MATB43: Introduction to Analysis
Lecture Notes

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Abstract. Shamelessly Stolen from the course description of MATB43 on the 2016/2017 UTSC Calendar:
Generalities of sets and functions, countability. Topology and analysis on the real line: sequences, compactness, completeness, continuity, uniform continuity. Topics from topology and analysis in metric and Euclidean spaces. Sequences and series of functions, uniform convergence.

Contents

Chapter 1

Administration Details

1.1. Pre-requisites

- MATA37: Calculus for Mathematical Sciencies II
- MATB24: Linear Algebra II
- Core-requisite - MATB42: Multivariable Calculus II

1.2. Professor

- Name: Raymond Grinnell
- Office: IC 466
- Office Phone Number: (416) 287 - 5655 (Please do not call his home phone)

1.3. Resources

- Real Analysis and Foundations (3^{rd} ed.) by Steven G. Kratz
- Class Website: http://www.math.utsc.utoronto.ca/b43/

1.4. Grading Scheme

- There are five tutorial quizzes with the lowest one dropped. This accounts for 18% of your grade.
- There are weekly practice problems posted on the class website. These are just to prepare for the quizzes and/or practice your skills. These account for 0% of your grade.
- There are two lecture assignments worth a total of 7% each. In two random lectures for this course, the professor will post problems

that are to be handed in for marking in the next lecture. This is to ensure that your mathematical writing is up to what is expected of this course.

- There is one midterm worth 30%.
- There is a final exam worth 45%.

Part 1

Sets and Number Systems

Chapter 2

Sets and their Operations

2.1. What is a Set?

Perhaps the simplest, most common and most useful object in mathematics is a set. Thus, one of the first thing a mathematician must learn - that is, if they ever hopes to learn math in a rigorous fashion - is some basic set theory. Unfortunately, despite a set being one of the simplest objects in mathematics, talking about it in a very rigorous fashion is a very challenging endeavour that is most commonly taken up at the graduate or senior undergraduate level. Fortunately, most of mathematics is content with taking the intuitive definition of a set. We introduce it now:

Definition 2.1. A **set** is a collection of arbitrary elements. If α is a set, and $\alpha_1, \alpha_2, \alpha_3$ are its elements, then we say $\alpha = \{\alpha_1, \alpha_2, \alpha_3\}$.

2.2. Listing a Set and Some Useful Sets

A set can contain finite or infinite elements. One example of a contains infinite elements is the set of all natural numbers or counting numbers. We denote this as:

$$\mathbb{N} = \{1, 2, 3, \ldots\}$$

The integers also form a set. We denote them as:

$$\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$$

Sometimes, it is not possible to explicitly list the elements of a set, instead we write the condition that an arbitrary element has to fullfill in order to be a part of the said set. For example:

$$\mathbb{Q} = \{\frac{p}{q} : p \in \mathbb{Z},\ q \in \mathbb{Z}\}$$

is the set of the rational numbers. Another example is the set:

$$\begin{aligned}\mathbb{R} = \{&\text{Set of Infinite Decimal Expansion with the Rule}\\ &\qquad\qquad i.999\ldots = i+1 \text{ where } i \in \mathbb{Z}\}\end{aligned}$$

This is the set of real numbers. A final, and also much simpler to comprehend example is the set:

$$\{x : x \geqslant 0, x \in \mathbb{Z}\}$$

which is the set of *non-negative* integers.

Definition 2.2. We denote the set of no elements by $\varnothing$ and call it the **empty set**.

If you have trouble believing or simply refuse to accept that such a set can exist, it is still very useful to have some notation that can represent an object with the properties of an empty set because it will be widely used in this text.

2.3. Some Common Set Notation

Definition 2.3. If a set α contains an element α_1, we say that $\alpha_1 \in \alpha$, and if α_1 is not in α, we write $\alpha_1 \notin \alpha$.

Definition 2.4. If every element in α is also in β, and every element of β is also in α, we say that, $\alpha = \beta$. If every element of the set α is contained in a set β, we say that $\alpha \subset \beta$ as in "Alpha is a **subset** of Beta". Note that, $\alpha = \beta$, if and only if $\alpha \subset \beta$ and $\beta \subset \alpha$. Consider sets α, β where $\alpha \subset \beta$. If we need to explicitly state that every element of α is in β but not vice verse, then we may write $\alpha \subseteq \beta$ and say that α is a **proper subset** of β. That being said, $\subset$ does not necessairly mean two sets are not proper subsets, however, it may be different in other texts.

Some common subsets of $\mathbb{R}$ that we will see in the text are *closed intervals* or *open intervals*.

Definition 2.5. We say a set I is a **closed interval** if it can be written in the form $\{x : \alpha \leqslant x \leqslant \beta,\ x \in \mathbb{R}\}$ where α and β are two real numbers such that $\alpha \leqslant \beta$. These sets are denoted as:

$$[\alpha,\ \beta]$$

Definition 2.6. Similarly to the definition above, we say that a set I is an **open interval** if it can be written in the form $\{x : \alpha < x < \beta,\ x \in \mathbb{R}\}$. These sets are denoted as:

$$(\alpha,\ \beta)$$

Remark 2.7. The notation used for the above two sets is very malleable. For example, $\{x : \alpha < x \leqslant \beta\}$ can be denoted as $(\alpha, \beta]$. Similarly, $\{x : \alpha \leqslant x < \beta\}$ can be denoted as $[\alpha, \beta)$

Remark 2.8. The set of one element α (given that α is in $\mathbb{R}$) can be written as a closed interval since it is in the form $\{x : \alpha \leqslant x \leqslant \alpha,\ x \in \mathbb{R}\}$.

2.4. Set Operations

Perhaps the most common operations on sets are *unions*, *intersections* and *cartesian product*. We define them now.

Definition 2.9. Let $\alpha = \{\alpha_1, \alpha_2, \ldots\}$ and $\beta = \{\beta_1, \beta_2, \ldots\}$, then we say that their **union** is the set where

$$\alpha \cup \beta = \{x : x \in \alpha \text{ or } x \in \beta\}$$

Definition 2.10. Using the same sets as definition 1.5, we say that the **intersection** of α and β is the set

$$\alpha \cap \beta = \{x : x \in \alpha \text{ and } x \in \beta\}$$

Definition 2.11. Again, let use the same sets as definition 1.5. Then, we say that the **cross product** of α and β is the set:

$$\alpha \times \beta = \{(x, y) : x \in \alpha, y \in \beta\}$$

We will do some quick examples, then quickly generalize the two definitions above as that is where most of the interesting results and behaviours lie.

Example 2.12. The intersection of the null set and any set is the null set. Similarly, the union of a null set and any arbitrary set α is equal to α.

Example 2.13. $\mathbb{Z} \cap \mathbb{R} = \mathbb{Z}$. To generalize this, let us say that $\alpha \subset \beta$ where α, β are arbitrary sets. Then, $\alpha \cap \beta = \alpha$.

Example 2.14. $\mathbb{Z} \cup \mathbb{R} = \mathbb{R}$. To generalize this, let us say similarly to example 1.8, that $\alpha \subset \beta$, then $\alpha \cup \beta = \beta$.

Example 2.15. We can construct $\mathbb{R}^2$ via taking the cross product of $\mathbb{R}$ with itself. This is because any element in $\mathbb{R}^2$ can be explicitly written as (x, y) where $x \in \mathbb{R}$ and $y \in \mathbb{R}$. By the definition of cross product (1.8), we know that every element in $\mathbb{R} \times \mathbb{R}$ can be written in this form, thus $\mathbb{R} \times \mathbb{R} = \mathbb{R}^2$.

The reader is invited to prove any of the results above (except perhaps example 1.12 since there is nothing much to prove), it is fairly easy and just requires for one to apply the appropriate definitions.

2.5. Generalized Set Operations

Generalizing definitions 1.6 and 1.7 mostly involves just being able to take the intersections, unions or cross products of just more than two sets. To talk about this efficiently, we introduce the idea of a *collection of sets*.

Definition 2.16. We say $\mathcal{A}$ is a **collection of sets** if its elements are sets. That is, if $\alpha_1, \alpha_2, \alpha_3, \ldots$ are arbitrary sets, and $\mathcal{A} = \{\alpha_1, \alpha_2, \alpha_3, \ldots\}$, then we say that $\mathcal{A}$ is a collection of sets.

When we say something like $\alpha = \{\alpha_1, \alpha_2, \alpha_3, \ldots\}$, we are *labelling* arbitrary elements through the counting numbers, this allows us to very easily talk about a large amount of arbitrary elements. We generalize this process of labelling.

Definition 2.17. A set α whose elements can associated to another set β in that β's elements are *labelled* or *indexed* by the elements of α is called a **index set**.

Example 2.18. Consider, $\alpha = \{x : x > 0, x \in \mathbb{Z}\}$. Then, α seems like it is an index set which could easily replace the natural numbers. For example, let us suppose

$$\beta = \{\beta_1, \beta_2, \beta_3, \ldots\}$$

Then, if we associate -1 to be the same element 1 is associated to and so continue the same way for the rest of the numbers, then saying:

$$\beta = \{\beta_{-1}, \beta_{-2}, \beta_{-3}, \ldots\}$$

is completely accurate.

Definition 2.19. Let $\mathcal{A}$ be an arbitrary index set onto A which is an arbitrary collection of sets. Then, the **union of arbitrary sets** is defined to be:

$$\bigcup_{\alpha \in \mathcal{A}} A_\alpha = \{x : x \in A_\alpha\}$$

Definition 2.20. Using the same index set and collection as definition 1.16, we say that the **intersection of arbitrary sets** is:

$$\bigcap_{\alpha \in \mathcal{A}} A_\alpha = \{x : \forall \ \alpha \in \mathcal{A},\ x \in A_\alpha\}$$

While we can introduce the what it means to take the cross product of an arbitrarily amount of sets, writing it down with the language that we have now, it may become messy. Therefore, we will introduce it later on in the text.

Now we give some examples of some of the definitions above.

Example 2.21. Let $i \in \mathbb{Z}$, and $A_i = \{x : i \leqslant x \leqslant i + 1,\ x \in \mathbb{R}\}$. Then,

$$\bigcup_{i \in \mathbb{Z}} A_i = \mathbb{R}$$

Also,

$$\bigcap_{i \in \mathbb{Z}} A_i = \mathbb{Z}$$

Example 2.22. Let $I = \{x : x > 0,\ x \in \mathbb{Z}\}$ be an index set, then:

$$\bigcap_{i \in I} (-\frac{1}{i},\ \frac{1}{i}) = \{0\} = [0]$$

Thus, we see that the *infinite* intersections of open intervals is not necessairly open.

Chapter 3

Fields

3.1. What is a Field?

Roughly put, a field is an abstract number system consisting of addition and multiplicative operations which work however one wants to and consisting of any elements which would otherwise represent numbers. The only limitation is that these rules and elements have to be consistent within itself, that is, we cannot do something which doesn't make sense i.e. obtaining $2 = 1$.

We now give this a rigorous definition.

Definition 3.1. A field F is a set on which two binary operations (denoted by $+$ and $\cdot$, called *addition* and *multiplication* respectively) are defined so that, for each pair of elements x, $y \in F$, there are unique elements $x + y$ and $x \circ y$ in F for which the following conditions hold for all elements a, b, c in F.

$F1$ $a + b = b + a$ and $a \cdot b = b \cdot a$. This is known as *commutativity* of addition and multiplication.

$F2$ $(a + b) + c = a + (b + c)$ and $(a \cdot b) \cdot c = a \cdot (b \cdot c)$.This is known as *associativity* of addition and multiplication.

$F3$ There exists distinct elements denoted by 0 and 1 in F such that

$$0 + a = a \quad \text{and} \quad 1 \cdot a = a$$

We call this the *existence of identity elements* for addition and multiplication.

$F4$ For each element a in F and each nonzero element b in F, there exist elements c and d in F such that

$$a + c = 0 \quad \text{and} \quad b \cdot d = 1$$

This is known as the *existence of inverses* for addition and multiplication.

$F5$ $a \cdot (b + c) = a \cdot b + a \cdot c$. This is known as the *distributivity of multiplication over addition.*

3.2. Examples of Fields

Example 3.2. The sets $\mathbb{Q}$ and $\mathbb{R}$ form a field over the normal rules of addition and multiplication.

Example 3.3. $\{a + \sqrt{2}b \in \mathbb{R} : a,\ b \in \mathbb{Q}\}$ forms a field over the "normal" rules of addition and multiplication operations that we see in $\mathbb{R}$ or $\mathbb{Q}$

Example 3.4. Not all fields have infinite elements, there are many examples of of what we call *finite fields*, that is fields which have a finite number of elements in them. Let $F = \{O,\ I,\ A,\ B\}$ with the operations

$\cdot$	O	I	A	B
O	O	O	O	O
I	O	I	A	B
A	O	A	B	I
B	O	B	I	A

$+$	O	I	A	B
O	O	I	A	B
I	I	O	B	A
A	A	B	O	I
B	B	A	I	O

Then, we say F forms a field over the operations stated above. Since F is finite and contains 4 elements, we denote F by F_4.

Example 3.5. There are other examples of finite fields as well. For example, consider the set $F = \{x \in \mathbb{Z} : 0 < x < n\}$ (where $n \in \mathbb{N}$) but we define its operations differently than what is normally defined on $\mathbb{Z}$. We define addition as the sum of two numbers modulo n, or in other words, if $,x y \in F$, then their sum is defined as: $x + y \mod n$. Furthermore, multiplication is defined as, $x \cdot y \mod n$. Similarly, multiplication is defined as $x \cdot y \mod n$. Then F forms a field if n is prime. The reason why n has to be prime is because if n is not prime, then there will exist some elements which have no multiplicative inverse. For example, if $x \in F$ divides $y \in F$ or otherwise stated $(x = y \cdot m)$ for some $m \in F$ with $1 < y < x$, then y has no multiplicative inverse.

Example 3.6. If $q \in \mathbb{N}$, there exists a finite field with q elements if and only if $q = p^m$, $m \in \mathbb{N}$ where p is prime.

Below are some useful elements in a field, that we may need to refer to.

Definition 3.7. Let F be a field. Then,

(i) For $a \in F$, denote by $-a \in F$, the unique element such that $a + (-a) = 0$. For $a,\ b \in F$, define $a - b := a + (-b)$.

(ii) For $a \in F$, $a \neq 0$ denote $a^{-1} \in F$ the unique element such that $a \cdot a^{-1} = 1$. For $a,\ b \in F$, $b \neq 0$, define $\frac{a}{b} := a \cdot b^{-1}$.

3.3. Properties of Fields

We now state some properties which hold for any general field.

Theorem 3.8 (The Cancellation Law)**.** *Let F be a field. Then,*

$$\forall\, a,\ b,\ c \in \mathbb{F} : a + b = c + b \implies a = c \tag{3.1}$$

$$\forall\, a,\ b,\ c \in \mathbb{F},\ b \neq 0 : a \cdot b = c \cdot b \implies a = c \tag{3.2}$$

Proof. Let F be a field with $a, b, c \in F$. Then,

(1) By F4, there exists $-b \in F$, so consider: $a + b + (-b) = a = c + b + (-b) = c$.

(2) By F4, there exists $b^{-1} \in F$, so consider: $a \cdot b \cdot b^{-1} = a = c \cdot b \cdot b^{-1} = c$.

□

Theorem 3.9 (Uniqueness of Inverse Element)**.** *Let F be a field. Then,*

(1) $\forall\ a \in F$ *there is a* *unique* $b \in F : a + b = 0$

(2) $\forall\ a \in F,\ a \neq 0$, *there is a* *unique* $b \in F : a \cdot b = 1$

Proof. We leave the second part as an exercise due to similarity.

(i) Suppose $b,\ b^{'} \in F$ with $a + b = 0 = a + b^{'}$. By cancellation, we obtain $b = b^{'}$

(ii) Exercise.

□

Theorem 3.10. *Let F be a field. Then if $a \in F$, $a \cdot 0 = 0$.*

Proof. Let $a \in F$. Consider:

$$\begin{aligned} a \cdot 0 &= a \cdot (0 + 0) && \text{By F3} \\ &= a \cdot 0 + a \cdot 0 && \text{By F5} \end{aligned}$$

So by F4 we have,

$$a \cdot 0 + (-a \cdot 0) = a \cdot 0 + a \cdot 0 + (-a \cdot 0)$$

which implies

$$0 = a \cdot 0$$

$\square$

Theorem 3.11. *Let F be a field. For any $a,\ b \in F$:*

$$a \cdot b = 0 \rightarrow a = 0 \text{ or } b = 0$$

Proof. Suppose $a \cdot b = 0$. If $a = 0$, we are done so assume $a \neq 0$. By $F4$, a^{-1} exists. Then consider,

$$\begin{aligned}
b &= 1 \cdot b \\
b &= (a^{-1} \cdot a) \cdot b \\
b &= a^{-1} \cdot (a \cdot b) \\
b &= a^{-1} \cdot 0 \\
b &= 0
\end{aligned}$$

$\square$

Theorem 3.12. *For $a, b, c, d \in F$ where F is a field, the following properties hold:*

(1) $-(-a) = a$
(2) $\frac{a}{b} = \frac{ac}{bd}$
(3) $(-a) \cdot b = -(a \cdot b)$
(4) $(a + b)(a - b) = a^2 - b^2$
(5) *If $a \neq 0$, then $(a^{-1})^{-1} = a$*

3.4. Ordered Fields

Definition 3.13. Let F be a field. Then F is an ordered field with order $\leqslant$ if and only if,

O1 For any $a, b \in F$: If $a \leqslant b$, then $a + c \leqslant b + c$.
O2 For any $a, b \in F$: If $0 \leqslant a$ and $0 \leqslant b$, then $0 \leqslant a \cdot b$.

Example 3.14. We define order on the real line using the concept of the number line. Similarly, order on the rational field is defined.

Remark 3.15. Consider a field where $\sum_{i=1}^{n} 1 = 0$ holds for some $n > 1$, then this field cannot be a ordered field. This follows from O1, specifically $1 < 1 + 1$.

We now describe some useful properties of ordered fields.

Theorem 3.16 (Properties of Ordered Fields)**.** *Suppose F is a ordered field with order $\leqslant$ and let $a, b, c, d \in F$. Then,*

1. *(Transitivity) If $a \leqslant b$ and $b \leqslant c$, then $a \leqslant c$.*
2. *(Summation of Inequalities) If $a \leqslant b$ and $c \leqslant d$, then $a + c \leqslant b + d$.*
3. *(Multiplication of an Inequality) If $a \leqslant b$, then $ac \leqslant bc$.*

Proof. □

Chapter 4

The Real Numbers

4.1. Finding a Suitable Number System

In order to do advanced mathematics, we must have a solid foundations. This can mean a variety of things, but in this context, we want a "good" number system to perform math on. Let's consider the natural numbers to be a candidate for this.

One way number systems can be "nice" is that performing operations of arithmetics yields you an element within the set you started from. So first, we consider addition. Obviously, any two elements in the natural numbers added together will give you back a natural numbers. However, we are not so fortunate in the case for subtraction. For example, consider $1 - 2 = -1$.

This is not within $\mathbb{N}$. Since $1 - 2 \in \mathbb{Z}$, let us consider $\mathbb{Z}$ as the viable candidate. Addition, Subtraction seem to work as we want them to and we see that any two integers multiplied give you back an integer. This victory is shortlived however, when we consider the case of division such as $1 \div 2$.

Since $1 \div 2 \in \mathbb{Q}$, let us consider $\mathbb{Q}$ as our number system of choice. It seems to be that all four mathematical operations give results back in $\mathbb{Q}$, does that mean that we have concluded our search? One other "nice" property that our choice of number system should have is that it should be yield rational numbers when performing *limiting* operations on the rational numbers.

This is where we begin to see that we have a problem. For example, consider the set $\{x : x > 0 \text{ and } x^2 < 2\}$. It seems obvious that the upper limit of this set seems to be tending to some number x that fulfills the equation $x^2 = 2$ but as we will show, it turns out that this number is not rational, but instead *irrational* (not rational).

Theorem 4.1. *There is no $x \in \mathbb{Q}$ such that it fulfills the equation $x^2 = 2$.*

Proof. Suppose x is rational. Then there exists $a, b \in \mathbb{Z}$ where $b \neq 0$ such that $x = \frac{a}{b}$. Suppose that a and b have no common factors (If they do, then divide the factor). Then consider,

$$\frac{a}{b}^2 = 2$$
$$a^2 = 2b^2$$

So that a^2 is even. But this can only be if a itself is even and that a^2 is divisble by 4. This in turn implies that b so that a and b are both even. This is a contradiction since we assumed they have no common factors. $\square$

This shows that there "holes" within the rational number system so we then use a number system which includes "holes" (i.e. numbers which fulfill equations such as $x^2 = 3$, $y^2 = 5$, $z^2 = 7, \ldots$. This is not all of the holes in the rational numbers, but filling in all of the holes, we obtain the real number system $\mathbb{R}$. When we construct the real numbers, we will show that we have indeed filled in all of the holes of the rational numbers.

4.2. Least Upper Bound Property of the Reals