \left[\because f(X) = \frac{e^{-m} m^X}{X!}\right]$$

$$= e^{-t\sqrt{m}} \; e^{-m} \sum_{X=0}^{\infty} \frac{\left(m\, e^{\frac{t}{\sqrt{m}}}\right)^X}{X!}$$

$$= e^{-m-t\sqrt{m}} \left[1 + \frac{m\, e^{\frac{t}{\sqrt{m}}}}{1!} + \frac{\left(m\, e^{\frac{t}{\sqrt{m}}}\right)^2}{2!} + \cdots\right]$$

$$= e^{-m-t\sqrt{m}} \; e^{m\, e^{\frac{t}{\sqrt{m}}}}$$

$$= e^{-m-t\sqrt{m}+m\,e^{\frac{t}{\sqrt{m}}}}$$

$$= e^{-m-t\sqrt{m}+m\left(1+\frac{\frac{t}{\sqrt{m}}}{1!}+\frac{\left(\frac{t}{\sqrt{m}}\right)^2}{2!}+\frac{\left(\frac{t}{\sqrt{m}}\right)^3}{3!}+\cdots\right)}$$

$$= e^{-m-t\sqrt{m}+m+t\sqrt{m}+\frac{m\frac{t^2}{m}}{2!}+\frac{m\frac{t^3}{m^{\frac{3}{2}}}}{3!}+\cdots}$$

$$= e^{\frac{t^2}{2!}+\frac{\frac{t^3}{\sqrt{m}}}{3!}+\cdots}$$

$$\therefore \quad M_Z(t) = e^{\frac{t^2}{2}+\frac{t^3}{3!\sqrt{m}}+\cdots}$$

Now, $\lim\limits_{m\to\infty} M_Z(t) = e^{\frac{t^2}{2}}$; which is the m.g.f of a standard normal variate.

But uniqueness property of m.g.f, standard Poisson variate tends to a standard normal variate as $m \to \infty$. Hence Poisson distribution tends to normal distribution for large values of the parameter m.

NB: $e^x = 1+\frac{x}{1!}+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots$

P-431, M. K. Roy (New book)

Q. Derive recursion formula of Poisson Distribution.

Ans: The probability function for the Poisson variate X with parameter λ is

$$f(x) = \frac{e^{-\lambda}\lambda^x}{x!};\; x = 0, 1, 2, \cdots, \infty$$

Now, $f(x+1) = \dfrac{e^{-\lambda}\lambda^{x+1}}{(x+1)!}$

$$\therefore \; \frac{f(x+1)}{f(x)} = \frac{\frac{e^{-\lambda}\lambda^{x+1}}{(x+1)!}}{\frac{e^{-\lambda}\lambda^{x}}{x!}}$$

$$= \frac{e^{-\lambda}\ \lambda^{x+1}}{(x+1)!} \times \frac{x!}{e^{-\lambda}\ \lambda^{x}}$$

$$= \frac{\lambda^{x}.\lambda}{(x+1)x!} \times \frac{x!}{\lambda^{x}}$$

$$= \frac{\lambda}{x+1}$$

$$\Rightarrow \frac{f(x+1)}{f(x)} = \frac{\lambda}{x+1}$$

$$\therefore\ f(x+1) = \frac{\lambda}{x+1} f(x)$$

which is the recursion formula of Poisson distribution.

Q. Derive recurrence relation for the moments of the Poisson distribution.

Or, If x is a Poisson variable with parameter λ, then show that $\mu_{r+1} = \lambda\left(r\ \mu_{r-1} + \frac{d\mu_r}{d\lambda}\right)$ **(NU-11)**

Or, Show that for Poisson distribution $\mu_{r+1} = r\, m\, \mu_{r-1} + m\, \frac{\partial \mu_r}{\partial m}$ **(NU-15)**

Proof: The probability function for Poisson variate X, with parameter λ is

$$f(x;\ \lambda) = \frac{e^{-\lambda}\ \lambda^{x}}{x!};\ \ x = 0, 1, 2, \cdots, \infty$$

By definition, r-th central moment of the distribution is,

$$\mu_r = E\left[x - E(x)\right]^r = E\left[x - \lambda\right]^r = \sum_{x=0}^{\infty} (x-\lambda)^r\ f(x)\ \text{...............(i)}$$

$$\therefore\quad \mu_r = \sum_{x=0}^{\infty} (x-\lambda)^r\ \frac{e^{-\lambda}\ \lambda^{x}}{x!}$$

$$= \sum_{x=0}^{\infty} \frac{e^{-\lambda}\ \lambda^{x}}{x!}\ (x-\lambda)^r$$

Now differentiating μ_r with respect to λ, we get,

$$\frac{d\mu_r}{d\lambda} = \sum_{x=0}^{\infty} \frac{e^{-\lambda}\ \lambda^{x}}{x!}.\ r\ (x-\lambda)^{r-1}\ (-1) + \sum_{x=0}^{\infty} \frac{(x-\lambda)^r}{x!} (e^{-\lambda}.\ x\ \lambda^{x-1} - \lambda^{x}\ e^{-\lambda})$$

$$= -r \sum_{x=0}^{\infty} (x-\lambda)^{r-1}\ f(x) + \sum_{x=0}^{\infty} (x-\lambda)^r\ \frac{e^{-\lambda}\ \lambda^{x}}{x!} \left(\frac{x}{\lambda} - 1\right)$$

$$= -\ r\ \mu_{r-1} + \sum_{x=0}^{\infty} (x-\lambda)^r\ f(x).\ \left(\frac{x-\lambda}{\lambda}\right) \quad \text{[according to (i)]}$$

$$= -r\,\mu_{r-1} + \frac{1}{\lambda}\sum_{x=0}^{\infty}(x-\lambda)^{r+1}\, f(x)$$

$$\Rightarrow \frac{d\mu_r}{d\lambda} = -r\,\mu_{r-1} + \frac{1}{\lambda}\,\mu_{r+1} \quad \text{[according to (i)]}$$

$$\Rightarrow -r\,\mu_{r-1} + \frac{1}{\lambda}\mu_{r+1} = \frac{d\mu_r}{d\lambda}$$

$$\Rightarrow \frac{1}{\lambda}\,\mu_{r+1} = r\,\mu_{r-1} + \frac{d\mu_r}{d\lambda}$$

$$\therefore\quad \mu_{r+1} = \lambda\left(r\,\mu_{r-1} + \frac{d\mu_r}{d\lambda}\right) \quad \text{(Proved)}$$

NB: r-th moment can be defined as:

r-th moment about origin: $\mu'_r = \dfrac{\sum x_i^{\,r}}{n}$, $\quad \mu'_r = E\left[x^r\right]$

r-th raw moment about point 'a' : $\mu'_r = \dfrac{\sum (x_i - a)^r}{n}$, $\quad \mu'_r = E\left[x-a\right]^r$

r-th central or corrected moment: $\mu_r = \dfrac{\sum (x_i - \bar{x})^r}{n}$, $\quad \mu_r = E\left[x - E(x)\right]^r$

Q. Mention the properties of Poisson distribution (NU-11)

Ans: The properties of Poisson distribution are as follows-

(i) Poisson distribution is a probability distribution of a discrete random variable.

(ii) The parameter of Poisson distribution is λ.

(iii) The mean and variance of Poisson distribution are same.

(iv) The standard deviation of Poisson distribution is $\sqrt{\lambda}$.

(v) If x is a Poisson variate with probability function $f(x;\lambda)$, then its m.g.f about origin is $M_x(t) = e^{\lambda}\left(e^t - 1\right)$.

(vi) The sum of all probabilities of a Poisson distribution is one.

(vii) The sum of two independent Poisson variates is also a Poisson.

(viii) The coefficient of skewness of Poisson distribution is $\dfrac{1}{\sqrt{\lambda}}$.

(ix) The coefficient of kurtosis of Poisson distribution is $3 + \dfrac{1}{\lambda}$.

Uses of Poisson distribution:

The distribution may be used in the following cases:

(i) To determine the number of suicides in a city per year.

(ii) To determine the number of machines breaking down during any one day.

(iii) In determining the number of telephone calls received at a switchboard per minute.

(iv) To find out the number defective items in quality control statistics.

(v) To count the number of rats per acre of cultivated land.

Short questions and answers:

(1) What is called Poisson variate? (NU-12)

Ans: A discrete random variable X is said to be Poisson variate if its probability function can be defined as

$$f(x, \lambda) = \frac{e^{-\lambda} \lambda^x}{x!};\; x = 0, 1, 2, \cdots, \infty \text{ and } \lambda > 0$$

(2) Give two real examples of Passion variate (NU-12)

Ans: The two real examples of Poisson variate are as follows-

1. Number of printing mistakes at each page of a book.
2. Number of air accidents in some unit of time.

(3) Write down the two uses of Poisson distribution (NU-12)

Ans: The distribution may be used in the following cases:

(i) To determine the number of suicides in a city per year.

(ii) To determine the number of machines breaking down during any one day.

(4) The mean of a Poisson distribution is 3, find its standard deviation (NU-11)

Ans: We know, the mean and variance of Poisson distribution are equal. So, the variance of this distribution is 3. Hence, standard deviation is $\sqrt{3}$.

(5) The mean of a Poisson distribution is 9, find its standard deviation.

Ans: We know, the mean and variance of Poisson distribution are equal. So, the variance of this distribution is 9. Hence, standard deviation $\sqrt{9} = 3$.

(6) The standard deviation of a Poisson distribution is 4, find its mean.

Ans: Here, standard deviation = 4. So, variance = 16. We know, the mean and variance of Poisson distribution are equal. So, the mean of this distribution is 16.

(7) If $P(x)=\dfrac{e^{-3}\,3^x}{x!}$ **, what is the value of mean and variance? (NU-12)**

Ans: $P(x)=\dfrac{e^{-3}\,3^x}{x!}$ is the probability function of Poisson distribution.

Here, the parameter of the distribution is 3. We know in case of Poisson distribution, mean and variance are equal to its parameter. So, mean and variance = 3.

(8) Write down the recursion formula for moment of Poisson distribution.

Ans: The recursion formula for moment of Poisson distribution is

$$\mu_{r+1}=\lambda\, r\,\mu_{r-1}+\lambda\,\frac{d\mu_r}{d\lambda}$$

Mathematical Problems:

(1) If $P(x=3)=\dfrac{1}{3}\,P(x=2)$ **in a Poisson distribution then finds its parameter and the value of** $P(x\geq 2)$ **.**

Solution: Let X is a Poisson variate with parameter λ.

$\therefore$ Probability function, $P(x)=f(x)=\dfrac{e^{-\lambda}\,\lambda^x}{x!};\ x=0,1,2,\cdots,\infty$ ....(i)

Now, $P(x=3)=\dfrac{1}{3}\,P(x=2)$

$$\Rightarrow \frac{e^{-\lambda}\,\lambda^3}{3!}=\frac{1}{3}\cdot\frac{e^{-\lambda}\,\lambda^2}{2!}$$

$$\Rightarrow \frac{\lambda}{6}=\frac{1}{3}\times\frac{1}{2}$$

$$\Rightarrow \frac{\lambda}{6}=\frac{1}{6}$$

$$\Rightarrow \lambda=1$$

$\therefore$ Parameter, $\lambda = 1$

Putting $\lambda = 1$ in (i), we get

$$P(x) = \frac{e^{-1}\ (1)^x}{x!};\ \ x = 0, 1, 2, \cdots, \infty$$

Now, $P(x \geq 2) = 1 - P(x = 0) - P(x = 1)$

$$= 1 - \frac{e^{-1}\ (1)^0}{0!} - \frac{e^{-1}\ (1)^1}{1!}$$

$$= 1 - e^{-1} - e^{-1}$$

$$\therefore\ \ P(x \geq 2) = 1 - 2\ e^{-1} = 1 - 2 \times 0.3679 = 0.2642$$

(2) In a company about 2% of the produced items are defective. Among the 100 produced items find the probability that (i) no item is defective (ii) at least 2 items are defective (iii) at best 1 item is defective.

Solution: Let X indicates the number of defective and to get defective means success.

$\therefore$ Probability of success, $p = 2\% = 0.02$

Total number of items, $n = 100$

Since p is very small compared to n, so the problem is to be solved through Poisson distribution.

Now, the probability function for the Poisson variate x is $f(x) = \dfrac{e^{-\lambda}\ \lambda^x}{x!}$

Here Parameter, $\lambda = n\ p = 100 \times 0.02 = 2$

$$\therefore\ \ f(x) = \frac{e^{-2}\ 2^x}{x!};\ \ x = 0, 1, 2, \cdots, 100$$

(i) The probability that no item is defective,

$$f(x = 0) = \frac{e^{-2}\ 2^0}{0!} = e^{-2} = 0.1353$$

(ii) The probability that at least 2 items are defective,

$$f(x \geq 2) = f(x = 2) + f(x = 3) + \cdots$$

$$= 1 - f(x = 0) - f(x = 1)$$

$$= 1 - 0.1353 - \frac{e^{-2}\ 2^1}{1!} = 1 - 0.1353 - 0.2706 = 0.5941$$

(iii) The probability that at best one item is defective,

$$f(x \leq 1) = f(x = 0) + f(x = 1)$$

$$= 0.1353 + 0.2706 = 0.4059$$

(3) In a factory about $\frac{1}{5}$% of the produced items are defective. If each box contains 200 items, then out of 20,000 boxes find the probability and the expected number of boxes of (i) no item is defective (ii) at least 1 item is defective.

Solution: Let X indicates the number of defective and to get defective means success.

$$\therefore \quad p = \frac{1}{5}\ \% = 0.2\ \% = 0.002$$

$$n = 200, \quad N = 20{,}000$$

Since p is very small, so we may use Poisson distribution.

Now, the probability function for the Poisson variate x is $f(x) = \frac{e^{-\lambda}\ \lambda^{x}}{x!}$

Here, Parameter, $\lambda = n\,p = 200 \times 0.002 = 0.4$

$$\therefore \quad f(x) = \frac{e^{-0.4}(0.4)^{x}}{x!}; \quad x = 0, 1, 2, \cdots, 200$$

(i) The probability that no item is defective,

$$f(x = 0) = \frac{e^{-0.4}\ (0.4)^{0}}{0!} = e^{-0.4} = 0.6703$$

$\therefore$ Expected number of boxes $= N\ f(x = 0) = 20{,}000 \times 0.6703 = 13406$

(ii) The probability that at least 1 item is defective,

$$f(x \geq 1) = 1 - f(x = 0) = 1 - 0.6703 = 0.3297$$

$\therefore$ Expected number of boxes $= N\ f(x \geq 1) = 20{,}000 \times 0.3297 = 6594$

(4) In a company 5 % of the manufactured bulbs are defective. 10 bulbs are supplied in a packet. Among 10,000 packets, find the probability and expected number of packets of (i) 0 bulbs; (ii) at best 2 defective bulbs (NU-11)

Solution: Let X indicates the number of defective bulbs and to get defective bulbs means success.

$$\therefore \quad p = 5\% = 0.05$$

$$n = 10, \quad N = 10{,}000$$

Since p is very small, so we may use Poisson distribution.

Now, the probability function for the Poisson variate x is $f(x)=\dfrac{e^{-\lambda}\ \lambda^{x}}{x!}$

Here, Parameter, $\lambda = n\ p = 10\times 0.05 = 0.5$

$$\therefore\quad f(x)=\frac{e^{-0.5}\ (0.5)^{x}}{x!};\quad x=0,1,2,\cdots,10$$

(i) The probability that '0' bulb is defective,

$$f(x=0)=\frac{e^{-0.5}\ (0.5)^{0}}{0!}=e^{-0.5}=0.6065$$

$\therefore$ Expected number of packets $= f(x=0) = 10,000\times 0.6065 = 6065$

(ii) The probability that at best 2 bulbs are defective,

$$f(x\le 2)=f(x=0)+f(x=1)+f(x=2)$$

$$=0.6065+\frac{e^{-0.5}(0.5)^{1}}{1!}+\frac{e^{-0.5}(0.5)^{2}}{2!}$$

$$=0.6065+0.3033+0.0758=0.9856$$

$\therefore$ Expected number of packets $= N\ f(x\le 2) = 10,000\times 0.9856 = 9856$

(5) The average number of misprints in one page of a book is 3.2. What is the probability that there will be 2 and 4 misprints respectively in the next two pages? (NU-10)

Solution: Let X denotes the number of misprints in one page.

Then the probability function for the Poisson variate x is

$$P(x)=f(x)=\frac{e^{-\lambda}\ \lambda^{x}}{x!};\quad x=0,1,2,\cdots,\infty$$

Here, Parameter, $\lambda = 3.2$

$\therefore$ Probability function, $f(x)=\dfrac{e^{-3.2}\ (3.2)^{x}}{x!};\ x=0,1,2,\cdots,\infty$

The probability that there will be 2 misprints,

$$f(x=2)=\frac{e^{-3.2}(3.2)^{2}}{2!}=\frac{0.0408\times 10.24}{2}=0.2089$$

The probability that there will be 4 misprints,

$$f(x=4)=\frac{e^{-3.2}(3.2)^{4}}{4!}=\frac{0.0408\times 104.86}{24}=0.1783$$

Negative Binomial Distribution

Definition: A discrete random variable X is said to have a negative binomial distribution if its probability function is given by

$$f(x;\, r,\, p) = \binom{x+r-1}{r-1} p^r\, q^x; \qquad x = 0, 1, 2, \cdots, \infty$$

Where $r > 0$ and $0 \le p \le 1$ are the parameters of the distribution such that $p + q = 1$

Q. Derive the probability function of negative binomial distribution stating necessary assumptions (NU-11)

Or, Derive the probability function of negative binomial distribution (NU-14, 16)

Ans: Suppose we have a succession (ধারাবাহিকতা) of Bernoulli trials until to get r-th success. We assume that

1. the trials are independent and
2. the probability of success p in a trial remains same from trial to trial.

Let $f(x;\, r,\, p)$ denote the probability that there are x failures preceding the r-th success in $(x+r)$ trials.

Now the last trial must be a success, whose probability is p. In the remaining $(x+r-1)$ trials, we must have $(r-1)$ successes and x failures whose probability is given by

$$\binom{x+r-1}{r-1} p^{r-1}\, q^x$$

Hence multiplying the two probabilities, we get

$$f(x;\, r,\, p) = \binom{x+r-1}{r-1} p^{r-1}\, q^x .\, p$$

$$= \binom{x+r-1}{r-1} p^r\, q^x; \quad x = 0, 1, 2, \cdots, \infty \quad \ldots\ldots\ldots\ldots\text{(i)}$$

Moreover, $\binom{x+r-1}{r-1} = \dfrac{(x+r-1)!}{(r-1)!\,(x+r-1-r+1)!}$

$$= \frac{(x+r-1)!}{(r-1)!\, x!}$$

$$=\frac{(x+r-1)\,(x+r-2)\cdots(x+r-1-x+2)\,(x+r-1-x+1)\,(x+r-1-x)!}{(r-1)!\;x!}$$

$$=\frac{(x+r-1)\,(x+r-2)\cdots(r+1)\;r\;(r-1)!}{(r-1)!\;x!}$$

$$=\frac{(x+r-1)\,(x+r-2)\cdots(r+1)\;r}{x!}$$

$$=\frac{r\,(r+1)\,(r+2)\cdots(r+x-2)\,(r+x-1)}{x!}$$

$$=\frac{(-1)^x\,(-r)\,(-r-1)\,(-r-2)\cdots(-r-x+2)\,(-r-x+1)}{x!}$$

$$=(-1)^x\binom{-r}{x}$$

So (i) can be written as

$$f(x;\,r,p)=(-1)^x\binom{-r}{x}p^r\;q^x$$

$$=\binom{-r}{x}p^r\,(-q)^x;\;\;x=0,1,2,\cdots,\infty$$

Hence, $f(x;\,r,\,p)=\binom{x+r-1}{r-1}p^r\,q^x;\;\;x=0,1,2,\cdots,\infty$

$$=\binom{-r}{x}p^r\,(-q)^x;\;\;x=0,1,2,\cdots,\infty$$

which is the probability function of negative binomial distribution.

NB:

(i) x indicates the failures preceding the r-th success (x হলো r-তম সফলতা প্রাপ্তির পূর্বে বিফলতার সংখ্যা).

(ii) Suppose we have a succession (ধারাবাহিকতা) of Bernoulli trials until to get second success. Let in 2 trials, we get $r=2$ successes successively where the last trial must be a success. That is after getting second success, we will stop our work. Here number of failures, $x=0$.

The series is : SS

Probability: p p

$$= \underbrace{\binom{0+2-1}{2-1} p^{2-1}\, q^{0}}_{\text{1st (0+2-1) trial probability}} . \underbrace{p}_{\text{last trial probability}}$$

i.e., we have to multiply the two probabilities.

$$= \underbrace{\binom{x+r-1}{r-1} p^{r-1}\, q^{x}}_{\text{1st (x+r-1) trial probability}} . \underbrace{p}_{\text{last trial probability}}$$

$$= \binom{x+r-1}{r-1} p^{r}\, q^{x}$$

(iii) Suppose we have a succession (ধারাবাহিকতা) of Bernoulli trials until to get 2nd success.

Let in 3 trials, we get $r = 2$ successes and $x = 1$ failure where the last trial must be a success.

The series may be: S F S, F S S

Probability: p q p, q p p

= p q p, p q p

By additive law of probability,

$$p\ q\ p + p\ q\ p$$

$$= 2\ p\ q\ p$$

$$= \underbrace{\binom{1+2-1}{2-1} p^{2-1}\, q^{1}}_{\text{1st (1+2-1) trials probability}} . \underbrace{p}_{\text{last trial probability}}$$

i.e., we have to multiply the two probabilities.

$$= \underbrace{\binom{x+r-1}{r-1} p^{r-1}\, q^{x}}_{\text{1st (x+r-1) trials probability}} . \underbrace{p}_{\text{last trial probability}}$$

$$= \binom{x+r-1}{r-1} p^{r}\, q^{x}$$

(iv) Suppose we have a succession (ধারাবাহিকতা) of Bernoulli trials until to get 2nd success.

Let in 4 trials, we get $r = 2$ successes and $x = 2$ failure where the last trial must be a success.

The series may be: S F F S, F S F S, F F S S

Probability: p q q p, q p q p, q q p p

$= p q^2 p,\quad p q^2 p,\quad p q^2 p$

By additive law of probability,

$$p\,q^2\,p + p\,q^2\,p + p\,q^2\,p$$

$$= 3\,p\,q^2\,p$$

$$= \underbrace{\binom{2+2-1}{2-1} p^{2-1}\;q^2}_{\text{1st (2+2-1) trials probability}} . \underbrace{p}_{\text{last trial probability}}$$

i.e., we have to multiply the two probabilities.

$$= \underbrace{\binom{x+r-1}{r-1} p^{r-1}\;q^x}_{\text{1st (x+r-1) trials probability}} . \underbrace{p}_{\text{last trial probability}}$$

$$= \binom{x+r-1}{r-1} p^r\;q^x$$

Q. Show that $\sum_{x=0}^{\infty} f(x;\,r,\,p) = 1$

Or, Show that the total probability of negative binomial distribution is one (NU-15)

Or, Prove that the sum of all probabilities of a negative binomial distribution is one.

Ans: We know the probability function of negative binomial distribution with parameters r and p is

$$f(x;\;r,\,p) = \binom{-r}{x} p^r (-q)^x;\; x = 0, 1, 2, \cdots, \infty$$

Now, $\sum_{x=0}^{\infty} f(x;\;r,\,p) = \sum_{x=0}^{\infty} \binom{-r}{x} p^r (-q)^x$

$$= p^r \sum_{x=0}^{\infty} \binom{-r}{x} (-q)^x$$

$$= p^r \sum_{x=0}^{\infty} \binom{-r}{x} (-q)^x\, 1^{-r-x}$$

$$= p^r \left[1 + (-q)\right]^{-r} \qquad \left[\because \sum_{x=0}^{n} \binom{n}{x} p^x\; q^{n-x} = (q+p)^n\right]$$

$$= p^r \left[1 - q\right]^{-r}$$

$$= p^r .\; p^{-r} \qquad \left[\because\; p + q = 1 \Rightarrow p = 1 - q\right]$$

$$= p^r . p^{-r}$$

$$= p^0 = 1$$

$$\therefore \sum_{x=0}^{\infty} f(x;\ r,\ p) = 1$$

Q. Find the moment generating function for negative binomial distribution and also find mean and variance of this distribution (NU-12)

Or, Find the moment generating function for negative binomial distribution and hence or otherwise find β_1 and β_2 **(NU-14)**

Or, Find the moment generating function of negative binomial distribution and hence find the mean and variance of the distribution **(NU-15)**

Or, Find the moment generating function for negative binomial distribution. Also find mean, variance, β_1 and β_2.

Ans: We know the probability function of negative binomial distribution with parameters r and p is

$$f(x) = \binom{-r}{x} p^r (-q)^x; \quad x = 0, 1, 2, \cdots, \infty$$

By definition of moment generating function (m.g.f), we get

$$M_x(t) = E(e^{tx}) = \sum_{x=0}^{\infty} e^{tx} f(x)$$

$$= \sum_{x=0}^{\infty} e^{tx} . \binom{-r}{x} p^r (-q)^x$$

$$= p^r \sum_{x=0}^{\infty} \binom{-r}{x} (-q\, e^t)^x$$

$$= p^r (1 - q\, e^t)^{-r} \qquad \left[\because \sum_{x=0}^{n} \binom{n}{x} p^x q^{n-x} = (q+p)^n\right]$$

$$= \frac{p^r}{(1 - q\, e^t)^r}$$

$$\therefore \quad M_x(t) = \left(\frac{p}{1-q\,e^t} \right)^r$$

Now, the cumulant generating function is

$$K_x(t) = \log M_x(t)$$

$$= \log \left(\frac{p}{1-q\ e^t} \right)^r$$

$$= r\ \log \left(\frac{p}{1-q\ e^t} \right)$$

$$= r\ \log \left(\frac{1-q\ e^t}{p} \right)^{-1}$$

$$= -r\ \ \log \left(\frac{p+q-q\,e^t}{p} \right) \qquad [\because\ p+q=1]$$

$$= -r\ \ \log \left(1 + \frac{q}{p} - \frac{q}{p}\ e^t \right)$$

$$= -r\ \log \left(1 - \frac{q}{p}\ e^t + \frac{q}{p} \right)$$

$$= -r\ \ \log \left[1 - \frac{q}{p} \left(e^t - 1 \right) \right]$$

$$= -r\ \ \log \left[1 - \frac{q}{p} \left(1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \frac{t^4}{4!} + \cdots - 1 \right) \right]$$

$$= -r\ \ \log \left[1 - \frac{q}{p} \left(t + \frac{t^2}{2!} + \frac{t^3}{3!} + \frac{t^4}{4!} + \cdots \right) \right]$$

$$= -r \left[-\frac{q}{p} \left(t + \frac{t^2}{2!} + \frac{t^3}{3!} + \frac{t^4}{4!} + \cdots \right) - \frac{q^2}{2p^2} \left(t + \frac{t^2}{2!} + \frac{t^3}{3!} + \cdots \right)^2 - \frac{q^3}{3p^3} \left(t + \frac{t^2}{2!} + \cdots \right)^3 - \frac{q^4}{4p^4} \left(t + \frac{t^2}{2!} + \cdots \right)^4 - \cdots \right]$$

$$= r \left[\frac{q}{p} \left(t + \frac{t^2}{2!} + \frac{t^3}{3!} + \frac{t^4}{4!} + \cdots \right) + \frac{q^2}{2p^2} \left(t^2 + \frac{t^4}{4} + 2.t.\frac{t^2}{2} + 2.t.\frac{t^3}{6} + \cdots \right) + \frac{q^3}{3p^3} \left(t^3 + 3.t^2.\frac{t^2}{2} + \cdots \right) + \frac{q^4}{4p^4} \left(t^4 + \cdots \right) + \cdots \right]$$

$$=r\left[\frac{q}{p}t+\left(\frac{q}{p}\frac{t^2}{2!}+\frac{q^2}{2p^2}t^2\right)+\left(\frac{q}{p}\frac{t^3}{3!}+\frac{q^2}{2p^2}t^3+\frac{q^3}{3p^3}t^3\right)+\left(\frac{q}{p}\frac{t^4}{4!}+\frac{q^2}{2p^2}\frac{t^4}{4}+\frac{q^2}{2p^2}\frac{t^4}{3}+\frac{q^3}{p^3}\frac{t^4}{2}+\frac{q^4}{4p^4}t^4\right)+\cdots\right]$$

$$=r\left[\frac{q}{p}t+\left(\frac{q}{p}+\frac{q^2}{p^2}\right)\frac{t^2}{2!}+\left(\frac{q}{p}+\frac{3q^2}{p^2}+\frac{2q^3}{p^3}\right)\frac{t^3}{3!}+\left(\frac{q}{p}+\frac{3q^2}{p^2}+\frac{4q^2}{p^2}+\frac{12q^3}{p^3}+\frac{6q^4}{p^4}\right)\frac{t^4}{4!}+\cdots\right]$$

Now, $K_r=\text{coefficient of } \dfrac{t^r}{r!} \text{ in } K_x(t)$

Putting $r=1, 2, 3, 4$ we get respectively

$$K_1=\text{coefficient of } \frac{t}{1!} \text{ in } K_x(t)=\frac{rq}{p}$$

$$K_2=\text{coefficient of } \frac{t^2}{2!} \text{ in } K_x(t)=r\left(\frac{q}{p}+\frac{q^2}{p^2}\right)=r\,\frac{q}{p}\left(1+\frac{q}{p}\right)=\frac{rq}{p}.\frac{p+q}{p}=\frac{rq}{p^2}\quad(\because\ p+q=1)$$

$$K_3=\text{coefficient of } \frac{t^3}{3!} \text{ in } K_x(t)$$

$$=r\left(\frac{q}{p}+\frac{3q^2}{p^2}+\frac{2q^3}{p^3}\right)$$

$$=r\,\frac{q}{p}\left(1+\frac{3q}{p}+\frac{2q^2}{p^2}\right)$$

$$=\frac{rq}{p}.\ \frac{p^2+3pq+2q^2}{p^2}$$

$$=\frac{rq}{p^3}\left(p^2+2pq+q^2+pq+q^2\right)$$

$$=\frac{rq}{p^3}\left[(p+q)^2+q(p+q)\right]$$

$$=\frac{rq}{p^3}(1+q)\quad(\because\ p+q=1)$$

$$K_4=\text{coefficient of } \frac{t^4}{4!} \text{ in } K_x(t)$$

$$=r\left(\frac{q}{p}+\frac{7q^2}{p^2}+\frac{12q^3}{p^3}+\frac{6q^4}{p^4}\right)$$

$$=r\,\frac{q}{p}\left(1+\frac{7q}{p}+\frac{12q^2}{p^2}+\frac{6q^3}{p^3}\right)$$

$$=\frac{rq}{p}.\ \frac{p^3+7p^2q+12pq^2+6q^3}{p^3}$$

$$=\frac{rq}{p^4}\left(p^3+3p^2q+3pq^2+q^3+4p^2q+4pq^2+5pq^2+5q^3\right)$$

$$=\frac{rq}{p^4}\left[(p+q)^3+4pq\,(p+q)+5q^2(p+q)\right]$$

$$=\frac{rq}{p^4}\left[1+4pq+5q^2\right]$$

$$=\frac{rq}{p^4}\left[1+4pq+4q^2+q^2\right]$$

$$=\frac{rq}{p^4}\left[1+4q\,(p+q)+q^2\right]$$

$$=\frac{rq}{p^4}\left[1+4q+q^2\right]\quad(\because\ p+q=1)$$

Here, Mean, $K_1=\frac{rq}{p}$

Variance, $\mu_2=K_2=\frac{rq}{p^2}$

$$\mu_3=K_3=\frac{rq\,(1+q)}{p^3}$$

and $\mu_4=K_4+3\,K_2{}^2=\frac{rq\left(1+4q+q^2\right)}{p^4}+\frac{3\,r^2q^2}{p^4}$

Now, $\beta_1=\frac{\mu_3{}^2}{\mu_2{}^3}=\frac{r^2q^2}{p^6}\,(1+q)^2\times\frac{p^6}{r^3q^3}=\frac{(1+q)^2}{rq}$

and $\beta_2=\frac{\mu_4}{\mu_2{}^2}=\left[\frac{rq\left(1+4q+q^2\right)}{p^4}+\frac{3\,r^2q^2}{p^4}\right]\times\frac{p^4}{r^2q^2}$

$$=3+\frac{1+4q+q^2}{rq}$$

The negative binomial distribution is positively skewed and leptokurtic.

Remarks: We know, $\sum_{x=0}^{n}\binom{n}{x}\ p^x\ q^{n-x}=(q+p)^n$

Similarly, $\sum_{x=0}^{\infty}\binom{-r}{x}\left(-qe^t\right)^x=\sum_{x=0}^{\infty}\binom{-r}{x}\left(-qe^t\right)^x\ 1^{-r-x}=\left\{1+\left(-qe^t\right)\right\}^{-r}=\left(1-qe^t\right)^{-r}$

NB:

1. $e^t = 1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \frac{t^4}{4!} + \cdots$

2. $\log(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \cdots$

Q. Find the characteristic function of negative binomial distribution.

Ans: Let X is a negative binomial variate with parameters r and p, then

$$f(x;\ r, p) = \binom{-r}{x}\ p^r(-q)^x; \qquad x = 0, 1, 2, \cdots, \infty$$

By the definition of characteristic function, we can write

$$\begin{aligned}
\phi_x(t) &= E\left(e^{itx}\right) \\
&= \sum_{x=0}^{\infty} e^{itx}\ f(x; r, p) \\
&= \sum_{x=0}^{\infty} e^{itx} \binom{-r}{x}\ p^r(-q)^x \\
&= p^r \sum_{x=0}^{\infty} \binom{-r}{x} \left(-qe^{it}\right)^x \\
&= p^r\ \left(1 - q\ e^{it}\right)^{-r} \\
&= \frac{p^r}{\left(1 - qe^{it}\right)^r}
\end{aligned}$$

$$\therefore \quad \phi_x(t) = \left(\frac{p}{1 - qe^{it}}\right)^r$$

which is the characteristic function of negative binomial distribution.

Theorem: State and prove the additive property of negative binomial distribution (NU-15)

Statement: Let x_1 and x_2 be two independent negative binomial variates with parameters (r_1, p) and (r_2, p), then $x = x_1 + x_2$ is also negative binomial variate with parameters $(r_1 + r_2, p)$.

Proof: The moment generating function (m.g.f) of negative binomial variate x_i is,

$$M_{x_i}(t) = \left(\frac{p}{1-q\ e^t}\right)^{r_i}$$

So, the m.g.f of x_1 is

$$M_{x_1}(t) = \left(\frac{p}{1-q\ e^t}\right)^{r_1}$$

and the m.g.f of x_2 is

$$M_{x_2}(t) = \left(\frac{p}{1-q\ e^t}\right)^{r_2}$$

Now, the m.g.f of $x = x_1 + x_2$ is

$$\begin{aligned}
M_x(t) &= M_{x_1+x_2}(t) \\
&= E\left\{e^{t(x_1+x_2)}\right\} \\
&= E\left\{e^{tx_1+tx_2}\right\} \\
&= E\left\{e^{tx_1}.e^{tx_2}\right\} \\
&= E\left(e^{tx_1}\right)E\left(e^{tx_2}\right) \\
&= M_{x_1}(t).\ M_{x_2}(t) \\
&= \left(\frac{p}{1-qe^t}\right)^{r_1} \cdot \left(\frac{p}{1-qe^t}\right)^{r_2} \\
&= \left(\frac{p}{1-qe^t}\right)^{r_1+r_2}
\end{aligned}$$

which is also the mgf of a negative binomial variate with parameters $(r_1 + r_2)$ and p. Since m.g.f uniquely determines a distribution, so $x = x_1 + x_2$ is also a negative binomial variate.

Theorem: Show that under certain conditions negative binomial distribution tends to Poisson distribution (NU-13)

Or, Prove that negative binomial distribution tends to Poisson distribution.

Proof: Let X be a negative binomial variate with probability function

$$f(x;\ r, p) = \binom{x+r-1}{r-1}\ p^r\ q^x; \qquad x = 0, 1, 2, \cdots, \infty$$

Then negative binomial distribution tends to Poisson distribution under the following conditions:

i) r-th number of success tends to infinity.

ii) q, the probability of failure tends to zero and

iii) r q, tends to a finite constant λ (say)

$$\text{i.e.,}\quad rq = \lambda\,,\quad q = \frac{\lambda}{r}\,,\quad p = 1 - \frac{\lambda}{r}$$

We have, $f(x;\ r, p) = \binom{x+r-1}{r-1} p^r\ q^x; \qquad x = 0, 1, 2, \cdots, \infty$

$$= \frac{(x+r-1)!}{(r-1)!\ x!}\ p^r\ q^x$$

$$= \frac{(x+r-1)\ (x+r-2)\cdots(r+1)\ r\ (r-1)!}{(r-1)!\ x!}\ p^r\ q^x$$

$$= \frac{(r+x-1)\ (r+x-2)\cdots(r+1)\ r}{x!}\ \left(1-\frac{\lambda}{r}\right)^r \left(\frac{\lambda}{r}\right)^x$$

$$= \frac{\lambda^x}{x!} \cdot \frac{r^x \left(1+\frac{x-1}{r}\right)\left(1+\frac{x-2}{r}\right)\cdots\left(1+\frac{1}{r}\right).1}{r^x}\ \left(1-\frac{\lambda}{r}\right)^r$$

$$\therefore\ \lim_{r\to\infty} f(x;\ r, p) = \frac{\lambda^x}{x!}\ \lim_{r\to\infty}\left\{\left(1+\frac{x-1}{r}\right)\ \left(1+\frac{x-2}{r}\right)\cdots\left(1+\frac{1}{r}\right).1\right\}.\ \lim_{r\to\infty}\left(1-\frac{\lambda}{r}\right)^r$$

$$= \frac{\lambda^x}{x!}\ (1.1\cdots1.1).\ \lim_{r\to\infty}\left(1-\frac{\lambda}{r}\right)^r \quad \left(\because\ \lim_{r\to\infty}\frac{1}{r} = 0\right)$$

$$\Rightarrow \lim_{r\to\infty} f(x;\ r, p) = \frac{\lambda^x}{x!}\ \lim_{r\to\infty}\left(1-\frac{\lambda}{r}\right)^r \ \ldots\ldots\ldots\text{(i)}$$

Now, Suppose $y = \left(1-\frac{\lambda}{r}\right)^r$

$$\Rightarrow \log y = r\ \log\left(1-\frac{\lambda}{r}\right)$$

$$= r\left(-\frac{\lambda}{r}-\frac{\lambda^2}{2r^2}-\frac{\lambda^3}{3r^3}-\cdots\right)$$

$$= -\lambda-\frac{\lambda^2}{2r}-\frac{\lambda^3}{3r^2}\cdots$$

$$\Rightarrow \lim_{r\to\infty}\log y = \lim_{r\to\infty}\left(-\lambda-\frac{\lambda^2}{2r}-\frac{\lambda^3}{3r^2}\cdots\right)$$

$$= -\lambda-0-0\cdots \qquad \left(\because \lim_{r\to\infty}\frac{1}{r}=0\right)$$

$$\Rightarrow \lim_{r\to\infty} \log\left(1-\frac{\lambda}{r}\right)^r = -\lambda$$

$$\Rightarrow \lim_{r\to\infty}\left(1-\frac{\lambda}{r}\right)^r = e^{-\lambda} \ldots\ldots\ldots\ldots\text{(ii)}$$

Putting the value from (ii) in (i), we get

$$f(x;\ r,p)=\frac{\lambda^x}{x!}.\ e^{-\lambda}=\frac{e^{-\lambda}\lambda^x}{x!};\qquad x=0,1,2,\cdots,\infty$$

which is the probability function of Poisson distribution with parameter λ. Hence the theorem is proved.

Q. Derive recursion formula or, recurrence relation of Negative Binomial Distribution.

Ans: Let X be a negative binomial variate with parameters r and p, then

$$f(x;\ r,\ p)=\binom{x+r-1}{r-1}p^r\ q^x;\qquad x=0,1,2,\cdots,\infty$$

$$=\frac{(x+r-1)!}{(r-1)!\,x!}\ p^r\ q^x \ldots\ldots\ldots\ldots\text{(i)}$$

Then, $f(x+1;\ r,\ p)=\dfrac{(x+1+r-1)!}{(r-1)!\,(x+1)!}\ p^r\ q^{x+1}$ [Using (i)]

$$=\frac{(x+r)!}{(r-1)!\,(x+1)!}\ p^r\ q^{x+1}$$

$$= \frac{(x+r)\,(x+r-1)!}{(r-1)!\,(x+1)\;x!}\;p^r\;q^x\;q$$

Now, $\dfrac{f(x+1;\ r,\ p)}{f(x;\ r,\ p)} = \dfrac{(x+r)\,(x+r-1)!\,p^r\;q^x\;q}{(r-1)!\,(x+1)\;x!} \times \dfrac{(r-1)!\;x!}{(x+r-1)!\;p^r\;q^x}$

$$= \frac{x+r}{x+1}\,q$$

$$\therefore \quad f(x+1;\ r,\ p) = \frac{x+r}{x+1}\,q \times f(x;\ r,\ p)$$

which is the recurrence relation of negative binomial distribution.

Q. Derive the recursion formula of negative binomial distribution for moments (NU-13, 16)

Proof: Let x be a negative binomial variate with parameter k and p, then

$$f(x) = \binom{x+k-1}{k-1}\,p^k\;q^x; \qquad x = 0, 1, 2, \cdots, \infty$$

Here, k = no. of success and p = probability of success.

We know Mean, $E(x) = \dfrac{k\,q}{p}$

Now, r-th central moment,

$$\mu_r = E\left[x - E(x)\right]^r = E\left[x - \frac{kq}{p}\right]^r$$

$$\Rightarrow \mu_r = \sum_{x=0}^{\infty}\left(x - \frac{kq}{p}\right)^r f(x) \text{ ....(i)}$$

$$\Rightarrow \mu_r = \sum_{x=0}^{\infty}\left(x - \frac{kq}{p}\right)^r \binom{x+k-1}{k-1}\,p^k\;q^x$$

$$\Rightarrow \mu_r = \sum_{x=0}^{\infty}\binom{x+k-1}{k-1}(1-q)^k\;q^x\left(x - \frac{q}{1-q}\,k\right)^r \quad [\because p = 1-q]$$

Now differentiating μ_r with respect of q, we get,

$$\frac{d\mu_r}{dq} = \sum_{x=0}^{\infty}\binom{x+k-1}{k-1}\left[(1-q)^k\,q^x.r\left(x - \frac{q}{1-q}k\right)^{r-1}.-k\frac{1-q-q\,(-1)}{(1-q)^2} + \left(x - \frac{q}{1-q}k\right)^r\left\{(1-q)^k\,xq^{x-1} + q^x k(1-q)^{k-1}(-1)\right\}\right]$$

$$=\sum_{x=0}^{\infty}\binom{x+k-1}{k-1}\left[p^k\, q^x\, r\left(x-\frac{kq}{p}\right)^{r-1}.-\frac{k}{p^2}+\left(x-\frac{kq}{p}\right)^{r}\left\{p^k\, x\, q^{x-1}-q^x\, k\, p^{k-1}\right\}\right]$$

$$=-\frac{rk}{p^2}\sum_{x=0}^{\infty}\binom{x+k-1}{k-1}\left(x-\frac{kq}{p}\right)^{r-1}p^k\, q^x+\sum_{x=0}^{\infty}\binom{x+k-1}{k-1}\left(x-\frac{kq}{p}\right)^{r}p^k\, q^x\left\{x\, q^{-1}-k\, p^{-1}\right\}$$

$$=-\frac{rk}{p^2}\sum_{x=0}^{\infty}\left(x-\frac{kq}{p}\right)^{r-1}f(x)+\sum_{x=0}^{\infty}\binom{x+k-1}{k-1}\left(x-\frac{kq}{p}\right)^{r}p^k\, q^x\left\{\frac{x}{q}-\frac{k}{p}\right\}$$

$$=-\frac{rk}{p^2}\,\mu_{r-1}+\sum_{x=0}^{\infty}\binom{x+k-1}{k-1}\left(x-\frac{kq}{p}\right)^{r}p^k\, q^x\,\frac{1}{q}\left\{x-\frac{kq}{p}\right\}\quad\text{[according to (i)]}$$

$$=-\frac{rk}{p^2}\,\mu_{r-1}+\frac{1}{q}\sum_{x=0}^{\infty}\left(x-\frac{kq}{p}\right)^{r+1}\binom{x+k-1}{k-1}p^k\, q^x$$

$$=-\frac{rk}{p^2}\,\mu_{r-1}+\frac{1}{q}\sum_{x=0}^{\infty}\left(x-\frac{kq}{p}\right)^{r+1}f(x)$$

$$\Rightarrow\frac{d\mu_r}{dq}=-\frac{rk}{p^2}\,\mu_{r-1}+\frac{1}{q}\,\mu_{r+1}$$

$$\Rightarrow\frac{1}{q}\,\mu_{r+1}=\frac{d\mu_r}{dq}+\frac{rk}{p^2}\,\mu_{r-1}$$

$$\therefore\quad \mu_{r+1}=q\left[\frac{d\mu_r}{dq}+\frac{rk}{p^2}\,\mu_{r-1}\right]$$

which is the recursion formula of negative binomial distribution for moments.

NB: (i) $\dfrac{d}{dx}(u.v)=u\,\dfrac{dv}{dx}+v\,\dfrac{du}{dx}$

(ii) $\dfrac{d}{dx}\left(\dfrac{u}{v}\right)=\dfrac{v\,\dfrac{du}{dx}-u\,\dfrac{dv}{dx}}{v^2}$

(iii) এখানে সফলতার সংখ্যা r এর পরিবর্তে k ধরা হয়েছে। কেননা r-তম মোমেন্টের r এবং সফলতার সংখ্যা r গুণ হয়ে $\mu_{r+1}=q\left[\dfrac{d\mu_r}{dq}+\dfrac{rk}{p^2}\,\mu_{r-1}\right]$ এ r k এর স্থলে r^2 হয়ে যাবে।

(iv) $\dfrac{d}{dx}\{f(x)g(x)h(x)\}=f(x)g(x)\dfrac{d}{dx}\{h(x)\}+g(x)h(x)\dfrac{d}{dx}\{f(x)\}+h(x)f(x)\dfrac{d}{dx}\{g(x)\}$

Q. Establish the relationship between negative binomial distribution and geometric distribution (NU-13)

Ans: We know the probability function of negative binomial distribution is

$$f(x;\ r, p) = \binom{x+r-1}{r-1}\ p^r\ q^x; \qquad x = 0,\ 1,\ 2, \cdots, \infty \ \ ....(i)$$

which denotes the probability of x failures preceding the r-th success.

If we put $r = 1$ in (i), we get

$$f(x;\ p) = \binom{x+1-1}{1-1}\ p\ q^x = \binom{x}{0}\ p\ q^x = p\ q^x; \quad x = 0, 1, 2, \cdots, \infty$$

which is the probability function of geometric distribution. It denotes the probability of x failures preceding the 1st success.

So, if $r = 1$, then negative binomial distribution reduces to geometric distribution.

Q. Write down the properties of negative binomial distribution (NU-12)

Or, Write down the characteristics of negative binomial distribution.

Ans: Some important properties or, characteristics of negative binomial distribution are as follows-

(a) If $r = 1$, negative binomial distribution reduces to geometric distribution with parameter p.

(b) If $r \to \infty$, $q \to 0$ and $rq \to \lambda$, then negative binomial distribution turns into Poisson distribution.

(c) The variance of this distribution is always greater than mean.

(d) The distribution is positively skewed and leptokurtic.

(e) Under certain conditions, negative binomial distribution follows additive property. That is if $x_1, x_2, \cdots, x_k$ are k independent negative binomial variates with parameters $(r_1, p), (r_2, p), \cdots, (r_k, p)$ then $x = x_1 + x_2 + \cdots + x_k$ is also a negative binomial variate with parameters $\left(r = \sum_{i=1}^{k} r_i,\ p\right)$.

Q. Discuss the uses of negative binomial distribution (NU-11)

Or, Write down the application of negative binomial distribution (NU-13)

Ans: The uses/ applications of negative binomial distribution are as follows:

(i) The negative binomial has extensive use in population count, health, family planning, accident statistics, medical science and communication and in biological experiments.

(ii) The negative binomial distribution has its application also in inverse sampling.

Q. Compare binomial distribution with negative binomial distribution.

Ans: The most important comparisons of binomial distribution with negative binomial distribution are as follows-

(a) Both the distributions are constructed on the basis of Bernoulli trials with the constant probability of success ‘p’ from trial to trial.

(b) Binomial distribution is the general term of the binomial expansion $(p+q)^n$, whereas negative binomial distribution is the general term of the binomial expansion with negative index i.e., $p^r(1-q)^{-r}$.

(c) Mean is always greater than variance for binomial distribution, whereas variance is always greater than mean for negative binomial distribution.

(d) The number of trials ‘n’ in binomial distribution is finite and predetermined, whereas the number of trials in negative binomial distribution depends on the number of success ‘r’ which is fixed and it may infinite.

Short questions and answers:

(1) Define negative binomial distribution (NU-11)

Ans: A discrete random variable x is said to have a negative binomial distribution if its probability function is given by

$$f(x;\ r,\ p)=\binom{x+r-1}{r-1}\ p^r\ q^x;\qquad x=0,1,2,\cdots,\infty$$

Where $r>0$ and $0\le p\le 1$ are the parameters of the distribution such that $p+q=1$

(2) Write down the probability function of negative binomial distribution (NU-12)

Ans: The probability function of negative binomial distribution with parameter r and p is

$$f(x;\ r,\ p) = \binom{x+r-1}{r-1} p^r\ q^x; \qquad x = 0, 1, 2, \cdots, \infty$$

(3) What is the mean and variance of negative binomial distribution? (NU-11, 13)

Ans: The mean and variance of negative binomial distribution are as follows:

$$\text{Mean } = \frac{r\,q}{p}$$

$$\text{and Variance, } \mu_2 = \frac{r\,q}{p^2}$$

Here, r = no. of success, p = probability of success and q = probability of failure.

(4) Under what conditions negative binomial distribution tends to Poisson distribution? (NU-11, 13)

Ans: If $r \to \infty$, $q \to 0$ and $r\,q \to \lambda$ then negative binomial distribution tends to Poisson distribution.

(5) Write down the moment generating function of negative binomial distribution (NU-14)

Ans: The moment generating function of negative binomial distribution is

$$M_x(t) = \left(\frac{p}{1-q\,e^t}\right)^r$$

(6) Under what condition negative binomial distribution tends to geometric distribution (NU-12, 14)

Ans: We know the probability function of negative binomial distribution with parameter r and p is

$$f(x;\ r,\ p) = \binom{x+r-1}{r-1} p^r\ q^x; \qquad x = 0, 1, 2, \cdots, \infty \quad \ldots\ldots\ldots..(i)$$

If $r = 1$ then (i) becomes

$$f(x;\ p) = p\ q^x; \qquad x = 0, 1, 2, \cdots, \infty$$

which is the probability function of geometric distribution with parameter p.

Hence, if $r = 1$ then negative binomial distribution tends to geometric distribution. Here r indicates the number of successes.

Mathematical Problems:

(1) The probability that T&T shall install a red telephone set in a residence is 0.3. What is the probability that the 10th telephone set installed in the area is the 5th red one.

Solution: Probability of red telephone set, $p = 0.3$.

So the probability of non-red telephone set, $q = 1 - p = 1 - 0.3 = 0.7$.

Let x be the number of non-red telephone set installed to get 5th red set. 10 telephone set installation is required to get 5th red one.

Here, $r = 5$ and $x + r = 10 \Rightarrow x + 5 = 10 \Rightarrow x = 5$.

Therefore, x is a negative binomial variate with parameters $r = 5$ and $p = 0.3$.

We know, $f(x;\ r, p) = \binom{x + r - 1}{r - 1} p^r\, q^x$

$$\therefore\quad f(x = 5;\ r = 5,\ p = 0.3) = \binom{5 + 5 - 1}{5 - 1} (0.3)^5\, (0.7)^5$$

$$= \binom{9}{4} (0.3)^5\, (0.7)^5$$

$$= 126 \times 0.00243 \times 0.16807$$

$$= 0.0515$$

(2) An item is produced in large numbers. The machine is known to produce 5% defectives. A quality control inspector is examining the items by taking them at random. What is the probability that at least 4 items are to be examined in order to get 2 defectives? (P-419, S.C. Gupta)

Solution: Probability of defective items, $p = 5\% = 0.05$.

$$\therefore\quad q = 1 - p = 1 - 0.05 = 0.95$$

Since two defectives are to be obtained then it can happen in 2 or more trials. It is a negative binomial distribution and probability function is

$$f(x;\ r, p) = \binom{x + r - 1}{r - 1} p^r\, q^x$$

Here, $r = 2$ and

$$x + r \geq 4$$

$$\Rightarrow x + 2 \geq 4$$

$$\therefore$$

Now the probability that at least 4 items are to be examined in order to get 2 defective is

Geometric Distribution

Definition: A discrete random variable X is said to have a geometric distribution if its probability function is defined as

$$f(x; p) = p\, q^x; \qquad x = 0, 1, 2, \cdots, \infty$$

Where p is the only parameter such that $0 \le p \le 1$ and $p + q = 1$.

Q. Derive probability function of geometric distribution (NU-11, 13, 16)

Ans: Suppose we have a series of independent Bernoulli trials where the probability of success is p and probability of failure is q such that $p + q = 1$. In each trial success and failure are denoted by S and F respectively. If $(x+1)$ trials are required to get the first success, then the succession (ধারাবাহিকতা, অনুক্রম) will be as follows:

$$\underbrace{F\ F \cdots F}_{x\ \text{times}}.\ \underbrace{S}_{1\ \text{time}}\ ; \quad x = 0, 1, 2, \cdots, \infty$$

Since the trials are independent, so according to the multiplicative law of probability, the probability of the above series is

$$P\left(\underbrace{F\ F \cdots F}_{x\ \text{times}}\ \underbrace{S}_{1\ \text{time}}\right) = \underbrace{P(F)\ P(F) \cdots P(F)}_{x\ \text{times}}\ \underbrace{P(S)}_{1\ \text{time}}$$

$$= \underbrace{q\ q \ldots q}_{x\ \text{times}}\ \underbrace{p}_{1\ \text{time}}$$

$$= q^x\ p$$

Hence the probability of first success in $(x+1)$ trials is

$$f(x; p) = q^x\, p$$

$$= p\, q^x; \quad x = 0, 1, 2, \cdots, \infty$$

which is the probability function of geometric distribution.

Actually $p q^x$ is the probability that there are x failures preceding the first success.

NB: The geometric distribution is often referred to as the failures time distribution.

Q. Show that in case of geometric distribution, total probability is unity.

Or, Show that $\sum_{x=0}^{\infty} f(x;\, p) = 1$

Proof: We know the probability function of geometric distribution with parameter p is

$$f(x;\, p) = p\, q^x;\;\; x = 0, 1, 2, \cdots, \infty$$

Now, $\sum_{x=0}^{\infty} f(x;\; p) = \sum_{x=0}^{\infty} p\, q^x$

$$= p \sum_{x=0}^{\infty} q^x$$

$$= p\left(1 + q + q^2 + \cdots\right)$$

$$= p\,(1-q)^{-1}$$

$$= p\, p^{-1} \qquad (\because\; p + q = 1 \Rightarrow p = 1 - q)$$

$$= p \times \frac{1}{p}$$

$$\therefore\; \sum_{x=0}^{\infty} f(x;\; p) = 1$$

NB: $(1-x)^{-1} = 1 + x + x^2 + \cdots$

Q. Find the moment generating function of geometric distribution and hence find mean, variance, β_1 and β_2.

Or, Find the mean and variance of geometric distribution (NU-11, 13, 16)

Or, Find the moment generating function of geometric distribution and hence or otherwise find β_1 and β_2 and comment (NU-15)

Ans: We know the probability function of geometric distribution with parameter p is

$$f(x) = p\, q^x;\quad x = 0, 1, 2, \cdots, \infty$$

By the definition of moment generating function (m.g.f), we have

$$M_x(t) = E\left(e^{tx}\right) = \sum_{x=0}^{\infty} e^{tx}\, f(x)$$

$$= \sum_{x=0}^{\infty} e^{tx} \; p q^{x}$$

$$= p \sum_{x=0}^{\infty} \left(q e^{t}\right)^{x}$$

$$= p \left\{1 + q\, e^{t} + \left(q\, e^{t}\right)^{2} + \cdots\right\}$$

$$= p \left(1 - q\, e^{t}\right)^{-1}$$

$$= \frac{p}{1 - q\, e^{t}}$$

which is the m.g.f of geometric distribution.

Now the cumulant generating function is

$$K_x(t) = \log M_x(t)$$

$$= \log\left(\frac{p}{1 - q\, e^{t}}\right)$$

$$= \log\left(\frac{1 - q\, e^{t}}{p}\right)^{-1}$$

$$= -\log\left(\frac{1}{p} - \frac{q}{p}\, e^{t}\right)$$

$$= -\log\left[\frac{1}{p} - \frac{q}{p}\left(1 + \frac{t}{1!} + \frac{t^{2}}{2!} + \frac{t^{3}}{3!} + \frac{t^{4}}{4!} + \cdots\right)\right]$$

$$= -\log\left[\frac{1}{p} - \frac{q}{p} - \frac{q}{p}\left(t + \frac{t^{2}}{2!} + \frac{t^{3}}{3!} + \frac{t^{4}}{4!} + \cdots\right)\right]$$

$$= -\log\left[\frac{1-q}{p} - \frac{q}{p}\left(t + \frac{t^{2}}{2!} + \frac{t^{3}}{3!} + \frac{t^{4}}{4!} + \cdots\right)\right]$$

$$= -\log\left[1 - \frac{q}{p}\left(t + \frac{t^{2}}{2!} + \frac{t^{3}}{3!} + \frac{t^{4}}{4!} + \cdots\right)\right] \qquad [\because\ p + q = 1 \Rightarrow p = 1 - q]$$

$$= -\left[-\frac{q}{p}\left(t + \frac{t^{2}}{2!} + \frac{t^{3}}{3!} + \frac{t^{4}}{4!} + \cdots\right) - \frac{q^{2}}{2p^{2}}\left(t + \frac{t^{2}}{2!} + \frac{t^{3}}{3!} + \cdots\right)^{2} - \frac{q^{3}}{3p^{3}}\left(t + \frac{t^{2}}{2!} + \cdots\right)^{3} - \frac{q^{4}}{4p^{4}}\left(t + \frac{t^{2}}{2!} + \cdots\right)^{4} - \cdots\right]$$

$$=\frac{q}{p}\left(t+\frac{t^2}{2!}+\frac{t^3}{3!}+\frac{t^4}{4!}+\cdots\right)+\frac{q^2}{2p^2}\left(t^2+\frac{t^4}{4}+2.t.\frac{t^2}{2}+2.t.\frac{t^3}{6}+\cdots\right)+\frac{q^3}{3p^3}\left(t^3+3.t^2.\frac{t^2}{2}+\cdots\right)+\frac{q^4}{4p^4}\left(t^4+\cdots\right)+\cdots$$

$$=\frac{q}{p}t+\left(\frac{q}{p}\frac{t^2}{2!}+\frac{q^2}{2p^2}t^2\right)+\left(\frac{q}{p}\frac{t^3}{3!}+\frac{q^2}{2p^2}t^3+\frac{q^3}{3p^3}t^3\right)+\left(\frac{q}{p}\frac{t^4}{4!}+\frac{q^2}{2p^2}\frac{t^4}{4}+\frac{q^2}{p^2}\frac{t^4}{6}+\frac{q^3}{p^3}\frac{t^4}{2}+\frac{q^4}{4p^4}t^4\right)+\cdots$$

$$=\frac{q}{p}t+\left(\frac{q}{p}+\frac{q^2}{p^2}\right)\frac{t^2}{2!}+\left(\frac{q}{p}+\frac{3q^2}{p^2}+\frac{2q^3}{p^3}\right)\frac{t^3}{3!}+\left(\frac{q}{p}+\frac{3q^2}{p^2}+\frac{4q^2}{p^2}+\frac{12q^3}{p^3}+\frac{6q^4}{p^4}\right)\frac{t^4}{4!}+\cdots$$

Now, $K_r =$ coefficient of $\frac{t^r}{r!}$ in $K_x(t)$

Putting $r=1, 2, 3, 4$ we get respectively-

$K_1 =$ coefficient of $\frac{t}{1!}$ in $K_x(t)=\frac{q}{p}$

$K_2 =$ coefficient of $\frac{t^2}{2!}$ in $K_x(t)=\frac{q}{p}+\frac{q^2}{p^2}=\frac{q}{p}\left(1+\frac{q}{p}\right)=\frac{q}{p}.\frac{p+q}{p}=\frac{q}{p^2}\quad (\because\ p+q=1)$

$K_3 =$ coefficient of $\frac{t^3}{3!}$ in $K_x(t)$

$$=\frac{q}{p}+\frac{3q^2}{p^2}+\frac{2q^3}{p^3}$$

$$=\frac{q}{p}\left(1+\frac{3q}{p}+\frac{2q^2}{p^2}\right)$$

$$=\frac{q}{p}.\ \frac{p^2+3pq+2q^2}{p^2}$$

$$=\frac{q}{p^3}\left(p^2+2pq+q^2+pq+q^2\right)$$

$$=\frac{q}{p^3}\left[(p+q)^2+q(p+q)\right]$$

$$=\frac{q}{p^3}(1+q)\quad [\because\ p+q=1]$$

$K_4 =$ coefficient of $\frac{t^4}{4!}$ in $K_x(t)$

$$=\frac{q}{p}+\frac{7q^2}{p^2}+\frac{12q^3}{p^3}+\frac{6q^4}{p^4}$$

$$=\frac{q}{p}\left(1+\frac{7\,q}{p}+\frac{12\,q^2}{p^2}+\frac{6\,q^3}{p^3}\right)$$

$$=\frac{q}{p}\cdot\ \frac{p^3+7\,p^2\,q+12\,pq^2+6q^3}{p^3}$$

$$=\frac{q}{p^4}\left(p^3+3\,p^2\,q+3\,pq^2+q^3+4\,p^2\,q+4\,pq^2+5\,pq^2+5\,q^3\right)$$

$$=\frac{q}{p^4}\left[\left(p+q\right)^3+4\,pq\left(p+q\right)+5\,q^2\left(p+q\right)\right]$$

$$=\frac{q}{p^4}\left[1+4\,pq+5\,q^2\right]$$

$$=\frac{q}{p^4}\left[1+4\,pq+4\,q^2+q^2\right]$$

$$=\frac{q}{p^4}\left[1+4\,q\left(p+q\right)+q^2\right]$$

$$=\frac{q}{p^4}\left[1+4\,q+q^2\right]\quad\left(\because\ p+q=1\right)$$

Here, Mean, $K_1=\dfrac{q}{p}$

Variance, $\mu_2=K_2=\dfrac{q}{p^2}$

$$\mu_3=K_3=\frac{q\ \left(1+q\right)}{p^3}$$

and $\mu_4=K_4+3\,{K_2}^2=\dfrac{q\left(1+4\,q+q^2\right)}{p^4}+\dfrac{3\,q^2}{p^4}$

Now, $\beta_1=\dfrac{{\mu_3}^2}{{\mu_2}^3}=\dfrac{q^2\left(1+q\right)^2}{p^6}\times\dfrac{p^6}{q^3}=\dfrac{\left(1+q\right)^2}{q}$

and $\beta_2=\dfrac{\mu_4}{{\mu_2}^2}=\left[\dfrac{q\left(1+4\,q+q^2\right)}{p^4}+\dfrac{3\,q^2}{p^4}\right]\times\ \dfrac{p^4}{q^2}$

$$=3+\frac{1+4\,q+q^2}{q}$$

The value of β_1 is always positive and $\beta_2>3$. So, the geometric distribution is positively skewed and leptokurtic.

NB:

1. $e^t = 1 + \frac{t}{1!} + \frac{t^2}{2!} + \frac{t^3}{3!} + \frac{t^4}{4!} + \cdots$
2. $\log(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \cdots$
3. $(1-x)^{-1} = 1 + x + x^2 + \cdots$

Theorem: State and prove the additive property of geometric distribution (NU-12, 14, 16)
Or, Show that sum of independent geometric variate is a negative binomial variate.

Statement: Let $x_1, x_2, \cdots, x_r$ be r independent geometric variate with parameter p, then $x_1 + x_2 + \cdots + x_r$ is a negative binomial variate.

Proof: The moment generating function (m.g.f) of geometric variate x_i is,

$$M_{x_i}(t) = p\left(1 - q\, e^t\right)^{-1}$$

$$\therefore \; M_{x_1}(t) = p\left(1 - q\, e^t\right)^{-1}$$

$$M_{x_2}(t) = p\left(1 - q\, e^t\right)^{-1}$$

....

$$M_{x_r}(t) = p\left(1 - q\, e^t\right)^{-1}$$

Now, the m.g.f of $x_1 + x_2 + \cdots + x_r$ is

$$M_{x_1+x_2+\cdots+x_r}(t) = E\left[e^{t\,(x_1+x_2+\cdots+x_r)}\right]$$

$$= E\left[e^{t\,x_1 + t\,x_2 + \cdots + t\,x_r}\right]$$

$$= E\left[e^{t\,x_1}\; e^{t\,x_2} \cdots e^{t\,x_r}\right]$$

$$= E\left[e^{t\,x_1}\right]\; E\left[e^{t\,x_2}\right] \cdots E\left[e^{t\,x_r}\right]$$

$$= M_{x_1}(t).\; M_{x_2}(t) \cdots M_{x_r}(t)$$

$$= p\left(1 - q\, e^t\right)^{-1}.\; p\left(1 - q\, e^t\right)^{-1} \cdots p\left(1 - q\, e^t\right)^{-1}$$

$$=\left\{p\ \left(1-q\ e^{t}\right)^{-1}\right\}^{r}$$

$$=p^{r}\ \left(1-q\ e^{t}\right)^{-r}$$

which is the m.g.f of negative binomial distribution with parameter r and p.

Hence the sum of independent geometric variate is a negative binomial variate.

NB: The m.g.f of negative binomial distribution with parameter r and p is

$$M_x(t)=\left(\frac{p}{1-q\ e^{t}}\right)^{r}=p^{r}\ \left(1-q\ e^{t}\right)^{-r}$$

Theorem: Let the two independent random variables X_1 and X_2 have the same geometric distribution. Show that the conditional distribution of $X_1|\left(X_1+X_2=n\right)$ is uniform.

Proof: Since we are given that the two independent random variables X_1 and X_2 have the same geometric distribution, so we can write

$$P\left(X_1=k\right)=P\left(X_2=k\right)=p\ q^{k};\quad k=0,1,2,\cdots,\infty$$

We know that sum of independent geometric variates is a negative binomial variate. Hence X_1+X_2 is a negative binomial variate with parameter $r=2$ and P.

$$\text{So, } P\left(X_1+X_2=n\right)=\binom{n+r-1}{r-1}\ p^{r}\ q^{n}$$

$$=\binom{n+2-1}{2-1}\ p^{2}\ q^{n}\quad\left[\because\ r=2\right]$$

$$=\binom{n+1}{1}\ p^{2}\ q^{n}$$

$$\Rightarrow P\left(X_1+X_2=n\right)=\left(n+1\right)\ p^{2}\ q^{n}$$

$$\text{Now, } P\left(X_1\ \middle|\ X_1+X_2=n\right)=\frac{P\left[X_1=k\cap X_1+X_2=n\right]}{P\left[X_1+X_2=n\right]};\quad k=0,1,2,\cdots,n$$

$$=\frac{P\left[X_1=k\cap\ X_2=n-k\right]}{\left(n+1\right)\ p^{2}\ q^{n}}$$

$$= \frac{P[X_1 = k].\ P[X_2 = n-k]}{(n+1)\, p^2\, q^n} \quad [\because X_1 \text{ and } X_2 \text{ are independent}]$$

$$= \frac{p\, q^k.\ p\, q^{n-k}}{(n+1)\, p^2\, q^n}$$

$$= \frac{p^2\, q^n}{(n+1)\, p^2\, q^n}$$

$$\therefore \quad P\left(X_1 \mid X_1 + X_2 = n\right) = \frac{1}{n+1}; \qquad k = 0, 1, 2, \cdots, n$$

which is the probability function of discrete uniform distribution.

Hence the conditional distribution of $X_1 \mid (X_1 + X_2 = n)$ is discrete uniform.

NB: Geometric distribution means there is only one success. If two independent random variables X_1 and X_2 distinctly follow geometric distribution, then $X_1 + X_2$ indicates two successes. One success for X_1 and another success for X_2.

Alternative method: P-455 (new book), M. K. Roy

Proof: We are given,

$$P(X_1 = k) = P(X_2 = k) = p\, q^k; \quad k = 0, 1, 2, \cdots, \infty;$$

By definition, the conditional distribution of $X_1 = r$ for given value of $X_1 + X_2 = n$ is

$$P(X_1 = r \mid X_1 + X_2 = n) = \frac{P[X_1 = r \cap X_1 + X_2 = n]}{P[X_1 + X_2 = n]}$$

$$= \frac{P[X_1 = r \cap X_2 = n - r]}{\sum_{s=0}^{n} P[X_1 = s \cap X_2 = n - s]}$$

$$= \frac{P[X_1 = r].\ P[X_2 = n - r]}{\sum_{s=0}^{n} P[X_1 = s].\ P[X_2 = n - s]} \quad [\because X_1 \text{ and } X_2 \text{ are independent}]$$

$$= \frac{p\, q^r.\ p\, q^{n-r}}{\sum_{s=0}^{n} p\, q^s.\ p\, q^{n-s}}$$

$$= \frac{p^2\ q^n}{\sum_{s=0}^{n} p^2\ q^n}$$

$$= \frac{p^2\ q^n}{(n+1)\ p^2\ q^n}$$

$$\therefore\ \ P\left(X_1 = r \mid X_1 + X_2 = n\right) = \frac{1}{n+1};\quad r = 0, 1, 2, \cdots, n$$

Hence the conditional distribution of X_1 for given value of $X_1 + X_2 = n$ is discrete uniform distribution.

Theorem: Let X and Y be independent random variable and $P[X = r] = P[Y = r] = p\ q^r$**;** $r = 0, 1, 2, \cdots, \infty,\quad p + q = 1$**.**

(i) Find the distribution of $X + Y$**.**

(ii) The conditional distribution of X given $X + Y = 3$

Solution: (i) We are given

$$P[X = r] = P[Y = r] = p\ q^r;\quad r = 0, 1, 2, \cdots, \infty,\quad p + q = 1.$$

i.e., X and Y are independent geometric variates with parameter p.

We know that sum of independent geometric variate is a negative binomial variate. Hence $X + Y$ is a negative binomial variate with parameter $r = 2$ and p.

Now, $P(X + Y = z) = \binom{z + r - 1}{r - 1}\ p^r\ q^z$

$$= \binom{z + 2 - 1}{2 - 1}\ p^2\ q^z\quad [\because\ r = 2]$$

$$= \binom{z + 1}{1}\ p^2\ q^z$$

$$= (z + 1)\ p^2\ q^z;\quad z = 0, 1, 2, \cdots, \infty$$

ii) The conditional distribution of X given $X + Y = 3$

i.e., $$P\left(X=r \mid X+Y=3\right)=\frac{P\left[X=r \cap X+Y=3\right]}{P\left[X+Y=3\right]}$$

$$=\frac{P\left[X=r \cap Y=3-r\right]}{P\left[X+Y=3\right]}$$

$$=\frac{P\left[X=r\right].\ P\left[Y=3-r\right]}{P\left[X+Y=3\right]}; \quad \left[\because \text{ X and Y are independent}\right]$$

$$=\frac{p\ q^{r}.\ p\ q^{3-r}}{(3+1)\ p^{2}\ q^{3}}$$

$$=\frac{p^{2}\ q^{3}}{4\ p^{2}\ q^{3}}$$

$$=\frac{1}{4}; \quad r=0, 1, 2, 3$$

Theorem: If X and Y be two independent random variables, each representing the number of failures preceding the 1st success in a sequence of Bernoulli trials with p as probability of success in a single trial and q as probability of failure. Show that $P\left(X=Y\right)=\frac{p}{1+q}$ (Similar as P: 377, Probability distribution theory, K C Bhuyan).

Solution: It is given that X and Y are independent geometric variate with parameter p.

$$\therefore \quad P(X=r)=P(Y=r)=p\ q^{r}; \quad r=0, 1, 2, \cdots, \infty$$

Now, $$P(X=Y)=P\left(X=0 \cap Y=0\right)+P\left(X=1 \cap Y=1\right)+P\left(X=2 \cap Y=2\right)+\cdots$$

$$=\sum_{r=0}^{\infty} P\left(X=r \cap Y=r\right)$$

$$=\sum_{r=0}^{\infty} P\left(X=r\right).\ P\left(Y=r\right); \quad \left[\because \text{ X and Y are independent}\right]$$

$$=\sum_{r=0}^{\infty} p\ q^{r}.\ p\ q^{r}$$

$$=p^{2} \sum_{r=0}^{\infty} q^{2r}$$

$$=p^{2}\left(1+q^{2}+q^{4}+\cdots\right)$$

$$=p^{2}\left(1-q^{2}\right)^{-1} \quad \left[\because (1-x)^{-1}=1+x+x^{2}+\cdots\right]$$

$$= \frac{p^2}{1-q^2}$$

$$= \frac{p^2}{(1+q)\,(1-q)}$$

$$= \frac{p^2}{(1+q)\,p} \qquad [\because\ p+q=1 \Rightarrow p=1-q]$$

$$\therefore \quad P(X=Y) = \frac{p}{1+q}$$

Hence the theorem is proved.

Q. State and prove the 'lack of memory' property of geometric distribution (NU-14)
(Hints: P- 8.55, S. C. Gupta)

Statement: The additional time to wait has the same distribution as initial time to wait. In this sense geometric distribution is known as the lack of memory.

Proof: Suppose an event 'E' can occur at one of the times (বার) $t = 0, 1, 2, \cdots$ and the occurrence (waiting) time X has a geometric distribution. Thus

$$P(X=t) = q^t\ p; \quad t = 0, 1, 2, \cdots$$
$$= p\ q^t$$

Suppose we know that the event 'E' has not occurred before k, i.e., $X \geq k$. Let $Y = X - K$, is the amount of additional time needed for 'E' to occur.

We can write,

$$P\left(Y=t \mid X \geq k\right) = P\left(Y \geq t \mid X \geq k\right) - P\left(Y \geq t+1 \mid X \geq k\right) \ldots\ldots\ldots\ldots.(i)$$

Now, $P\left(X \geq r\right) = \sum_{x=r}^{\infty} p\ q^x$

$$= p\ \left(q^r + q^{r+1} + q^{r+2} + \cdots\right)$$

$$= p\ q^r\ \left(1+q+q^2+\cdots\right)$$

$$= p\ q^r\ (1-q)^{-1}$$

$$= p\ q^r.\ p^{-1} \quad (\because\ p+q=1 \Rightarrow 1-q=p)$$

$$\therefore \quad P\left(X \geq r\right) = q^r \ldots\ldots\ldots\ldots(ii)$$

Again, $P(Y \geq t \mid X \geq k) = \dfrac{P(Y \geq t \cap X \geq k)}{P(X \geq k)}$

$$= \frac{P(X - k \geq t \cap X \geq k)}{P(X \geq k)} \qquad [\because Y = X - k]$$

$$= \frac{P(X \geq k + t \cap X \geq k)}{P(X \geq k)}$$

$$= \frac{P(X \geq k + t)}{P(X \geq k)}$$

$$= \frac{q^{K+t}}{q^{K}} \quad \text{[Using (ii)]}$$

$\therefore \quad P(Y \geq t \mid X \geq k) = q^{t}$

Similarly, $P(Y \geq t + 1 \mid X \geq k) = q^{t+1}$

Now, (i) can be written as

$$P(Y = t \mid X \geq k) = q^{t} - q^{t+1}$$

$$= q^{t} - q^{t}\, q = q^{t}\,(1 - q) = q^{t}\, p$$

So, $P(Y = t \mid X \geq k) = p\, q^{t}$

which implies that the additional time to wait has the same distribution as initial time to wait. Since the distribution does not depend upon k, it, in a sense, 'lacks memory' of how much we shifted the time origin.

NB: (i) Suppose we know that the event 'E' has not occurred before k, i.e., $X \geq k$. Let $Y = X - k$, is the additional amount of time needed for 'E' to occur.

Now, $X \geq k$ i.e., $X = k, k+1, k+2, \cdots$

So additional time, $Y = t = X - k$

When, $X = k$ then, $Y = t = k - k = 0$

$X = k + 1$ then, $Y = t = k + 1 - k = 1$

$X = k + 2$ then, $Y = t = k + 2 - k = 2$

Hence additional time, $Y = t = 0, 1, 2, \cdots$

(ii) $P(x \geq 2) = P(x = 2) + P(x = 3) + P(x = 4) + \cdots$

$$P(x \geq 3) = P(x = 3) + P(x = 4) + P(x = 5) + \cdots$$

So, $P(x \geq 2) - P(x \geq 3) = P(x = 2)$

Similarly, $P(x \geq t) - P(x \geq t+1) = P(x = t)$

Theorem: Let X be a geometric variate with probability function defined as

$$f(x;\ p) = p\ q^x; \qquad x = 0, 1, 2, \cdots, \infty$$

If $Y = X - K$**, where K is an integer, then**

$$P\left[Y = t \mid X \geq K\right] = P\left[X = t\right]$$

Proof: It is given that

$$f(x;\ p) = p\ q^x;\ \ x = 0, 1, 2, \cdots, \infty$$

We can write

$$P(Y = t \mid X \geq K) = P(Y \geq t \mid X \geq K) - P(Y \geq t+1 \mid X \geq K) \text{............(i)}$$

Now, $P(X \geq r) = \sum_{x=r}^{\infty} p\ q^x$

$$= p\ \left(q^r + q^{r+1} + q^{r+2} + \cdots\right)$$

$$= p\ q^r\ \left(1 + q + q^2 + \cdots\right)$$

$$= p\ q^r\ (1-q)^{-1} \qquad \left[\because\ (1-x)^{-1} = 1 + x + x^2 + \cdots\right]$$

$$= p\ q^r.\ p^{-1} \qquad \left[\because\ p + q = 1 \Rightarrow p = 1 - q\right]$$

$$\therefore\ P(X \geq r) = q^r \text{.............(ii)}$$

and $P(Y \geq t \mid X \geq K) = \dfrac{P\left[Y \geq t\ \cap\ X \geq K\right]}{P\left[X \geq K\right]}$

$$= \frac{P\left[X - K \geq t\ \cap\ X \geq K\right]}{P\left[X \geq K\right]} \qquad \left[\because\ Y = X - K\right]$$

$$= \frac{P\left[X \geq K + t\ \cap\ X \geq K\right]}{P\left[X \geq K\right]}$$

$$= \frac{P\left[X \geq K + t\right]}{P\left[X \geq K\right]}$$

$$= \frac{q^{K+t}}{q^k} \quad \text{[Using (ii)]}$$

$$\therefore \quad P\left(Y \ge t \mid X \ge K\right) = q^t$$

Similarly, $P\left(Y \ge t+1 \mid X \ge K\right) = q^{t+1}$

Now, (i) can be written as

$$P\left(Y = t \mid X \ge K\right) = q^t - q^{t+1} = q^t \left(1-q\right) = q^t \ p = p \ q^t$$

$$\therefore \quad P\left[Y = t \mid X \ge K\right] = p \ q^t = P\left[X = t\right] \quad \textbf{(Proved)}$$

Short questions and answers:

(i) Define geometric distribution (NU-11)

Ans: A discrete random variable x is said to have a geometric distribution if its probability function is defined as

$$f\left(x;\ p\right) = p\ q^x; \qquad x = 0, 1, 2, \cdots, \infty$$

where p is the only parameter such that $0 \le p \le 1$ and $p + q = 1$.

(ii) What is the "lack of memory" property of geometric distribution? (NU-15)

Ans: The lack of memory property of geometric distribution means that the additional time to wait has the same distribution as initial time to wait.

(iii) Write down the applications of geometric distribution.

Ans: The applications of geometric distributions are as follows-

(i) The geometric distribution is used in Markov chain models, particularly meteorological (আবহাওয়া) modes of weather cycles and precipitation (বৃষ্টিপাতের পরিমাণ).

(ii) The geometric distribution may be used to describe the number of interviews that have to be conducted by a selection board to appoint the first acceptable candidate.

Mathematical Problems:

(1) A couple decides that they will take children until a son. If the probability of son in that community (সম্প্রদায়) is $\frac{1}{3}$. P-456, M. K. Roy

a) What is the probability that the fourth child is a son?

b) Also find the expected number of children to get a son.

Solution: Here the probability of getting a son in that community, $P = \frac{1}{3}$.

So, the probability of getting a female child in that community, $q = 1 - p = 1 - \frac{1}{3} = \frac{2}{3}$

Now, let x be a random variable which is defined by the number of female children required to get a son i.e., x follows geometric distribution. So, the probability function of x is

$$f(x; p) = p\, q^x$$

$$\therefore\ f\left(x; \frac{1}{3}\right) = \frac{1}{3}\left(\frac{2}{3}\right)^x$$

a) Probability that the fourth child will be a son

$$P(x = 3) = f\left(3; \frac{1}{3}\right) = \frac{1}{3}\left(\frac{2}{3}\right)^3 = \frac{1}{3}.\frac{8}{27} = \frac{8}{81}$$

b) The expected number of female children to get a first son is the mean of the distribution is

$$\frac{q}{p} = \frac{2}{3} \times \frac{3}{1} = 2$$

Hence, the number of children will be $(2+1) = 3$ to get a son.

(2) A die is cast until 6 appear. What is the probability that it must be cast more than 5 times.

Solution: Let X denote a variable which represent the failure preceding 6 appear (i.e., 1^{st} success) in a die casting experiment.

Here X follows geometric distribution with parameter, $p = \frac{1}{6}$

$$\therefore\ q = 1 - p = 1 - \frac{1}{6} = \frac{5}{6}$$

Now, $P(X = x) = p\, q^x$

$$= \frac{1}{6}\left(\frac{5}{6}\right)^{x}; \quad x = 0, 1, 2, \cdots, \infty$$

So, the probability that 6 appear after casting the die more than 5 times i.e.,

$$P(x > 5) = P(x = 6) + P(x = 7) + \cdots$$

$$= 1 - P(x \le 5)$$

$$= 1 - \sum_{x=0}^{5} \frac{1}{6}\left(\frac{5}{6}\right)^{x}$$

Hypergeometric (পরাজ্যামিতিক) Distribution

If a sample is drawn without replacement from a finite population which contains two types of things, we get hypergeometric distribution.

Q. Define hyper-geometric distribution (NU-13, 15)

Ans. A discrete random variable X is said to have a hypergeometric distribution if its probability function is defined as

$$f(x;\ N, M, n) = \frac{\binom{M}{x}\binom{N-M}{n-x}}{\binom{N}{n}}; \qquad x = 0, 1, 2, \cdots, n$$

Where N is a positive integer, M is a non-negative integer that is at most N and n is a positive integer that is at most M.

Q. Derive the probability function of hypergeometric distribution (NU-12, 14, 16)

Ans: Suppose an urn (পাত্র) contains N balls among which M balls are white and $(N-M)$ balls are black. A sample of n balls are drawn at random from the urn without replacement. Let x be the number of white balls in n balls, then x can take values $0, 1, 2, \cdots, n.$

Now n balls can be drawn from N balls in $\binom{N}{n}$ ways and x white balls and $(n-x)$ black balls can be drawn from M white and $(N-M)$ black balls in $\binom{M}{x}\binom{N-M}{n-x}$ ways.

Then the probabilities of x white balls among the selected n balls is

$$f(x;\ N, M, n) = \frac{\binom{M}{x}\binom{N-M}{n-x}}{\binom{N}{n}}; \quad x = 0, 1, 2, \cdots, n$$

which is the probability function of the hypergeometric distribution with parameters N, M and n.

NB: Suppose total number of balls, $N = 5$ in which

White balls, $M = 3$

$\therefore$ Black balls, $N - M = 5 - 3 = 2$

A sample of $n = 3$ balls are drawn at random from $N = 5$ balls without replacement.

Let $x = 2$ be the number of white balls in $n = 3$ balls.

$\therefore$ $n - x = 3 - 2 = 1$ be the number of black balls in $n = 3$ balls.

Now, $n = 3$ balls can be drawn from $N = 5$ balls in $\binom{N}{n} = \binom{5}{3} = 10$ ways.

$x = 2$ white balls can be drawn from $M = 3$ white balls in $\binom{M}{x} = \binom{3}{2} = 3$ ways.

i.e., $W_1, W_2, W_3 \to W_1\ W_2,\quad W_1\ W_3,\quad W_2\ W_3$.

Similarly, $n - x = 1$ black ball can be drawn from $N - M = 2$ black balls in $\binom{N-M}{n-x} = \binom{2}{1} = 2$ ways.

i.e., $B_1, B_2 \to B_1,\ B_2$.

Now, $x = 2$ white balls and $n - x = 1$ black ball can be drawn together from $M = 3$ white and $N - M = 2$ black balls in $\binom{M}{x}\binom{N-M}{n-x} = 3 \times 2 = 6$ ways.

	W_1W_2	W_1W_3	W_2W_3
B_1	$B_1W_1W_2$	$B_1W_1W_3$	$B_1W_2W_3$
B_2	$B_2W_1W_2$	$B_2W_1W_3$	$B_2W_2W_3$

Then the probabilities of $x = 2$ white balls among the selected $n = 3$ balls is

$$f(x = 2;\ N = 5, M = 3, n = 3) = \frac{\binom{M}{x}\binom{N-M}{n-x}}{\binom{N}{n}} = \frac{\binom{3}{2}\binom{5-3}{3-2}}{\binom{5}{3}} = \frac{\binom{3}{2}\binom{2}{1}}{\binom{5}{3}} = \frac{3 \times 2}{10} = \frac{3}{5};\quad x = 0, 1, 2$$

In general, we can write

$$f(x;\ N, M, n) = \frac{\binom{M}{x}\binom{N-M}{n-x}}{\binom{N}{n}};\quad x = 0, 1, 2, \cdots, n$$

Again, $\binom{N}{n} - \binom{M}{x}\binom{N-M}{n-x} = \binom{5}{3} - \binom{3}{2}\binom{2}{1} = 10 - 3\times 2 = 4$ ways

The above 4 ways can be occurred in the following way:

$$W_1W_2W_3,\ B_1B_2W_1,\ B_1B_2W_2,\ B_1B_2W_3$$

Q. Show that $\sum_{x=0}^{n} f(x;\ N, M, n) = 1$

Or, Show that in case of hypergeometric distribution, total probability is unity.

Proof: We know the probability function of hypergeometric distribution with parameters N, M and n is

$$f(x;\ N, M, n) = \frac{\binom{M}{x}\binom{N-M}{n-x}}{\binom{N}{n}}; \quad x = 0, 1, 2, \cdots, n$$

Now, $$\sum_{x=0}^{n} f(x;\ N, M, n) = \sum_{x=0}^{n} \frac{\binom{M}{x}\binom{N-M}{n-x}}{\binom{N}{n}} \text{..............(i)}$$

Using the result, $\sum_{i=0}^{n}\binom{a}{i}\binom{b}{n-i} = \binom{a+b}{n}$, the equation (i) can be written as

$$\sum_{x=0}^{n} f(x;\ N, M, n) = \frac{\binom{N}{n}}{\binom{N}{n}} = 1 \quad \text{(Showed)}$$

NB: $$\sum_{i=0}^{2}\binom{3}{i}\binom{4}{2-i} = \binom{3+4}{2}$$

$$\Rightarrow \binom{3}{0}\binom{4}{2} + \binom{3}{1}\binom{4}{1} + \binom{3}{2}\binom{4}{0} = \binom{7}{2}$$

$$\Rightarrow (1\times 6) + (3\times 4) + (3\times 1) = \frac{7!}{2!\ 5!}$$

$$\Rightarrow 6+12+3=\frac{7\times 6}{2}$$

$$\therefore\ 21=21$$

So, $$\sum_{i=0}^{n}\binom{a}{i}\binom{b}{n-i}=\binom{a+b}{n}$$

Similarly, $$\sum_{x=1}^{n}\binom{M-1}{x-1}\binom{N-M}{n-x}=\binom{N-1}{n-1}$$

and $$\sum_{x=2}^{n}\binom{M-2}{x-2}\binom{N-M}{n-x}=\binom{N-2}{n-2}$$

Theorem: If x is a hypergeometric variate with parameters N, M and n then

$E(x)=np$**, where** $p=\dfrac{M}{N}$

and $V(x)=npq\ \dfrac{N-n}{N-1}$**, when** $p+q=1$

Or, Find mean and variance of hypergeometric distribution (NU-14, 15)

Proof: We know the probability function of hypergeometric distribution with parameters N, M and n is

$$f\left(x;\ N, M, n\right)=\frac{\binom{M}{x}\binom{N-M}{n-x}}{\binom{N}{n}};\qquad x=0, 1, 2,\cdots,n$$

By definition,

$$E(x)=\sum_{x=0}^{n}x.\ f\left(x;\ N, M, n\right)$$

$$=\sum_{x=0}^{n}x.\ \frac{\binom{M}{x}\binom{N-M}{n-x}}{\binom{N}{n}}$$

$$=\sum_{x=1}^{n}x.\frac{M!}{x!\ (M-x)!}\times\binom{N-M}{n-x}\times\frac{n!\ (N-n)!}{N!}$$

$$= \sum_{x=1}^{n} x. \frac{M\,(M-1)!}{x\,(x-1)!\,(M-x)!} \times \binom{N-M}{n-x} \times \frac{n\,(n-1)!\;(N-n)!}{N\,(N-1)!}$$

$$= \frac{n\,M}{N} \sum_{x=1}^{n} \frac{(M-1)!}{(x-1)!\;(M-x)!} \times \binom{N-M}{n-x} \times \frac{(n-1)!\;(N-n)!}{(N-1)!}$$

$$= \frac{n\,M}{N} \sum_{x=1}^{n} \frac{\binom{M-1}{x-1}\binom{N-M}{n-x}}{\binom{N-1}{n-1}}$$

$$= \frac{n\,M}{N} \frac{\binom{N-1}{n-1}}{\binom{N-1}{n-1}}$$

$$= n\,\frac{M}{N}$$

$$= n\,p, \quad \text{when } p = \frac{M}{N}$$

$$\therefore \quad E(x) = n\,p$$

Again, $V(x) = E(x^2) - \{E(x)\}^2$

$$= E\{x\,(x-1) + x\} - \{E(x)\}^2$$

$$= E\{x\,(x-1)\} + E(x) - \{E(x)\}^2$$

$$\Rightarrow V(x) = E\{x\,(x-1)\} + \frac{n\,M}{N} - \frac{n^2\,M^2}{N^2} \text{ ....(i)}$$

Now, $E\{x\,(x-1)\} = \sum_{x=0}^{n} x\,(x-1)\, \frac{\binom{M}{x}\binom{N-M}{n-x}}{\binom{N}{n}}$

$$= \sum_{x=2}^{n} x\,(x-1)\, \frac{M!}{x!\;(M-x)!} \times \binom{N-M}{n-x} \times \frac{n!\;(N-n)!}{N!}$$

$$= \sum_{x=2}^{n} x\,(x-1)\, \frac{M\,(M-1)\,(M-2)!}{x\,(x-1)\,(x-2)!\;(M-x)!} \times \binom{N-M}{n-x} \times \frac{n\,(n-1)\,(n-2)!\;(N-n)!}{N\,(N-1)\,(N-2)!}$$

$$= \frac{n(n-1)\,M(M-1)}{N(N-1)} \sum_{x=2}^{n} x\,(x-1)\,\frac{(M-2)!}{x\,(x-1)\,(x-2)!\,(M-x)!} \times \binom{N-M}{n-x} \times \frac{(n-2)!\,(N-n)!}{(N-2)!}$$

$$= \frac{n(n-1)\,M(M-1)}{N(N-1)} \sum_{x=2}^{n} \frac{(M-2)!}{(x-2)!\,(M-x)!} \times \binom{N-M}{n-x} \times \frac{(n-2)!\,(N-n)!}{(N-2)!}$$

$$= \frac{n(n-1)\,M(M-1)}{N(N-1)} \sum_{x=2}^{n} \frac{\binom{M-2}{x-2}\binom{N-M}{n-x}}{\binom{N-2}{n-2}}$$

$$= \frac{n(n-1)\,M(M-1)}{N(N-1)}\,\frac{\binom{N-2}{n-2}}{\binom{N-2}{n-2}}$$

$$\therefore\quad E\{x\,(x-1)\} = \frac{n\,M\,(n-1)(M-1)}{N(N-1)}$$

Now (i) can be written as

$$V(x) = \frac{n\,M\,(n-1)(M-1)}{N(N-1)} + \frac{n\,M}{N} - \frac{n^2\,M^2}{N^2}$$

$$= \frac{n\,M}{N}\left[\frac{(n-1)(M-1)}{N-1} + 1 - \frac{n\,M}{N}\right]$$

$$= \frac{n\,M}{N}\left[\frac{N\,(n\,M-n-M+1) + N\,(N-1) - n\,M\,(N-1)}{N(N-1)}\right]$$

$$= \frac{n\,M}{N}\left[\frac{N\,n\,M - Nn - NM + N + N^2 - N - Nn\,M + n\,M}{N(N-1)}\right]$$

$$= \frac{n\,M}{N}\left[\frac{N^2 - N\,M - N\,n + n\,M}{N(N-1)}\right]$$

$$= \frac{n\,M}{N}\left[\frac{N\,(N-M) - n\,(N-M)}{N(N-1)}\right]$$

$$= \frac{n\,M}{N}\,\frac{(N-M)(N-n)}{N(N-1)}$$

$$= n.\ \frac{M}{N}.\ \frac{N-M}{N}.\ \frac{N-n}{N-1}$$

$$= n\,p\,q\;\frac{N-n}{N-1} \qquad \left[\because\;\; p=\frac{M}{N}\quad \text{and}\quad q=1-p=1-\frac{M}{N}=\frac{N-M}{N}\right]$$

So, Mean, $E(x)=n\,p$ and Variance, $V(x)=n\,p\,q\;\dfrac{N-n}{N-1}$

NB: (i) $\displaystyle\sum_{x=1}^{n}\binom{M-1}{x-1}\binom{N-M}{n-x}=\binom{N-1}{n-1}$

$$\Rightarrow \sum_{x=1}^{3}\binom{3-1}{x-1}\binom{5-3}{3-x}=\binom{5-1}{3-1};\quad \text{when}\;\; N=5,\; M=3,\; n=3$$

$$\Rightarrow \sum_{x=1}^{3}\binom{2}{x-1}\binom{2}{3-x}=\binom{4}{2}$$

$$\Rightarrow \binom{2}{0}\binom{2}{2}+\binom{2}{1}\binom{2}{1}+\binom{2}{2}\binom{2}{0}=6$$

$$\Rightarrow 1+4+1=6$$

$$\therefore\;\; 6=6$$

So, $\displaystyle\sum_{x=1}^{n}\binom{M-1}{x-1}\binom{N-M}{n-x}=\binom{N-1}{n-1}$

Similarly, $\displaystyle\sum_{x=2}^{n}\binom{M-2}{x-2}\binom{N-M}{n-x}=\binom{N-2}{n-2}$

Remarks: (i) $\displaystyle\sum_{x=1}^{n}\binom{M-1}{x-1}\binom{N-M}{n-x}$ ....(i)

Let $y=x-1$

When $x=1$ then $y=0$ and

When $x=n$ then $y=n-1$

So, (i) can be written as

$$\sum_{y=0}^{n-1}\binom{M-1}{y}\binom{N-M}{n-1-y} \qquad \left[\begin{array}{l}\because\;\; y=x-1\\ \Rightarrow x=1+y\end{array}\right]$$

$$=\binom{N-1}{n-1} \qquad \left[\because\;\; \sum_{i=0}^{n}\binom{a}{i}\binom{b}{n-i}=\binom{a+b}{n}\right]$$

$$\therefore \sum_{x=1}^{n}\binom{M-1}{x-1}\binom{N-M}{n-x}=\binom{N-1}{n-1}$$

(ii) $\sum_{x=2}^{n}\binom{M-2}{x-2}\binom{N-M}{n-x}$ ....(ii)

Let $z=x-2$

When $x=2$ then $z=0$ and

When $x=n$ then $z=n-2$

So, (ii) can written as

$$\sum_{z=0}^{n-2}\binom{M-2}{z}\binom{N-M}{n-2-z} \quad \left[\begin{array}{l}\because\ z=x-2\\ \Rightarrow x=2+z\end{array}\right]$$

$$=\binom{N-2}{n-2} \quad \left[\because \sum_{i=0}^{n}\binom{a}{i}\binom{b}{n-i}=\binom{a+b}{n}\right]$$

$$\therefore \sum_{x=2}^{n}\binom{M-2}{x-2}\binom{N-M}{n-x}=\binom{N-2}{n-2}$$

Theorem: Let x be a hypergeometric variate with parameters N, M and n and if N tends to infinity then hypergeometric distribution reduces to binomial distribution.

Or, Show that hypergeometric distribution tends to binomial distribution if $N\to\infty$.

Or, Show that under certain conditions, the hypergeometric distribution tends to binomial distribution (NU-12)

Or, How the hypergeometric distribution reduces to binomial distribution? **(NU-13)**

Or, Establish the relation between hyper-geometric distribution and binomial distribution **(NU-15)**

Proof: The probability function of hypergeometric distribution with parameters N, M and n is

$$f(x;\ N,\ M,\ n)=\frac{\binom{M}{x}\binom{N-M}{n-x}}{\binom{N}{n}}; \qquad x=0,1,\ 2,\cdots,n$$

$$=\frac{M!}{x!\,(M-x)!}\cdot\frac{(N-M)!}{(n-x)!\,(N-M-n+x)!}\cdot\frac{n!\ (N-n)!}{N!}$$

$$=\frac{n!}{x!\,(n-x)!}\cdot\frac{M!}{(M-x)!}\;\frac{(N-M)!}{(N-M-n+x)!}\cdot\frac{(N-n)!}{N!}$$

$$=\binom{n}{x}\cdot\frac{M^{(x)}\,(M-x)!}{(M-x)!}\;\frac{(N-M)^{(n-x)}\{N-M-(n-x)\}!}{\{N-M-(n-x)\}!}\cdot\frac{(N-n)!}{N^{(n)}\,(N-n)!}\quad\left[\because\ n!=n^{(x)}\,(n-x)!\right]$$

$$=\binom{n}{x}\cdot\frac{M^{(x)}\,(N-M)^{(n-x)}}{N^{(n)}}$$

$$=\binom{n}{x}\cdot\frac{M\,(M-1)\cdots(M-x+1)\;\;(N-M)\,(N-M-1)\cdots(N-M-n+x+1)}{N\,(N-1)\,(N-2)\cdots(N-n+1)}\;\ldots\ldots\ldots\ldots\text{(i)}$$

Let $p=\dfrac{M}{N}$

$\Rightarrow M=Np$

and $1-p=1-\dfrac{M}{N}$

$\Rightarrow q=\dfrac{N-M}{N}\qquad\begin{bmatrix}\because\ p+q=1\\ \Rightarrow q=1-p\end{bmatrix}$

$\Rightarrow N-M=Nq$

Now (i) can be written as

$$f(x;\ N,M,n)=\binom{n}{x}\cdot\frac{Np\,(Np-1)\cdots\{Np-(x-1)\}.\ Nq\,(Nq-1)\cdots\{Nq-(n-x-1)\}}{N\,(N-1)\,(N-2)\cdots(N-n+1)}$$

$$=\binom{n}{x}\cdot\frac{N.\,N^{x-1}\left\{p\left(p-\frac{1}{N}\right)\cdots\left(p-\frac{x-1}{N}\right)\right\}.\ N.\,N^{n-x-1}\left\{q\left(q-\frac{1}{N}\right)\cdots\left\{q-\frac{n-x-1}{N}\right\}\right\}}{N.\,N^{n-1}\left(1-\frac{1}{N}\right)\left(1-\frac{2}{N}\right)\cdots\left(1-\frac{n-1}{N}\right)}$$

Now, $\lim_{N\to\infty} f(x;\ N,M,n)=\dbinom{n}{x}\dfrac{N^n\,(p.\,p\ldots p)\,(q.\,q\ldots q)}{N^n\,(1.\,1\cdots1)}\qquad\left[\because\ \lim\limits_{N\to\infty}\dfrac{1}{N}=0\right]$

$$=\binom{n}{x}\,p^x\,q^{n-x};\qquad x=0,1,2,\cdots,n$$

which is the probability function of binomial distribution with parameters n and p.

NB: We know,

$$x^{(n)}=x\,(x-h)\,(x-2h)\cdots(x-\overline{n-1}h)$$

In particular, if h = 1

$$x^{(n)} = x\ (x-1)\ (x-2)\cdots(x-n+1)$$

Similarly, $n^{(x)} = n\ (n-1)\ (n-2)\cdots(n-x+1)$

Now, $n! = n\ (n-1)\ (n-2)\cdots(n-x+1)\ (n-x)! = n^{(x)}\ (n-x)!$

$$\therefore\ \ M! = M^{(x)}\ (M-x)!$$

$$\therefore\ \ (N-M)! = (N-M)^{(n-x)}\{N-M-(n-x)\}!$$

$$\therefore\ \ N! = N^{(n)}(N-n)!$$

Theorem: Let X be a binomial variate with parameters n_1 and p and Y be another binomial variate with parameters n_2 and p. If X and Y are independent, then the conditional distribution of X, given $X+Y$ is a hypergeometric distribution.

Proof: Here $X \sim B(n_1,\ p)$ and $Y \sim B(n_2,\ p)$ are two independent binomial variates with common probability of success p. Then $X+Y=z$ is also a binomial variate with parameters $n_1+n_2=n$ and p.

By definition,

$$P\left[X=r \mid X+Y=z\right] = \frac{P\left[X=r \cap X+Y=z\right]}{P\left[X+Y=z\right]}$$

$$= \frac{P\left[X=r \cap Y=z-X\right]}{P\left[X+Y=z\right]}$$

$$= \frac{P\left[X=r\right]\ \ P\left[Y=z-r\right]}{P\left[X+Y=z\right]};\quad \left[\because \text{ X and Y are independent}\right]$$

$$= \frac{\binom{n_1}{r}\ p^r\ (1-p)^{n_1-r}\ .\ \binom{n_2}{z-r}\ p^{z-r}\ (1-p)^{n_2-z+r}}{\binom{n_1+n_2}{z}\ p^z\ (1-p)^{n_1+n_2-z}}$$

$$= \frac{\binom{n_1}{r}\binom{n_2}{z-r}\ p^{r+z-r}\ (1-p)^{n_1-r+n_2-z+r}}{\binom{n_1+n_2}{z}\ p^z\ (1-p)^{n_1+n_2-z}}$$

$$= \frac{\binom{n_1}{r}\binom{n_2}{z-r}}{\binom{n_1+n_2}{z}}; \qquad r = 0,\ 1,\ 2$$

which is the probability function of a hypergeometric distribution.

Q. Distinguish between binomial distribution with hypergeometric distribution (NU-12)

Or, Compare between binomial distribution with hypergeometric distribution (NU-13)

Ans: The comparisons of binomial distribution with hypergeometric distribution are as follows:

Binomial distribution		**Hypergeometric distribution**	
(i)	In binomial distribution, the probability of success 'p' remains same from trial to trial.	(i)	In hypergeometric distribution, probability of success varies from trial to trial.
(ii)	The sampling is done with replacement.	(ii)	The sampling is done without replacement.
(iii)	The distribution is hypothetical.	(iii)	It is not hypothetical.

NB: If the sampling is done with replacement, then hypergeometric distribution turns into binomial distribution or, if the population size from which the sample drawn is infinite, then the hypergeometric distribution turns into binomial distribution.

Q. Write down the properties of hypergeometric distribution (NU-13)

Ans: The properties of hypergeometric distribution are as follows:

(i) The probability function of hypergeometric distribution is with parameters N, M, n is

$$f(x;\ N, M, n) = \frac{\binom{M}{x}\binom{N-M}{n-x}}{\binom{N}{n}}; \qquad x = 0, 1,\ 2, \cdots, n$$

Where N is a positive integer, M is a non-negative integer that is at most N and n is a positive integer that is at most M.

(ii) The mean of hypergeometric distribution is np, where $p = \frac{M}{N}$

(iii) The variance of hypergeometric distribution is $V(x) = npq\ \dfrac{N-n}{N-1}$, when $p+q=1$

(iv) Let x be a hypergeometric variate with parameters N, M and n and if N tends to infinity, then hypergeometric distribution reduces to binomial distribution.

Short questions and answers:

(i) Define hypergeometric distribution. Why it is so called? (NU-13)

Ans. A discrete random variable x is said to have a hypergeometric distribution if its probability function is defined as

$$f\left(x;\ N, M, n\right) = \frac{\binom{M}{x}\binom{N-M}{n-x}}{\binom{N}{n}}; \qquad x = 0, 1, 2, \cdots, n$$

Where N is a positive integer, M is a non-negative integer that is at most N and n is a positive integer that is at most M.

This is called hypergeometric distribution because the distribution can be obtained from the hypergeometric function.

(ii) What is mean and variance of hypergeometric distribution? (NU-13)

Ans: The mean and variance of hypergeometric distribution are as follows:

$$\text{Mean, } E(x) = np\text{, where } p = \frac{M}{N}$$

$$\text{and Variance, } V(x) = npq\ \frac{N-n}{N-1}$$

(iii) What are the parameters of hypergeometric distribution? (NU-11, 13)

Ans: The parameters of hypergeometric distribution are N, M and n.

Here N is a positive integer, M is a non-negative integer that is at most N and n is a positive integer that is at most M.

(iv) When hypergeometric distribution is symmetric?

Ans: Hypergeometric distribution is symmetrical if $p = \dfrac{1}{2}$.

NB: Hypergeometric distribution is positively skewed if $p < \frac{1}{2}$ and is negatively skewed if $p > \frac{1}{2}$.

Mathematical Problems:

1. **A bag contains 5 white and 5 red balls. 4 balls are drawn from this bag without replacement.**
   (a) What is the probability that the selected balls are 2 white and 2 red.
   (b) Also find the probability that the selected balls contains at least one white ball.

Solution: Here we have $N = 10,\ M = 5$ and $n = 4$.

Let x be the number of white balls, then x follows hypergeometric distribution with parameters $N = 10$, $M = 5$ and $n = 4$ and the probability function of x is

$$f(x;\ 10, 5, 4) = \frac{\binom{5}{x}\binom{5}{4-x}}{\binom{10}{4}}; \qquad x = 0, 1, 2, 3, 4$$

(a) The probability that the selected balls will contain 2 white ball and 2 red ball is

$$P(x = 2) = f(2;\ 10, 5, 4) = \frac{\binom{5}{2}\binom{5}{2}}{\binom{10}{4}} = \frac{10 \times 10}{210} = \frac{10}{21}$$

(b) The probability that the selected balls will contain at least one white ball is

$$\begin{aligned} P(x \geq 1) &= 1 - P(x = 0) \\ &= 1 - f(x = 0; 10, 5, 4) \\ &= 1 - \frac{\binom{5}{0}\binom{5}{4}}{\binom{10}{4}} = 1 - \frac{5}{210} = 1 - \frac{1}{42} = \frac{41}{42} \end{aligned}$$

2. **A bag contains 6 white and 4 red balls. 4 balls are drawn from the bag without replacement. Find the probability that the selected balls are (i) 3 white and 1 red; (ii) at least one white ball (NU-12)**

Solution: Here we have $N = 10$, $M = 6$ and $n = 4$.

Let x be the number of white balls, then x follows hypergeometric distribution with parameters $N = 10,\ M = 6$ and $n = 4$ and the probability function of x is

$$f(x;\ 10,\ 6,\ 4) = \frac{\binom{6}{x}\binom{4}{4-x}}{\binom{10}{4}}; \qquad x = 0,\ 1,\ 2,\ 3,\ 4$$

(a) The probability that the selected balls will contain 3 white and 1 red ball is

$$P(x = 3) = f(3;\ 10, 6, 4) = \frac{\binom{6}{3}\binom{4}{1}}{\binom{10}{4}} = \frac{20 \times 4}{210} = \frac{8}{21}$$

(b) The probability that the selected ball is at least one white ball

$$P(x \geq 1) = 1 - P(x = 0)$$

$$= 1 - f(x = 0;\ 10, 6, 4)$$

$$= 1 - \frac{\binom{6}{0}\binom{4}{4}}{\binom{10}{4}} = 1 - \frac{1}{210} = \frac{219}{210}$$

Discrete Uniform Distribution/ Rectangular Distribution

Uniform distribution is the simplest distribution among all the discrete distributions. This is the only distribution which may be appropriate for discrete and continuous variable.

Q. Define uniform distribution (NU-15)

Ans: A discrete random variable x is said to have a rectangular or uniform distribution if its probability function is given by

$$f(x;\quad N)=\frac{1}{N};\qquad x=1,\ 2,\ 3,\cdots,N$$

Where N is the only parameter of the distribution.

For example- Suppose a fair die is thrown and the number of points on the faces of the die is a random variable x. Then $x=1,\ 2,\ 3,\ 4,\ 5$ and 6 with probability $\frac{1}{6}$ for each. The probability function of the random variable x is

$$f(x;\ \ 6)=\frac{1}{6};\qquad x=1,\ 2,\ 3,\ 4,\ 5,\ 6$$

Q. Find the mean, variance, β_1 and β_2 of discrete uniform distribution.

Or, Find mean and variance of discrete uniform distribution (NU-12)

Ans: We know the probability function of discrete uniform distribution with parameter N is

$$f(x;\ \ N)=\frac{1}{N};\qquad x=1,\ 2,\ 3,\cdots,N$$

Now, Mean $\mu_1'=E(x)=\sum_{x=1}^{N}x\ \ f(x;\ N)$

$$=\sum_{x=1}^{N}x.\ \frac{1}{N}$$

$$=\frac{1}{N}\ \sum_{x=1}^{N}x$$

$$=\frac{1}{N}\ (1+2+\cdots+N)$$

$$=\frac{1}{N}.\ \frac{N\ (N+1)}{2}$$

$$\therefore \quad \mu_1' = E(x) = \frac{N+1}{2}$$

Again, $\mu_2' = E(x^2) = \sum_{x=1}^{N} x^2 \; f(x;\, N)$

$$= \sum_{x=1}^{N} x^2 .\ \frac{1}{N}$$

$$= \frac{1}{N} \sum_{x=1}^{N} x^2$$

$$= \frac{1}{N} \left(1^2 + 2^2 + \cdots + N^2\right)$$

$$= \frac{1}{N} .\ \frac{N\,(N+1)\,(2\,N+1)}{6}$$

$$\therefore \quad \mu_2' = E(x^2) = \frac{(N+1)\,(2\,N+1)}{6}$$

So, Variance $\mu_2 = \mu_2' - \mu_1'^2$

$$= \frac{(N+1)\,(2\,N+1)}{6} - \left(\frac{N+1}{2}\right)^2$$

$$= \frac{N+1}{2} \left(\frac{2\,N+1}{3} - \frac{N+1}{2}\right)$$

$$= \frac{N+1}{2} .\ \frac{4\,N+2-3\,N-3}{6}$$

$$= \frac{N+1}{2} .\ \frac{N-1}{6}$$

$$\therefore \quad \mu_2 = \frac{N^2-1}{12}$$

Further, $\mu_3' = E(x^3) = \sum_{x=1}^{N} x^3 \; f(x;\, N)$

$$= \sum_{x=1}^{N} x^3 .\ \frac{1}{N}$$

$$= \frac{1}{N} \sum_{x=1}^{N} x^3$$

$$= \frac{1}{N} \left(1^3 + 2^3 + \cdots + N^3\right)$$

$$= \frac{1}{N} \cdot \left\{ \frac{N\,(N+1)}{2} \right\}^2$$

$$\therefore \quad \mu_3' = E(x^3) = \frac{N\,(N+1)^2}{4}$$

and $\quad \mu_4' = E(x^4) = \sum_{x=1}^{N} x^4 \; f(x;\, N)$

$$= \sum_{x=1}^{N} x^4 \cdot \frac{1}{N}$$

$$= \frac{1}{N} \sum_{x=1}^{N} x^4$$

$$= \frac{1}{N} \left(1^4 + 2^4 + \cdots + N^4\right)$$

$$= \frac{1}{N} \cdot \frac{N\,(N+1)\;(2\,N+1)\;\left(3\,N^2 + 3\,N - 1\right)}{30}$$

$$\therefore \quad \mu_4' = E(x^4) = \frac{(N+1)\;(2\,N+1)\;\left(3\,N^2 + 3\,N - 1\right)}{30}$$

Now, $\mu_3 = \mu_3' - 3\,\mu_2'\;\mu_1' + 2\,\mu_1'^3$

$$= \frac{N\,(N+1)^2}{4} - 3\,\frac{(N+1)\;(2\,N+1)}{6} \cdot \frac{N+1}{2} + 2\left(\frac{N+1}{2}\right)^3$$

$$= \frac{N\,(N+1)^2}{4} - \frac{(N+1)^2\;(2\,N+1)}{4} + 2\left(\frac{N+1}{2}\right)^2 \cdot \frac{N+1}{2}$$

$$= \frac{(N+1)^2}{4}\left(N - 2\,N - 1 + N + 1\right)$$

$$= \frac{(N+1)^2}{4} \times 0$$

$$\therefore \quad \mu_3 = 0$$

$$\mu_4 = \mu_4' - 4\;\mu_3'\;\mu_1' + 6\;\mu_2'\;\mu_1'^2 - 3\,\mu_1'^4$$

$$= \frac{(N+1)\,(2\,N+1)\;\left(3\,N^2 + 3\,N - 1\right)}{30} - 4\,\frac{N\,(N+1)^2}{4} \cdot \frac{N+1}{2} + 6\,\frac{(N+1)\,(2\,N+1)}{6} \cdot \frac{(N+1)^2}{4} - 3\,\frac{(N+1)^4}{16}$$

$$=\frac{(N+1)\ (2\ N+1)\ \left(3\ N^2+3\ N-1\right)}{30}-\frac{N\ (N+1)^3}{2}+\frac{(N+1)^3\ (2\ N+1)}{4}-\frac{3\ (N+1)^4}{16}$$

$$=\frac{\left(N^2-1\right)\left(N^2-\frac{7}{3}\right)}{80} \quad \text{(on simplification)}$$

Now, $\beta_1=\frac{\mu_3{}^2}{\mu_2{}^3}=\frac{0}{\mu_2{}^3}=0$

and $\beta_2=\frac{\mu_4}{\mu_2{}^2}=\frac{\left(N^2-1\right)\left(N^2-\frac{7}{3}\right)}{80}\times\frac{144}{\left(N^2-1\right)^2}=\frac{9\left(N^2-\frac{7}{3}\right)}{5\left(N^2-1\right)}$

The discrete uniform distribution is symmetrical since $\beta_1=0$

NB: (i) The discrete uniform distribution is sometimes defined as

$$f(x;\ \ N)=\frac{1}{N+1}; \qquad x=0, 1, 2, \cdots, N$$

for N, a non-negative integer. If such is the case, the formula for the mean and variance have to be modified accordingly.

(ii) $1+2+\cdots+N=\frac{N\ (N+1)}{2}$

$$1^2+2^2+\cdots+N^2=\frac{N\ (N+1)\ (2\ N+1)}{6}$$

$$1^3+2^3+\cdots+N^3=\left\{\frac{N\ (N+1)}{2}\right\}^2$$

$$1^4+2^4+\cdots+N^4=\frac{N\ (N+1)\ (2\ N+1)\ \left(3\ N^2+3\ N-1\right)}{30}$$

(iii) $\mu_r'=E[x-a]^r$

$$\mu_r'=E[x-0]^r=E(x^r) \text{ (w.r.t origin)}$$

$$\mu_r=E[x-E(x)]^r$$

(iv) For μ_4, see page-469, M.K. Roy (New)

(v) $\mu_4=\mu_4'-4\ \mu_3'\ \mu_1'+6\ \mu_2'\ \mu_1'^2-3\ \mu_1'^4$

$$= \frac{(N+1)\,(2\,N+1)\,\left(3\,N^2+3\,N-1\right)}{30} - 4\,\frac{N\,(N+1)^2}{4}\cdot\frac{N+1}{2} + 6\,\frac{(N+1)\,(2\,N+1)}{6}\cdot\frac{(N+1)^2}{4} - 3\,\frac{(N+1)^4}{16}$$

$$= \frac{(N+1)\,(2\,N+1)\,\left(3\,N^2+3\,N-1\right)}{30} - \frac{N\,(N+1)^3}{2} + \frac{(N+1)^3\,(2\,N+1)}{4} - \frac{3\,(N+1)^4}{16}$$

$$= \frac{N+1}{2}\left[\frac{(2\,N+1)\,\left(3\,N^2+3\,N-1\right)}{15} - N\,(N+1)^2 + \frac{(N+1)^2\,(2\,N+1)}{2} - \frac{3\,(N+1)^3}{8}\right]$$

$$= \frac{N+1}{2}\left[\frac{8\left(6N^3+6N^2-2N+3N^2+3N-1\right)-120\,N\left(N^2+2N+1\right)+60\left(N^2+2N+1\right)(2N+1) - 45\left(N^3+3N^2+3N+1\right)}{120}\right]$$

$$= \frac{N+1}{240}\left[8\left(6N^3+9N^2+N-1\right)-120\left(N^3+2N^2+N\right)+60\left(2N^3+4N^2+2N+N^2+2N+1\right) - 45\left(N^3+3N^2+3N+1\right)\right]$$

$$= \frac{N+1}{240}\left[48N^3+72N^2+8N-8-120\,N^3-240\,N^2-120\,N+120\,N^3+300\,N^2+240\,N+60 - 45\,N^3-135N^2-135\,N-45\right]$$

$$= \frac{N+1}{240}\left[3\,N^3-3N^2-7N+7\right]$$

$$= \frac{N+1}{240}\left[3\,N^2(N-1)-7(N-1)\right]$$

$$= \frac{N+1}{240}\,(N-1)\left(3\,N^2-7\right)$$

$$= \frac{(N^2-1).\,3\left(N^2-\frac{7}{3}\right)}{240}$$

$$= \frac{(N^2-1)\left(N^2-\frac{7}{3}\right)}{80}$$

Now, $\beta_1 = \frac{{\mu_3}^2}{{\mu_2}^3} = \frac{0}{{\mu_2}^3} = 0$

and $\beta_2 = \frac{\mu_4}{{\mu_2}^2} = \frac{(N^2-1)\left(N^2-\frac{7}{3}\right)}{80} \times \frac{144}{(N^2-1)^2} = \frac{9\left(N^2-\frac{7}{3}\right)}{5\,(N^2-1)}$

The discrete uniform distribution is symmetrical since $\beta_1 = 0$

Short questions and answers:

(1) Define discrete uniform distribution (NU-12)

Ans: A discrete random variable x is said to have a rectangular or uniform distribution if its probability function is given by

$$f(x;\ N) = \frac{1}{N}; \qquad x = 1,\ 2,\ 3, \cdots, N$$

Where N is the only parameter of the distribution.

For example- Suppose a fair die is thrown and the number of points of the faces of the die is a random variable x. Then $x = 1,\ 2,\ 3,\ 4,\ 5$ and 6 with probability $\frac{1}{6}$ for each. Then the probability function of the random variable x is

$$f(x;\ 6) = \frac{1}{6}; \qquad x = 1,\ 2,\ 3,\ 4,\ 5,\ 6$$

Multinomial Distribution
(বহুপদিক বিন্যাস)

Multinomial distribution is the generalization of the binomial distribution. In case of binomial distribution, each trial consists of two outcomes with probabilities p and q. But in case of multinomial distribution, each trial consists more than two outcomes.

Q. Define multinomial distribution (NU-12, 16)

Ans: k discrete random variables $x_1,\ x_2,\cdots,x_k$ is said to have a multinomial distribution if its probability function is defined as

$$f\left(x_1,\ x_2,\cdots,x_k\right)=\frac{n!}{x_1!\,x_2!\cdots x_k!}\,p_1^{x_1}\ p_2^{x_2}\cdots p_k^{x_k}$$

$$\text{such that } \sum_{i=1}^{k}x_i=n \text{ and } \sum_{i=1}^{k}p_i=1.$$

The distribution contains k parameters which are n and $(k-1)$ of p's, since $\sum_{i=1}^{k}p_i=1$.

Q. Derive the probability function of multinomial distribution (NU-12, 14, 16)

Ans: Let in an experiment each trial consists of k mutually exclusive outcomes $E_1,\ E_2,\cdots,E_k$ with probability $p_1,\ p_2,\cdots,p_k$ such that $\sum_{i=1}^{k}p_i=1$. Since the trials are independent so the probability of a definite order where E_1 occurs x_1 times, E_2 occurs x_2 times,..., E_k occurs x_k times is $p_1^{x_1}\ p_2^{x_2}\ \ldots\ p_k^{x_k}$.

In n independent trials, E_1 occurs x_1 times, E_2 occurs x_2 times, ... , E_k occurs x_k times in any order. Here this can happen $\frac{n!}{x_1!\ x_2!\cdots x_k!}$ mutually (পরস্পর) exclusive (স্বতন্ত্র) ways, where $\sum_{i=1}^{k}x_i=n$ and the probability of each way is $p_1^{x_1}\ p_2^{x_2}\ \ldots\ p_k^{x_k}$.

Hence if this experiment is repeated n times, then the probability that E_1 occurs x_1 times, E_2 occurs x_2 times, ... , E_k occurs x_k times is

$$f\left(x_1,\ x_2,\cdots,x_k;\ \ n,\ p_1,\ p_2,\cdots,p_k\right)=\frac{n!}{x_1!\ x_2!\cdots x_k!}\ p_1^{x_1}\ \ p_2^{x_2}\cdots p_k^{x_k}$$

such that $\sum_{i=1}^{k} x_i = n$ and $\sum_{i=1}^{k} p_i = 1$.

which is the probability function of multinomial distribution.

NB-1: The probability function of multinomial distribution is

$$f(x_1,\ x_2,\cdots,x_k) = \frac{n!}{x_1!\ x_2!\cdots x_k!}\ p_1^{x_1}\ \ p_2^{x_2}\cdots p_k^{x_k}$$

This distribution contains k parameters which are n and $(k-1)$ of p's i.e., $p_1,\ p_2,\cdots,p_{k-1}$. If $p_1,\ p_2,\cdots,p_{k-1}$ is known, then $p_k = 1 - p_1 - p_2 - \cdots - p_{k-1}$. So, to find the probability by multinomial distribution, we need to know the value of n and $p_1,\ p_2,\cdots,p_{k-1}$.

Therefore, the number of parameters of multinomial distribution is k $\left(\text{i.e., } 1+k-1=k\right)$

NB-2: Let in an experiment each trial consists of 3 mutually exclusive outcomes $E_1,\ E_2$ and E_3 with probability p_1, p_2 and p_3 such that $\sum_{i=1}^{3} p_i = 1$.

Suppose in $n\ (=4)$ independent trials E_1 occurs $x_1\ (=2)$ times, E_2 occurs $x_2\ (=1)$ time and E_3 occurs $x_3\ (=1)$ time in any order. This can happen $\frac{4!}{2!\ 1!\,1!} = 12$ mutually exclusive ways, where $\sum_{i=1}^{k} x_i = n$.

At first, we consider the combinations of E_1 and E_2 which are-

$$E_1\ E_1\ E_2,\quad E_1\ E_2\ E_1,\quad E_2\ E_1\ E_1$$

Now, E_1 occurs $x_1\ (=2)$ times, E_2 occurs $x_2\ (=1)$ time and E_3 occurs $x_3\ (=1)$ time any order in the following 12 ways:

	E_3	E_3	E_3	E_3
$E_1\ E_1\ E_2$	$E_1\ E_1\ E_2\ E_3$	$E_1\ E_1\ E_3\ E_2$	$E_1\ E_3\ E_1\ E_2$	$E_3\ E_1\ E_1\ E_2$
$E_1\ E_2\ E_1$	$E_1\ E_2\ E_1\ E_3$	$E_1\ E_2\ E_3\ E_1$	$E_1\ E_3\ E_2\ E_1$	$E_3\ E_1\ E_2\ E_1$
$E_2\ E_1\ E_1$	$E_2\ E_1\ E_1\ E_3$	$E_2\ E_1\ E_3\ E_1$	$E_2\ E_3\ E_1\ E_1$	$E_3\ E_2\ E_1\ E_1$

The probability each way of above series is $p_1\ p_1\ p_2\ p_3 = p_1^{\,2}\, p_2^{\,1}\, p_3^{\,1}$

Now by additive law of probability, we get

$$p_1^{\,2}\, p_2^{\,1}\, p_3^{\,1} + p_1^{\,2}\, p_2^{\,1}\, p_3^{\,1} + \cdots + p_1^{\,2}\, p_2^{\,1}\, p_3^{\,1}$$

$$= 12\ p_1^{\,2}\, p_2^{\,1}\, p_3^{\,1}$$

$$= \frac{4!}{2!\ 1!\, 1!}\ p_1^{\,2}\ \ p_2^{\,1}\ \ p_3^{\,1}$$

Similarly, we can write

$$f\left(x_1,\ x_2,\ x_3\right) = \frac{n!}{x_1!\ x_2!\ x_3!}\ p_1^{\,x_1}\ p_2^{\,x_2}\ p_3^{\,x_3} \qquad \left[\because\ n=4,\ \ x_1=2,\ \ x_2=1,\ \ x_3=1\right]$$

Hence for k mutually exclusive outcomes, we can write

$$f\left(x_1,\ x_2,\cdots,x_k\right) = \frac{n!}{x_1!\ \ x_2!\cdots x_k!}\ \ p_1^{\,x_1}\ \ p_2^{\,x_2}\cdots p_k^{\,x_k}\text{ ; where } n = x_1 + x_2 + \cdots + x_k$$

Remarks (লক্ষ্য করা, মন্তব্য):

(i) When there are more than two mutually exclusive outcomes of a trial, the observations lead to multinomial distribution.

(ii) The distribution is called multinomial because the probability function $f(x_1,\ x_2,\cdots,x_n)$ is the general term in the multinomial expansion of $\left(p_1+p_2+\cdots+p_k\right)^n;\quad \sum_{i=1}^{k} p_i = 1$.

Q. Find the moment generating function and also find mean and variance of the multinomial distribution.

Or, Find mean and variance of multinomial distribution (NU-12, 16)

Ans: The probability function of multinomial distribution is

$$f\left(x_1,\ x_2,\cdots,x_k\right) = \frac{n!}{x_1!\ \ x_2!\cdots x_k!}\ \ p_1^{\,x_1}\ \ p_2^{\,x_2}\cdots p_k^{\,x_k}$$

$$\text{such that } \sum_{i=1}^{k} x_i = n \text{ and } \sum_{i=1}^{k} p_i = 1.$$

By definition of moment generating function, we can write

$$M_x(t) = M_{x_1\ x_2 \ldots x_k}\left(t_1,\ \ t_2,\cdots,t_k\right)$$

$$= E\left[e^{t_1 x_1 + t_2 x_2 + \cdots + t_k x_k}\right]$$

$$= \sum e^{t_1 x_1 + t_2 x_2 + \cdots + t_k x_k} . \quad f\left(x_1, x_2, \cdots, x_k\right)$$

$$= \sum e^{t_1 x_1} . \quad e^{t_2 x_2} \cdots e^{t_k x_k} \frac{n!}{x_1! \; x_2! \cdots x_k!} \; p_1^{x_1} \; p_2^{x_2} \cdots p_k^{x_k}$$

$$= \sum \frac{n!}{x_1! \; x_2! \cdots x_k!} \left(p_1 e^{t_1}\right)^{x_1} \left(p_2 e^{t_2}\right)^{x_2} \cdots \left(p_k e^{t_k}\right)^{x_k}$$

$$= \left(p_1 e^{t_1} + p_2 e^{t_2} + \cdots + p_k e^{t_k}\right)^n$$

$$\Rightarrow M_x(t) = \left(\sum_{i=1}^{k} p_i e^{t_i}\right)^n \text{.................(i)}$$

Now differentiating (i) w.r.t t_i, we get

$$\frac{d M_x(t)}{d t_i} = n \left(\sum_{i=1}^{k} p_i e^{t_i}\right)^{n-1} p_i e^{t_i} \text{...............(ii)}$$

$$\therefore \text{ Mean, } \mu_1' = \left.\frac{d M_x(t)}{d t_i}\right|_{t_i=0} \qquad (i = 1, 2, \cdots, k)$$

$$= n \left.\left(\sum_{i=1}^{k} p_i e^{t_i}\right)^{n-1} p_i e^{t_i}\right|_{t_i=0}$$

$$= n \left(\sum_{i=1}^{k} p_i\right)^{n-1} p_i$$

$$= n \, p_i \qquad \left[\because \sum_{i=1}^{k} p_i = 1\right]$$

Again, differentiating (ii) w.r.t t_i, we get

$$\frac{d^2 M_x(t)}{d t_i^2} = n p_i \left[\left(\sum_{i=1}^{k} p_i e^{t_i}\right)^{n-1} e^{t_i} + e^{t_i} (n-1) \left(\sum_{i=1}^{k} p_i e^{t_i}\right)^{n-2} p_i e^{t_i}\right]$$

$$\therefore \quad \mu_2' = \left.\frac{d^2 M_x(t)}{d t_i^2}\right|_{t_i=0} \qquad (i = 1, 2, \cdots, k)$$

$$= n \, p_i \left[\left(\sum_{i=1}^{k} p_i\right)^{n-1} + (n-1)\left(\sum_{i=1}^{k} p_i\right)^{n-2} p_i\right]$$

$$= n\ p_i\ \left[1+(n-1)\ p_i\right] \qquad \left(\because\ \sum_{i=1}^{k} p_i = 1\right)$$

$$= n\ p_i + n\ (n-1)\ p_i^{\ 2}$$

$$= n\ p_i + n\ (n-1)\ p_i^{\ 2}$$

$\therefore$ Variance, $\mu_2 = \mu_2{}' - \mu_1{}'^2$

$$= n\ p_i + n^2\ p_i^{\ 2} - n\ p_i^{\ 2} - n^2\ p_i^{\ 2}$$

$$= n\ p_i - n\ p_i^{\ 2}$$

$$= n\ p_i\ (1-p_i)$$

NB (Nota Bene-দ্রষ্টব্য, দেখ): The probability function for the binomial variate x is,

$$f(x;\ \ n,\ p) = \binom{n}{x}\ p^x\ q^{n-x}; \qquad x = 0, 1, 2, \cdots, n$$

Now, the m.g.f of x is,

$$M_x(t) = E(e^{tx})$$

$$= \sum_{x=0}^{n} e^{tx}\ \ f(x;\ n, p)$$

$$= \sum_{x=0}^{n} e^{tx} \cdot \binom{n}{x}\ p^x\ q^{n-x}$$

$$= \sum_{x=0}^{n} \binom{n}{x}\ q^{n-x}\ (p\ e^t)^x$$

$$= \sum_{x=0}^{n} \frac{n!}{x!\,(n-x)!}\ q^{n-x}\ (p\ e^t)^x$$

$$= \frac{n!}{0!\ n!}\ q^{n-0}\ (p\ e^t)^0 + \frac{n!}{1!\ (n-1)!}\ q^{n-1}\ (p\ e^t)^1 + \frac{n!}{2!\,(n-2)!}\ q^{n-2}\ (p\ e^t)^2 + \cdots$$

$$= \binom{n}{0}\ q^{n-0}\ (p\ e^t)^0 + \binom{n}{1}\ q^{n-1}\ (p\ e^t)^1 + \binom{n}{2}\ q^{n-2}\ (p\ e^t)^2 + \cdots$$

$$= \left(q + p\ e^t\right)^n$$

$$\therefore \sum_{x=0}^{n} \frac{n!}{x!\,(n-x)!}\; q^{n-x}\,(p\,e^{t})^{x} = \left(q + p\,e^{t}\right)^{n}$$

Similarly, $$\sum \frac{n!}{x_1!\; x_2! \cdots x_k!} \left(p_1\, e^{t_1}\right)^{x_1} \left(p_2\, e^{t_2}\right)^{x_2} \cdots \left(p_k\, e^{t_k}\right)^{x_k}$$

$$= \left(p_1\, e^{t_1} + p_2\, e^{t_2} + \cdots + p_k\, e^{t_k}\right)^{n}$$

Theorem: The trinomial distribution of two random variables X and Y is given by

$$f(x,y) = \frac{n!}{x!\,y!\,(n-x-y)!}\; p^{x}\, q^{y}\, (1-p-q)^{n-x-y} \text{ for } x,\ y = 0, 1, 2, \cdots, n \text{ and } x + y \le n$$

where, $0 \le p$, $0 \le q$ and $p + q \le 1$.

(i) Find the marginal distribution of X and Y.

(ii) Find the conditional distribution of X and Y and obtain $E\left(Y \mid X = x\right)$ and $E\left(X \mid Y = y\right)$.

(iii) Find the correlation coefficient between X and Y.

Solution: The trinomial distribution of two random variables X and Y is given by

$$f(x,\ y) = \frac{n!}{x!\ y!\ (n-x-y)!}\ p^{x}\ q^{y}\ (1-p-q)^{n-x-y}$$

The joint m.g.f of X and Y is

$$M_{x,y}(t_1,\ t_2) = E\left[e^{t_1 x + t_2 y}\right]$$

$$= \sum_{x=0}^{n} \sum_{y=0}^{n-x} e^{t_1 x + t_2 y}\ f(x,\ y)$$

$$= \sum_{x=0}^{n} \sum_{y=0}^{n-x} e^{t_1 x + t_2 y} \frac{n!}{x!\ y!\ (n-x-y)!}\ p^{x}\ q^{y}\ (1-p-q)^{n-x-y}$$

$$= \sum_{x=0}^{n} \sum_{y=0}^{n-x} \frac{n!}{x!\ y!\ (n-x-y)!}\ e^{t_1 x}\ e^{t_2 y}\ p^{x}\ q^{y}\ (1-p-q)^{n-x-y}$$

$$= \sum_{x=0}^{n} \sum_{y=0}^{n-x} \frac{n!}{x!\ y!\ (n-x-y)!} \left(p\, e^{t_1}\right)^{x} \left(q\, e^{t_2}\right)^{y} (1-p-q)^{n-x-y}$$

$$\therefore \quad M_{x,y}(t_1, t_2) = \left(p\, e^{t_1} + q\, e^{t_2} + 1 - p - q\right)^n$$

Now, the m.g.f of X is

$$M_x(t_1) = M_{x,y}(t_1,\ 0) = \left(p\, e^{t_1} + q + 1 - p - q\right)^n = \left[(1-p) + p\, e^{t_1}\right]^n$$

which is the m.g.f of binomial distribution.

$\therefore\ X \sim B\ (n,\ p)$

So, the marginal distribution of X is binomial with probability function

$$f(x;\ n,\ p) = \binom{n}{x}\ p^x\ (1-p)^{n-x}$$

Similarly, the m.g.f of Y is

$$M_y(t_2) = M_{x,\,y}(0,\ t_2) = \left(p + q\, e^{t_2} + 1 - p - q\right)^n = \left[(1-q) + q\, e^{t_2}\right]^n$$

which is the m.g.f of binomial distribution.

$\therefore\ Y \sim B\ (n,\ q)$

So, the marginal distribution of Y is binomial with probability function

$$f(y;\ n,\ q) = \binom{n}{y}\ q^y\, (1-q)^{n-y}$$

(ii) The conditional distribution of X given $Y = y$ is

$$f\left(X \mid Y = y\right) = \frac{f(x,\ y)}{f(y)}$$

$$= \frac{f(x, y)}{\binom{n}{y}\ q^y\ (1-q)^{n-y}} \qquad [\because\ Y \sim B\ (n,\ q)]$$

$$= \frac{\dfrac{n!}{x!\ y!\ (n-x-y)!}\ \ p^x\ q^y\ (1-p-q)^{n-x-y}}{\dfrac{n!}{y!\ (n-y)!}\ \ q^y\ (1-q)^{n-y}}$$

$$= \frac{n!}{x!\ y!\ (n-x-y)!} \times \frac{y!\ (n-y)!}{n!} \times \frac{p^x\ (1-p-q)^{n-x-y}}{(1-q)^{n-y}\ (1-q)^x\ (1-q)^{-x}}$$

$$= \frac{(n-y)!}{x!\,(n-y-x)!} \times \frac{p^x\,(1-p-q)^{n-x-y}}{(1-q)^x\,(1-q)^{n-x-y}}$$

$$= \binom{n-y}{x} \left(\frac{p}{1-q}\right)^x \left(1-\frac{p}{1-q}\right)^{n-y-x}; \qquad x = 0,\ 1,\ 2, \cdots, n$$

which is the probability function of binomial distribution.

i.e., $(X \mid Y = y) \sim B\left(n-y,\ \frac{p}{1-q}\right)$

$$\therefore\ E(X \mid Y = y) = (n-y)\ \frac{p}{1-q} = \frac{p\,(n-y)}{1-q}$$

Similarly, the conditional distribution of Y given $X = x$ is

$$f\left(Y \middle| X = x\right) = \frac{f(x, y)}{f(x)}$$

$$= \frac{f(x,\, y)}{\binom{n}{x}\ p^x\,(1-p)^{n-x}} \qquad [\because\ X \sim B\ (n,\, p)]$$

$$= \frac{\dfrac{n!}{x!\ y!\ (n-x-y)!}\ p^x\ q^y\ (1-p-q)^{n-x-y}}{\dfrac{n!}{x!\ (n-x)!}\ p^x\ (1-p)^{n-x}}$$

$$= \frac{n!}{x!\ y!\ (n-x-y)!} \times \frac{x!\ (n-x)!}{n!} \times \frac{q^y\,(1-p-q)^{n-x-y}}{(1-p)^{n-x}\ (1-p)^y\ (1-p)^{-y}}$$

$$= \frac{(n-x)!}{y!\ (n-x-y)!} \times \frac{q^y\ (1-p-q)^{n-x-y}}{(1-p)^y\ (1-p)^{n-x-y}}$$

$$= \binom{n-x}{y} \left(\frac{q}{1-p}\right)^y \left(1-\frac{q}{1-p}\right)^{n-x-y}; \qquad y = 0,\ 1,\ 2, \cdots, n$$

which is the probability function of binomial distribution.

i.e., $(Y \mid X = x) \sim B\left(n-x,\ \frac{q}{1-p}\right)$

$$\therefore\ E(Y \mid X = x) = (n-x)\ \frac{q}{1-p} = \frac{q\,(n-x)}{1-p}$$

(iii) Correlation coefficient between X and Y:

The correlation coefficient between X and Y is

$$r=\frac{COV\,(X,\,Y)}{\sqrt{V(X)\;V(Y)}}\;...............(i)$$

Since, $X \sim B\,(n,\;p)$

$$\therefore\;E(X)=n\,p\quad\text{and}\quad V(X)=n\,p\,(1-p)$$

and $Y \sim B(n, q)$

$$\therefore\;E(Y)=n\,q\quad\text{and}\quad V(Y)=n\,q\,(1-q)$$

We know,

$$COV(X,\,Y)=E(X\,Y)-E(X)\;E(Y)\;................(ii)$$

Now, $$E(X\,Y)=\left.\frac{d^2\;M_{x,y}(t_1,\,t_2)}{d\;t_1\,t_2}\right|_{t_1=t_2=0}$$

$$=\left.\frac{d^2}{dt_1t_2}\left(p\,e^{t_1}+q\,e^{t_2}+1-p-q\right)^n\right|_{t_1=t_2=0}$$

$$=\left.\frac{d}{d\,t_2}\left[\frac{d}{dt_1}\left(p\,e^{t_1}+q\,e^{t_2}+1-p-q\right)^n\right]\right|_{t_1=t_2=0}$$

$$=\left.\frac{d}{dt_2}\left[n\left(p\,e^{t_1}+q\,e^{t_2}+1-p-q\right)^{n-1}.\;\;p\,e^{t_1}\right]\right|_{t_1=t_2=0}$$

$$=\left.n\,(n-1)\,\left(p\,e^{t_1}+q\,e^{t_2}+1-p-q\right)^{n-2}.\quad p\,e^{t_1}.\;\;q\,e^{t_2}\right|_{t_1=t_2=0}$$

$$=n\,(n-1)\;\left(p+q+1-p-q\right)^{n-2}.\;p.\;\;q$$

$$=n\,(n-1)\,p\,q$$

Now, (ii) can be written as

$$COV(X,\,Y)=n\,(n-1)\,p\,q-n\,p.\;n\,q$$

$$=(n^2-n)\,p\,q-n^2\;p\,q$$

$$=n^2\;p\,q-n\,p\,q-n^2\;p\,q$$

$$=-\,n\,p\,q$$

So (i) can be written as

$$r = \frac{-npq}{\sqrt{np\ (1-p).\ nq\ (1-q)}}$$

$$= \frac{-npq}{n\sqrt{pq}.\ \sqrt{(1-p)\ (1-q)}}$$

$$= -\frac{\sqrt{pq}}{\sqrt{(1-p)\ (1-q)}}$$

$$= -\sqrt{\frac{p\ q}{(1-p)\ (1-q)}}$$

$$= -\left[\frac{p\ q}{(1-p)\ (1-q)}\right]^{\frac{1}{2}}$$

NB: (i) If X and Y are not independent, then

$$M_{x,y}(t_1,\ t_2)\ \neq\ M_x(t_1).\ \ M_y(t_2)$$

(ii) $x + y \leq n$

i.e.,

$x =$	0	1	2	3	...	n	$\sum_{x=0}^{n} \sum_{y=0}^{n-x}$
$y =$	n	$n-1$	$n-2$	$n-3$	...	0	

Theorem: If $X_1, X_2, \cdots, X_k$ are k independent Poisson variates with parameters $\lambda_1, \lambda_2, \cdots, \lambda_k$ respectively. Prove that the conditional distribution $P\left[X_1 \cap X_2 \cap \cdots \cap X_k | X\right]$ where $X = X_1 + X_2 + \cdots + X_k$ is fixed, is multinomial (P-439, S. C. Gupta)

Proof: Here, $X_i\left(i = 1, 2, \cdots, k\right)$ are independent Poisson variate with parameters λ_i respectively. By additive property of Poisson distribution, we have $X = X_1 + X_2 + \cdots + X_k$ is also a Poisson variate with parameter $\lambda = \lambda_1 + \lambda_2 + \cdots + \lambda_k$ (Say).

Now, $\quad P\left[X_1 \cap X_2 \cap \cdots \cap X_k \ | \ X = n\right]$

$$= P\left[X_1 = r_1 \cap X_2 = r_2 \cap \cdots \cap X_k = r_k \ | \ X = n\right]$$

$$= \frac{P\left[X_1 = r_1 \cap X_2 = r_2 \cap \cdots \cap X_k = r_k \cap X = n\right]}{P\left[X = n\right]}$$

$$= \frac{P\left[X_1 = r_1\right].\;\; P\left[X_2 = r_2\right]\cdots\; P\left[X_{k-1} = r_{k-1}\right]\;\; P\left[X_k = n - r_1 - r_2 - \cdots - r_{k-1}\right]}{P\left[X = n\right]}$$

$\because\;$ $X_1,\; X_2, \cdots, X_k$ are independent.

$$= \frac{\dfrac{e^{-\lambda_1}\,\lambda_1^{r_1}}{r_1!}.\;\dfrac{e^{-\lambda_2}\,\lambda_2^{r_2}}{r_2!}\cdots\dfrac{e^{-\lambda_{k-1}}\,\lambda_{k-1}^{r_{k-1}}}{r_{k-1}!}.\;\;\dfrac{e^{-\lambda_k}\,\lambda_k^{\,n-r_1-r_2\cdots-r_{k-1}}}{\left(n - r_1 - r_2 - \cdots - r_{k-1}\right)!}}{\dfrac{e^{-\lambda}\,\lambda^n}{n!}}$$

$$= \frac{e^{-\left(\lambda_1+\lambda_2+\cdots+\lambda_{k-1}+\lambda_k\right)}\;\;\lambda_1^{r_1}\;\;\lambda_2^{r_2}\;\cdots\;\lambda_k^{\,n-r_1-r_2\cdots-r_{k-1}}}{r_1!\;\;r_2!\;\cdots\;r_{k-1}!\;\left(n - r_1 - r_2 - \cdots r_{k-1}\right)!} \times \frac{n!}{e^{-\lambda}\,\lambda^n}$$

$$= \frac{n!}{r_1!\;\;r_2!\;\cdots\;r_{k-1}!\;\;r_k!}.\;\;\frac{e^{-\lambda}}{e^{-\lambda}}.\;\;\frac{\lambda_1^{r_1}\;\;\lambda_2^{r_2}\;\cdots\;\lambda_k^{r_k}}{\lambda^{r_1+r_2+\cdots r_k}} \qquad \left[\begin{array}{l}\because\;\; r_1 + r_2 + \cdots + r_{k-1} + r_k = n \\ \Rightarrow\;\; r_k = n - r_1 - r_2 - \cdots - r_{k-1} \\ \text{and }\; \lambda_1 + \lambda_2 + \cdots + \lambda_k = \lambda\end{array}\right]$$

$$= \frac{n!}{r_1!\;\;r_2!\;\cdots\;r_k!}\;\;\frac{\lambda_1^{r_1}\;\;\lambda_2^{r_2}\;\cdots\;\lambda_k^{r_k}}{\lambda^{r_1}\;\;\lambda^{r_2}\;\cdots\;\lambda^{r_k}}$$

$$= \frac{n!}{r_1!\;\;r_2!\;\cdots\;r_k!}\left(\frac{\lambda_1}{\lambda}\right)^{r_1}\left(\frac{\lambda_2}{\lambda}\right)^{r_2}\cdots\left(\frac{\lambda_k}{\lambda}\right)^{r_k}$$

$$= \frac{n!}{r_1!\;\;r_2!\;\cdots\;r_k!}\;\;p_1^{r_1}\;\;p_2^{r_2}\;\cdots\;p_k^{r_k}$$

where, $p_i = \dfrac{\lambda_i}{\lambda};\qquad i = 1,\; 2, \cdots, k$

and $\displaystyle\sum_{i=1}^{k} r_i = n, \qquad \sum_{i=1}^{k} p_i = \sum_{i=1}^{k}\frac{\lambda_i}{\lambda} = \frac{1}{\lambda}\left(\lambda_1 + \lambda_2 + \cdots + \lambda_k\right) = \frac{\lambda}{\lambda} = 1$

which is the probability function of multinomial distribution.

Hence the conditional distribution $P\left[X_1 \cap X_2 \cap \cdots \cap X_k \mid X\right]$ is multinomial with

$p_i = \dfrac{\lambda_i}{\lambda};\qquad i = 1,\; 2, \cdots, k.$

Short questions and answers:

(1) Write down the probability function of multinomial distribution (NU-11)

Ans: The probability function of multinomial distribution is

$$f(x_1,\ x_2,\cdots,x_k)=\frac{n!}{x_1!\ x_2!\ \cdots\ x_k!}\ p_1^{x_1}\ p_2^{x_2}\ \cdots\ p_k^{x_k}$$

such that $\sum_{i=1}^{k}x_i=n$ and $\sum_{i=1}^{k}p_i=1.$

The distribution contains k parameters which are n and $(k-1)$ of p's, since $\sum_{i=1}^{k}p_i=1.$

(2) Define multinomial distribution (NU-12, 14)

Ans: k discrete random variables $x_1,\ x_2,\cdots,x_k$ is said to have a multinomial distribution if its probability function is defined as

$$f(x_1,\ x_2,\cdots,x_k)=\frac{n!}{x_1!\ x_2!\cdots\ x_k!}\ p_1^{x_1}\ p_2^{x_2}\ \cdots\ p_k^{x_k}$$

such that $\sum_{i=1}^{k}x_i=n$ and $\sum_{i=1}^{k}p_i=1.$

The distribution contains k parameters which are n and $(k-1)$ of p's, since $\sum_{i=1}^{k}p_i=1.$

Mathematical Problems:

(1) In rolling 12 dice what is the probability of getting 1, 2, 3, 4, 5 and 6 point exactly twice? (P-409, M.K. Roy)

Solution: Here, we have

$$p_i=\frac{1}{6};\qquad i=1,\ 2,\cdots,6$$

$$n=12$$

and $x_1=x_2=x_3=x_4=x_5=x_6=2$

Hence the required probability is

$$f(x_1, x_2, x_3, x_4, x_5, x_6) = \frac{n!}{x_1!\,x_2!\,x_3!\,x_4!\,x_5!\,x_6!} p_1^{x_1}\, p_2^{x_2}\, p_3^{x_3}\, p_4^{x_4}\, p_5^{x_5}\, p_6^{x_6}$$

$$= \frac{12!}{2!\;2!\;2!\;2!\;2!\;2!} \left(\frac{1}{6}\right)^2 \left(\frac{1}{6}\right)^2 \left(\frac{1}{6}\right)^2 \left(\frac{1}{6}\right)^2 \left(\frac{1}{6}\right)^2 \left(\frac{1}{6}\right)^2$$

$$= \frac{12!}{64} \left(\frac{1}{6}\right)^{12}$$

(2) What is the probability of obtaining a sum of 15 points by throwing five dice together?

Or, Five unbiased dice are thrown simultaneously. What is the probability of getting sum of fifteen points on the head appeared? **(NU-14)**

Solution: The number of exhaustive (সম্পূর্ণ) cases in throwing of 5 dice is 6^5.

The number of ways in which the 5 dice thrown will give a sum of 15 points is the coefficient of x^{15} in the expansion of $(x^1 + x^2 + x^3 + \cdots + x^6)^5$.

Favorable number of cases

= Coefficient of x^{15} in $(x + x^2 + x^3 + \cdots + x^6)^5$

= Coefficient of x^{10} in $(1 + x + x^2 + \cdots + x^5)^5$

= Coefficient of x^{10} in $\left(\dfrac{1-x^6}{1-x}\right)^5$

= Coefficient of x^{10} in $(1-x^6)^5 (1-x)^{-5}$

Now, $(1-x^6)^5 = 1 - 5_{c_1}\, x^6 + 5_{c_2}\, x^{12} - \cdots - x^{30}$

$$= 1 - 5\, x^6 + 10\, x^{12} - \cdots - x^{30}$$

and $(1-x)^{-5} = 1 + 5x + \dfrac{5\times6}{2!}x^2 + \dfrac{5\times6\times7}{3!}x^3 + \dfrac{5\times6\times7\times8}{4!}x^4 + \cdots + \dfrac{5\times6\times7\times8\times9\times10\times11\times12\times13\times14}{10!}x^{10} + \cdots$

$$= 1 + 5\,x + 15\,x^2 + 35\,x^3 + 70\,x^4 + \cdots + 1001\,x^{10} + \cdots$$

$\therefore$ Favorable number of cases

= Coefficient of x^{10} in $\left(1 - 5\,x^6 + 10\,x^{12} - \cdots - x^{30}\right)\left(1 + 5\,x + 15\,x^2 + 35\,x^3 + 70\,x^4 + \cdots + 1001\,x^{10} + \cdots\right)$

$= 1001 - 5 \times 70$

Hence the required probability $= \dfrac{651}{6^5} = \dfrac{651}{7776}$

NB: (i) $\sum_{k=0}^{n} z^k = \frac{z^{n+1}-1}{z-1} = \frac{1-z^{n+1}}{1-z}$

$$\Rightarrow 1+z+z^2+\cdots+z^n = \frac{z^{n+1}-1}{z-1}$$

(ii) $\sum_{k=0}^{2} 3^k = 3^0+3^1+3^2 = 1+3+9 = 13$

Again, $\sum_{k=0}^{2} 3^k = \frac{1-3^{2+1}}{1-3}$ $\left[\because \sum_{k=0}^{n} z^k = \frac{1-z^{n+1}}{1-z}\right]$

$$= \frac{1-27}{1-3} = \frac{-26}{-2} = 13$$

(iii) Coefficient of x^3 in $(x+x^2)^2 = x^2+2x^3+x^4$ is 2.

Coefficient of x in $(1+x)^2 = 1+2x+x^2$ is 2.

$\therefore$ Coefficient of x^3 in $(x+x^2)^2 = x^2+2x^3+x^4$ = Coefficient of x in $(1+x)^2 = 1+2x+x^2$ is 2.

(iv) $(1-x)^{-3}$

$$= \{1+(-x)\}^{-3}$$

$$= \binom{-3}{0}(-x)^0 + \binom{-3}{1}(-x)^1 + \binom{-3}{2}(-x)^2 + \binom{-3}{3}(-x)^3 + \cdots$$

$$= 1+(-3)(-x) + \frac{-3!}{2!\,(-3-2)!}x^2 + \frac{-3!}{3!\,(-3-3)!}(-x)^3 + \cdots$$

$$= 1+3x + \frac{-3(-3-1)}{2!}x^2 + \frac{-3(-3-1)(-3-2)}{3!}(-x)^3 + \cdots$$

$$= 1+3x + \frac{3\times 4}{2!}x^2 + \frac{3\times 4\times 5}{3!}x^3 + \cdots$$

পরিসংখ্যান (তত্ত্বীয়)-২০১৫

(Probability Distribution)

বিষয় কোড: 223601

সময়-৪ ঘণ্টা

পূর্ণমান-৮০

[দ্রষ্টব্যঃ- প্রত্যেক বিভাগ হতে ধারাবাহিকভাবে প্রশ্নের উত্তর দিতে হবে।]

ক-বিভাগ

(যে কোন দশটি সংক্ষিপ্ত প্রশ্নের উত্তর দাও)

মান- ১ x ১০ = ১০

১। (ক) বার্ণোলী বিন্যাসের সংজ্ঞা দাও।
[Define Bernoulli distribution.]

(খ) সুষম বিন্যাসের সংজ্ঞা দাও।
[Define uniform distribution.]

(গ) দেখাও যে, দ্বিপদী বিন্যাসের গড় > ভেদাংক।
[Show that, mean > variance of binomial distribution.]

(ঘ) পরিমিত বিন্যাসের ক্ষেত্রে ৫ম কেন্দ্রীয় পরিঘাতের মান কত?
[What is the value of 5^{th} central moment of normal distribution?]

(ঙ) জ্যামিতিক বিন্যাসের "স্মৃতি ভ্রম" ধর্মটি কি?
[What is the "lack of memory" property of geometric distribution?]

(চ) পরিমিত বিন্যাসের ক্ষেত্রে $\overline{X}, Me$ ও Mo এর মধ্যে সম্পর্ক কি?
[What is the relation among $\overline{X}, Me$ and Mo of normal distribution?]

(ছ) পরা-জ্যামিতিক বিন্যাসের সংজ্ঞা দাও।
[Define hyper-geometric distribution.]

(জ) প্রথম প্রকার বিটা বিন্যাসের পরিঘাত উৎপাদকী ফাংশনটি লিখ।
[Write down the moment generating function of beta distribution of first kind.]

(ঝ) পরিমিত রেখা কি?
[What is normal curve?]

(ঞ) দ্বিতীয় প্রকার বিটা বিন্যাসের সম্ভাবনা অপেক্ষকটি লিখ।
[Write down the p.d.f of beta distribution of 2^{nd} kind.]

(ট) আয়তাকার বিন্যাসের মধ্যমা বের কর।
[Find the median of rectangular distribution.]

(ঠ) দ্বি-চলক পৈঁসু বিন্যাসের সংজ্ঞা দাও।
[Define bi-variate Poission distribution.]

খ বিভাগ

(যে কোন পাঁচটি প্রশ্নের উত্তর দাও)

মান- ৪x ৫ = ২০

২। দ্বিপদী বিন্যাসের পরিঘাত উৎপাদনকারী অপেক্ষকটি উদ্ভাবন কর এবং এখান থেকে বিন্যাসটির ভেদাংক নির্ণয় কর।
[Derive the m.g.f of binomial distribution, hence find out variance of the distribution.]

৩। অবিচ্ছিন্ন আয়তাকার বিন্যাসের পরিঘাত উৎপাদকী ফাংশন বের কর এবং সেখান থেকে বিন্যাসটির গড় ও ভেদাংক বের কর।
[Find the moment generating function of continuous uniform distribution and hence find mean and variance of the distribution.]

৪। পরাজ্যামিতিক বিন্যাসের গড় ও ভেদাংক নির্ণয় কর।
[Find out the mean and variance of hyper-geometric distribution.]

৫। দেখাও যে, $\beta(m,n)=\dfrac{\Gamma m\Gamma n}{\Gamma(m+n)}$

[Show that $\beta(m,n)=\dfrac{\Gamma m\Gamma n}{\Gamma(m+n)}$]

৬। দেখাও যে, পৈঁসু বিন্যাসের ক্ষেত্রে, $\mu_{r+1}=rm\mu_{r-1}+m\dfrac{\partial\mu_r}{\partial m}$

[Show that for Poission distribution $\mu_{r+1}=rm\mu_{r-1}+m\dfrac{\partial\mu_r}{\partial m}$.]

৭। মনে কর, X ও Y দুটি স্বাধীন গামা চলক যাদের পরামিতি l এবং m, তাহলে $Z=\dfrac{X}{Y}$ এর বিন্যাস নির্ণয় কর।

[Let X and Y be two independent gamma variates with parameters l and m respectively, then find the distribution of $Z=\dfrac{X}{Y}$.]

৮। যদি $f(x)=be^{-b(x-a)}$; $a\le x\le\infty$, $b>0$ সম্ভাবনা ঘনত্ব অপেক্ষক বিশিষ্ট x একটি অবিচ্ছিন্ন দৈব চলক হয়, তবে দেখাও যে, $a=m-\sigma$ যেখানে m ও σ যথাক্রমে বিন্যাসটির গড় ও পরিমিত ব্যবধান।

[If x is a continuous random variable with pdf $f(x)=be^{-b(x-a)}$; $a\le x\le\infty$, $b>0$, then show that $a=m-\sigma$, where m and σ are mean and standard deviation of the distribution respecitvely .]

৯। দ্বিপদী পরিমিত বিন্যাসের প্রান্তীয় বিন্যাসগুলো নির্ণয় কর।

[Find out the marginal distributions of bi-varitate normal distribution.]

গ বিভাগ

(যে কোন পাঁচটি প্রশ্নের উত্তর দাও)

মান- ১০ x ৫ = ৫০

১০। (ক) দ্বিপদী বিন্যাসের সম্ভাবনা অপেক্ষক উদ্ভাবন কর।

(খ) দ্বিপদী বিন্যাসের ক্ষেত্রে দেখাও যে, $\mu_{r+1}=pq\left(nr\mu_{r-1}+\dfrac{d\mu_{r-1}}{dp}\right)$ যেখানে প্রতীকসমূহ সচরাচর অর্থ বহন করে। এখান থেকে μ_2, μ_3 এবং μ_4 এর মান বের কর।

[(a) Derive the probability function of binomial distribution.

(b) For the binomial distribution show that $\mu_{r+1}=pq\left(nr\mu_{r-1}+\dfrac{d\mu_{r-1}}{dp}\right)$, where the symbols have their usual meanings. Hence find μ_2, μ_3 and μ_4 .]

১১। (ক) পৈঁসু বিন্যাসের কুমুল্যান্ট উৎপাদনকারী ফাংশন বের কর এবং সেখান থেকে β_1 ও β_2 বের কর ও মন্তব্য কর।

(খ) যদি X একটি m পরামানবিশিষ্ট পৈঁসু চলক হয়, তবে দেখাও যে, $Z=\dfrac{X-m}{\sqrt{m}}$ একটি আদর্শ পরিমিত চলক, যখন $m\to\infty$.

[(a) Find the cumulant generating function of Poission distribution and hence find β_1 and β_2 and comment.

(b) If X is a Poission variate with parameter m then show that $Z=\dfrac{X-m}{\sqrt{m}}$ is a standard normal variate, when $m\to\infty$.]

১২। (ক) ঋণাত্মক দ্বিপদী বিন্যাসের যোগবোধক ধর্মটি বিবৃতিসহ প্রমাণ কর। দেখাও যে, ঋণাত্মক দ্বিপদী বিন্যাসের মোট সম্ভাবনা এক।

১২। (খ) ঋণাত্মক দ্বিপদী বিন্যাসের পরিঘাত উৎপাদকী ফাংশন বের কর এবং সেখান থেকে বিন্যাসটির গড় ও ভেদাংক বের কর।

[(a) State and prove the additive property of negative binomial distribution. Show that the total probability of negative binomial distribution is one.

(b) Find the moment generating function of negative binomial distribution and hence find the mean and variance of the distribution.]

১৩। (ক) পরা-জ্যামিতিক বিন্যাসের সাথে দ্বিপদী বিন্যাসের সম্পর্ক স্থাপন কর।

(খ) জ্যামির্তিক বিন্যাসের পরিঘাত উৎপাদকী ফাংশন বের কর এবং সেখান থেকে বা অন্যভাবে β_1 ও β_2 বের কর এবং মন্তব্য কর।

[(a) Establish the relation between hyper-geometric distribution and binomial distribution

(b) Find the moment generating function of geometric distribution and hence or otherwise find β_1 and β_2 and comment.]

১৪। (ক) পৈঁসু বিন্যাস হতে পরিমিত বিন্যাস উদ্ভাবন কর।

(খ) দেখাও যে, পরিমিত বিন্যাসের গড়, মধ্যমা ও প্রচুরক সমান।

[(a) Derive normal distribution from Poission distribution.

(b) Show that mean, median and mode of normal distribution coincides.]

১৫। (ক) গামা বিন্যাসের তরঙ্গ গড় ও প্রচুরক বের কর।

(খ) অক্ষাভিসারী বিন্যাসের মধ্যমা এবং গড় হতে গড় ব্যবধান বের কর।

[(a) Find the harmonic mean and mode of gamma distribution.

(b) Find the median and mean deviation from mean of exponential distribution.]

১৬। (ক) বিটা বিন্যাসের গড় ও ভেদাংক বের কর।

(খ) তুমি কিভাবে প্রথম প্রকার বিটা বিন্যাস হতে দ্বিতীয় প্রকার বিটা বিন্যাস এবং দ্বিতীয় প্রকার বিটা বিন্যাস হতে প্রথম প্রকার বিটা বিন্যাস বের করবে?

[(a) Find the mean and variance of beta distribution.

(b) How will you find the beta distribution of second kind from beta distribution of first kind and beta distribution of first kind from beta distribution of second kind?]

১৭। (ক) দ্বি-চলক পরিমিত বিন্যাসের সংজ্ঞা দাও। বিন্যাসটির পরিঘাত উৎপাদকী ফাংশন বের কর।

(খ) মনে কর, $(X,Y) \sim BVN\left(\mu_1,\mu_2,\sigma_1^2,\sigma_2^2,\rho\right)$, তাহলে দেখাও যে, X ও Y স্বাধীন হবে যদি ও কেবল যদি $\rho=0$ হয়।

[(a) Define bi-variate normal distribution. Find the moment generating function of this distribution.

(b) Let $(X,Y) \sim BVN\left(\mu_1,\mu_2,\sigma_1^2,\sigma_2^2,\rho\right)$, then show that X and Y are independent if and only if $\rho=0$.]

পরিসংখ্যান (তত্ত্বীয়)-২০১৬

[২০১৩-২০১৪ সালের সিলেবাস অনুযায়ী]

বিষয় কোডঃ 223601

(Probability Distribution)

সময়-৪ ঘণ্টা

পূর্ণমান-৮০

[দ্রষ্টব্যঃ- প্রত্যেক বিভাগ হতে ধারাবাহিকভাবে প্রশ্নের উত্তর দিতে হবে।]

ক-বিভাগ

(যে কোন দশটি সংক্ষিপ্ত প্রশ্নের উত্তর দাও)

মান- ১ x ১০ = ১০

১। (ক) ঋণাত্মক দ্বিপদী বিন্যাসের গড় ও ভেদাংক কত?
[What is the mean and variance of negative binomial distribution?]

(খ) জ্যামিতিক বিন্যাসের সংজ্ঞা দাও।
[Define Geometric distribution.]

(গ) পরাজ্যামিতিক বিন্যাসের পরামিতিগুলো কি কি?
[What are the parameters of Hyper Geometric distribution?]

(ঘ) কি কি শর্তে গামা বিন্যাস পরিমিত বিন্যাসে পরিণত হয়?
[Under what conditions gamma distribution tends to normal distribution?]

(ঙ) প্রথম প্রকার বেটা বিন্যাসের সম্ভাবনা অপেক্ষকটি লিখ।
[Write the pdf of beta distribution of 1st kind.]

(চ) পরিমিত বিন্যাসের সম্ভাবনা ঘনত্ব অপেক্ষকটি লিখ।
[Write down the probability density function of normal distribution.]

(ছ) দ্বিচলক পরিমিত বিন্যাসের গড় ও ভেদাংক কত?
[What is the mean and variance of bivariate normal distribution?]

(জ) যদি $f(x)=\theta e^{-\theta x}$, যেখানে $0<x<\infty$ হয়, তবে x চলকের গড় ও ভেদাংক কত?
[If $f(x)=\theta e^{-\theta x}$,where $0<x<\infty$, then what is the mean and variance of x variable?]

(ঝ) অবিচ্ছিন্ন আয়তাকার বিন্যাসের গড় কত, যার ব্যাপ্তি a থেকে b পর্যন্ত?
[What is the mean of a continuous uniform distribution within the interval a to b?]

(ঞ) আদর্শায়িত গামা বিন্যাসের সম্ভাবনা অপেক্ষকটি লিখ।
[Write the pdf of standard gamma distribution.]

(ট) পরাজ্যামিতিক বিন্যাসের গড় ও ভেদাংক কত?
[What is the mean and variance of Hyper Geometric distribution.]

(ঠ) কৌশি বিন্যাসের তিনটি বৈশিষ্ট্য লিখ।
[Write the three properties of Cauchy distribution.]

খ বিভাগ

(যে কোন পাঁচটি প্রশ্নের উত্তর দাও)

মান- ৪x ৫ = ২০

২। পৈঁসু বিন্যাসের পরিঘাত উৎপাদনকারী ফাংশনটি উদ্ভাবন কর এবং এখান থেকে গড় ও ভেদাংক বের কর।
[Derive the mgf of Poission distribution, hence find mean and variance.]

৩। ঋণাত্মক দ্বিপদী বিন্যাসের সম্ভাবনা অপেক্ষকটি উদ্ভাবন কর।
[Derive the probability function of negative binomial distribution.]

৪। পরাজ্যামিতিক বিন্যাসের সম্ভাবনা ফাংশনটি উদ্ভাবন কর।
[Derive the probability function of Hyper Geometric distribution.]

৫। অক্ষাভিসারী বিন্যাসের গড় ও ভেদাংক নির্ণয় কর।
[Find mean and variance of exponential distribution.]

৬। দ্বিতীয় প্রকার বিটা বিন্যাসের তরঙ্গ গড় নির্ণয় কর।

[Find the harmonic mean of beta distribution of 2^{nd} kind.]

৭। গামা বিন্যাসের β_1 ও β_2 নির্ণয় কর।

[Find β_1 and β_2 of gamma distribution.]

৮। জ্যামিতিক বিন্যাসের গড় ও ভেদাংক নির্ণয় কর।

[Find mean and variance of Geometric distribution.]

৯। পরিমিত বিন্যাসের সংজ্ঞা দাও। পরিমিত বিন্যাসের গড়, মধ্যমা ও প্রচুরকের মান কত? পরিমিত বিন্যাসের ব্যবহারগুলো লিখ

[Define normal distribution. What are the values of mean, median and mode of normal distribution? Write the uses of normal distributoin.]

গ বিভাগ

(যে কোন পাঁচটি প্রশ্নের উত্তর দাও)

মান- ১০ X ৫ = ৫০

১০। (ক) পৈঁসু বিন্যাসের সম্ভাবনা অপেক্ষক উদ্ভাবন কর।

(খ) পৈঁসু বিন্যাসের প্রচুরক নির্ণয় কর।

[(a) Derive the probability function of Poission distribution.

(b) Find mode of Poission distribution.]

১১। (ক) জ্যামিতিক বিন্যাসের সম্ভাবনা অপেক্ষক উদ্ভাবন কর।

(খ) জ্যামিতিক বিন্যাসের যোজন ধর্মটি বিবৃতিসহ প্রমাণ কর।

[(a) Derive probability function of Geometric distribution.

(b) State and prove the additive property of Geometric distribution.]

১২। (ক) ঋণাত্মক দ্বিপদী বিন্যাসের প্রচুরক নির্ণয় কর।

(খ) এ বিন্যাসের পরিঘাতের ক্ষেত্রে পৌনঃপুনিক সূত্রটি উদ্ভাবন কর।

[(a) Find the mode of negative binomial distribution.

(b) Derive the recursion formula of negative binomial distribution for moments.]

১৩। (ক) বহুপদী বিন্যাসের সংজ্ঞা দাও। বহুপদী বিন্যাসের সম্ভাবনা অপেক্ষক উদ্ভাবন কর।

(খ) বহুপদী বিন্যাসের গড় ও ভেদাংক বের কর।

[(a) Define multinomial distribution. Derive probability function of this distribution.

(b) Find mean and variance of multinomial distribution.]

১৪। (ক) প্রথম প্রকার বিটা বিন্যাস কি? এই বিন্যাসের তরঙ্গ গড় নির্ণয় কর।

(খ) দ্বিতীয় প্রকার বিটা বিন্যাসের r-তম পরিঘাত নির্ণয় কর এবং এখান থেকে গড় ও ভেদাংক নির্ণয় কর।

[(a) What is beta distribution of first kind? Find the harmonic mean of this distribution.

(b) Find the r-th moment of beta distribution of 2^{nd} kind and hence find mean and variance.]

১৫। (ক) কৌশি বিন্যাসের মধ্যমা নির্ণয় কর।

(খ) যদি x একটি কৌশি চলক হয় তবে দেখাও যে, x^2 একটি $\beta_2\left(\frac{1}{2},\frac{1}{2}\right)$ চলক হবে।

[(a) Find median of Cauchy distribution.

(b) If x is a Cauchy variate, show that x^2 is $\beta_2\left(\frac{1}{2},\frac{1}{2}\right)$ variate.]

১৬। দ্বিচলক পরিমিত বিন্যাস কি? দ্বিচলক পরিমিত বিন্যাসের প্রান্তীয় বিন্যাস $f(x)$ এবং শর্তাধীন বিন্যাস $f\left(\frac{y}{x}\right)$ বের কর।

[What is bi-variate normal distribution? Find marginal distribution of $f(x)$ and condition distribution of $f\left(\frac{y}{x}\right)$ of bi-variate normal distribution.]

১৭। (ক) গামা বিন্যাসের সংজ্ঞা দাও। ইহার গড় ও ভেদাংক বের কর।

(খ) x ও y যথাক্রমে ও পরামানবিশিষ্ট স্বাধীন গামা চলক হলে, দেখাও যে,এবং স্বাধীন হবে।

[(a) Define gamma distribution. Find its mean and variance.

(b) If x and y are two independent gamma variates with parameters m and n respectively, show that variates $u = x + y$ and $v = \frac{x}{y}$ are independent.

পরিসংখ্যান (তত্ত্বীয়)-২০১৭

বিষয় কোড: 223601

(Probability Distribution)

সময়—৪ ঘণ্টা

পূর্ণমান—৮০

[দ্রষ্টব্য:—একই বিভাগের বিভিন্ন প্রশ্নের উত্তর ধারাবাহিকভাবে লিখতে হবে।]

ক বিভাগ

(যে কোন দশটি প্রশ্নের উত্তর দাও)

মান—১×১০=১০

১। (ক) 'বার্ণোলী চেষ্টা' কি?

[What is a 'Bernoulli trial'?]

(খ) পৈঁসু বিন্যাসের গড় ও ভেদাংক কত?

[What is the mean and varriance of Poisson distribution?]

(গ) বিচ্ছিন্ন সুষম বিন্যাসের সংজ্ঞা দাও।

[Define discrete uniform distribution.]

(ঘ) কি কি শর্তে ঋণাত্মক দ্বিপদী বিন্যাস পৈঁসু বিন্যাসের রূপান্তরিত হয়?

[Under what conditions negative binomial distribution becomes to Poisson distribution.]

(ঙ) ঋণাত্মক দ্বিপদী বিন্যাসের সংজ্ঞা দাও।

[Define negative binomial distribution.]

(চ) পরিমিত বিন্যাসের গড় ও ভেদাংক কত?

[What is the mean and variance of normal distribution?]

(ছ) দ্বিতীয় প্রকারের বিটা বিন্যাসের সম্ভাবনা ঘনত্ব অপেক্ষকটি লিখ।

[Write down the probability density function of beta distribution of 2nd kind.]

(জ) দ্বি-চলক পরিমিত বিন্যাসের পরিমিতিগুলো কি কি?

[Wht are the parameters of bi-variate normal distribution?]

(ঝ) অক্ষাভিসারী বিন্যাসের সংজ্ঞা দাও।

[Define exponential distribution?]

(ঞ) আদর্শ পরিমিত বিন্যাসের লেখচিত্র অংকনপূর্বক বৈশিষ্ট্যগুলো দেখাও।

[Sketch a standard normal probability curve and show its area properties.]

(ট) কৌশি বিন্যাসের সম্ভাবনা ঘনত্ব অপেক্ষকটি লিখ।

[Write down the probability density function of Cauchy distribution.]

(ঠ) পরা-জ্যামিতিক বিন্যাসের একটি ব্যবহার লিখ।

[Write down any one use of hyper-geometric distribution.]

খ বিভাগ

(যে কোন পাঁচটি প্রশ্নের উত্তর দাও)

মান—৪×৫=২০

২। বার্ণোলী বিন্যাসের গড় ও ভেদাংক নির্ণয় কর।

[Find the mean and variance of Bernouli distribution.]

৩। দ্বিপদী বিন্যাসের পরিঘাত উৎপাদক ফাংশনটি উদ্ভাবন কর এবং এখান থেকে বিন্যাসটির গড় নির্ণয় কর।

[Derive the moment generating function of binomial distribution and hence find mean of the distribution.]

৪। পরা-জ্যামিতিক বিন্যাসের গড় ও ভেদাংক নির্ণয় কর।

[Find out the mean and variance of hyper-geometric distribution.]

৫। পৈঁসু বিন্যাসের ধর্ম ও ব্যবহার লিখ।

[Write down the properties and uses of Poisson distribution.]

৬। অবিচ্ছিন্ন আয়তাকার বিন্যাসের গড় ও ভেদাংক বের কর।

[Find out the mean and variance of continuous uniform distribution.]

৭। ঋণাত্মক দ্বিপদী বিন্যাসের পৌনঃপুনিক সূত্রটি উদ্ভাবন কর।

[Derive the recursion formula of negative binomial distribution.]

৮। পরিমিত বিন্যাসের গড় কেন্দ্রিক গড় ব্যবধান নির্ণয় কর।

[Find mean deviation about mean of normal distribution.]

৯। প্রচলিত প্রতীকে প্রমাণ কর যে, $\beta(m,n)=\dfrac{\Gamma m\,\Gamma n}{\Gamma m+n}$ ।

[In usual notations, prove that, $\beta(m,n)=\dfrac{\Gamma m\,\Gamma n}{\Gamma m+n}$]

গ বিভাগ

(যে কোন পাঁচটি প্রশ্নের উত্তর দাও)

মান—১০×৫ = ৫০

১০। (ক) দ্বিপদী বিন্যাসের কুমুল্যান্ট উৎপাদনী অপেক্ষক নির্ণয় কর। এখান থেকে বিন্যাসটির β_1 ও β_2 নির্ণয় কর।

(খ) দ্বিপদী বিন্যাসের ধর্ম ও ব্যবহার আলোচনা কর।

[(a) Find the cumulant generating function of binomial distribution. Hence find β_1 and β_2 of the distribution.

(b) Discuss the uses and properties of binomial distribution.]

১১। (ক) দেখাও যে, যদি N -এর মান '∞' হয় তবে পরা-জ্যামিতিক বিন্যাস দ্বিপদী বিন্যাসে রূপান্তরিত হয়।

(খ) পরা-জ্যামিতিক বিন্যাসের সংজ্ঞা দাও। এ বিন্যাসের ধর্মগুলো লিখ ।

[(a) Show that, if N tends to '∞' then hyper-geometric distribution tends to binomial distribution.

(b) Define hyper-geometric distribution. Write down the properties of this distribution.]

১২। (ক) পৈঁসু বিন্যাস হতে পরিমিত বিন্যাস উদ্ভাবন কর।

(খ) প্রমাণ কর যে, পরিমিত বিন্যাসের জোড় কেন্দ্রীয় পরিঘাতের মান

$$\mu_{2r}=1\times3\times5\times\cdots\times(2r-1)\sigma^{2r};\ \ r=1,\ 2,\ 3\cdots$$

[(a) Derive normal distribution from Poisson distribution.

(b) Prove that, even order central moment of a normal distribution is:

$$\mu_{2r}=1\times3\times5\times\cdots\times(2r-1)\sigma^{2r};\ \ r=1,\ 2,\ 3\cdots$$

১৩। (ক) সমরূপ বিন্যাসের সংজ্ঞা দাও। দেখাও যে, এটি একটি অনতি সূঁচালো বিন্যাস।

(খ) যদি $X_i\sim U(0,1);\ i=1,2,3,\cdots,n$ পরস্পর স্বাধীন হয় তবে $V=2\sum_{i=1}^{n}\log_e X_i$ এর সম্ভাবনা বিন্যাস নির্ণয় কর।

[(a) Define uniform distribution. Show that, it is a platykurtic distribution.

(b) If $X_i\sim U(0,1);\ i=1,2,3,\cdots,n$ are independent then find the distribution of $V=2\sum_{i=1}^{n}\log_e X_i$.]

১৪। (ক) দেখাও যে, পরামিতি $m\to\infty$ এর জন্য গামা বিন্যাস পরিমিত বিন্যাসে রূপান্তরিত হয়।

(খ) যদি x একটি পরিমিত চলক হয় তবে দেখাও যে, $u=\dfrac{1}{2}\left(\dfrac{x-\mu}{\sigma}\right)^2$ একটি '$\dfrac{1}{2}$' পরামানবিশিষ্ঠ গামা চলক।

[(a) Show that for parameter $m\to\infty$, gamma distribution tends to normal distribution.

(b) If x is a normal variate then show that $u=\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2$ is a gamma variate with parameter '$\frac{1}{2}$'.]

১৫। (ক) প্রথম প্রকার বিটা বিন্যাসের প্রচুরক বের কর।

(খ) যদি x ও y যথাক্রমে m এবং n পরামানবিশিষ্ট দু'টি স্বাধীন গামা চলক হয় তবে দেখাও যে, (i) $u=x+y$ এবং $v=\frac{x}{x+y}$ চলকদ্বয় স্বাধীন হবে; (ii) u একটি $\gamma(m+n)$ চলক এবং v একটি $\beta_1(m,n)$ চলক হবে।

[(a) Find the mode of beta distribution of first kind.

(b) If x and y are two independent gamma variates with parameter m and n respectively, then show that (i) $u=x+y$ and $v=\frac{x}{x+y}$ are independent variates; (ii) u is a $\gamma(m+n)$ variate and v is a $\beta_1(m,n)$ variate.]

১৬। (ক) কৌশি বিন্যাসের সংজ্ঞা দাও। যদি $x_1,x_2,\cdots,x_n$ পরিমিত কৌশি বিন্যাস হতে গৃহীত n সংখ্যক দৈব পর্যবেক্ষণ হয় তবে $\overline{X}=\frac{1}{n}\sum_{i=1}^{n}x_i$ এর বিন্যাস নির্ণয় কর।

(খ) অক্ষাভিসারী বিন্যাসের সংজ্ঞা দাও। θ পরামিতিবিশিষ্ট একটি অক্ষাভিসারী বিন্যাসের গড় কেন্দ্রীক গড় ব্যবধান এবং মধ্যমা নির্ণয় কর।

[(a) Define Cauchy distribution. If $x_1,x_2,\cdots,x_n$ are randomly taken n observations from standard Cauchy distribution then find the distribution of $\overline{X}=\frac{1}{n}\sum_{i=1}^{n}x_i$.

(b) Define exponential distribution. Find the median and mean deviation about mean of an exponential distribution with parameter θ.]

১৭। (ক) দ্বি-চলক পরিমিত বিন্যাসের সংজ্ঞা দাও। ইহার mgf নির্ণয় কর। এখান থেকে বিন্যাসটির গড় বের কর।

(খ) যদি x ও y দু'টি আদর্শ পরিমিত চলক হয় যাদের সংশ্লেষাংক ρ, তবে প্রমাণ কর যে, $cor(x^2,y^2)=\rho^2$.

[(a) Define bi-variate normal distribution. Find it mgf. Hence find mean of this distribution.

(b) If x and y are two standard normal variates with corelation co-efficient ρ, then prove that, $\text{cor}(x^2,y^2)=\rho^2$.]

পরিসংখ্যান-২০১৮

বিষয় কোড ঃ 223601

(Probability Distribution)

সময় - ৪ ঘন্টা

পূর্ণমান - ৮০

[দ্রষ্টব্য ঃ – একই বিভাগের প্রশ্নের উত্তর ধারাবাহিকভাবে লিখতে হবে।]

ক বিভাগ

(যে কোন দশটি প্রশ্নের উত্তর দাও)

মান – ১ × ১০ = ১০

১। (ক) বার্নোলী বিন্যাস কি?

[What is Bernoulli distribution?]

(খ) দ্বিপদী বিন্যাসের গড় ও ভেদাংক যথাক্রমে ২০ ও ৪ হলে চেষ্টার সংখ্যা কত?

[If the mean and variance of a binomial distribution are 20 and 4 respectively then find the number of trials.]

(গ) জ্যামিতিক বিন্যাস সংজ্ঞায়িত কর।

[Define Geometric distribution.]

(ঘ) ঋণাত্মক দ্বিপদী বিন্যাসের পরিঘাত উৎপাদক অপেক্ষকটি লিখ।

[Write down the moment generating function of negative binomial distribution.]

(ঙ) পরাজ্যামিতিক বিন্যাসের ভেদাংক কত?

[What is the variance of Hyper Geometric distribution?]

(চ) একটি অক্ষাভিসারী বিন্যাসের গড় ৪ হলে উহার সম্ভাবনা ঘনত্ব অপেক্ষকটি লিখ।

[Write down the probability density function of an exponential distribution with mean 4.]

(ছ) অবিচ্ছিন্ন আয়তাকার বিন্যাসের $E(x)$ বের কর যখন $2 \leq x \leq 7$।

[Find $E(x)$ of a continuous uniform distribution when $2 \leq x \leq 7$.]

(জ) প্রথম প্রকার বিটা বিন্যাসের সম্ভাবনা ঘনত্ব অপেক্ষকটি লিখ।

[Write the probability density function of beta distributio of 1^{st} kind.]

(ঝ) পরিমিত বিন্যাসের গড় ব্যবধান ও পরিমিত ব্যবধানের সম্পর্কটি লিখ।

[Write down the relationship between the standard deviation and mean deviation of normal distribution.]

(ঞ) Weibull বিন্যাসের সংজ্ঞা দাও।

[Define Weibull distribution.]

(ট) Maxwell বিন্যাসের সম্ভাবনা ঘনত্ব অপেক্ষকটি লিখ।

[Write down the probability density function of Maxwell distribution.]

(ঠ) দ্বিচলক পরিমিত বিন্যাসের পরামিতি কতটি?

[How many parameters have a bi-variate normal distribution?]

খ বিভাগ

(যে কোনো পাঁচটি প্রশ্নের উত্তর দাও)

মান – ৪ × ৫ = ২০

২। একটি দ্বিপদী চলক x-এর জন্য $p(x = 0) = 2p(x = 1) = 9p(x = 2)$ হলে সম্ভাবনা অপেক্ষকটি নির্ণয় কর।

[For a binomial variate x, if $p(x = 0) = 2p(x = 1) = 9p(x = 2)$ then find the probability function.]

৩। পৈসুঁ বিন্যাসের সম্ভাবনা অপেক্ষকটি উদ্ভাবন কর।

[Drive the probability function of Poisson distribution.]

৪। ঋণত্মক দ্বিপদী বিন্যাসের গড় ও ভেদাংক নির্ণয় কর।

[Find the mean and variance of Negative binomial distribution.]

৫। জ্যামিতিক বিন্যাসের পরিঘাত উৎপাদকী অপেক্ষক বের কর। অতঃপর গড় ও ভেদাংক নির্ণয় কর।

[Determine the probability generating function of Geometric distribution. Hence find the mean and variance.]

৬। পরিমিত বিন্যাস কি? এর ধর্মাবলি উল্লেখ কর।

[What is Normal distribution? Mention its properties.]

৭। গামা চলক কি? গামা বিন্যাসের নিয়ামক অপেক্ষকটি নিরূপণ কর।

[What id Gamma Variate? Determine the characteristic function of Gamma distribution.]

৮। দ্বিতীয় প্রকারের বিটা বিন্যাস কি? এর তরঙ্গ গড় নির্ণয় কর।

[What is beta distribution of 2nd kind? Find its harmonic mean.]

৯। কৌশি বিন্যাসের নিয়ামক অপেক্ষক নির্ণয় কর।

[Find out the characteristic function of Cauchy distribution.]

গ বিভাগ

(যে কোনো পাঁচটি প্রশ্নের উত্তর দাও)

মান − ১০ × ৫ = ৫০

১০। (ক) দ্বিপদী বিন্যাসের সম্ভাবনা অপেক্ষকটি উদ্ভাবন কর।

(খ) দ্বিপদী বিন্যাসের ক্ষেত্রে প্রচলিত প্রতীকে দেখাও যে, $\mu_{r+1} = pq\left[\frac{d}{dp}\mu_r + nr\,\mu_{r-1}\right]$। অতঃপর μ_2 এবং μ_3 নির্ণয় কর।

[(a) Drive the probability function of binomial distribution.

(b) For the binomial distribution in usual notation show that, $\mu_{r+1} = pq\left[\frac{d}{dp}\mu_r + nr\,\mu_{r-1}\right]$. Hence find μ_2 and μ_3.]

১১। (ক) পৈসোঁ বিন্যাসের কুমুল্যান্ট উৎপাদকী অপেক্ষক নির্ণয় কর। অতঃপর β_1 ও β_2 বের কর।

(খ) পৈসোঁ বিন্যাসের যোগবোধক ধর্মটি বিবৃতিসহ প্রমাণ কর।

[(a) Find the cumulant generating function of Poission distribution. Hence find β_1 & β_2.

(b) State and prove the additive property of Poission distribution.]

১২। (ক) জ্যামিতিক বিন্যাসের সম্ভাবনা অপেক্ষকটি উদ্ভাবন কর।

(খ) জ্যামিতিক বিন্যাসের 'স্মৃতিভ্রম' ধর্মটি বিবৃতিসহ প্রমাণ কর।

[(a) Derive the probability function of Geometric distribution.

(b) State and prove the "Lack of Memory" property Geometric distribution]

১৩। (ক) পরাজ্যামিতিক বিন্যাসের সম্ভাবনা অপেক্ষকটি উদ্ভাবন কর।

(খ) বহুপদী বিন্যাসের পরিঘাত উৎপাদকী অপেক্ষক নিরূপণ কর। অতঃপর এ বিন্যাসের গড় ও ভেদাংক নির্ণয় কর।

[(a) Derive the probability function of Hyper Geometric distribution.

(b) Determine the moment generating function of multinomial distribution. Hence find the mean & variance of this distribution.]

১৪। (ক) পরিমিত বিন্যাসের মধ্যমা ও প্রচুরক নির্ণয় কর।

(খ) দেখাও যে, পরিমিত বিন্যাসের ক্ষেত্রে $\beta_1 = 0$ এবং $\beta_2 = 3$।

[(a) Find the median and mode of normal distribution.

(b) Show that in case of normal distribution $\beta_1 = 0$ and $\beta_2 = 3$.]

১৫। (ক) আয়তাকার বিন্যাস কি? অবিচ্ছিন্ন আয়তাকার বিন্যাসের ক্ষেত্রে গড় হতে গড় ব্যবধান নির্ণয় কর।

(খ) অক্ষাভিসারী বিন্যাসের কুমুল্যান্ট উৎপাদকী অপেক্ষক বের কর এবং অতঃপর দেখাও যে, অক্ষাভিসারী বিন্যাসটি হলো অতি সূচাঁলো বিন্যাস।

[(a) What is Rectangular distribution? Find the mean deviation from mean in case of continuous rectangular distribution.

(b) Find out the cumulant generating function of exponential distribution and hence show that exponential distribution is platykurtic distribution.]

১৬। (ক) বিটা বিন্যাস কি? কিভাবে প্রথম প্রকার বিটা বিন্যাস হতে ২য় প্রকার বিটা বিন্যাস এবং ২য় প্রকার বিটা বিন্যাস হতে

প্রথম প্রকার বিটা বিন্যাস পাওয়া যায়?

(খ) যদি x ও y দুইটি স্বাধীন গামা চলক হয় তবে দেখাও যে, $(x + y)$ গামা বিন্যাস মেনে চলে কিন্তু $\left(\frac{x}{y}\right)$ ২য় প্রকার বিটা বিন্যাস মেনে চলে।

[(a) What is Beta distribution? How will you get Beta distribution of 2nd kind from Beta distribution of 1st kind and Beta distribution of 1st kind from Beta distribution of 2nd kind?

(b) If x and y are two independent Gamma variates, then show that $(x + y)$ follows Gamma distribution, but $\left(\frac{x}{y}\right)$ follows Beta distribution of 2nd kind.]

১৭। (ক) দ্বিচলক পরিমিত বিন্যাসের সম্ভাবনা ঘনত্ব অপেক্ষকটি উপস্থাপন কর এবং এ বিন্যাসের ধর্মাবলি বিবৃত কর।

(খ) যদি $(x_1, x_2) \sim$ BVN $(\mu_1, \mu_2, \sigma_1^2, \sigma_2^2, \rho)$ হয় তবে x_1 দৈবচলকের প্রান্তীয় বিন্যাসের সম্ভাবনা ঘনত্ব অপেক্ষক নির্ণয় কর।

[(a) Represent the probability density function of bivariate normal distribution and state the properties of this distribution.

(b) If $(x_1, x_2) \sim$ BVN $(\mu_1, \mu_2, \sigma_1^2, \sigma_2^2, \rho)$ then determine the p.d.f of the marginal distribution of the random variable x_1.]

পরিসংখ্যান (তত্ত্বীয়)-২০১৯

বিষয় কোড ঃ 223601

Probability Distribution

সময় - ৪ ঘন্টা

পূর্ণমান - ৮০

[**দ্রষ্টব্য ঃ –** বিভিন্ন বিভাগের প্রশ্নের উত্তর ধারাবাহিকভাবে লিখতে হবে।]

ক বিভাগ

(যে কোন **দশটি** প্রশ্নের উত্তর দাও)

মান- ১ x ১০ = ১০

১। (ক) বার্ণোলী চেষ্টা কী?
[What is a Bernoulli trial?]

(খ) পৈঁসু বিন্যাসের গড় ৫ হলে সম্ভাবনা অপেক্ষকটি লিখ।
[Suppose mean of a Poisson distribution be 5, then write down the probability function.]

(গ) ঋণাত্মক দ্বিপদী বিন্যাসের সম্ভাবনা অপেক্ষকটি লিখ।
[Write down the probability function of negation binomial distribution.]

(ঘ) জ্যামিতিক বিন্যাসের পরিঘাত উৎপাদকী অপেক্ষকটি লিখ।
[Write down the moment generation function of geometric distribution?]

(ঙ) কখন পরাজ্যামিতিক বিন্যাস দ্বিপদী বিন্যাসে রূপান্তরিত হয়?
[When hyper geometric distribution converted to a binomial distribution?]

(চ) বহুপদিক বিন্যাসে সম্ভাবনা অপেক্ষকটি লিখ।
[Write down the probability function of multinomial distribution.]

(ছ) পরিমিতি বিন্যাসের ৪র্থ কেন্দ্রীয় পরিঘাতের মান কত?
[What is the value of the 4th central moment of normal distribution?]

(জ) অক্ষাভিসারী বিন্যাস সংজ্ঞায়িত কর।
[Define exponential distribution.]

(ঝ) ২য় প্রকার বিটা বিন্যাসের সম্ভাবনা ঘনত্ব অপেক্ষকটি লিখ।
[Write down the p.d.f. of beta distribution of 2nd kind.]

(ঞ) দ্বি-পরামিতিক গামা বিন্যাসের সম্ভাবনা ঘনত্ব ফাংশনটি লিখ।
[Write down the probability density function of a bi-parametric gamma distribution.]

(ট) দ্বি-চলক পরিমিত বিন্যাসের সম্ভাবনা ঘনত্ব অপেক্ষকটি লিখ।
[Write down the probability density function of bi-variate normal distribution.]

(ঠ) সাধারণ কৌশি বিন্যাসের গড় কত?
[What is the mean of general Cauchy distribution?]

খ বিভাগ

(যে কোন **পাঁচটি** প্রশ্নের উত্তর দাও)

মান- ৪ x ৫ = ২০

২। দ্বি-পদী বিন্যাসের পরিঘাত উৎপাদনকারী অপেক্ষক বের কর এবং এখান থেকে উক্ত বিন্যাসের গড় ও ভেদাংক বের কর।
[Find the moment generating function of bi-nominal distribution and hence find its mean and variance.]

৩। দেখাও যে, ঋণাত্মক দ্বিপদী বিন্যাসের সীমাস্থ রূপ হলো পৈঁসু বিন্যাস।
[Show that, Poisson distribution is a limiting form of the negative binomial distribution.]

৪। জ্যামিতিক বিন্যাসের ক্ষেত্রে দেখাও যে, $P(x \geq r+j \mid x \geq j) = P(x \geq r) = q^r$.
[In case of geometric distribution show that, $P(x \geq r+j \mid x \geq j) = P(x \geq r) = q^r$.]

৫। পরা-জ্যামিতিক বিন্যাস কি? দ্বিপদী বিন্যাসের সাথে এর সম্পর্ক প্রতিষ্ঠা কর।
[What is hyper-geometric distribution? Establish its relation with binomial distribution.]

৬। যদি দৈব চলক $x \sim N(\mu, \sigma^2)$ হয় তবে, দেখাও যে, $u = \frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2$ হবে "$\frac{1}{2}$" পরামিতি বিশিষ্ট গামা চলক।

[If random variable $x \sim N(\mu, \sigma^2)$, then show that $u = \frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2$ be a gamma variate with parameter " $\frac{1}{2}$ " .]

৭। অবিচ্ছিন্ন আয়তাকার বিন্যাসের গড় ও ভেদাংক নির্ণয় কর।

[Find the mean and variance of continuous uniform distribution.]

৮। অক্ষাভিসারী বিন্যাসের মধ্যমা ও তরঙ্গ গড় নির্ণয় কর।

[Find median and harmonic mean of exponential distribution.]

৯। দ্বি-চলক পরিমিত বিন্যাসের পরিঘাত উৎপাদকী অপেক্ষক নির্ণয় কর।

[Find the moment generating function of bi-variate normal distribution.]

গ বিভাগ

(যে কোনো **পাঁচটি** প্রশ্নের উত্তর দাও)

মান- ১০ x ৫ = ৫০

১০। (ক) পরিঘাতের ক্ষেত্রে পৈঁসু বিন্যাসের পৌনঃপুনিক সূত্রটি প্রতিষ্ঠা কর।

(খ) পরা-জ্যামিতিক বিন্যাসের গড় ও ভেদাংকের মান বের কর।

[(a) Establish the recurrence relation for the moments of Poisson distribution.

(b) Determine the mean and variance of hyper-geometric distribution.]

১১। (ক) ঋণাত্মক দ্বিপদী বিন্যাসের সম্ভাবনা অপেক্ষকটি উদ্ভাবন কর।

(খ) ঋণাত্মক দ্বিপদী বিন্যাসের কুমুল্যান্ট উৎপাদকী অপেক্ষক নির্ণয় কর এবং এখান থেকে β_1 ও β_2 নির্ণয় কর।

[(a) Derive the probability function of negative binomial distribution.

(b) Determine the cumulant generating function of negative binomial distribution and hence find β_1 and β_2 .]

১২। (ক) অক্ষাভিসারী বিন্যাসের গড় ও ভেদাংক নির্ণয় কর।

(খ) অবিচ্ছিন্ন আয়তাকার বিন্যাসের r -তম কেন্দ্রীয় পরিঘাত নির্ণয় কর। অতঃপর β_1 ও β_2 বের কর।

[(a) Find the mean and variance of exponential distribution.

(b) Determine r-th central moment of the continuous rectangular distribution. Hence find β_1 and β_2 .]

১৩। (ক) পরিমিত বিন্যাসের গড় কেন্দ্রিক গড় ব্যবধান নির্ণয় কর।

(খ) প্রমাণ কর যে, পরিমিত বিন্যাসের জোড় স্থানীয় কেন্দ্রীয় পরিঘাত হলো-

$$\mu_{2r} = (2r-1)(2r-3)\ldots\ldots 5.3.1.\sigma^{2r}; r = 1, 2, 3, \ldots\ldots$$

[(a) Find the mean deviation about mean for normal distribution.

(b) Prove that the even order central moment of normal distribution is

$$\mu_{2r} = (2r-1)(2r-3)\ldots\ldots 5.3.1.\sigma^{2r}; r = 1, 2, 3, \ldots\ldots]$$

১৪। (ক) ১ম প্রকার বিটা বিন্যাস বলতে কী বুঝ? এর তরঙ্গ গড় নির্ণয় কর।

(খ) ২য় প্রকার বিটা বিন্যাসের r -তম পরিঘাত নির্ণয় কর এবং এখান থেকে বিন্যাসের গড় ও ভেদাঙ্ক নির্ণয় কর।

[(a) What do you mean by beta distribution of 1st kind? Find its harmonic mean.

(b) Determine the r-th moment of beta distribution of 2nd kind and hence find the mean and variance of the distribution.]

১৫। (ক) সমরূপ বিন্যাসের সংজ্ঞা দাও। দেখাও যে, সমরূপ বিন্যাস একটি অনতিসূঁচালো বিন্যাস।

(খ) যদি $X_i \sim U(0,1), i = 1, 2, 3, \ldots\ldots, n$ পরস্পর স্বাধীন হয়, তবে, $W = -2\sum_{i=1}^{n}\log_e X_i$ - এর সম্ভাবনা বিন্যাস নির্ণয় কর।

[(a) Define uniform distribution. Show that, uniform distribution is platykurtic.

(b) If $X_i \sim U(0,1), i = 1, 2, 3, \ldots\ldots, n$ are independent, then find the probability distribution of $W = -2\sum_{i=1}^{n}\log_e X_i$.]

১৬। (ক) কৌশি বিন্যাস কি? এর মধ্যমা নির্ণয় কর।

(খ) যদি দৈবচলক x আদর্শ কৌশি বিন্যাস অনুসরণ করে, তবে দেখাও যে, $x^2 \sim \beta_2\left(\frac{1}{2}, \frac{1}{2}\right)$.

[(a) What is Cauchy distribution? Find its median.

(b) If the random variable x follows standard Cauchy distribution, then show that $x^2 \sim \beta_2\left(\frac{1}{2}, \frac{1}{2}\right)$.]

১৭। (ক) যদি x ও y দুটি আদর্শ পরিমিত চলক হয় যাদের সংশ্লেষাংক ρ তবে প্রমাণ কর যে cor $(x^2, y^2) = \rho^2$.

(খ) x ও y এর দ্বিচলক পরিমিত বিন্যাসের ক্ষেত্রে শর্তাধীন সম্ভাবনা ঘনত্ব অপেক্ষক $f(y|x)$ নির্ণয় কর।

[(a) If x and y are two standard normal variate with correlation co-efficient ρ then prove that $\operatorname{cor}(x^2, y^2) = \rho^2$.

(b) In case of bi-variate normal distribution of x and y determine the conditional probability density function $f(y|x)$.]

পরিসংখ্যান (তত্ত্বীয়)-২০২০

বিষয় কোড ঃ 223601

Probability Distribution

সময় - ৪ ঘন্টা

পূর্ণমান - ৮০

[**দ্রষ্টব্য ঃ –** প্রতিটি বিভাগের বিভিন্ন প্রশ্নের উত্তর ধারাবাহিকভাবে লিখতে হবে।]

ক বিভাগ

(যে কোন **দশটি** প্রশ্নের উত্তর দাও)

মান- ১ x ১০ = ১০

১। (ক) বার্ণোলি বিন্যাস কী?
[What is Bernoulli distribution?]

(খ) কখন দ্বিপদী বিন্যাস অতিসূচাল হয়?
[When binomial distribution is leptokurtic?]

(গ) পৈঁসো বিন্যাসের পরিঘাতের পুনঃপৌনিক সূত্রটি লেখ।
[Write down the recurrence relation for moments of Poisson distribution.]

(ঘ) কী শর্তে ঋণাত্মক দ্বিপদী বিন্যাস জ্যামিতিক বিন্যাসে পরিণত হয়?
[Under what conditions negative binomial distribution reduces to geometric distribution?]

(ঙ) ঋণাত্মক দ্বিপদী বিন্যাসের একটি বাস্তব উদাহরণ দাও।
[Give a real-life example of negative binomial distribution.]

(চ) জ্যামিতিক বিন্যাসের যোগবোধক ধর্মটি বর্ণনা কর।
[State the additive property of geometric distribution.]

(ছ) জ্যামিতিক বিন্যাস বলতে কী বুঝ?
[What do you mean by geometric distribution?]

(জ) পরাজ্যামিতিক বিন্যাসের ভেদাংক কত?
[What is the variance of hypergeometric distribution?]

(ঝ) পরিমিত বিন্যাসের জোড় কেন্দ্রীয় পরিঘাতের মান কত?
[What is the value of ever order central moments of normal distribution?]

(ঞ) অক্ষাভিসারী বিন্যাসের সম্ভাবনা ঘনত্ব অপেক্ষকটি লেখ।
[Write down the probability density function of exponential distribution?]

(ট) Weibul বিন্যাস সংজ্ঞায়িত কর।
[Define Weibul distribution.]

(ঠ) দ্বিচলক বিশিষ্ট পরিমিত বিন্যাসের পরামানসমূহ উল্লেখ কর।
[Mention the parameters of bi-variate distribution.]

খ বিভাগ

(যে কোন **পাঁচটি** প্রশ্নের উত্তর দাও)

মান- ৪ x ৫ = ২০

২। বার্ণোলি বিন্যাসের ১ম চারটি কেন্দ্রীয় পরিঘাতের মান নির্ণয় কর।
[Find out the first four central moments of Bernoulli distribution.]

৩। দেখাও যে গামা বিন্যাস হলো পরিমিত বিন্যাসের একটি বিশেষ রূপ।
[Show that gamma distribution is a special case of normal distribution.]

৪। ঋণাত্মক দ্বিপদী বিন্যাসের সংজ্ঞা দাও। এ বিন্যাসের ব্যবহার আলোচনা কর।
[Define negative binomial distribution. Discuss the uses of this distribution.]

৫। জ্যামিতিক বিন্যাসের স্মৃতিভ্রম ধর্মটি প্রমাণ কর।
[Prove the lack of memory property of geometric distribution.]

৬। ১ম প্রকারের বিটা বিন্যাসের সংজ্ঞা দাও। এ বিন্যাসের গড় নির্ণয় কর।
[Define beta distribution of 1^{st} kind. Find out the mean of this distribution.]

৭। পরিমিত বিন্যাসের বৈশিষ্ট্য আলোচনা কর।

[Discuss the characteristics of normal distribution.]

৮। আয়তাকার বিন্যাসের ক্ষেত্রে প্রচলিত প্রতীকে দেখাও যে, $MD = \frac{\sqrt{3}}{2} SD$.

[In case of rectangular distribution, in usual notation show that $MD = \frac{\sqrt{3}}{2} SD$.]

৯। দেখাও যে, দুটি প্রমিত পরিমিত চলকের অনুপাত প্রমিত বিন্যাসকে মেনে চলে।
[Show that the ratio of two independent standard normal variates follow standard Cauchy distribution.]

গ বিভাগ

(যে কোনো **পাঁচটি** প্রশ্নের উত্তর দাও)

মান- ১০ x ৫ = ৫০

১০। (ক) প্রয়োজনীয় অনুমিতি উল্লেখপূর্বক দ্বিপদী বিন্যাসের সম্ভাবনা অপেক্ষক উদ্ভাবন কর।
(খ) দ্বিপদী বিন্যাসের পরিঘাতের পুনঃপৌনিক সূত্রটি উদ্ভাবন কর। এখান থেকে বিন্যাসটির β_1 নির্ণয় কর।
[(a) Derive the probability function of binomial distribution mentioning the necessary assumptions.
(b) Derive the recurrence relation for moments of binomial distribution. Hence find out β_1 of the distribution.]

১১। (ক) পৈঁসু চলক কী? পৈঁসু চলকের ব্যবহার লেখ।
(খ) পৈঁসু বিন্যাসের কুমুল্যান্ট উৎপাদনকারী অপেক্ষক নির্ণয় কর এবং এখান থেকে গড় ও ভেদাংক নির্ণয় কর।
[(a) What is Poisson variate? Write down the uses of Poisson variate.
(b) Determine the cumulant generating function of Poisson distribution and hence find its mean and variance.]

১২। (ক) বহুপদী বিন্যাস কাকে বলে? বহুপদী বিন্যাসের সম্ভাবনা অপেক্ষক উদ্ভাবনা কর।
(খ) যদি দৈব চলক $x_1, x_2, \cdots, x_k$ এর বিন্যাস n ও $p_i \; (i = 1, 2, \cdots, k)$ পরামিতি বিশিষ্ট বহুপদী বিন্যাস হয় তবে দেখাও যে,
(i) x_i এর প্রান্তীয় বিন্যাস n ও $p_i \; (i = 1, 2, \cdots, k)$ পরামিতি বিশিষ্ট দ্বিপদী বিন্যাস এবং
(ii) $Cov(x_i, x_j) = -n p_i p_j; \; \forall \; i \neq j = 1, 2, \cdots, k$.
[(a) What is multinomial distribution? Derive the probability function of multinomial distribution.
(b) If the distribution of random variables $x_1, x_2, \cdots, x_k$ is a multinomial distribution with parameters n and $p_i \; (i = 1, 2, \cdots, k)$ then show that
(i) The marginal distribution of x_i is binomial with parameters n and $p_i \; (i = 1, 2, \cdots, k)$, (ii) $Cov(x_i, x_j) = -n p_i p_j; \; \forall \; i \neq j = 1, 2, \cdots, k$]

১৩। (ক) জ্যামিতিক বিন্যাসের পরিঘাত উৎপাদনকারী অপেক্ষক বের কর এবং এখান থেকে উক্ত বিন্যাসের গড় ও ভেদাংক নির্ণয় কর।
(খ) পরাজ্যামিতিক বিন্যাসের মডেল ব্যবহার করে একটি লেকের মাছের সংখ্যা কীভাবে পরিমাপ করা যায়, ব্যাখ্যা কর।
[(a) Find the moment generating function of geometric distribution and hence find its mean and variance.
(b) Explain how you will use hyper geometric model to estimate the number of fish in a lake.]

১৪। (ক) অক্ষাভিসারী বিন্যাস কী? ইহার মধ্যমা এবং গড় কেন্দ্রিক গড় ব্যবধান নির্ণয় কর।
(খ) যদি দুটি স্বাধীন দৈব চলক X ও Y এর সাধারণ অক্ষাভিসারী pdf-

$$f(x) = \begin{cases} e^{-x} & ; \; x \geq 0 \\ 0 & ; \; \text{Otherwise} \end{cases}$$

থাকে তবে $(Y - X)$ এর pdf নির্ণয় কর এবং মন্তব্য কর।

[(a) What is an exponential distribution? Find its median and mean deviation about mean.

(b) If X and Y are two independent random variables with a common exponential pdf-

$$f(x)=\begin{cases} e^{-x} & ; \ x \ge 0 \\ 0 & ; \ \text{Otherwise} \end{cases}$$

Then find the pdf of $(Y-X)$ and comment.]

১৫। (ক) দ্বিতীয় প্রকার বিটা বিন্যাসের প্রচুরক নির্ণয় কর। প্রচলিত প্রতীকে প্রমাণ কর যে, $\beta(m,\ n)=\frac{\Gamma m\ \Gamma n}{\Gamma m+n}$.

(খ) $\beta\left(\frac{1}{2},\ \frac{1}{2}\right)$ এর মান বের কর। এখান থেকে $\Gamma\frac{1}{2}$ এর মান বের কর।

[(a) Find out the mode of beta distribution of 2^{nd} kind. In usual notation prove that,

$$\beta(m,\ n)=\frac{\Gamma m\ \Gamma n}{\Gamma m+n}.$$

(b) Find out the value of $\beta\left(\frac{1}{2},\ \frac{1}{2}\right)$. Hence find out the value of $\Gamma\frac{1}{2}$.]

১৬। (ক) সাধারণ গামা বিন্যাসের সংজ্ঞা দাও। ইহার পরিঘাত উৎপাদনী অপেক্ষক নির্ণয় কর। এখান থেকে বা অন্যভাবে বিন্যাসটির ভেদাংক, β_1 ও β_2 নির্ণয় কর।

(খ) দেখাও যে, পরিমিত বিন্যাসের গড়, মধ্যমা ও প্রচুরক ইহার পরামিতি μ -এর সমান।

[(a) Define generalized gamma distribution. Find its moment generating function. Hence or otherwise find its variance, β_1 and β_2.

(b) Show that, the mean, median and mode of normal distribution is equal to its parameter μ.]

১৭। (ক) দ্বি-চলক পরিমিত বিন্যাস কী? ইহার প্রান্তীয় বিন্যাস নির্ণয় কর।

(খ) দুটি দৈব চলক X ও Y-এর গড় শূন্য, ভেদাংক যথাক্রমে σ_1^2 ও σ_2^2 এবং সংশ্লেষাংক ρ. চলক দুটি পরিমিত বিন্যাসে বিন্যস্ত হলে দেখাও যে, $U=\frac{X}{\sigma_1}+\frac{Y}{\sigma_2}$ এবং $V=\frac{X}{\sigma_1}-\frac{Y}{\sigma_2}$ পরস্পর স্বাধীন পরিমিত চলক যাদের ভেদাংক যথাক্রমে $2(1+\rho)$ এবং $2(1-\rho)$.

[(a) What is bi-variate normal distribution? Find its marginal distribution.

(b) Two random variables X and Y have zero mean, variances σ_1^2 and σ_2^2 respectively with correlation co-efficient ρ. If the two variables are normally distributed then show that, $U=\frac{X}{\sigma_1}+\frac{Y}{\sigma_2}$ and $V=\frac{X}{\sigma_1}-\frac{Y}{\sigma_2}$ are independent normal variableswith variances $2(1+\rho)$ and $2(1-\rho)$.]

পরিসংখ্যান (তত্ত্বীয়)-২০২১

বিষয় কোড ঃ 223601

(Probability Distribution)

সময় - ৪ ঘন্টা

পূর্ণমান - ৮০

[**দ্রষ্টব্য ঃ –** প্রতিটি বিভাগের বিভিন্ন প্রশ্নের উত্তর ধারাবাহিকভাবে লিখতে হবে।]

ক বিভাগ

(যে কোন **দশটি** প্রশ্নের উত্তর দাও)

মান- ১ X ১০ = ১০

১। (ক) দ্বিপদী বিন্যাসের কুমুল্যান্টের পৌনঃপুনিক সূত্রটি লেখ।
[Write down the recurrence formula for the cumulants of binomial distribution.]

(খ) কী ধরনের ঘটনায় পোঁসো বিন্যাস ব্যবহার করা হয়?
[For what type of events Poisson distribution is used?]

(গ) ঋণাত্মক দ্বিপদী বিন্যাসের সীমাস্থ ধর্মটি বিবৃত কর।
[State the limiting property of negative binomial distribution.]

(ঘ) কখন পরাজ্যামিতিক বিন্যাস দ্বিপদী বিন্যাসে রূপান্তরিত হয়?
[When hypergeometric distribution converted to a binomial distribution?]

(ঙ) ১ম প্রকার বিটা বিন্যাসের পরিঘাত উৎপাদকী ফাংশনটি লেখ।
[Write down the moment generating function of beta distribution of 1^{st} kind.]

(চ) বিটা ও গামা ফাংশনের সম্পর্কটি উল্লেখ কর।
[State the relationship between Beta and Gamma function.]

(ছ) একটি অক্ষাভিসারী বিন্যাসের গড় ৩ হলে উহার সম্ভাবনা ঘনত্ব অপেক্ষকটি লেখ।
[Write down the probability density function of an exponential distribution with mean 3.]

(জ) পরিমিত বিন্যাসের গড় ব্যবধান ও পরিমিত ব্যবধানের সম্পর্কটি লেখ।
[Write down the relationship between the standard deviation and the mean deviation of normal distribution.]

(ঝ) Maxwell বিন্যাসের সম্ভাবনা ঘনত্ব অপেক্ষকটি লেখ।
[Wrtie down the probability density function of Maxwell distribution.]

(ঞ) Laplace বিন্যাসের সম্ভাবনা ঘনত্ব অপেক্ষকটি লেখ।
[Wrtie down the probability density function of Laplace distribution.]

(ট) দ্বিচলক পরিমিত বিন্যাসের সম্ভাবনা ঘনত্ব অপেক্ষকটি লেখ।
[Write down the probability density function of bivariate normal distribution.]

(ঠ) বহুপদী বিন্যাস কাকে বলে?
[What is multinomial distribution?]

খ বিভাগ

(যে কোন **পাঁচটি** প্রশ্নের উত্তর দাও)

মান- ৪ X ৫ = ২০

২। দ্বিপদী বিন্যাসের প্রচুরক নির্ণয় কর।
[Find the mode of binomial distribution.]

৩। যদি X একটি একক গড় বিশিষ্ট পোঁসো চলক হয়, তবে দেখাও যে, বিন্যাসটির গড়ভিত্তিক গড় ব্যবধান বিন্যাসটির পরিমিত ব্যবধানের $\frac{2}{e}$ গুণ।
[If X be a Poisson variate with unit mean, then show that the mean deviation about mean of this distribution is $\frac{2}{e}$ times the standard deviation.]

৪। পরাজ্যামিতিক বিন্যাসের সম্ভাবনা অপেক্ষকটি উদ্ভাবন কর।
[Derive the probability function of hypergeometric distribution.]

৫। পরিমিত বিন্যাসের পরিঘাত উৎপাদকী অপেক্ষক নির্ণয় কর।

[Find moment generating function of normal distribution.]

৬। অক্ষাভিসারী বিন্যাসের কুমুল্যান্ট উৎপাদকী অপেক্ষক নির্ণয় কর। অতঃপর দেখাও যে, বিন্যাসটির গড় ও পরিমিত ব্যবধান সমান।

[Obtain the cumulant generating function of exponential distribution. Hence, show that mean and standard deviation of this distirbution is same.]

৭। মনে কর X ও Y দুটি স্বাধীন গামা চলক যাদের পরামিতি যথাক্রমে m ও n, তবে $U=\frac{X}{Y}$ চলকটির বিন্যাস নির্ণয় কর।

[Let, X and Y be two independent gamma variate with parameters m and n respectively, then find the distribution of variable, $U=\frac{X}{Y}$.]

৮। দ্বি-চলক পৈঁসো বিন্যাসের সংজ্ঞা দাও। এর সম্ভাবনা অপেক্ষকটি উদ্ভাবন কর।

[Define bi-variate Poisson distribution. Derive its probability function.]

৯। কৌশি বিন্যাসের যৌগিক ধর্মটি লেখ ও প্রমাণ কর।

[State and prove additive property of Cauchy distribution.]

গ বিভাগ

(যে কোনো **পাঁচটি** প্রশ্নের উত্তর দাও)

মান- ১০ X ৫ = ৫০

১০। (ক) দ্বিপদী বিন্যাসের পরিঘাত উৎপাদকী অপেক্ষক নির্ণয় কর এবং তা হতে বিন্যাসটির গড় ও ভেদাংক নির্ণয় কর।

(খ) যদি x ও y দুটি স্বাধীন পৈঁসো চলক হয়, তবে দেখাও যে, দেয় $x+y$ এর জন্য x এর শর্তাধীন সম্ভাবনা বিন্যাসটি হবে দ্বিপদী।

[(a) Determine the moment generating function of binomial distribution and hence find mean and variance of this distribution.

(b) If x and y are independent Poisson variates, then show that the conditional distribution of x given $x+y$ is binomial.]

১১। (ক) ঋণাত্মক দ্বিপদী বিন্যাসের সম্ভাবনা অপেক্ষকটি উদ্ভাবন কর।

(খ) ঋণাত্মক দ্বিপদী বিন্যাসের কুমুল্যান্ট উৎপাদকী অপেক্ষক নির্ণয় কর এবং তা হতে β_1 ও β_2 নির্ণয় কর।

[(a) Derive the probability function of negative binomial distribution.

(b) Determine the cumulant generating function of negative binomial distribution and hence find β_1 and β_2.]

১২। (ক) জ্যামিতিক বিন্যাসের সংজ্ঞা দাও। p পরামিতি বিশিষ্ট k সংখ্যক স্বাধীন জ্যামিতিক চলকের যোগফলের বিন্যাস নির্ণয় কর।

(খ) জ্যামিতিক বিন্যাসের ধর্ম বর্ণনা কর। প্রচলিত প্রতীকে প্রমাণ কর যে, $\mu_{r+1}=\frac{rq}{p^2}\mu_{r-1}+q\frac{d\mu_r}{dq}$.

[(a) Define geometric distribution. Find the distribution of sum of k independent geometric variables with parameter p.

(b) State the properties of geomtric distribution. In usual notation prove that, $\mu_{r+1}=\frac{rq}{p^2}\mu_{r-1}+q\frac{d\mu_r}{dq}$.]

১৩। (ক) পরিমিত বিন্যাসের সংজ্ঞা দাও। এর গড় কেন্দ্রিক গড় ব্যবধান এবং চতুর্থক ব্যবধান নির্ণয় কর।

(খ) পরিমিত বিন্যাসের প্রচুরক নির্ণয় কর। দেখাও যে, পরিমিত বিন্যাসের স্ফীতি বিন্দু দুটি যথাক্রমে $(\mu-\sigma)$ ও $(\mu+\sigma)$.

[(a) Define normal distribution. Find its mean deviation about mean and quartile deviation.

(b) Find the mode of normal distribution. Show that, two inflexion points of normal distribution are $(\mu-\sigma)$ and $(\mu+\sigma)$ respectively.]

১৪। (ক) প্রথম প্রকার বিটা বিন্যাস কাকে বলে? এর প্রচুরক ও তরঙ্গ গড় নির্ণয় কর।

(খ) দেখাও যে, প্রথম প্রকার বিটা বিন্যাস হতে নির্দিষ্ট রূপান্তরের মাধ্যমে দ্বিতীয় প্রকার বিটা বিন্যাস পাওয়া যায়।

[(a) What is beta distribution of first kind? Find its mode and hermonic mean.

(b) Show that, beta distribution of second kind can be obtained from beta distribution of first kind through certain transformation.]

১৫। (ক) আদর্শ গামা বিন্যাস কী? দেখাও যে, আদর্শ গামা বিন্যাসের গড় ও ভেদাংক বিন্যাসটির পরামানের সমান।

(খ) যদি X ও Y যথাক্রমে l ও m পরামান বিশিষ্ট দুটি স্বাধীন গামা চলক হয় তবে দেখাও যে, $U = X + Y,\ Z = \dfrac{X}{X+Y}$ স্বাধীন এবং U একটি $l + m$ পরামান বিশিষ্ট গামা চলক, z একটি l ও m পরামান বিশিষ্ট প্রথম প্রকারের বিটা চলক।

[(a) What is standard gamma distribution. Show that, mean and variance of standard gamma distribution are equal to parameter of the distribution.

(b) If X and Y are independent gamma variates with parameters l and m respectively, then show that $U = X + Y,\ Z = \dfrac{X}{X+Y}$ are independent and U is a gamma variate with parameter $l + m$ and z is a beta variate of first kind with parameters l and m.]

১৬। (ক) কৌশি বিন্যাস কী? এর মধ্যমা নির্ণয় কর।

(খ) যদি দৈব চলক X আদর্শ কৌশি বিন্যাস অনুসরণ করে তবে দেখাও যে, $X^2 \sim \beta_2\left(\frac{1}{2}, \frac{1}{2}\right)$

[(a) What is Cauchy distribution? Find its median.

(b) If the random variable X follows standard Cauchy distribution, then show that $X^2 \sim \beta_2\left(\frac{1}{2}, \frac{1}{2}\right)$.]

১৭। (ক) দ্বি-চলক পরিমিত বিন্যাসের সংজ্ঞা দাও। দুটি আদর্শ পরিমিত চলক X_1 ও X_2 এর সংশ্লেষাংক ρ হলে দেখাও যে, $\mathrm{Cor}\left(X_1^2, X_2^2\right) = \rho^2$.

(খ) দ্বি-চলক পরিমিত বিন্যাসের সম্ভাবনা ঘনত্ব অপেক্ষক উদ্ভাবন কর।

[(a) Define bi-variate normal distribution. If the correlation coefficient between two standard normal variables X_1 and X_2 is ρ, then show that $\mathrm{Cor}\left(X_1^2, X_2^2\right) = \rho^2$.

(b) Derive the probability density function of bi-variate normal distribution.]

পরিসংখ্যান (তত্ত্বীয়)-২০২২

বিষয় কোড ঃ 223601

(Probability Distribution)

সময় - ৪ ঘন্টা

পূর্ণমান - ৮০

[**দ্রষ্টব্য ঃ –** প্রতিটি বিভাগের বিভিন্ন প্রশ্নের উত্তর ধারাবাহিকভাবে লিখতে হবে।]

ক বিভাগ

(যে কোনো **দশটি** প্রশ্নের উত্তর দাও)

মান- ১ x ১০ = ১০

১। (ক) বার্ণোলী চলক কী?

[What is Bernoulli variate?]

(খ) দ্বিপদী বিন্যাসের পরিঘাত উৎপাদনকারী অপেক্ষকটি লিখ।

[Write down the moment generating function of binomial distribution.]

(গ) পৈঁসু বিন্যাসের পরিঘাতের পৌনঃপুনিক সূত্রটি লেখ।

[Write down the recurrence relation for moments of Poisson distribution.]

(ঘ) ঋণাত্মক দ্বিপদী বিন্যাসের একটি উদাহরণ দাও।

[Give an example of negative binomical distribution.]

(ঙ) বিচ্ছিন্ন আয়তাকার বিন্যাসের গড় ও ভেদাংকের মান লেখ।

[Write down the mean and variance of discrete uniform distribution.]

(চ) জ্যামিতিক বিন্যাসের প্রচুরকের মান লেখ।

[Write down the mode of geometric distribution.]

(ছ) পরিমিত বিন্যাসের সন্ধিবিন্দুর মান কত?

[What is the value of point of inflexion of normal distribution.]

(জ) a এবং λ পরামানবিশিষ্ট গামাবিন্যাসের সম্ভাবনা ঘনত্ব অপেক্ষক লেখ।

[Write down the probability density function of gamma distribution with parameters a and λ.]

(ঝ) ৫ ও ৩ পরামানবিশিষ্ট প্রথম প্রকারের বিটা বিন্যাসের উল্টন গড়ের মান কত?

[What is the value of harmonic mean of beta distribution of 1^{st} kind with parameters 5 and 3.]

(ঞ) Laplace বিন্যাসের নির্দেশক অপেক্ষক এর মান লেখ।

[Write down the characteristic function of Laplace distribution.]

(ট) কৌশি বিন্যাসের সংজ্ঞা দাও।

[Define Cauchy distribution.]

(ঠ) দ্বিচলক পরিমিত বিন্যাসের পরামানসমূহ লেখ।

[Write down the parameters of bi-variate normal distribution.]

খ বিভাগ

(যে কোন **পাঁচটি** প্রশ্নের উত্তর দাও)

মান- ৪ x ৫ = ২০

২। দ্বিপদী বিন্যাসের সম্ভাবনা অপেক্ষকটি উদ্ভাবন কর।

[Derive the probability function of binomial distribution.]

৩। পৈঁসু বিন্যাসের প্রচুরকের মান বের কর।

[Find the mode of Poisson distribution.]

৪। দেখাও যে, ঋণাত্মক দ্বিপদী বিন্যাসের সীমান্তরূপ হলো পৈঁসু বিন্যাস।

[Show that Poisson distribution is the limiting case of negative binomidal distribution.]

৫। কোনো একটি হ্রদের মাছের সংখ্যা প্রাক্কলন করতে তুমি কীভাবে পরাজ্যামিতিক বিন্যাস মডেল ব্যবহার করবে-ব্যাখ্যা কর।

[Explain how you will use hypergeometric distribution model to estimate the number of fish in a lake.]

৬। পরিমিত বিন্যাসের সম্ভাবনা ঘনত্ব অপেক্ষকটি উদ্ভাবন কর।

[Derive the probability density function of normal distribution.]

৭। জ্যামিতিক বিন্যাসের স্মৃতিভ্রম ধর্মটি প্রমাণ কর।

[Prove the lack of memory property of geometric distribution.]

৮। অক্ষাভিসারি বিন্যাসের মধ্যমা এবং গড় হতে নির্ণীত গড় ব্যবধান বের কর।

[Find the median and mean deviation from mean of an exponential distribution.]

৯। অবিচ্ছিন্ন আয়তাকার বিন্যাসের r-তম কাঁচা পরিঘাতের মান বের কর এবং এখান থেকে উহার গড় ও ভেদাংকের মান বের কর।

[Find the r-th raw moments of continuous uniform distribution and hence find its mean and variance.]

গ বিভাগ

(যে কোনো **পাঁচটি** প্রশ্নের উত্তর দাও)

মান- ১০ X ৫ = ৫০

১০। (ক) দেখাও যে, r-তম কেন্দ্রীয় পরিঘাতের ক্ষেত্রে দ্বিপদী বিন্যাস নিম্নোক্ত পৌনঃপুনিক সম্পর্কটি মেনে চলে

$$\mu_{r+1} = pq\left[n r \mu_{r-1} + \frac{d}{dp}\mu_r \right];\ r \geq 1.$$ এখান থেকে দ্বিপদী বিন্যাসের ভেদাংকের মান বের কর।

(খ) দ্বিপদী বিন্যাসের প্রচুরক নির্ণয় কর।

[(a) Show that in case of r-th central moment, binomial distribution follows the following recurrence relation

$$\mu_{r+1} = pq\left[n r \mu_{r-1} + \frac{d}{dp}\mu_r \right];\ r \geq 1$$ and hence find the variance of binomial dsitribution.

(b) Find the mode of binomial distribution.]

১১। (ক) পৈঁসু বিন্যাসের সম্ভাবনা উৎপাদনকারী অপেক্ষকের মান বের কর এবং এখান থেকে উক্ত বিন্যাসের গড় ও ভেদাংক নির্ণয় কর।

(খ) দেখাও যে, পরিমিত বিন্যাস হল পৈঁসু বিন্যাসের একটি সীমান্ত রূপ।

[(a) Find the probability generating function of Poisson distribution and hence determine the mean and variance of the distribution.

(b) Show that normal distribution is the limiting case of Poisson distribution.]

১২। (ক) জ্যামিমিক বিন্যাসের পরিঘাত উৎপাদকী অপেক্ষকটি বের কর এবং এর গড় ও ভেদাংক নির্ণয় কর।

(খ) পরাজ্যামিতিক বিন্যাসের r -তম গৌণিক পরিঘাত এর মান বের কর। এখান থেকে উহার গড় ও ভেদাংকের মান বের কর।

[(a) Find the moment generating fucntion of geometric distribution and hence find its mean and variance.

(b) Find the r-th factorial moment of hypergeometric distribution. Hence find its mean and variance.]

১৩। (ক) বহুপদী বিন্যাসের পরিঘাত উৎপাদনকারী অপেক্ষকের মান বের কর। এখান থেকে উহার গড়, ভেদাংক ও সহভেদাংকের মান বের কর।

(খ) যদি $f(xy) = \dfrac{n!}{x!\,y!\,(n-x-y)!}\, p^x q^y (1-p-q)^{n-x-y}$ যেখানে $x, y = 0, 1, 2, \cdots, n$ এবং $x + y \leq n,\ 0 \leq p,\ 0 \leq q$ এবং $p + q \leq 1$ হয় তবে x ও y এর প্রান্তীয় সম্ভাবনা অপেক্ষক বের কর।

[(a) Find moment generating function of multinomial distribution. Hence find its mean, variance and covariance.

(b) If $f(xy) = \dfrac{n!}{x!\,y!\,(n-x-y)!}\, p^x q^y (1-p-q)^{n-x-y}$ for $x, y = 0, 1, 2, \cdots, n$ and $x + y \leq n$ where $0 \leq p,\ 0 \leq q$ and $p + q \leq 1$ then find the marginal distribution of x and y.]

১৪। (ক) দেখাও গড়ের সাপেক্ষে নির্ণীত পরিমিত বিন্যাসের সকল বিজোড় ঘাতের পরিঘাতের মান শূন্য এবং সকল জোড় ঘাতের পরিঘাত নিম্নোক্ত সম্পর্ক মেনে চলে-

$$\mu_{2n} = 1, 3, 5, \cdots, (2n-1)\,\sigma^{2n}$$

(খ) দেখাও যে, কতগুলো স্বাধীন পরিমিত চলকের সরলরৈখিক সমাবেশও একটি পরিমিত চলক।

[(a) Show that all order moments about the mean of normal distribution is zero and all even order moments about the mean of normal distribution follows the relation- $\mu_{2n} = 1, 3, 5, \cdots, (2n-1)\,\sigma^{2n}$.

(b) Show that a linear combination of independent normal variates is also a normal variate.]

১৫। (ক) দ্বিতীয় প্রকারের বিটা বিন্যাসের সংজ্ঞা দাও। এর প্রচুরকের মান বের কর।

(খ) গামা বিন্যাসের পরিঘাত উৎপাদকী অপেক্ষক নির্ণয় কর এবং এর গড় ও ভেদাংক নির্ণয় কর।

[(a) Define beta distribution of 2nd kind. Find its mode.

(b) Find the moment generating function of gamma distribution and find its mean and variance.]

১৬। (ক) দেখাও যে, দুটি স্বাধীন প্রমিত পরিমিত চলকের অনুপাত প্রমিত কৌশি বিন্যাস মেনে চলে।

(খ) যদি দৈব চলক $x \sim N(\mu, \sigma^2)$ তবে দেখাও যে, $u = \frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2$ হবে $\frac{1}{2}$ পরামিত বিশিষ্ট গামা চলক।

[(a) Show that the ratio of two independent standard normal variates follows standard Cauchy distribution.

(b) If random variable $x \sim N(\mu, \sigma^2)$ then show that $u = \frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2$ be a gamma variable with parameter $\frac{1}{2}$.]

১৭। (ক) দ্বিচলক পরিমিত বিন্যাসের পরিঘাত উৎপাদনকারী অপেক্ষকের মান বের কর।

(খ) y এর দেয় মানের জন্য দ্বিচলক পরিমিত বিন্যাসের ক্ষেত্রে x এর শর্তাধীন সম্ভাবনা ঘনত্ব অপেক্ষকের মান বের কর।

[(a) Find the moment generating function of bi-variate normal distribution.

(b) Find the conditional distribution of x for given y of bi-variate normal distribution.]

পরিসংখ্যান (তত্ত্বীয়)-২০২৩

বিষয় কোড ঃ 223601

Probability Distribution

সময় - ৪ ঘন্টা

পূর্ণমান - ৮০

[**দ্রষ্টব্য ঃ –** প্রতিটি বিভাগের প্রশ্নসমূহের উত্তর ধারাবাহিকভাবে লিখতে হবে।]

ক বিভাগ

(যে কোন **দশটি** প্রশ্নের উত্তর দাও)

মান- ১ x ১০ = ১০

১। (ক) বার্ণোলী বিন্যাস কী?
[What is Bernoulli distribution?]

(খ) দ্বিপদী বিন্যাসের প্রচুরকের মান লেখ।
[Write down the mode of binomial distribution.]

(গ) দ্বিপদী বিন্যাসের গড় ও ভেদাংক যথাক্রমে ২০ ও ৪ হলে চেষ্টার সংখ্যা বের কর।
[If the mean and variance of a binomial distribution are 20 and 4 respectively, find the number of trials.]

(ঘ) জ্যামিতক বিন্যাসের 'স্মৃতি ভ্রম' ধর্মটি কী?
[What is the 'lack of memory' property of geometric distribution?]

(ঙ) পৈঁসু বিন্যাসের গড় ও ভেদাংক কত?
[What is the mean and variance of Poisson distribution?]

(চ) পরাজ্যামিতেক বিন্যাসের তম এর মান লেখ।
[Write down the r-th factorial moment of hyper-geometric distribution.]

(ছ) কী কী শর্তে ঋণাত্মক দ্বিপদী বিন্যাস পৈঁসু বিন্যাসে রূপান্তরিত হয়?
[Under what conditions negative binomial distribution becomes to Poisson distribution?]

(জ) একটি অক্ষাভিসারী বিন্যাসের গড় ৪ হলে উহার সম্ভাবনা ঘনত্ব অপেক্ষকটি লেখ।
[Write down the probability density function of an exponential distribution with mean 4.]

(ঝ) পরিমিত বিন্যাসের গড় ব্যবধান ও পরিমিত ব্যবধানের সম্পর্কটি লেখ।
[write down the relation between standard deviation and mean deviation of normal distribution.]

(ঞ) কৈশি বিন্যাসের নির্দেশক অক্ষেকটি লেখ।
[Write down the characteristic function of Cauchy distribution.]

(ট) Weibull বিন্যাস সংজ্ঞায়িত কর।
[Define Weibull distribution.]

(ঠ) দ্বিপদী পরিমিত বিন্যাসের সম্ভাবনা ঘনত্ব অপেক্ষকটি লেখ।
[Write down the probability density function of bi-variate normal distribution.]

খ বিভাগ

(যে কোন **পাঁচটি** প্রশ্নের উত্তর দাও)

মান- ৪ x ৫ = ২০

২। একটি দ্বিপদী চলক x এর জন্য $P(x=0)=2P(x=1)=9P(x=2)$ হলে সম্ভাবনা অপেক্ষকটি নির্ণয় কর।
[For a binomial variate x, if $P(x=0)=2P(x=1)=9P(x=2)$ then find the probability function.]

৩। যদি x ও y স্বাধীন পৈঁসু চলক হয় তবে দেখাও যে $x+y$ এর দেয় মানের জন্য x এর শর্তাধীন বিন্যাস হবে দ্বিপদী।
[If x and y are independent Poisson variates then show that the conditional distribution of x given $x+y$ is binomial.]

৪। ঋণাত্মক দ্বিপদী বিন্যাসের গড় ও ভেদাংক নির্ণয় কর।
[Find the mean and variance of negative binomial distribution.]

৫। পরাজ্যামিতিক বিন্যাসের সাথে দ্বিপদী বিন্যাসের সম্পর্ক স্থাপন কর।
[Establish the relation between hyper-geometric distribution and binomial distribution.]

৬। পরিমিত বিন্যাসের মধ্যমা ও প্রচুরক নির্ণয় কর।

[Find the median and mode of normal distribution.]

৭। গামা বিন্যাসের β_1 ও β_2 নির্ণয় কর।

[Find β_1 and β_2 of gamma distribution.]

৮। Laplace বিন্যাসের নির্দেশক অপেক্ষকের মান বের কর, এখান থেকে উহার β_1 ও β_2 নির্ণয় কর।

[Find the characteristic function of Laplace distribution. Hence find its β_1 and β_2 .]

৯। বিন্যাস সংজ্ঞায়িত কর। বিন্যাসের গড় ও ভেদাংকের মান বের কর।

[Define Maxwel distribution. Find the mean and variance of Maxwel distribution.]

গ বিভাগ

(যে কোনো **পাঁচটি** প্রশ্নের উত্তর দাও)

মান- ১০ x ৫ = ৫০

১০। (ক) দ্বিপদী বিন্যাসের কুমুল্যান্ট উৎপাদনকারী অপেক্ষকটি বের কর। এখান থেকে উক্ত বিন্যাসের β_1 ও β_2 এর মান বের কর।

(খ) দ্বিপদী বিন্যাসের গুরুত্বপূর্ণ ধর্মাবলি লেখ। দ্বিপদী বিন্যাসের যৌগিক ধর্মটি বিবৃতিসহ প্রমাণ কর।

[(a) Find the cumulant generating function of binomial distribution. Hence find β_1 and β_2 of the distribution.

(b) Write down the important properties of binomial distribution. State and prove the additive property of binomial distribution.]

১১। (ক) পৈঁসু বিন্যাসের সম্ভাবনা অপেক্ষকটি উদ্ভাবন কর।

(খ) পৈঁসু বিন্যাসের যোগবোধক ধর্মটি বিবৃতিসহ প্রমাণ কর।

[(a) Derive the probability function of Poisson distribution.

(b) State and prove the additive property of Poisson distribution.]

১২। (ক) ঋণাত্মক দ্বিপদী বিন্যাস সংজ্ঞায়িত কর। ঋণাত্মক দ্বিপদী বিন্যাসের সম্ভাবনা অপেক্ষক উদ্ভাবন কর।

(খ) ঋণাত্মক দ্বিপদী বিন্যাসের পরিঘাত উৎপাদন অপেক্ষক বের কর। এখান থেকে উক্ত বিন্যাসের β_1 ও β_2 বের কর এবং মন্তব্য কর।

[(a) Define negative binomial distribution. Derive the proobability function of negative binomial distribution.

(b) Find the moment generating function of negative binomial distribution. Hence find β_1 and β_2 of this distribution and comment.]

১৩। (ক) পরা-জ্যামিতিক বিন্যাস সংজ্ঞায়িত কর। পরা-জ্যামিতিক বিন্যাসের সম্ভাবনা অপেক্ষক উদ্ভাবন কর।

(খ) দেখাও যে পরা-জ্যামিতিক বিন্যাস হলো দ্বিপদী বিন্যাসের সীমায়িত রূপ।

[(a) Define hyper-geometric distribution. Derive the probability function of hyper-geometric distribution.

(b) Show that hyper-geometric distribution is the limiting case of binomial distribution.]

১৪। (ক) দেখাও যে, পরামিতি $m \to \infty$ এর জন্য গামা বিন্যাস পরিমিত বিন্যাসে রূপান্তরিত হয়

(খ) যদি x একটি পরিমিত চলক হয়, তবে দেখাও যে, $u = \frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2$ একটি $\frac{1}{2}$ পরামানবিশিষ্ট গামা চলক।

[(a) Show that for parameter $m \to \infty$, gamma distribution tends to normal distribution.

(b) If x is a normal variate, then show that $u = \frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2$ is a gamma variate with parameter $\frac{1}{2}$.]

১৫। (ক) কৌশি বিন্যাসের সংজ্ঞা দাও। যদি $x_1, x_2, \cdots, x_n$ প্রমিত কৌশি বিন্যাস হতে গৃহীত n সংখ্যক দৈব পর্যবেক্ষণ হয়, তবে $\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i$ এর বিন্যাস নির্ণয় কর।

(খ) বহুপদী বিন্যাসের সংজ্ঞা দাও। বহুপদী বিন্যাসের সম্ভাবনা অপেক্ষক উদ্ভাবন কর।

[(a) Define Cauchy distribution. If $x_1, x_{2,} \cdots, x_n$ are randomly taken n observations from standard Cauchy distribution then find the distribution of $\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i$.

(b) Define multinomial distribution. Derive probability function of multinomial distribution.]

১৬। (ক) বিটা বিন্যাস কী? কীভাবে প্রথম প্রকারের বিটা বিন্যাস হতে ২য় প্রকার বিটা বিন্যাস এবং ২য় প্রকার বিটা বিন্যাস হতে প্রথম প্রকার বিটা বিন্যাস পাওয়া যায়?

(খ) যদি x ও y দুইটি স্বাধীন গামা চলক হয় তবে দেখাও যে, $(x+y)$ গামা বিন্যাস মেনে চলে কিন্তু $\frac{x}{y}$ ২য় প্রকার বিটা বিন্যাস মেনে চলে।

[(a) What is Beta distribution? How wil you get Beta distribution of 2^{nd} kind from Beta distribution of 1^{st} kind and Beta distribution of 1^{st} kind from Beta distribution of 2^{nd} kind?

(b) If x and y are two independent gamma variates, then show that $(x+y)$ follow gamma distribution but $\frac{x}{y}$ follows Beta distribution of 2^{nd} kind.]

১৭। (ক) দ্বিচলক পরিমিত বিন্যাস কী? দ্বিচলক পরিমিত বিন্যাসের শর্তাধীন বিন্যাস $\int\left(\frac{x}{y}\right)$ বের কর।

(খ) যদি x ও y দুটি আদর্শ পরিমিত চলক হয় যাদের সংশ্লেষাংক ρ, তবে প্রমাণ কর যে, $\operatorname{cor}\left(x^2, y^2\right) = \rho^2$.

[(a) What is bivariate normal distribution? Find the conditional distribution $\int\left(\frac{x}{y}\right)$ of bivariate normal distribution.

(b) If x and y are two standard normal variates with correlation coefficient ρ, then prove that $\operatorname{cor}\left(x^2, y^2\right) = \rho^2$.]