CSF Notes

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1 Basics

Suppose $\{\Gamma_t \subset \mathbb{R}^2\}$ is one-parameter family of embedded (i.e. simple) curves. If this family moves by curve shortening flow (CSF), by definition, it satisfies:

$$\partial_t(p) = \vec{\kappa}(p) \tag{1.1}$$

where p is some point on Γ_t. In order to compute $\partial_t(p)$ and $\vec{\kappa}(p)$, first, we parameterize Γ_t by some parameteric equation $\phi_t : [0, 2\pi] \to \mathbb{R}^2$. Suppose this is in arc-length parameterization. Then, $\partial_t(p)$ is just computed by differentiating $\phi_t(x)$ with respect to t (x some element of domain) and $\vec{\kappa}(p)$ is computed by $\partial_x^2[\phi_t(x)]$. Suppose ϕ_t is not given in arc-length parameterization. Note also that if $\alpha : I \to \mathbb{R}^2$ is a plane curve $\alpha(s) = (x(s), y(s))$, the signed curvature is given by:

$$k(s) = \frac{x'y'' - x''y'}{((x')^2 + (y')^2)^{3/2}}$$

so that we can then compute $\vec{k}$ by evaluating $k(s)N(s)$ where N denotes the normal vector at of α at s.

1.1 Round shrinking circles

We show that if $\Gamma_t = \partial B^2_{r(t)} \subset \mathbb{R}^2$, then (1.1) reduces to the ODE:

$$\dot{r} = -1/r$$

and if we give it the initial value $r(0) = R$, then $r(t) = \sqrt{R^2 - 2t}$ for $t \in (-\infty, R^2/2)$.

Proof. Let $r(t)$ be a C^1 function that gives a radius dependent on the parameter t. Assume $B^2_{r(t)} = \{x \in \mathbb{R}^2 : |x| < r(t)\}$. Then $\partial B^2_{r(t)} = \{x \in \mathbb{R}^2 : |x| = r(t)\}$. Fix t. We let $\partial B^2_{r(t)}$ be given by the parameterization $\phi_t : [0, 2\pi r(t)] \to \mathbb{R}^2$,

$$\phi_t(s) = r(t)(\cos(s), \sin(s)), \qquad s = x/r(t).$$

where we assume $r(t) \neq 0$. Then s is the parameter which makes ϕ_t into an arclength parameterization since,

$$\begin{aligned} |\phi_t'(s)| &= |\phi_t'(s) \cdot ds/dx| \\ &= |r(t)(-\sin(s), \cos(s)) \cdot 1/r(t)| \\ &= 1 \end{aligned}$$

Now, we compute $\partial_t(\phi_t(s))$. Consider:

$$\begin{aligned} \partial_t(\phi_t(s)) &= \partial_t(r(t)(\cos(s), \sin(s))) \\ &= \dot{r}(t)(\cos(s), \sin(s)) - [\dot{r}(t) \cdot r(t)/r^2(t)](-\sin(s), \cos(s)) \\ &= \dot{r}(t)(\cos(s), \sin(s)) - [\dot{r}(t)/r(t)](-\sin(s), \cos(s)) \end{aligned}$$

and also,

$$\begin{aligned} \vec{\kappa}(\phi_t(s)) &= \partial_s^2(\phi_t(s)) \\ &= \partial_s[\partial_s(\phi_t(s))] \\ &= \partial_s[r(t)(-\sin(s)1/r(t), \cos(s)1/r(t))] \\ &= \partial_s[(-\sin(s), \cos(s))] \\ &= [-1/r(t)](\cos(s), \sin(s)) \end{aligned}$$

so that (1.1) reduces to,

$$\dot{r}(t)(\cos(s), \sin(s)) - [\dot{r}(t)/r(t)](-\sin(s), \cos(s)) = [-1/r(t)](\cos(s), \sin(s))$$

or equivalently,

$$\begin{aligned} \dot{r}(t)(\cos(s), \sin(s)) + [1/r(t)](\cos(s), \sin(s)) &= [\dot{r}(t)/r(t)](-\sin(s), \cos(s)) \\ [\dot{r}(t) + 1/r(t)](\cos(s), \sin(s)) &= [\dot{r}(t)/r(t)](-\sin(s), \cos(s)) \\ \frac{\dot{r}(t) + 1/r(t)}{\dot{r}(t)/r(t)}(\cos(s), \sin(s)) &= (-\sin(s), \cos(s)) \\ \frac{r(t)\dot{r}(t) + 1}{\dot{r}(t)}(\cos(s), \sin(s)) &= (-\sin(s), \cos(s)) \\ \left[r(t) + \frac{1}{\dot{r}(t)}\right](\cos(s), \sin(s)) &= (-\sin(s), \cos(s)) \end{aligned}$$

Might have done something wrong here. Let us try another way.

Let our next attempt be to try and evaluate (1.1) using the parameterization $\psi_t(x) = r(t)(\cos(x), \sin(x))$ i.e. a parameterization which is not an arclength-parameterization. Then $\partial_t(\psi_t(x)) = \dot{r}(t)(\cos(x), \sin(x))$ and the curvature is computed by the formula:

$$\begin{aligned}
k(x,t) &= \frac{x'y'' - x''y'}{((x')^2 + (y')^2)^{3/2}} \\
&= \frac{(-r(t)\sin(x))(-r(t)\sin(x)) - (-r(t)\cos(x))(r(t)\cos(x))}{((-r(t)\sin(x))^2 + (r(t)\cos(x))^2)^{3/2}} \\
&= \frac{r^2(t)[\sin^2(x) + \cos^2(x)]}{(r^2(t)[\sin^2(x) + \cos^2(x)])^{3/2}} = \frac{r^2(t)}{(r^2(t))^{3/2}} = \frac{1}{r(t)}
\end{aligned}$$

and $N(x,t)$ should be a unit normal pointing towards the center of the circle. Not sure if I am even computing my curvature and my normal correctly... $\square$