MATD11 OVERVIEW

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1 Hilbert Space Preliminaries

1.1 Normed Linear Spaces

Definition. Let X be a vector space over either the scalar field $\mathbb{R}$ of real numbers or the scalar field $\mathbb{C}$ of complex numbers. Suppose we have a function $\|\cdot\| : X \to [0, \infty)$ such that

1. $\|x\| = 0$ if and only if $x = 0$.
2. $\|x + y\| \leq \|x\| + \|y\|$ for all $x, y \in X$, and
3. $\|\alpha x\| = |\alpha| \, \|x\|$ for all scalars α and vectors x.

We call $(X, \|\cdot\|)$ a **normed linear space**.

From property 2, we can derive the reverse triangle inequality

$$\|x + y\| \geq | \, \|x\| - \|y\| \, |$$

Example 1. Let $X = \mathbb{C}^n$ with $\|(z_1, z_2, \ldots, z_n)\| = (\sum_{j=1}^{\infty} |z_j|^2)^{1/2}$; this is called the **Euclidean norm**. The Euclidean space $\mathbb{R}^n$ is similarly defined.

Example 2. Let $X = \mathbb{C}^n$ with $\|(z_1, z_2, \ldots, z_n)\| = \max\{|z_j| : 1 \leq j \leq n\}$.

Example 3. Let $Y = [0, 1]$, or more generally any compact Hausdorff space, and let $C(Y)$ be the vector space of continuous, complex-valued functions on Y, under pointwise addition and scalar multiplication. Define a norm on $C(Y)$ by $\|f\| = \max\{|f(y)| : y \in Y\}$.

Example 4. Choose a value $p \geq 1$, and let $\ell^p = \ell^p(\mathbb{N})$ denote the set of all sequences $\{a_n\}_1^{\infty}$ of all complex numbers for which $\sum_1^{\infty} |a_n|^p < \infty$. Define the norm of $\{a_n\} \in \ell^p$ by

$$\|a_n\|_p = (\sum_1^{\infty} |a_n|^p)^{1/p}$$

We can include the choice $p = \infty$ by saying:

$$\ell^{\infty} = \{\{a_n\}_1^{\infty} : \ \sup_n |a_n| < \infty\}, \quad \|a_n\| = \ \sup_n |a_n|$$

The triangle inequality for $1 < p < \infty$ is non trivial and is called **Minkowski's inequality**:

$$(\sum_1^n |a_j + b_j|^p)1/p \leq (\sum_1^n |a_j|^p)^{1/p} + (\sum_1^n |b_j|^p)^{1/p}$$

which is proven from Hölder's inequality:

$$(\sum_1^n |a_i b_i|^p) \leq (\sum_1^n |a_i|^p)^{1/p} (\sum_1^n |b_i|^q)^{1/q}$$

where p and q are **conjugate indicies** i.e. $1/p + 1/q = 1$.

Example 5. We can generalize ℓ^p spaces as follows. Consider a positive measure space (Y, M, μ), where Y is a non-empty set, M is a σ-algebra of Y and μ is a positive measure. Choose $1 \leq p < \infty$ and denote $L^p(Y, \mu)$ the collection of μ-measurable functions such that $\int_Y |f|^p d\mu < \infty$ and its norm is given by

$$\|f\|_p = (\int_Y |f|^p d\mu)^{1/2}$$

We can also define $L^\infty(X, \mu)$ of **essentially bounded functions**. We say that a measurable function is essentially bounded if there exists $M < \infty$ such that $\mu(\{x : |f(x)| > M\}) = 0$. To prove the triangle inequality for $1 < p < \infty$, we need Minkowski's inequality:

$$(\int_Y |f + g|^p d\mu)^{1/p} \leq (\int_Y |f|^p d\mu)^{1/p} + (\int_Y |g|^p d\mu)^{1/p}$$

which follows from Hölder's inequality

$$(\int_Y |fg| d\mu) \leq (\int_Y |f|^p d\mu)^{1/p} (\int_Y |g|^q d\mu)^{1/q}$$

for conjugate indices p, q.

Example 6. Let $X = \mathbb{N}$, $M = \mathcal{P}(\mathbb{N})$, and let μ assign to each finite subset of $\mathbb{N}$ its cardinality, and to each infinite subset of $\mathbb{N}$, the value ∞. This is called the **counting measure** on the positive integers. With the convention $a + \infty = \infty + a$, we have countable additivity.

The L^p spaces then generalize ℓ^p spaces as follows: Choose $Y = \mathbb{N}$ and $\mu =$ counting measure on $\mathbb{N}$. Then $L^p(\mathbb{N}, \mathcal{P}(\mathbb{N}), \mu) = \ell^p(\mathbb{N})$.

Definition. A **metric space** is a set X with a function $d(\cdot, \cdot) : X \to [0, \infty)$ satisfying for $x, y, z \in X$:

1. $d(x, y) = 0$ if and only if $x = y$
2. $d(x, y) = d(y, x)$

3. $d(x, y) + d(y, z) \geq d(x, z)$

On every normed linear space, we can define a metric via $d(x, y) = \|x - y\|$.

Definition. A metric space is said to be **complete** if every Cauchy sequence in X converges in X.

Definition. Let X be a normed space. If X is complete in the metric d defined by the norm $d(x, y) = \|x - y\|$, we call X a **Banach space**.

Theorem 1 (Riesz-Fischer Theorem)**.** For every positive measure μ and $1 \leq p \leq \infty$, $L^p(\mu)$ is a Banach space.

$C(Y)$ and $\mathbb{C}^n$ with the max and Euclidean norm are also complete.

Definition. Let X be a vector space over $\mathbb{C}$. An **inner product** is a map $\langle \cdot, \cdot \rangle : X \times X \to \mathbb{C}$ satisfying, for $x, y \in X$ and $z \in X$ and scalars $\alpha \in \mathbb{C}$:

1. $\langle x, y \rangle = \overline{\langle y, x \rangle}$
2. $\langle x, x \rangle \geq 0$ with $\langle x, x \rangle = 0$ if and only if $x = 0$
3. $\langle x + y, z \rangle = \langle x, z \rangle + \langle y, z \rangle$
4. $\alpha \langle x, y \rangle = \langle \alpha x, y \rangle$

The bar denotes complex conjugation.

Example 7. Define an inner product on $L^2(X, \mu)$ for a positive measure space (X, μ) by

$$\langle f, g \rangle = \int_X f \overline{g} d\mu$$

On $\mathbb{C}^n$, we have:

$$\langle z = (z_1, \ldots, z_n), w = (w_1, \ldots, w_n) \rangle = \sum_1^n z_j \overline{w_j}$$

On ℓ^2:

$$\langle z, w \rangle = \sum_1^\infty z_j \overline{w_j}$$

Proposition 1 (Cauchy-Schwarz Inequality)**.** If $\langle \cdot, \cdot \rangle$ is an inner product on a vector space X, then for all $x, y \in X$, we have

$$|\langle x, y \rangle|^2 \leq \langle x, x \rangle \langle y, y \rangle$$

Proof. For all $t \in \mathbb{R}$ and $\xi \in \mathbb{C}$, with $|\xi| = 1$ compute $\langle x + t\xi y, x + t\xi y \rangle$ to get a polynomial of order 2 $p(t) = \langle x, x \rangle + 2t|\langle x, y \rangle| + t^2 \langle y, y \rangle$

Argue $p(t) \geq 0$ and $p(0) = 0$ and show this implies the discriminant is,

$$(2t|\langle x, y \rangle|)^2 - 4t^2 \langle y, y \rangle \langle x, x \rangle \leq 0$$

From this, derive the inequality. $\square$

Proposition 2. If $\langle \cdot, \cdot \rangle$ is an inner product on a vector space X, then

$$\|x\|^2 = \langle x, x \rangle$$

is a norm on X.

Proof. Expand $\|x+y\|^2$ to prove the triangle inequality. $\square$

Definition. A (complex) **Hilbert space** $\mathcal{H}$ is a vector space over $\mathbb{C}$ with an inner product such that $\mathcal{H}$ is complete in the metric,

$$d(x, y) = \|x - y\| = \langle x - y, x - y \rangle^{1/2}$$

Example 8. $L^2(X, \mu)$ is a Hilbert space, thus so is $\mathbb{C}^n$ and ℓ^2.

1.2 Orthogonality

Definition. Given vectors f, g in a Hilbert space $\mathcal{H}$, we say that f is **orthogonal** to g written $f \perp g$, if $\langle f, g \rangle = 0$. For sets A, B in $\mathcal{H}$ we write $A \perp B$ if $\langle f, g \rangle = 0$ for all $f \in A$ and $g \in B$. Finally, $A^\perp$ is the set of all vectors of $f \in \mathcal{H}$ such that $f \perp g$ for all $g \in A$; for any set A this is always a subspace of $\mathcal{H}$, moreover $A^\perp = \cap_{a \in A} \{a\}^\perp$, $A^\perp$ is a closed subspace by continuity of the inner product.

We have $A \cap A^\perp = \{\emptyset\}$.

Example 9. An example of a subspace which is not closed: In ℓ^2, the set of all sequences with finitely many non-zero terms.

Proposition 3. In $f_1, f_2, \ldots, f_n$ are pairwise orthogonal vectors in a Hilbert space, then

$$\|f_1 + f_2 + \ldots + f_n\|^2 = \|f_1\|^2 + \ldots + \|f_n\|^2$$

In general, for any vectors f and g in a Hilbert space, we have

$$\|f + g\|^2 = \|f\|^2 + 2\mathrm{Re}\langle f, g \rangle + \|g\|^2$$

and

$$\|f - g\|^2 = \|f\|^2 - 2\mathrm{Re}\langle f, g \rangle + \|g\|^2$$

The **Parallelogram equality** is then obtain:

$$\|f + g\|^2 + \|f - g\|^2 = 2\|f\|^2 + 2\|g\|^2$$

In any inner product space, the inner product can be recovered from the norm:

$$\langle f, g \rangle = 1/4 \sum_1^3 i^k \left\| f + i^k g \right\|^2$$

which is called the **polarization identity**.

Given a normed linear space in which the parallelogram equality holds, there is an inner product that gives the norm.

1.3 Hilbert Space Geometry

A **convex set** in a vector space V is a subset S of V with the property that whenever a, b are in S, so is $ta + (1-t)b$ for any $0 \leq t \leq 1$. Clearly every subspace is convex, every ball in a normed linear space is also convex, and any translate $x + S$ of a convex set is also convex.

Theorem 2 (Nearest Point Property)**.** Every nonempty, closed convex set K in a Hilbert space $\mathcal{H}$ contains a unique element of smallest norm. Moreover, given any $h \in \mathcal{H}$, there is a unique k_0 in K such that

$$\|h - k_0\| = \operatorname{dist}(h, K) = \inf\{\|h - k\| : k \in K\}$$

Proof. Let $d = \inf\{\|y\| : y \in K\}$ so that $\|x_n\| \to d$ for $x_n \in K$. Then, by parallelogram equality:

$$\|x_n - x_m/2\|^2 = \|x_n/2\|^2 + \|x_m/2\|^2 - \|x_n + x_m/2\|^2$$

for n, m. Show $1/2(x_n + x_m)$ is in K and thus, it implies

$$\|x_n + x_m/2\|^2 \geq d^2$$

and,

$$0 \leq \|x_n - x_m\|^2 \leq 2(\|x_n\|^2 + \|x_m\|^2) - 4d^2$$

so that $\{x_n\}$ is a Cauchy sequence and thus it converges to some $x \in H$ and importantly, it converges to some $x \in K$. Show $d = \|x\|$. Continuity says that if $x_n \to x$, then $\|x_n\| \to \|x\|$ so that $\|x\| = d$. Now, prove uniqueness by using parallelogram equality again i.e. for $\|x\| = \|z\|$, take $\|x + z/2\|^2$ and show $d^2 - \|x + z/2\|^2 \leq 0$.

Second part, find the point x of smallest norm in $K - h$ for any $h \in H$. Then $x + h$ has the smallest norm in K, so that $\|x + h - h\| = \|x\|$ is the closest point to h. $\square$

Theorem 3 (Projection Theorem)**.** Let M be a closed subspace of a Hilbert space $\mathcal{H}$. There is a unique pair of mappings $P : \mathcal{H} \to M$ and $Q : \mathcal{H} \to M^\perp$ such that $x = Px + Qx$ for all $x \in \mathcal{H}$. Furthermore, P and Q have the following additional properties:

1. $x \in M \implies Px = x$ and $Qx = 0$.
2. $x \in M^\perp \implies Px = 0$ and $Qx = x$.
3. Px is the closest vector in M to x.
4. Qx is the closest vector in $M^\perp$ to x.
5. $\|Px\|^2 + \|Qx\|^2 = \|x\|^2$ for all x.
6. P and Q are linear maps.

Proof. Let Px be the closest point to x in M. Let $Qx = x - Px$. $\square$

Corollary 1. If M is a closed, proper, subspace of $\mathcal{H}$, then there exists a non-zero vector $y \in \mathcal{H}$ with $y \perp M$.

1.4 Linear Functionals

Definition. If X is a normed linear space over $\mathbb{C}$, a **linear functional** on X is a map $\Lambda : X \to \mathbb{C}$ satisfying $\Lambda(\alpha x + \beta y) = \alpha\Lambda(x) + \beta\Lambda(y)$ for all vectors x and y in X and all scalars α and β.

Definition. A **bounded linear functional** on a normed linear space X is a linear functional $\Lambda : X \to \mathbb{C}$ for which there exists a finite constant C satisfying $|\Lambda(x)| \leq C \|x\|$ for all $x \in X$.

Example 10. The set of all bounded linear functionals on X forms a normed linear space with norm

$$\|\Lambda\| = \sup\{|\Lambda(x)| : \|x\| \leq 1\}$$

Proposition 4. If X is a normed linear space, and $\Lambda : X \to \mathbb{C}$ is a linear functional, then the following are equivalent:

1. Λ is continuous
2. Λ is continuous at 0
3. Λ is bounded

Theorem 4 (Riesz Representation Theorem)**.** Every bounded linear functional Λ on a Hilbert space $\mathcal{H}$ is given by inner product with a (unique) fixed vector h_0 in $\mathcal{H}$: $\Lambda(h) = \langle h, h_0\rangle$. Moreover, the norm of the linear functional is $\|h_0\|$.

Proof. 1. Handle the case for $\Lambda = 0$.
2. Consider the kernel of M and choose some non-zero vector from the orthogonal of the kernel (why should one exist?)
3. Argue that $\Lambda(h)z - h$ is perpendicular to z for any $h \in \mathcal{H}$.
4. Derive $\Lambda(h) = \langle h, z/\|z\|^2\rangle$ and choose an appropriate h_0 from this.
5. Argue uniqueness. $\square$

Example 11. The dual space of $L^p[a,b]$ for $1 \leq p < \infty$ is $L^q[a,b]$ where p and q are conjugate indices. If $p = 1$, we set $q = \infty$. There are no continuous linear functionals on $L^p[a,b]$ for $0 < p < 1$.

Lemma 1. Let $P : \mathcal{H} \to M$ be the orthogonal projection of a Hilbert space $\mathcal{H}$ onto a closed subspace M of $\mathcal{H}$. We have $\langle f, Pg\rangle = \langle Pf, g\rangle$ for all vectors f and g in $\mathcal{H}$.

1.5 Orthonormal Bases

Definition. An **orthonormal set** in a Hilbert space $\mathcal{H}$ is a set ϵ with the properties:

1. for every $e \in \epsilon$, $\|e\| = 1$, and
2. for distinct vectors e and f in ϵ, $\langle e, f \rangle = 0$.

Definition. An **orthonormal basis** for a Hilbert space $\mathcal{H}$ is a maximal orthonormal set; that is, an orthonormal set that is not properly contained in any orthonormal set.

Example 12. For the Hilbert space ℓ^2, take the set of all vectors $\{e_j : j \geq 1\}$ where e_j has 1 in the jth coordinate and zeros elsewhere. As a second example, consider the Hilbert space $L^2[0, 2\pi]$, with respect to normalized Lebesgue measure $dt/(2\pi)$. The collection of functions e^{int} for any integer n forms an orthonormal set in this Hilbert space.

It is a fact that every Hilbert space has an orthonormal basis.

Given a linearly independent sequence $\{f_n\}_1^\infty$ in a Hilbert space $\mathcal{H}$, there always exists an orthonormal sequence $\{e_n\}_1^\infty$ such that

$$\text{span}\{f_1, \ldots, f_k\} = \text{span}\{e_1, \ldots, e_k\}$$

for each positive integer k.

Theorem 5 (Bessel's inequality)**.** When $\{e_k\}$ is a finite or countably infinite orthonormal set in $\mathcal{H}$, then for every vector $h \in \mathcal{H}$ we have

$$\sum |\langle h, e_k \rangle|^2 \leq \|h\|^2$$

Proof. Let $r_n = x - \sum_1^n \langle x, e_k \rangle e_k$ and compute $\langle r_n, e_j \rangle = \langle x - \sum_1^n \langle x, e_k \rangle e_k, e_j \rangle$ and show it is equal to 0 for all $j \in \mathbb{N}$.

Write $x = r_n + (\sum_1^n \langle x, e_k \rangle e_k)$ and compute the squared norm of both sides and conclude the inequality holds. ☐

Theorem 6. If $\{e_n\}_1^\infty$ is an orthonormal sequence in a Hilbert space $\mathcal{H}$, then the following conditions are equivalent:

1. $\{e_n\}_1^\infty$ is an orthonormal basis
2. If $h \in \mathcal{H}$ and $h \perp e_n$ for all n, then $h = 0$.
3. For every $h \in \mathcal{H}$, $h = \sum_1^\infty \langle h, e_n \rangle e_n$
4. For every $h \in \mathcal{H}$, there exist complex numbers a_n so that $h = \sum_1^\infty a_n e_n$.

5. For every $h \in \mathcal{H}$, $\sum_1^\infty |\langle h, e_n \rangle|^2 = \|h\|^2$.

6. For all h and g in $\mathcal{H}$, $\sum_1^\infty \langle h, e_n \rangle \langle e_n, g \rangle = \langle h, g \rangle$.

Proof. Equivalence of (1) and (2) follows from the fact that if $0 \neq h$ and $h \perp e_n$ for all n, then $\{e_n\} \cup \{h/\|h\|\}$ is an orthonormal sequence. $\square$

2 Operator Theory Basics

2.1 Bounded Linear Operators

Definition. If X and Y are normed linear spaces, a map $T : X \to Y$ is **linear** if

$$T(\alpha x_1 + \beta x_2) = \alpha T(x_1) + \beta T(x_2)$$

for all x_1, x_2 in X and scalars α and β. We say the linear map T is **bounded linear operator** from X to Y if there is a finite constant C such that $\|Tx\|_Y \leq C\|x\|_X$ for all $x \in X$.

Proposition 5. If $T : X \to Y$ is a linear map from a normed linear space X to a normed linear space Y, the following are equivalent:

1. T is bounded

2. T is continuous

3. T is continuous at 0

Proposition 6. The collection $\mathcal{B}(X, Y)$ of all bounded linear operators from a normed linear space X to a Banach space Y forms a Banach space with norm

$$\|T\| = \sup\{\|Tx\|_Y : \|x\|_X \leq 1\}$$

Example 13. Suppose M is a closed subspace in a Hilbert space $\mathcal{H}$. Let $P_M : \mathcal{H} \to M$ be the orthogonal projection of $\mathcal{H}$ onto M. This is a bounded linear operator with norm 1.

Example 14. The **forward shift** is a bounded linear operator $S : \ell^2 \to \ell^2$ with

$$S(x_1, x_2, \ldots) = (0, x_1, x_2, \ldots)$$

This has norm 1 and is clearly linear. In fact, it is an isometry with $\|Sx\| = \|x\|$. The **backward shift** is the operator ℓ^2 to ℓ^2 which takes $(x_1, x_2, \ldots) \mapsto (x_2, x_3, \ldots)$. It has norm 1 but is not an isometry.

Definition. If $\mathcal{H}$ and $\mathcal{K}$ are both Hilbert spaces, a **sesquilinear form** $u : \mathcal{H} \times \mathcal{K} \to \mathbb{C}$ is a mapping satisfying

1. $u(\alpha h + \beta g, k) = \alpha u(h, k) + \beta u(g, k)$, and

2. $u(h, \alpha k + \beta f) = \overline{\alpha} u(h, k) + \overline{\beta} u(h, f)$

for all $h, g \in \mathcal{H}$, all $k, f \in \mathcal{K}$ and all scalars α and β. A sesquilinear form u is bounded if there is a finite constant M such that $|u(h,k)| \leq M \|h\| \|k\|$ for all $h \in \mathcal{H}$ and $k \in \mathcal{K}$.

Theorem 7. Let $\mathcal{H}$ and $\mathcal{K}$ be Hilbert spaces and suppose that $u : \mathcal{H} \times \mathcal{K} \to \mathbb{C}$ is a bounded sesquilinear form. There exists a unique $A \in \mathcal{B}(\mathcal{H}, \mathcal{K})$ such that

$$u(h,k) = \langle Ah, k \rangle_{\mathcal{K}}$$

for all $h \in \mathcal{H}$ and $k \in \mathcal{K}$.

Theorem 8. Given Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$ and $A \in \mathcal{B}(\mathcal{H}, \mathcal{K})$, there is a unique $A^* \in \mathcal{B}(\mathcal{K}, \mathcal{H})$ so that

$$\langle Ah, k \rangle_{\mathcal{K}} = \langle h, A^* k \rangle_{\mathcal{H}}$$

for all $h \in \mathcal{H}$ and $k \in \mathcal{K}$.

Proof. Claim that $\langle k, Ah \rangle$ is a sesquilinear form. Use the previous theorem to claim that the sesquilinear form is equal to $\langle A^* k, h \rangle$. Thus, $\langle k, Ah \rangle = \langle A^* k, h \rangle$ and take conjugates. $\square$

Proposition 7. For A and B in $\mathcal{B}(\mathcal{H})$ we have

1. $A^{**} = A$ where $A^{**} = (A^*)^*$.
2. $(A + B)^* = A^* + B^*$
3. $(\alpha A)^* = \overline{\alpha} A^*$ for $\alpha \in \mathbb{C}$
4. $(AB)^* = B^* A^*$

Proof. 1. Combine $\langle A^* x, y \rangle = \langle x, A^{**} y \rangle$ and $\langle Ay, x \rangle = \langle y, A^* x \rangle$ so that $\langle x, A^{**} y \rangle = \langle x, Ay \rangle$ which shows that for all x, y: $\langle x, A^{**} y - Ay \rangle = 0$. Choose $x = A^{**} y - Ay$, so that we get $A^{**} y = A^* y$ as wanted. $\square$

Proposition 8. If $A \in \mathcal{B}(\mathcal{H})$, then $\|A\| = \|A^*\|$ and $\|A^* A\| = \|A\|^2$.

Proof.

$$\|Ah\|^2 = \langle Ah, Ah \rangle = \langle h, A^* Ah \rangle \leq \|h\| \|A^* Ah\| \leq \|A^* A\| \leq \|A\| \|A^*\|$$

This shows $\|A\|^2 \leq \|A*\| \|A\|$ which shows $\|A\| \leq \|A^*\|$. Then, apply this for the operator $\|A*\|$ to get, $\|A*\| \leq \|A^{**}\| = \|A\|$ to get that $\|A\| = \|A^*\|$. We know, $\|AA^*\| \leq \|A\| \|A^*\| = \|A\| \|A\| = \|A\|^2$ already. $\square$

Definition. An operator A in $\mathcal{B}(\mathcal{H})$ is **normal** if $AA^* = A^* A$, and **self adjoint** if $A = A^*$. If $U : \mathcal{H} \to \mathcal{K}$ is a linear surjection that preserves inner products i.e. $\langle Uh_1, Uh_2 \rangle = \langle h_1, h_2 \rangle$ for all h_1, h_2 in $\mathcal{H}$, we say that U is a **Hilbert space isomorphism**.

Proposition 9. If $U : \mathcal{H} \to \mathcal{H}$ is an isomorphism, then $U^* U = I_{\mathcal{H}}$ and $UU^* = I_{\mathcal{K}}$.

Definition. An operator A in $\mathcal{B}(\mathcal{H}, \mathcal{K})$ is said to be **invertible** if there exists B in $\mathcal{B}(\mathcal{K}, \mathcal{H})$ with $AB = I_{\mathcal{K}}$ and $BA = I_{\mathcal{H}}$. We write $B = A^{-1}$.

We can then rephrase the last proposition as: If U is an isomorphism, then $U^* = U^{-1}$.

Definition. If H and K are Hilbert spaces and if $U : H \to K$ is bijective linear map with

$$\langle Uh_1, Uh_2 \rangle_K = \langle h_1, h_2 \rangle$$

for all h_1 and h_2 in H, then U is said to be a **Unitary operator**.

A linear and surjective isometry is always unitary.

Proposition 10. If A is invertible, then so is A^*, and $(A^*)^{-1} = (A^{-1})^*$.

Proposition 11. If $U \in \mathcal{B}(H, K)$ with U invertible and $U^{-1} = U^*$, then U is an isomorphism.