Vehicle Routing Problem (VRP)

A typical VRP is defined on a directed graph $G = (D, C, A)$, where node set D represents distribution facility nodes, node set C represents pickup and delivery customer nodes, and arc set A represents the set of arcs connecting these nodes, wherein an arc connecting nodes i and j spans a length l_{ij}. Note, each distribution facility node $d \in D$ has an associated set of delivery vehicles V_d, capacity q_d, service start and end time t_d^s and t_d^e, respectively, as well as fixed cost π_d^f, and operational cost π_d^o per package. Further, each customer node $c \in C$ has an associated pickup/delivery node n_c, demand q_c, service time τ_c, and must be catered within a specified time-window $[t_c^e, t_c^l]$ from a distribution facility using a delivery vehicle. These delivery vehicles have an associated set of delivery routes R_v encompassing at most $r_v^{\bar{k}}$ routes, capacity q_v, range l_v, speed s_v, refueling time τ_v^f, loading time per package τ_v^d, parking time τ_v^c, driver working hours τ_v^w, fixed cost π_v^f, and operational costs π_v^{od} per unit distance and π_v^{ot} per unit time, respectively. Note, here we formulate a mixed pickup and delivery vehicle routing problem, i.e., some delivery customer nodes have a specific associated pickup customer node, while other delivery customer nodes must be catered via a distribution facility and hence the associated pickup node is a distribution facility node.

The objective of a VRP is to develop least cost routes from the distribution facilities using select vehicles such that every customer node is visited exactly once while also accounting for depot capacity, vehicle capacity, vehicle range, driver working-hours, and customers' service constraints and time-window.

$$min\ \Pi = \sum_{d \in D} \left(\pi_d^f + \pi_d^o \sum_{v \in V_d} \sum_{r \in R_v} \sum_{c \in C} \psi_c^d z_{cr} q_c \right) + \sum_{v \in V} \left(\pi_v^f + \sum_{r \in R_v} \sum_{(i,j) \in A} \pi_v^{od} x_{ij}^r l_{ij} + \pi_v^{ot} (t_v^e - t_v^s) \right) y_v \qquad (1)$$

Subject to,

$$\sum_{r \in R} z_{cr} = 1 \qquad \forall\, c \in C \qquad (2)$$

$$\sum_{j \in H_c} x_{cj}^r = z_{cr} \qquad \forall\, c \in C, r \in R \qquad (3)$$

$$z_{n_c r} \varphi_{n_c}^p = z_{cr}(1 - \psi_c^d) \qquad \forall\, c \in C \qquad (4)$$

$$\sum_{i \in T_j} x_{ij}^r = \sum_{k \in H_j} x_{jk}^r \qquad \forall\, j \in N, r \in R \qquad (5)$$

$$\sum_{v \in V_d} \sum_{r \in R_v} \sum_{c \in C} \psi_c^d z_{cr} q_c \leq q_d y_d \qquad \forall\, d \in D \qquad (6)$$

$$q_c^a + M(1 - x_{dc}^r) \geq x_{dc}^r \sum_{c \in C} \psi_c^d z_{cr} q_c \qquad \forall\, c \in C, r \in R_v, v \in V_d, d \in D \qquad (7)$$

$$q_c^a + M(1 - x_{ic}^r) \geq x_{ic}^r\, q_i^d \qquad \forall\, i \in T_c, c \in C, r \in R_v, v \in V \qquad (8)$$

$$q_c^d = q_c^a + \varphi_c^p q_c - \varphi_c^d q_c \qquad \forall\, c \in C \qquad (9)$$

$$q_c^a \varphi_c^d \leq \sum_{r \in R_v} z_{cr} q_v \qquad \forall\, c \in C \tag{10}$$

$$q_c^d \varphi_c^p \leq \sum_{r \in R_v} z_{cr} q_v \qquad \forall\, c \in C \tag{11}$$

$$t_c^a + M(1 - x_{dc}^r) \geq x_{dc}^r \left(t_r^s + \frac{l_{dc}}{s_v}\right) \qquad \forall\, c \in C, r \in R_v, v \in V_d, d \in D \tag{12}$$

$$t_c^a + M(1 - x_{ic}^r) \geq x_{ic}^r \left(t_i^d + \frac{l_{ic}}{s_v}\right) \qquad \forall\, i \in T_c, c \in C, r \in R_v, v \in V \tag{13}$$

$$t_c^d \geq t_c^a + \tau_v^c + max(0, t_c^e - t_c^a - \tau_v^c) + \tau_c \qquad \forall\, c \in C \tag{14}$$

$$t_c^a \leq t_c^l \qquad \forall\, c \in C \tag{15}$$

$$t_{n_c}^a \varphi_{n_c}^p \leq t_c^a (1 - \psi_c^d) \qquad \forall\, c \in C \tag{16}$$

$$t_{r_v^1}^s = t_d^s \qquad \forall\, r_v \in R_v, v \in V_d, d \in D \tag{17}$$

$$t_{r_v^k}^s = t_{r_v^{k-1}}^e + \tau_v^d \sum_{c \in C} z_{c r_v^k}\, q_c \qquad \forall\, r_v^{k-1}, r_v^k \in R_v, v \in V \tag{18}$$

$$t_r^e + M(1 - x_{cd}^r) \geq x_{cd}^r \left(t_c^d + \frac{l_{cd}}{s_v}\right) \qquad \forall\, r_v^{k-1}, r_v^k \in R_v, v \in V \tag{19}$$

$$t_v^s = t_d^s \qquad \forall\, v \in V \tag{20}$$

$$t_v^e = t_{r_v^{\bar{k}_v}}^e \qquad \forall\, v \in V \tag{21}$$

$$t_v^e \leq min(t_v^s + \tau_v^w, t_d^e) \qquad \forall\, v \in V_d, d \in D \tag{22}$$

$$\sum_{r \in R_v} \sum_{(i,j) \in A} x_{ij}^r\, l_{ij} \leq l_v \qquad \forall\, v \in V \tag{23}$$

$$x_{ij}^r \in \{0,1\} \qquad \forall\, (i,j) \in A, r \in R \tag{24}$$

$$y_v \in \{0,1\} \qquad \forall\, v \in V \tag{25}$$

$$y_d \in \{0,1\} \qquad \forall\, d \in D \tag{26}$$

$$z_{cr} \in \{0,1\} \qquad \forall\, c \in C; r \in R \tag{27}$$

$$\varphi_{\mathrm{c}}^{\mathrm{d}} = \begin{cases} 1, & \text{if } c \text{ is a pickup node} \\ 0, & \text{else} \end{cases} \qquad \forall\, c \in C \tag{28}$$

$$\varphi_{\mathrm{c}}^{d} = \begin{cases} 1, & \text{if } c \text{ is a delivery node} \\ 0, & \text{else} \end{cases} \qquad \forall\, c \in C \tag{29}$$

$$\psi_{\mathrm{c}}^{\mathrm{d}} = \begin{cases} 1, & \text{if } \varphi_{\mathrm{c}}^{\mathrm{d}} = 1 \text{ and } n_c \in D \\ 0, & \text{else} \end{cases} \qquad \forall\, c \in C \tag{30}$$

Note, the current implementation of the Vehicle Routing Problem in Julia is a pure pickup and delivery problem with time-windows, unlike the formulation, which is a mixed pickup and delivery problem with time-windows. The formulation is set up for future development of the Julia tool.