Problem Set 14

MAT237: Advanced Calculus
January 29, 2017 Anmol Bhullar

Problem 0.1. *Using any equivalent definition of integration, show that integration is a linear operator; that is, show that if f_1, f_2 are integrable on $[a, b]$, then $c_1 f_1 + c_2 f_2$ are integrable on $[a, b]$ and*

$$\int_a^b [c_1 f_1(x) + c_2 f_2(x)]dx = c_1 \int_a^b f_1(x)dx + c_2 \int_a^b f_2(x)dx$$

Solution. We will use the following definition of integration. Suppose f is a bounded function $f : [a, b] \to \mathbb{R}$ that satisfies the following property: $\forall\, \epsilon > 0$, $\exists$ a partition $P_{[a,b]}$ such that:

$$U_f(P) - L_f(P) < \epsilon$$

Assume $c_1 \neq 0$ (if it is, then the integral of $c_1 f$ where f is any function is simply 0), then:

$$cU_f(P) - cL_f(P)$$

to which we can choose any number $|c_1|x > 0$ as our epsilon so that,

$$c_1(U_f(P) - L_f(P)) = c_1 U_f(P) - c_1 L_f(P) < |c_1|x = \epsilon$$

Similarly if c_2 is zero, then we are just integrating $c_1 f_1$, so assume $c_2 \neq 0$. Then consider:

$$c_1(U_{f_1}(P) - L_{f_1}(P)) + c_2(U_{f_2}(P) - L_{f_2}(P)) < \epsilon_1 + \epsilon_2 = \epsilon$$

so that $c_1 f_1 + c_2 f_2$ is integrable. Expanding the left most side of the equation/inequality above lets us see that:

$$\int_a^b [c_1 f_1(x) + c_2 f_2(x)]dx = c_1 \int_a^b f_1(x)dx + c_2 \int_a^b f_2(x)dx$$

■

Problem 0.2. *Show that the set $S = \{\frac{1}{n}\}_{n=1}^{\infty} \subseteq \mathbb{R}$ has Jordan Measure zero.*

Solution. Refer to problem 0.3 ■

Problem 0.3. *More generally, show that if $(a_n)_{n=1}^{\infty}$ is any convergent sequence, then (a_n) has Jordan Measure Zero.*

Solution. For every $n \in \mathbb{N}$, cover a_n with an interval I_n of length $\frac{\epsilon}{\epsilon+1}$ for any $\epsilon > 0$. Observe that this interval is smaller than 1. Then, by the geometric series test, we have that the sum of the interval of all terms a_n is equal to ϵ. This shows that the set $\{a_n\}_{n=1}^{\infty}$ is measure zero. ■