Location Routing Problem (LRP)

A typical two-echelon LRP is defined on a directed graph $G = (N, A)$ with node set N encompassing customer nodes C, and potential distribution facility nodes $D = \{P \cup S\}$, where P and S represent the set of primary and secondary distribution facilities, respectively; while A represents the set of arcs connecting these nodes, with a vehicle spanning the arc connecting nodes i and j traversing a length l_{ij}. Further, each distribution facility node $d \in D$ has an associated set of delivery vehicles V_d, capacity q_d, service start and end time t_d^s and t_d^e, respectively, as well as fixed cost π_d^f, and operational cost π_d^o per package. And each customer node $c \in C$ has an associated service time τ_c^d, and demand q_c, which must be delivered within a specified time-window $[t_c^e, t_c^l]$ from a delivery vehicle from one of the primary or secondary distribution facilities. These delivery vehicles have an associated set of delivery routes R_v, capacity q_v, range l_v, refueling time τ_v^f, loading time per package τ_v^d, driver working hours w_v, fixed cost π_v^f, and operational costs π_v^{od} per unit distance and π_v^{ot} per unit time, respectively.
The objective of a LRP is to develop least cost routes from select distribution facilities using select vehicles such that every customer node is visited exactly once while also accounting for depot capacity, vehicle capacity, vehicle range, driver working-hours, and customers' time-windows.

$$\min \Pi = \sum_{p\in P}\left(\pi_p^f + \sum_{s\in S}\pi_p^o f_{ps}\right)y_p + \sum_{s\in S}\left(\pi_s^f + \sum_{p\in P}\pi_s^o f_{ps}\right)y_s + \sum_{v\in V}\left(\pi_v^f + \sum_{r\in R_v}\sum_{(i,j)\in A}\pi_v^{od}x_{ij}^r l_{ij} + \pi_v^{ot}(t_v^e - t_v^s)\right)y_v \quad (1)$$

Subject to,

$$\sum_{r\in R} z_{cr} = 1 \qquad \forall\, c \in C \qquad (2)$$

$$\sum_{j\in H_c} x_{cj}^r = z_{cr} \qquad \forall\, c \in C, r \in R \qquad (3)$$

$$\sum_{i\in T_j} x_{ij}^r = \sum_{k\in H_j} x_{jk}^r \qquad \forall\, j \in N, r \in R \qquad (4)$$

$$\sum_{p\in P} f_{ps} = \sum_{v\in V_s}\sum_{r\in R_v}\sum_{c\in C} z_{cr}q_c \qquad \forall\, s \in S \qquad (5)$$

$$\sum_{c\in C} z_{cr}q_c \le q_v y_v \qquad \forall\, r \in R_v, v \in V \qquad (6)$$

$$\sum_{v\in V_s}\sum_{r\in R_v}\sum_{c\in C} z_{cr}q_c \le q_s y_s \qquad \forall\, s \in S \qquad (7)$$

$$\sum_{s\in S} f_{ps} + \sum_{v\in V_p}\sum_{r\in R_v}\sum_{c\in C} z_{cr}q_c \le q_p y_p \qquad \forall\, p \in P \qquad (8)$$

$$t_c^a + M(1 - x_{ic}^r) \ge \begin{cases} t_r^s; & i \in D \\ t_i^d; & i \in C \end{cases} + x_{ic}^r \tau_{ic} \qquad \forall\, i \in T_c, c \in C, r \in R \qquad (9)$$

$$t_c^d \geq t_c^a + \max(0, t_c^e - t_c^a) + \tau_c^v \qquad \forall\, c \in C \tag{10}$$

$$t_c^a \leq t_c^l \qquad \forall\, c \in C \tag{11}$$

$$t_{r_v^1}^s = t_d^s \qquad \forall\, r_v \in R_v, v \in V_d \tag{12}$$

$$t_{r_v^k}^s = t_{r_v^{k-1}}^e + \tau_v^f \sum_{(i,j)\in A} \frac{x_{ij}^{r_v^k}\, l_{ij}}{s_v} + \tau_v^d \sum_{c\in C} z_{cr_v^k}\, q_c \qquad \forall\, r_v^{k-1}, r_v^k \in R_v, v \in V \tag{13}$$

$$t_v^s = t_d^s \qquad \forall\, v \in V \tag{14}$$

$$t_v^e = t_{r_v^{\bar{k}_v}}^e \qquad \forall\, v \in V \tag{15}$$

$$t_v^e \leq min(t_v^s + w_v, t_d^e) \qquad \forall\, v \in V_d, d \in D \tag{16}$$

$$\sum_{r\in R_v} \sum_{(i,j)\in A} x_{ij}^r\, l_{ij} \leq l_v \qquad \forall\, v \in V \tag{17}$$

$$f_{ps} \in I^+ \qquad \forall\, p \in P, s \in S \tag{18}$$

$$x_{ij}^r \in \{0,1\} \qquad \forall\, (i,j) \in A, r \in R \tag{19}$$

$$y_v \in \{0,1\} \qquad \forall\, v \in V \tag{20}$$

$$y_s \in \{0,1\} \qquad \forall\, s \in S \tag{21}$$

$$y_p \in \{0,1\} \qquad \forall\, p \in P \tag{22}$$

$$z_{cr} \in \{0,1\} \qquad \forall\, c \in C; r \in R \tag{23}$$