Heat Equation Problems

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The heat equation is given by:

$$u_t - \triangle u = 0$$

where $t > 0$ and $x \in U \subset \mathbb{R}^n$ is open and the unknown is $u : \bar{U} \times [0, \infty) \to \mathbb{R}$, $u = u(x,t)$, the laplacian $\triangle$ is taken with respect to the spatial variables $x = (x_1, \dots, x_n)$ (i.e. not including t)

Problem 1. The fundamental solution of the heat equation is given by:

$$\Phi(x,t) := \begin{cases} \frac{1}{(4\pi t)^{n/2}} e^{-\frac{|x|^2}{4t}} & (x \in \mathbb{R}^n, t > 0) \\ 0 & (x \in \mathbb{R}^n, t < 0) \end{cases}$$

Show that the fundamental solution of the heat equation solves the heat equation.

Solution s

Problem 2. Show that $\int_{\mathbb{R}^n} \Phi(x,t) d^n x = 1$.

Solution

Problem 3. Show $\Phi(x,t) \to \delta_0(x)$ as $t \to 0$ i.e. for all $\psi \in C_c^\infty$, we have $\int \Phi(x,t)\psi(x) d^n x \to \phi(0)$ as $t \to 0$.

Problem 4. Check that the solution of the initial value problem

$$\begin{cases} \partial_t u = \triangle u \\ u|_{t=0} = f \end{cases}$$

is given by,

$$u(x,t) = \int_{\mathbb{R}^n} \Phi(x-y,t) f(y) d^n y = \int_{\mathbb{R}^n} \frac{1}{(4\pi t)^{n/2}} e^{-\frac{|x-y|^2}{4t}} f(y) d^n y$$

Problem 5. For $t > 0$ show $u(x,t)$ is C^∞ and compute formulas for $\partial_x u$ and $\partial_t u$.

Problem 6. [The Maximum Principle] If $u(x,t) \le C$ at $t = 0$, then $u(x,t) \le C$ for all $t \ge 0$.