Origin-based formulation of multi-class traffic assignment

To begin, authors introduce the network as a directed graph - $G(N, A)$, with set of nodes N and a set of arcs $A = \{(i,j); i,j \in N\}$. Authors further introduce a set of origins R, a set of destination S, and a set of vehicle class K, with demand q_{rs}^k between origin $r \in R$ and destination $s \in S$ for vehicle class $k \in K$. A vehicle traversing through this network observes costs pertaining to travel-related parameters from the set P, which manifest on the arc at a rate defined by the polynomial function on arc speed, given by, $\sum_n \eta_n^p v_{ij}^n$. This generalized cost function, as given in **Equation 1**, must be strictly positive, continuously differentiable, and monotonically non-decreasing to guarantee existence of the traffic assignment solution, uniqueness of the equilibrium and absence of infinite loops, respectively. Authors develop these properties for the generalized cost function later in this work. Additionally, this work assumes arc costs to be separable, i.e., cost on arc only depends on the flow of that arc and none other.

N:	Set of nodes
A:	Set of arcs
R:	Set of origins
S:	Set of destinations
K:	Set of vehicle classes
P:	Set of parameters
d_{ij}:	Length of arc (i,j)
v_{ij}:	Vehicle speed on arc (i,j)
t_{ij}:	Travel time on arc (i,j)
x_{ij}:	Flow on arc (i,j)
x_{ij}^{kr}:	Flow of vehicle class k from origin r on arc (i,j)
x_{ij}^{kr-}:	Flow of vehicle class k from all origins but r on arc (i,j)
V_{ij}:	Volume capacity for arc (i,j)
α_{ij}, β_{ij}:	BPR parameters for arc (i,j)
c_{ij}^k:	Cost of traversing arc (i,j) for vehicle class k
r, s:	Origin, Destination
q_{rs}^k:	Demand for vehicle class k between origin r and destination s
θ_p:	Cost of parameter p
η_n^p:	Coefficient of v_{ij}^n for parameter p
φ_p:	Binary variable indicating inclusion of parameter p in the generalized cost function

$$c_{ij}^k(x_{ij}) = \sum_{p \in P} \varphi_p^k \theta_p t_{ij}(x_{ij}) \sum\nolimits_n \eta_n^p v_{ij}^n \tag{1}$$

$$t_{ij}(x_{ij}) = t_{ij}^o \left(1 + \alpha_{ij} \left(\frac{x_{ij}}{V_{ij}}\right)^{\beta_{ij}}\right) \tag{2}$$

This study develops an origin-based formulation for the multi-class traffic assignment problem, with an objective,

$$\min_{x_{ij}^{kr}; (i,j) \in A} \quad z_k(\boldsymbol{x}) = \sum_{(i,j) \in A} \int_0^{x_{ij}^{kr} + x_{ij}^{kr-}} c_{ij}^k(u) du \quad \forall \ k \in K \tag{3}$$

Subject to flow conservation,

$$x_{ij} = \sum_{r \in R} \sum_{k \in K} x_{ij}^{kr} \tag{4}$$

$$x_{ij} = x_{ij}^{kr} + x_{ij}^{kr\_} \tag{5}$$

$$q_{ri}^{k} + \sum_{n \in T(i)} x_{ni}^{kr} = \sum_{s \in S} q_{rs}^{k} + \sum_{n \in H(i)} x_{in}^{kr} \tag{6}$$

Equation 4 and **Equation 5** presents arc flow as sum of origin-based vehicle class flows. **Equation 6** balances flow incoming to node i with flow outgoing from node i. To solve this optimization problem, authors Lagrange transform the above formulation and consequently apply Karush-Kuhn-Tucker (KKT) conditions rendering,

$$\mathcal{L}_k(\boldsymbol{x}, \boldsymbol{u}) = z_k(\boldsymbol{x}) + \sum_{r \in R} \sum_{i \in N} u_i^{kr} \left(\sum_{s \in S} q_{rs}^{k} + \sum_{n \in H(i)} x_{in}^{kr} - \sum_{n \in T(i)} x_{ni}^{kr} - q_{ri}^{k} \right) \tag{7}$$

$$x_{ij}^{kr} \frac{\partial \mathcal{L}_k}{\partial x_{ij}^{kr}} = 0; \quad \frac{\partial \mathcal{L}_k}{\partial x_{ij}^{kr}} \geq 0, x_{ij}^{kr} \geq 0 \tag{8}$$

$$\frac{\partial \mathcal{L}_k}{\partial x_{ij}^{kr}} = c_{ij}^{k}\left(x_{ij}\right) + u_i^{kr} - u_j^{kr} \tag{9}$$

$$x_{ij}^{kr} \pi_{ij}^{kr} = 0; \quad \pi_{ij}^{kr} \geq 0, x_{ij}^{kr} \geq 0 \tag{10}$$

Where, π_{ij}^{kr} is the reduced cost for a vehicle, belonging to class $k \in K$, origination from node $r \in R$, for traversing arc $(i, j) \in A$, defined as $u_i^{kr} + c_{ij}^{k}\left(x_{ij}\right) - u_j^{kr}$. Here, u_i^{kr} is the minimum travel cost for vehicle class $k \in K$ from origin $r \in R$ to node $i \in N$. The above-developed KKT conditions (**Equation 10)** are akin to the classical Wardrop's equilibrium condition, wherein the cost of all traversed paths between an origin and a destination must be equal, and less than the cost of all untraversed paths between this origin and destination. Here, an analogous interpretation of the KKT condition suggests that all traversed arcs at equilibrium must have zero reduced cost (corresponding to the particular origin and vehicle class). Such an origin-based traffic assignment model was first formulated by Bar-Gera (*26)*, who later developed the Traffic Assignment by Paired Alternative Segments (TAPAS) algorithm (*27*) to get the assignment solution. Xie and Xie (*28)* made further advancements, developing the improved TAPAS (iTAPAS). As show in this section, authors expand the origin-based framework for multi-class traffic assignment, thus developing multi-class TAPAS (mTAPAS). The fundamental idea behind the mTAPAS algorithm is to identify potential arcs, i.e., arcs that have a non-zero/substantial origin-based flow ($x_{ij}^{kr} > \epsilon$) and non-zero/substantial origin-based reduced cost ($\pi_{ij}^{kr} > \theta$), and to consequently adjust flow on these arcs. This flow shift occurs between Paired Alternative Segments (PAS), which are sequence of arcs sharing a tail and a head node. Starting from the head node of the potential arc, the first segment traces back the least-cost path between origin and the head-node of the potential arc. The second segment on the other hand backtracks from the tail node of the potential arc, choosing predecessor nodes with maximum cost, until it converges with the least-cost path between origin and the head node of the potential arc. This renders a pair of sequence of arcs sharing a tail node at the point of intersection and a head node at the head of the potential arc. Once identified, flow is adjusted on the PAS based on Newton method (*29*). This process of identifying potential arcs and adjusting flow on the associated PAS continues until the relative gap (rg), as established in **Equation 11**, falls below a pre-defined tolerance level - tol. Below is a brief description of the multi-class TAPAS algorithm and the accompanying Maximum

Cost Search (MCS) and Newton Flow Shift (NFS) algorithms. The implementation in this work employs $\epsilon = 10^{-12}$, $\theta = 10^{-16}$ and $tol = 10^{-6}$.

$$rg = 1 - \frac{\sum_{r \in R} \sum_{s \in S_r} \sum_{k \in K} q_{rs}^k . u_{rs}^k}{\sum_{r \in R} \sum_{k \in K} \sum_{(i,j) \in A} x_{ij}^{kr} . c_{ij}^k \left(\sum_{r \in R} \sum_{k \in K} x_{ij}^{kr} \right)} \tag{11}$$

Properties of the generalized cost function

To develop the necessary properties of the generalized cost function, authors further generalize the polynomial cost function (**Equation 12**) as a quadratic function on arc speed (**Equation 13**) and consequently redefine the function on arc travel time (**Equation 14**), given by the BPR function (**Equation 15**). Authors here assume the quadratic function on arc speed to be strictly convex, rendering $A > 0$. Such vehicle behavior is typical in context of vehicle efficiency, fuel consumption and emission rate.

$$\boldsymbol{c_{ij} = \sum_{p \in P} \theta_p t_{ij} \sum_{k=0}^{2} \eta_k^p v_{ij}^k} \tag{12}$$

$$\boldsymbol{c_{ij} = \left(A v_{ij}^2 + B v_{ij} + C\right) t_{ij}} \tag{13}$$

$$\boldsymbol{c_{ij} = \frac{A d_{ij}^2}{t_{ij}} + B + C t_{ij}} \tag{14}$$

$$\boldsymbol{t_{ij} = t_{ij}^o \left(1 + \alpha_{ij} \left(\frac{x_{ij}}{V_{ij}}\right)^{\beta_{ij}}\right) ;\, t_{ij}^o, V_{ij} > 0;\ \alpha_{ij}, \beta_{ij} \geq 0} \tag{15}$$

In order to guarantee existence of the traffic assignment solution, uniqueness of the equilibrium and absence of infinite loops, the generalized cost function must be continuously differentiable, monotonically non-decreasing and strictly positive, respectively. This work ensures these properties for the generalized cost function as shown below,

1. *Continuously differentiable*

To establish continuous differentiability of c_{ij}, the analysis here establishes differentiability for c_{ij} and continuity of c'_{ij} in the domain of $c_{ij}: \mathbb{R}^+ \to \mathbb{R}$, where $\mathbb{R}^+: \{x_{ij} \in \mathbb{R}: x_{ij} \geq 0\}$.

$$c_{ij} = \frac{A d_{ij}^2}{t_{ij}} + B + C t_{ij} \tag{16}$$

$$c'_{ij} = \frac{d c_{ij}}{d x_{ij}} \tag{17}$$

$$c'_{ij} = \frac{t_{ij}^o \alpha_{ij}}{V_{ij}^{\beta_{ij}}} \left(C - \frac{A d_{ij}^2}{t_{ij}^2}\right) x_{ij}^{\beta_{ij}-1} \tag{18}$$

Since $t_{ij} > 0$, c_{ij} is continuous in its domain, while c'_{ij} is continuous in the domain of c_{ij} for $\beta_{ij} \geq 1$. Note, for $\beta_{ij} \in (0,1)$, c'_{ij} is undefined at $x_{ij} = 0$, hence the condition, $\beta_{ij} \geq 1$ is essential for the generalized cost function to be continuously differentiable.

2. *Monotonically non-decreasing*

$$c'_{ij} \geq 0 \tag{19}$$

$$\frac{t_{ij}^o \alpha_{ij}}{V_{ij}^{\beta_{ij}}} \left(C - \frac{A d_{ij}^2}{t_{ij}^2}\right) x_{ij}^{\beta_{ij}-1} \geq 0 \tag{20}$$

$$C - \frac{Ad_{ij}^2}{t_{ij}^2} \geq 0 \tag{21}$$

$$\frac{C}{A} \geq v_{ij}^2 \tag{22}$$

The above condition must hold true for the maximum possible speed in the network, thus,

$$\frac{C}{A} \geq \max_{(i,j) \in A} {v_{ij}^o}^2 \tag{23}$$

3. Strictly positive

Since the generalized cost function is monotonically non-decreasing, $c_{ij} > 0$ at $x_{ij} = 0$ ensures strict positivity, thus,

$$\frac{Ad_{ij}^2}{t_{ij}^o} + B + Ct_{ij}^o > 0 \tag{24}$$

$$B > -\left(\frac{Ad_{ij}^2}{t_{ij}^o} + Ct_{ij}^o\right) \tag{25}$$

For above condition to hold true for all arcs,

$$B > \max_{(i,j) \in A} -\left(\frac{Ad_{ij}^2}{t_{ij}^o} + Ct_{ij}^o\right) \tag{26}$$

$$B > \min_{(i,j) \in A} \left(\frac{Ad_{ij}^2}{t_{ij}^o} + Ct_{ij}^o\right) \tag{27}$$