MATC37 Assignment 4

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Problem 1. Given a collection of Lebesgue measurable sets $F_1, \ldots, F_n \subset \mathbb{R}^d$, construct another collection $F_1^*, \ldots, F_N^*$ of Lebesgue measurable sets with $N = 2^n - 1$ such that $\cup_1^n F_k = \cup_1^N F_j^*$, the F_j^*'s are disjoint and $F_k = \cup_{F_j^* \subset F_k} F_j^*$.

Solution The collection $A := \{F_1^{'} \cap \ldots \cap F_n^{'} : F_i^{'} = F_i \text{ or } F_i^{'} = F_i^c\}$ is clearly of cardinality 2^n since in each element of A, there are n intersections of elements from $\{F_1, \ldots, F_n, F_1^c, \ldots, F_n^c\}$ where each element can be one of two choices F_i^c or F_i. Remove $F_1^c \cap F_2^c \cap \ldots \cap F_n^c$ from A so that A then has cardinality of $2^n - 1$. We claim that A is the desired collection of Lebesgue measurable sets. First, of all it is clear that each element of A is measurable since it is the intersection of measurable sets with intersections of the complement of a measurable set (which is still measurable).

First, we show that the union of all elements in A is equal to $\cup_1^n F_k$.
Choose $x \in F_k \subseteq \cup_1^n F_k$. If $x \in F_k$, then for all $1 \leq i \leq n$, $x \in F_i$ or $x \in F_i^c$ (recall $B \cup B^c = \mathbb{R}^d$ for any $B \subseteq \mathbb{R}^d$). Thus, we can find an element in A such that $x \in F_1^{'} \cap F_2^{'} \cap \ldots \cap F_n^{'}$ with the property that for each $F_i^{'}$, if $x \in F_i$, then $F_i^{'} = F_i$ and similarly for the complement case. Therefore, $x \in \cup A$.
Now, choose $x \in F_j^* \subseteq \cup A$. Write $F_j^* = F_1^{'} \cap \ldots \cap F_n^{'} = F_j^*$. Since we removed the set $F_1^c \cap \ldots \cap F_n^c$ from A, there exists some i such that $F_i^{'} = F_i$ and so $x \in F_i$. But then since $F_i \in \cup_1^n F_k$, we have that $x \in \cup_1^n F_k$.
Therefore, $\cup_1^n F_k = \cup A$ as wanted.

Now, we show A is a collection of pairwise disjoint elements. Pick F_j^* and F_k^* in A such that $j \neq k$. Write $F_j^* = a_1 \cap \ldots \cap a_n$ and $F_k^* = b_1 \cap \ldots \cap b_n$ where for each a_i, b_i, we have that $a_i = F_i^c$ or $a_i = F_i$ and similarly for each b_i. If $F_j^* \neq F_k^*$, then by this construction, there exists some $1 \leq i \leq n$ such that $a_i \neq b_i$. Thus, if $x \in F_k^*$, then $x \in b_i$ and consequently $x \notin a_i$ so that $x \notin a_1 \cap \ldots \cap a_n$. In particular, for every $x \in F_k^*$, $x \notin F_j^*$ and a quick repetition of this argument shows that if $x \in F_j^*$, then $x \notin F_k^*$ implying that $F_j^* \cap F_k^* =$ as wanted. Thus, all the F_j^* (i.e. elements of A) are disjoint.

It is left to show $F_k = \cup_{F_j^* \subset F_k} F_j^*$ for all $k \neq n$. For any F_j^* in A, $F_j^* \subset F_k$ if and only if the kth intersecting element of F_j^* is equal to F_k i.e. if we can write $F_j^* = F_1^{'} \cap \ldots \cap F_n^*$, then $F_k^{'} = F_k$. Choose some $x \in F_k$. If $x \in F_j^*$, then we're done. So, assume $x \notin F_j^*$ which implies for some i, $x \notin F_i^{'}$. Note $i \neq k$ by the discussion above. Thus, choose some element $F_q^* = a_1 \cap \ldots \cap a_n$ such that $F_q^* = F_j^*$ except at all choices of i for which $x \notin F_i^{'}$, let $a_i = (F_i^{'})^c$. Then $x \in F_q^*$ and $F_q^* \subset F_k$ since $a_k = F_k$. Thus, $F_k \subseteq \cup_{F_j^* \subset F_k} F_j^*$ and the subset containment for the other direction follows directly from the fact that $\cup_{F_j^* \subset F_k} F_j^*$ is a union of sets contained in F_k. Thus, $\cup_{F_j^* \subset F_k} F_j^* = F_k$ as wanted.

Problem 2. Let $\phi : \mathbb{R}^d \to \mathbb{R}$ be a simple function, and let $\phi = \sum_1^n \hat{a}_i \chi_{\hat{E}_i}$ be its canonical representation.

1. Prove that if $\phi = \sum_1^N a_k \chi_{E_k}$ is another representation of the simple function ϕ, where the E_k's are disjoint finite measure sets, but the a_k's are not necessarily distinct or nonzero, then $\sum_1^N a_k m(E_k) = \sum_1^n \hat{a}_i m(\hat{E}_i)$.

2. Prove that if $\phi = \sum_1^N a_k \chi_{E_k}$ is another representation of the simple function ϕ, where the E_k's are finite measure sets but not necessarily disjoint and the a_k's are not necessarily distinct or nonzero, then $\sum_1^N a_k m(E_k) = \sum_1^n \hat{a}_i m(\hat{E}_i)$.

Solution Let $\phi = \sum_{k=1}^N a_k \chi_{E_k}$ where the collection $E_1, \ldots, E_k$ of lebesgue measurable sets are disjoint but not all a_k's are distinct and some may be zero. If some a_i is zero, then we can simply remove the set E_i and the constant a_i from our summation since it does not impact the sum. Thus, we have a new collection $E_1, \ldots, E_M$ of disjoint Lebesgue measurable sets (for $M \leq N$) such that $\phi = \sum_{k=1}^M a_k \chi_{E_k}$. Next, since every E_k is disjoint from E_j $(j \neq k)$, then $\phi(x)$ for $x \in \mathbb{R}^d$ is in some specific E_i for some i i.e. $\phi(x) = a_i \chi_{E_i}$. Now, suppose that there is some $a_i = a_j$ for $i \neq j$. We can again simplify our summation by setting $E_i := E_{i'} \cup E_{j'}$ and $a_i = a_{i'}$ (the apostrophe signifies that we are referring to the previous, unsimplified collection) and completely removing E_j and a_j. Clearly the function is not affected since for any $y \in E_{j'}$, $y \in E_i$ so that $\phi(y) = a_i \chi_{E_i} = a_{j'} \chi_{E_{j'}}$ and $a_i = a_{j'}$ from assumption that $a_{i'} = a_{j'}$. We can repeat this process until every a_i is distinct. Thus, we have that every a_i is distinct and non-zero and every E_k is disjoint. Thus, $\phi = \sum_{k=1}^F a_k \chi_{E_k}$ (for some F given by removal of non-distinct constants) is a canonical representation of ϕ but the canonical representation of a simple function is unique. Thus, the collection $a_1, \ldots, a_F$ and $\hat{a_1}, \ldots, \hat{a_n}$ are equivalent (perhaps upto reordering of indices) and similarly for the collections $E_1, \ldots, E_F$ and $\hat{E}_1, \ldots, \hat{E}_n$. Thus, it follows that the following equality holds:

$$\sum_{k=1}^F a_k m(E_k) = \sum_{i=1}^n \hat{a}_i m(\hat{E}_i)$$

To see that the sum on the left side is equal to the sum where we hadn't removed the non-distinct constants, it suffices to notice that $a_k m(E_k) = a_k m(E_{k'} + E_{j'})$ and since $E_{k'}$ and $E_{j'}$ are disjoint, then: $a_k m(E_k) = a_k(m(E_{k'} + E_{j'})) = a_k m(E_{k'}) + a_k m(E_{j'}) = a_{k'} m(E_{k'}) + a_{j'} m(E_{j'})$. And this sum is equivalent to the summation where we hadn't removed zero-valued constants because 0 is the additive identity and does not affect summations. Thus,

$$\sum_{k=1}^N a_k m(E_k) = \sum_{i=1}^n \hat{a}_i m(\hat{E}_i)$$

as wanted.

For the second part, we know $E_1, \ldots, E_N$ are lebesgue measurable sets (otherwise $m(E_k)$ is meaningless). By question 1, we know there exists some other collection of lebesgue measurable sets $E_1^{'}, \ldots, E_F^{'}$ where $F = 2^N - 1$ and has all of the properties question 1 requires. We can then write $\phi = \sum_{i=1}^{F} g_i m(E_i^{'})$ for constants g_i which are yet to be defined.

We know $E_k = \cup_{E_j^{'} \subset E_k} E_j^{'}$ so $m(E_k) = \sum_{j:E_j^{'} \subset E_k} E_j^{'}$. Thus,

$$\phi = \sum_{k=1}^{N} a_k m(E_k) = \sum_{k=1}^{N} a_k \Big(\sum_{j:E_j^{'} \subset E_k} m(E_j^{'}) \Big) = \sum_{k=1}^{N} \Big(\sum_{j:E_j^{'} \subset E_k} a_k m(E_j^{'}) \Big)$$

In the right side of the equation above, we are summing over finitely many indices, thus, we can rewrite this as

$$\sum_{j=1}^{R} a_j m(E_j^{'})$$

where this new collection of all $E_j^{'}$ is not disjoint. However, if for some $k \neq j$, $E_j^{'} \cap E_j^{'} \neq \emptyset$, then $E_j^{'} = E_k^{'}$. Thus, we can write this as $(a_j + a_k) m(E_j^{'})$ since $m(E_k^{'}) = m(E_j^{'})$. Thus, we can again rewrite the sum above as

$$\phi = \sum_{j=1}^{R'} a_j E_j^{'}$$

where then all $E_1^{'}, \ldots, E_{R'}^{'}$ are all disjoint but the constants a_j are not and may not be all non-zero. By the previous part of this question, this is enough to imply that

$$\sum_{j=1}^{R'} a_j m(E_j^{'}) = \sum_{i=1}^{n} \hat{a}_i m(\hat{E}_i)$$

Problem 3. Let $E \subset \mathbb{R}$ be a measurable set wtih $m(E) < \infty$. Recall from class that if $f : E \to \mathbb{R}$ is a measurable function with $|f| \leq M$, then we defined $\int_E f := \int_E \phi_n$ where ϕ_n is any sequence of simple functions which converges point wise to f and satisfies $|\phi_n| \leq M$.

1. Prove that if $f, g : E \to \mathbb{R}$ are bounded measurable functions with $f \leq g$ then $\int_E f \leq \int_E g$
2. Prove that $f : E \to \mathbb{R}$ is a bounded Lebesgue measurable function then $|\int_E f| \leq \int_E |f|$

Solution For the first part, take some sequence of functions $\phi_n \to f$ and $\psi_n \to g$. Fix $x \in E$. If $f(x) < g(x)$, then let $r = (g - f)(x)$. Then, for any $0 < \epsilon < r$, there exists some $N > 0$ such that $|\psi_n(x) - g(x)| < \epsilon$ and $|\phi_n(x) - f(x)| < \epsilon$ (take N to be the max of the integers which makes the inequalities of both sequences hold). By choice of ϵ, we know that for all $n > N$, $\phi_n(x) < \psi_n(x)$ (if $f(x) = g(x)$ i.e. $r = 0$, then we can simply let $\phi_n = \psi_n$ for all $x \in E$. Thus, for all $n > N$ we have that $\phi_n(x) \leq \psi_n(x)$ for all $x \in E$. Since ϕ_n and ψ_n are simple functions, then $\int_E \phi_n \leq \int_E \psi_n$. Limits preserves inequalities so we know the following holds:

$$\lim_{n\to\infty} \int_E \phi_n \leq \lim_{n\to\infty} \int_E \psi_n$$

But by our choice of ϕ_n and ψ_n, we know this is just equivalent to saying:

$$\int_E f \leq \int_E g$$

as wanted.

Let ϕ_n be a sequence of functions so that $\phi_n \to f$. Then by a simple $\epsilon - N$ proof, we know $|\phi_n| \to |f|$ since

$$||\phi_n(x)| - |f(x)|| \leq |\phi_n(x) - f(x)|$$

by the reverse triangle inequality. Therefore, if $\lim_{n\to\infty} \int_E \phi_n = \int_E f$, we know that $\lim_{n\to\infty} \int_E |\phi_n| = \int_E |f|$. Additionally, note for simple functions we have that:

$$|\int_E \phi_n| \leq \int_E |\phi_n|$$

Thus,

$$|\int_E f| = \lim_{n\to\infty} |\int_E \phi_n| \leq \lim_{n\to\infty} \int_E |\phi_n| = \int_E |f|$$

as wanted.

Problem 4. Let $f : E \to \mathbb{R}$ be a bounded measurable function defined on a measurable domain $E \subset \mathbb{R}^d$ with $m(E) < \infty$.

1. Show that if $g : E \to \mathbb{R}$ is bounded and $g(x) = f(x)$ for a.e. $x \in E$, then $\int_E g = \int_E f$.

2. Show that if $f \geq 0$ and $\int_E f = 0$, then $f(x) = 0$ for a.e. $x \in E$.

Solution For the first part, define $h(x) := f(x) - g(x)$. Then $h(x) = 0$ for a.e. $x \in E$ by choice of f and g. If $\phi_n \to f$ and $\psi_n \to g$, then

$$h(x) = f(x) - g(x) = (\lim_n \phi_n(x)) - (\lim_n \psi_n(x)) = \lim_n (\phi_n - \psi_n)(x)$$

so that $\phi_n - \psi_n \to h$. In particular, by Lemma 1.2 in Chapter 2, we have that since h is 0 a.e. for every $x \in E$, then:

$$\lim_n \int_E (\phi_n - \psi_n) \to 0 \qquad \Longrightarrow \qquad \int_E h = 0$$

Since ϕ_n and ψ_n are simple functions, we know that $\int_E (\phi_n - \psi_n) = \int_E \phi_n - \int_E \psi_n$ so in particular,

$$\lim_n [(\int_E \phi_n) - (\int_E \psi_n)] \to 0 \qquad \Longrightarrow \lim_n \int_E \phi_n - \lim_n \int_E \psi_n \to 0$$

so that in particular, $\int_E f - \int_E g = 0$ so that $\int_E f = \int_E g$ as wanted.

For the second part, we know f is non-negative so if we write $E = E' \cup E^2$ where E^2 is the set where f is non-zero and E' is the set where f is identically zero, then we have $\int_E f = \int_{E'} f + \int_{E^2} f$. Clearly, we have $\int_{E'} f = 0$ since f is exactly zero on E' (if one wishes to be rigorous in this, then choose the sequence of identically zero simple functions, these converge to f on E' and all have integral zero). Thus, we obtain $\int_{E^2} f = 0$ since $\int_E f = 0$. Find some sequence of functions $\{\phi_n\} \to f$ on E^2 such that each ϕ_n is non-negative, and the sequence is increasing (possible by Theorem 4.1 in Chapter 1). Since f is non-zero (and thus, positive) on E^2, if we assume $m(E^2) > 0$, then by the Lemma below, we obtain: $\int_{E^2} f > 0$. This is absurd as we have already established $\int_{E^2} f = 0$. Thus $m(E^2) = 0$. This implies $f(x) = 0$ for a.e. $x \in E$.

Problem 5. Compute the following limits and justify the calculations:

1. $\lim_{n\to\infty}\int_0^1 n\sin(x/n)/xdx$
2. $\lim_{n\to\infty}\int_0^\infty (1-x/n)^n x^2 dx$

Solution For the first calculation: Recall that for Lebesgue integrals: $\int_E \lim_{n\to\infty} f_n = \lim_{n\to\infty}\int_E f_n$. Thus, it suffices to compute $\int_0^1 \lim_{n\to\infty} n\sin(x/n)/xdx$. First, we compute the limit. Note that we can write this as: $\lim_{n\to\infty}\sin(x/n)/(x/n)$ where then both sides of the fraction go to zero. Thus, by L'hopital's rule, we have that this limit is equivalent to: $\lim_{n\to\infty}\cos(x/n)(x/n)'/(x/n)'$ where the $'$ denotes the derivative opeartor. Cancelling out terms, we arrive at $\int_0^1 \lim_{n\to\infty} n\sin(x/n)/xdx = \int_0^1 \lim_{n\to\infty}\cos(x/n)dx = \int_0^1 \cos(0)dx = \int_0^1 1 = 1$.

Again as before, we switch the limit and integral operator to arrive at the computation:

$$\int_0^\infty \lim_{n\to\infty}(1-x/n)^n x^2 dx = \int_0^\infty x^2 e^x dx$$

Recalling that the Riemann and Lebesgue integrals agree on finite domains, we write the integral above as:

$$\int_0^\infty e^x x^2 dx = \lim_{n\to\infty}\int_0^n e^x x^2 dx$$

and then compute the integral on the right side using techniques from first year Calculus.

Problem 6. Let $f : \mathbb{R}^d \to \mathbb{R}$ be an integrable function such that $\int_E f(x)dx \geq 0$ for every measurable set E

1. Prove that $f(x) \geq 0$ for a.e. x

2. Prove that if we assume in addition that f is continuous, then $f(x) \geq 0$ for all x.

Solution For the first part, the proof is done in a somewhat similar style to question 4.2. Thus, write $E = E' \cup E^2$ where f is non-negative on E' and negative on E^2. Note that f is measurable on E' and E^2 because f is measurable on $\{f \geq 0\}$ and $\{f < 0\}$. Thus, we have that $0 \leq \int_E f = \int_{E'} f + \int_{E^2} f$. Note since f is non-negative on E', it follows that $0 \leq \int_{E'} f$ so we must have that $0 \leq \int_{E^2} f$. Intuitively, we must have that $m(E^2) = 0$ since if it were not, then f is negative on every $x \in E^2$ but $\int_{E^2} f \geq 0$ which is absurd. Assume $m(E^2) \neq 0$. We know that f is negative on E^2 so that $-f$ is positive on E^2. By the lemma, it follows that $\int_{E^2} -f > 0$. By linearity, $-(\int_{E^2} f) > 0$ so that $\int_{E^2} f < 0$ which is absurd. Thus, $m(E^2) = 0$ so that $f \geq 0$ for a.e. $x \in E$.

We know $m(E^2) = 0$ so in particular, E^2 does not contain any intervals (not including the trivial $[x, x]$). Thus, if f is continuous on E, then it is continuous on $x \in E^2$. Thus, for any neighborhood $N_1(f(x))$, there is a neighborhood $N_2(x)$ such that $f(y) \in N_1(f(x))$ whenever $y \in N_2(x)$. Since E^2 does not contain any intervals, we have that $N_1(f(x))$ is not contained in E^2. In particular, there exists some $y \in N_1(x)$ such that $f(y) \geq 0$. Thus, $|f(x) - f(y)| \geq |f(x) - 0| = |f(x)|$. Thus, choose $0 < \epsilon < |f(x)|$ to get that $|f(x) - f(y)| \geq \epsilon$ which is absurd since we have established that f is continuous on x. Thus, no such $x \in E^2$ exists. Thus, $E^2 = \emptyset$ and so $f(x) \geq 0$ for all $x \in E$ as wanted.

Lemma I. If f is positive on some measurable domain F (with $m(F) > 0$), then $\int_F f > 0$.

Proof. Let $F_n = F \cap \{f > 1/n\}$. Then $\cup_n F_n = F$ since as $n \to \infty$, we know $\{f > 1/n\} \to \{f > 0\} = F$ since f is positive on F. We know $m(F) > 0$ so it follows for some n, $m(F_n) > 0$. For this n, we have that if $x \in F_n$, then $f(x) \geq 1/n$ just by definition of $\{f > 1/n\}$. Thus, $\int_{F_n} f \geq 1/n \cdot m(F_n) > 0$. Since $F_n \subseteq F$ with both F_n and F being measurable, then by monotonicity $\int_F f \geq \int_{F_n} f > 0$ as wanted. □