**Origin-based multi-class traffic assignment formulation**

For the purpose of traffic assignment analysis, again, the authors define the network as a directed graph - , with and denoting sets of nodes and arcs in the network, respectively. Further, this section introduces a set of origins and a set of destination with demand between origin and destination . To develop multi-class traffic assignment, this work assumes classes of vehicles in the network, wherein vehicle class traverses arc at cost , defined on the parameters from the set . These parameters (such as distance, time, energy, and emissions) are assumed to be generated/consumed/emitted at a rate defined by the polynomial function on arc speed . This generalized arc cost for vehicle class , as expressed in *equation 1* is assumed to be strictly positive, continuously differentiable, monotonically non-decreasing and separable (i.e., cost on arc is only dependent on the flow of that arc, and no other arc). These assumptions are further explained, enforced, and explored in the Appendix - A. The arc travel time here is assumed to be deterministic, defined by the BPR function.

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| --- | --- |
| : | Set of nodes |
| *:* | Set of arcs |
| : | Set of parameters |
| : | Set of origins |
| : | Set of destinations |
| : | Set of vehicle classes |
| : | Demand for vehicle class between origin and destination |
| : | Set of destinations with non-zero demand from origin |
| : | Cost of arc for vehicle class |
| : | Length of arc |
| : | Vehicle speed on arc |
| : | Travel time on arc |
| : | Flow on arc |
| : | Flow of vehicle class on arc from origin |
| : | Volume capacity for arc |
| : | BPR parameters for arc |
| : | BPR parameter for arc |
| : | Cost of parameter |
| : | Coefficient of for parameter |

Subject to,

This optimization problem can be significantly simplified by applying the Karush-Kuhn-Tucker (KKT) conditions to the Lagrange transformed problem. The Lagrange transformed objective is,

Developing the KKT conditions,

Where, is the reduced cost defined as , with having an interpretation of least travel cost for vehicle class from origin to node . Thus, the above-developed KKT condition can be interpreted like the classical Wardrop’s equilibrium condition, wherein the cost of all traversed paths between an origin and a destination must be less than the cost of all untraversed paths between this origin and destination. Here, an analogous interpretation of the KKT condition suggests that a traversed arc at equilibrium must have non-zero reduced cost. Such an origin-based traffic assignment model was first formulated by Bar-Gera (2002), who later developed the Traffic Assignment by Paired Alternative Segments (TAPAS) algorithm (Bar-Gera, 2010) to get the assignment solution. Xie and Xie (2016) made further advancements, developing the improved TAPAS (iTAPAS). This work extends the previous developments on iTAPAS to solve a multi-class traffic assignment problem with the multi-class iTAPAS (m-iTAPAS). The fundamental idea behind the TAPAS algorithm is to identify potential arcs, i.e., arcs that have a non-zero origin-based flow and non-zero origin-based reduced cost, and to consequently adjust flow on these arcs. This flow shift process involves the development of a paired alternative segment (PAS), wherein the first segment entails the least-cost path from the origin to the head node of the potential arc, while the other segment is developed backtracking from the tail node of the potential arc for maximum cost node until this second segment converges with the least-cost path between the origin and the head node of the potential arc. Once identified, flow is adjusted on the PAS based on Newton method (Dial, 2006). This process of identifying potential arcs and adjusting flow on the associated PAS continues until a convergence criterion is met.

To develop the necessary properties of the generalized cost function, it is further generalized and simplified as a function on arc travel time, which itself is defined by the BPR function.

Before, the assumptions are imposed on the generalized cost function, this work makes an additional assumption on the nature of the function . In particular, the authors here assume this function on arc speed to be strictly convex, rendering . Such vehicle behavior is typical in context of vehicle efficiency, fuel consumption and emission rate. Now, in order to guarantee existence of the traffic assignment solution, uniqueness of the equilibrium and absence of infinite loops the generalized cost function is assumed to be continuously differentiable, monotonically non-decreasing and strictly positive, respectively, each of which are analyzed or imposed below.

1. *Continuously differentiable*

To establish continuous differentiability of , the analysis here establishes differentiability for and continuity of in the domain of , where .

Since , is continuous in its domain, while is continuous in the domain of for . Note, for , is undefined at , hence the condition, is essential for the generalized cost function to be continuously differentiable.

1. *Monotonically non-decreasing*

The above condition must hold true for the maximum possible speed in the network, thus,

1. *Strictly positive*

Since the generalized cost function is monotonically non-decreasing, at ensures strict positivity, thus,

For above condition to hold true for all arcs,