MATC27 Exam Study

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Chapter 2: Section 12.

Definition 0.1. A **topology** on a set X is a collection τ of subsets of X having the following properties:

1. $\emptyset$ and X are in τ
2. The union of the elements of any subcollection of τ is in τ
3. The intersection of the elements of any finite subcolelction of τ is in τ

A set X for which a topology τ has been specified is called a **topological space**. We might write this as (X, τ) or as X if the topology put on X is unambigious.

Definition 0.2. If X is a topological space with topology τ, we say that a subset U of X is an **open set** of X if U belongs to the collection τ.

Examples 1. *If X is any set, the collection of all subsets of X is a topology on X; it is called the* ***discrete topology****. The collection consisting of X and $\emptyset$ only is also a topology on X; we shall call it the* ***indiscrete topology****, or the* ***trivial topology****. Let X be a set; let τ_f be the collection of all subsets U of X such that $X - U$ either is finite or is all of X. Then τ_f is a topology on X, called the* ***finite complement topology****.*

Definition 0.3. Suppose that τ and τ' are two topologies on a given set X. If $\tau' \supseteq \tau$, we say that τ' is **finer** than τ; if τ' *properly* contains τ, we say that τ' is **strictly finer** than τ. We also say that τ is **coarser** than τ', or **strictly coarser**, in these two respective situations. We say τ is **comparable** with τ' if either $\tau' \supseteq \tau$ or $\tau \supseteq \tau'$.

Chapter 2: Section 13.

Definition 0.4. If X is a set, a **basis** for a topology on X is a collection $\mathcal{B}$ of subsets of X (called **basis elements**) such that

1. For each $x \in X$, there is at least one basis element B containing x.
2. If x belongs to the intersection of two basis elements B_1 and B_2, then there is a basis element B_3 containing x such that $B_3 \subseteq B_1 \cap B_2$.

If $\mathcal{B}$ satisfies these two conditions, then we define the **topology τ generated by $\mathcal{B}$** as follows: A subset U of X is said to be open in X (that is, to be an element of τ) if for each $x \in U$, there is a basis element $B \in \mathcal{B}$ such that $x \in B$ and $B \subseteq U$. Note that each basis element is itself an element of τ.

Lemma 1 (Lemma 13.1). Let X be a set; let $\mathcal{B}$ be a basis for a topology on X. Then τ equals the collection of all unions of of elements of $\mathcal{B}$.

Lemma 2 (Lemma 13.2). Let X be a topological space. Suppose that $\mathcal{C}$ is a collection of open sets of X such that for each open set U of X and each x in U, there is an element C of $\mathcal{C}$ such that $x \in C \subseteq U$. Then $\mathcal{C}$ is a basis for the topology of X.

Lemma 3 (Lemma 13.3). Let $\mathcal{B}$ and $\mathcal{B}'$ be bases for the topologies τ and τ', respectively, on X. Then the following are equivalent:

1. τ' is finer than τ
2. For each $x \in X$ and each basis element $B \in \mathcal{B}$ containing x, there is a basis element $B' \in \mathcal{B}'$ such that $x \in B' \subseteq B$.

Definition 0.5. If $\mathcal{B}$ is the collection of all open intervals in the real line,

$$(a, b) = \{x : a < x < b\}$$

the topology generated by $\mathcal{B}$ is called the **standard topology** on the real line. Whenever we consider $\mathbb{R}$, we shall suppose it is given this topology unless we specifically state otherwise. If $\mathcal{B}'$ is the collection of all half open intervals of the form

$$[a, b) = \{x : a \leq x < b\}$$

where $a < b$, the topology generated by $\mathcal{B}'$ is called the **lower limit topology** on $\mathbb{R}$. When $\mathbb{R}$ is given the lower limit topology, we denote it by $\mathbb{R}_l$. Finally let K denote the set of all numbers of the form $\frac{1}{n}$, for $n \in \mathbb{Z}_+$, and let $\mathcal{B}''$ be the collection of all open intervals (a, b), along with the sets of the form $(a, b) - K$. The topology generated by $\mathcal{B}''$ will be called the **K-topology** on $\mathbb{R}$.

Lemma 4 (Lemma 13.4). The topologies of $\mathbb{R}_l$ and $\mathbb{R}_K$ are strictly finer than the standard topology on $\mathbb{R}$, but are not comparable with one another.

Definition 0.6. A **subbasis** $\mathcal{S}$ for a topology on X is a collection of subsets of X whose union equals X. The **topology generated by the subbasis $\mathcal{S}$** is defined to be the collection τ of all unions of finite intersections of elements of $\mathcal{S}$.

Chapter 2: Section 15.

Definition 0.7. Let X and Y be topological spaces. The **product topology** on $X \times Y$ is the topology having as basis the collection $\mathcal{B}$ of all sets of the form $U \times V$, where U is an open subset of X and V is an open subset of Y.

Theorem 5 (Theorem 15.1)**.** *If $\mathcal{B}$ is a basis for the topology of X and $\mathcal{C}$ is basis for the topology of Y, then the collection*

$$\mathcal{D} = \{\mathcal{B} \times \mathcal{C} : B \in \mathcal{B} \, and \, C \in \mathcal{C}\}$$

is a basis for the topology of $X \times Y$.

Definition 0.8. Let $\pi_1 : X \times Y \to X$ be defined by the equation

$$\pi_1(x, y) = x;$$

let $\pi_2 : X \times Y \to Y$ be defined by the equation

$$\pi_2(x, y) = y.$$

The maps π_1 and π_2 are called **projections** of $X \times Y$ onto its first and second factors, respectively.

Theorem 6 (Theorem 15.2)**.** *The collection*

$$\mathcal{S} = \{\pi_1^{-1}(U) : U \text{ open in } X\} \cup \{\pi_2^{-1}(V) : V \text{ open in } Y\}$$

is a subbasis for the product topology on $X \times Y$.

Chapter 2: Section 16.

Definition 0.9. Let X be a topological space with topology τ. If Y is a subset of X, the collection

$$\tau_Y = \{Y \cap U : U \in \tau\}$$

is a topology on Y, called the **subspace topology**. With this topology, Y is called a **subspace** of X; its open sets consist of all intersections of open sets of X with Y.

Lemma 7 (Lemma 16.1). If $\mathcal{B}$ is a basis for the topology of X then the collection

$$\mathcal{B}_Y = \{B \cap Y : B \in \mathcal{B}\}$$

is a basis for the subspace topology on Y.

If Y is a subspace of X, we say that a set U is **open in** Y (or open *relative* to Y) if it belongs to the topology on Y; this implies in particular that it is a subset of Y. We say that U is **open in** X if it belongs to the topology of X.

Lemma 8 (Lemma 16.2). Let Y be a subspace of X. if U is open in Y and Y in X, then U is open in X.

Theorem 9 (Theorem 16.3)**.** *If A is a subspace of X and B is a subspace of Y, then the product topology on $A \times B$ is the same as the topology $A \times B$ inherits as a subspace of $X \times Y$.*

Chapter 2: Section 17.

Definition 0.10. A subset A of a topological space X is said to be **closed** if the set $X - A$ is open.

Theorem 10 (Theorem 17.1)**.** *Let X be a topological space. Then the following conditions hold:*

1. *$\emptyset$ and X are closed*
2. *Arbitrary intersections of closed sets are closed*
3. *Finite unions of closed sets are closed*

If Y is a subspace of X, we say that a set A is **closed in** Y if A is a subset of Y and if A is closed in the subspace topology of Y (that is, if $Y - A$ is open in Y).

Theorem 11 (Theorem 17.2)**.** *Let Y be a subspace of X. Then a set A is closed in Y if and only if it equals the intersection of a closed set of X with Y.*

Theorem 12 (Theorem 17.3)**.** *Let Y be a subspace of X. If A is closed in Y and Y is closed in X, then A is closed in X.*

Definition 0.11. Given a subset A of a topological space X, the **interior** of A is defined as the union of all open sets contained in A i.e. the largest open set contained in A, and the **closure** of A is defined as the intersection of all closed sets contained in A i.e. the smallest closed set which contains A.

The interior of A is denoted by int(A) or A° and the closure of A is denoted Cl A or by $\overline{A}$.

Theorem 13 (Theorem 17.4)**.** *Let Y be a subspace of X; let A be a subset of Y; let $\overline{A}$ denote the closure of A in X. Then the closure of A in Y equals $\overline{A} \cap Y$.*

Say that a set A **intersects** a set B if the intersection $A \cap B$ is not empty.

Theorem 14 (Theorem 17.5)**.** *Let A be a subset of the topological space X.*

1. *Then $x \in \overline{A}$ if and only if every open set U containing x intersects A.*
2. *Supposing the topology of X is given by a basis, then $x \in \overline{A}$ if and only if every basis element B containing x intersects A.*

Proof. Prove (a). We prove the contrapositive of (a):

$$x \notin \overline{A} \Leftrightarrow \text{there exists an open set } U \text{ containing } x \text{ that does not intersect } A$$

If x is not in $\overline{A}$, the set $U = X - \overline{A}$ is an open set containing x that does not intersect A, as desired. Conversely, if there exists an open set U containing x which does not intersect A, then $X - U$ is a closed set containing A. By definition of closure $\overline{A}$, the set $X - U$ must contain $\overline{A}$; therefore, x cannot be in A. □

Say "U is a **neighbourhood** of x" if U is some open set containing x.

Definition 0.12. If A is a subset of the topological space X and if x is a point of X, we say that x is a **limit point** (or "cluster point," or "point of accumulation") of A if every neighborhood of x intersects A in some point *other than x itself* i.e. x is a limit point of A if it belongs to the closure of $A - \{x\}$

Theorem 15 (Theorem 17.6)**.** *Let A be a subset of the topological space X; let A' be the set of all limit points of A. Then*

$$\overline{A} = A \cup A'$$

We may also denote A' by ∂A.

Corollary 15.1 (Corollary 17.7)*.* A subset of a topological space is closed if and only if it contains all limit points.

In an arbitrary topological space, one says that a sequence $x_1, x_2, \ldots$ of points of the space X **converges** to the point x of X provided that, corresponding to each neighborhood U of x, there is a positive integer N such that $x_n \in U$ for all $n \geq N$.

Definition 0.13. A topological space X is called a **Hausdorff space** if for each pair x_1, x_2 of distinct points of X, there exists a neighborhoods U_1, and U_2 of x_1 and x_2, respectively, that are disjoint.

Theorem 16 (Theorem 17.8)**.** *Every finite point set in a Hausdorff space X is closed.*

The condition that finite point sets are closed is weaker than Hausdorff, if a topological space X has the condition that all finite point sets are closed, then it is said to be $\mathbf{T}_1$. The property itself is called the T_1 axiom.

Theorem 17 (Theorem 17.9)**.** *Let X be a space satisfying the T_1 axiom; let A be a subset of X. Then the point x is a limit point of A if and only if every neighborhood of x contains infinitely many points of A.*

Theorem 18 (Theorem 17.10)**.** *If X is a Hausdorff space, then a sequence of points of X converges to at most one point of X.*

Chapter 2: Section 18.

Definition 0.14. Let X and Y be topological spaces. A function $f : X \to Y$ is said to be **continuous** if for each open subset V of Y, the set $f^{-1}(V)$ is an open subset of X. Since the continuity on the definition of f *and* on the topologies on X and Y, we can also say f is continous *relative* to specific topologies X and Y.

Example. *Let us consider a function like those studied in analysis, a "real-valued function of a real variable,"*

$$f : \mathbb{R} \to \mathbb{R}$$

In analysis, one defines continuity of f via the "$\epsilon - \delta$ definition". To prove that our definition implies the $\epsilon - \delta$ definition, for instance, we proceed as follows: Given $x_0 \in \mathbb{R}$, and given $\epsilon > 0$, the interval $V = (f(x_0) - \epsilon, f(x_0) + \epsilon)$ is an open set of the range space $\mathbb{R}$. Therefore, $f^{-1}(V)$ is an open set of the domain space $\mathbb{R}$ (we assumed our topological definition holds). Because $f^{-1}(V)$ contains the point x_0, it contains some basic element (a, b) about x_0. We choose δ to be the smaller of the two numbers $x_0 - a$ and $b - x_0$. Then if $|x - x_0| < \delta$, the point x must be in (a, b), so that $f(x) \in V$, and $|f(x) - f(x_0)| < \epsilon$, as desired.

Example. *Let*

$$f : \mathbb{R} \to \mathbb{R}_l$$

be the identity function; $f(x) = x$ for every real number x. Then f is not a continuous function; the inverse image of the open set $[a, b)$ of $\mathbb{R}_l$ equals itself, which is not open in $\mathbb{R}$. On the other hand, the identity function

$$g : \mathbb{R}_l \to \mathbb{R}$$

is continuous, because the inverse image of (a, b) is itself, which is open in $\mathbb{R}_l$.

Theorem 19 (Theorem 18.1)**.** *Let X and Y be topological spaces; let $f : X \to Y$. Then the following are equivalent:*

1. *f is continuous.*
2. *For every subset A of X, one has $f(\overline{A}) \subseteq \overline{f(A)}$.*
3. *For every closed set B of Y, the set $f^{-1}(B)$ is closed in X.*
4. *For each $x \in X$ and each neighborhood V of $f(x)$, there is a neighborhood U of x such that $f(U) \subseteq V$.*

*If the condition in (4) holds for the point x of X, we say that f is **continuous at the point** x.*

Definition 0.15. Let X and Y be topological spaces; let $f : X \to Y$ be a bijection. If both the function f and the inverse function

$$f^{-1} : Y \to X$$

are continuous (and exist), then f is called a **homeomorphism**.

These functions have the property that any property that can be expressed solely via the open sets of a space are preserved when mapped onto a different space. Such a property (that can be expressed solely via open sets) is called a **topological property**.

Example. *The function* $F : (-1, 1) \to \mathbb{R}$ *defined by*

$$F(x) = \frac{x}{1 - x^2}$$

is a homeomorphism. A bijective continuous function does not necessairly have to be continuous, for example, the identity map $g : \mathbb{R}_l \to \mathbb{R}$ *is bijective continuous but it's inverse is not continuous as noted earlier.*

Theorem 20 (Theorem 18.2)**.** *Let* X, Y *and* Z *be topological spaces.*

1. *(Constant function) If* $f : X \to Y$ *mapps all of* X *into the single point* y_0 *of* Y*, then* f *is continuous.*
2. *(Inclusion) If* A *is a subspace of* X*, the inclusion function* $j : A \to X$ *is continuous.*
3. *(Composites) If* $f : X \to Y$ *and* $g : Y \to Z$ *are continuous, then the map* $g \circ f : X \to Z$ *is continuous.*
4. *(Restricting the domain) If* $f : X \to Y$ *is continuous, and if* A *is a subspace of* X*, then the restricted function* $f\big|_A : A \to Y$ *is continuous.*
5. *(Restricting or expanding the range) Let* $f : X \to Y$ *be continuous. If* Z *is a subspace of* Y *containing the image set* $f(X)$*, then the function* $g : X \to Z$ *obtained by restricting the range of* f *is continuous. If* Z *is a space having* Y *as a subspace, then the function* $h : X \to Z$ *obtained by expanding the range of* f *is continuous.*
6. *(Local formulation of continuity) The map* $f : X \to Y$ *is continuous if* X *can be written as the union of open sets* U_α *such that* $f\big|_{U_\alpha}$ *is continuous for each* α*.*

Theorem 21 (Theorem 18.3)**.** *Let* $X = A \cup B$*, where* A *and* B *are closed in* X*. Let* $f : A \to Y$ *and* $g : B \to Y$ *be continuous. If* $f(x) = g(x)$ *for every* $x \in A \cap B$*, then* f *and* g *combine to give a continuous function* $h : X \to Y$*, defined by setting* $h(x) = f(x)$ *if* $x \in A$*, and* $h(x) = g(x)$ *if* $x \in B$*.*

Theorem 22 (Theorem 18.4)**.** *Let* $f : A \to X \times Y$ *be given by the equation*

$$f(a) = (f_1(a), f_2(a))$$

Then f *is continuous if and only if the functions*

$$f_1 : A \to X \qquad \textit{and} \qquad f_2 : A \to Y$$

are continuous.

Chapter 2: Section 19.

Definition 0.16. Let $\{A_\alpha\}_{\alpha \in J}$ be an indexed family of sets; let $X = \bigcup_{\alpha \in J} A_\alpha$. The **cartesian product** of this indexed family, denoted by

$$\prod_{\alpha \in J} A_\alpha$$

is defined to be the set of all J-tuples $(x_\alpha)_{\alpha \in J}$ of elements of X such that $x_\alpha \in A_\alpha$ for each $\alpha \in J$. That is, it is the set of all functions

$$\mathbf{x} : J \to \bigcup_{\alpha \in J} A_\alpha$$

such that $\mathbf{x}(\alpha) \in A_\alpha$ for each $\alpha \in J$. If all $A_\alpha = X$, then the cartesian product is denoted by X^J.

Definition 0.17. Let $\{X_\alpha\}_{\alpha \in J}$ be an indexed family of topological spaces. Let us take as a basis for a topology on the product space

$$\prod_{\alpha \in J} X_\alpha$$

the collection of all sets of the form

$$\prod_{\alpha \in J} U_\alpha$$

where U_α is open in X_α, for each $\alpha \in J$. The topology generated by this basis is called the **the box topology**.

Now we generalize the subbasis formulation of this definition. Let

$$\pi_\beta : \prod_{\alpha \in J} X_\alpha \to X_\beta$$

be the function assigning to each element of the product space its βth coordinate,

$$\pi_\beta((x_\alpha)_{\alpha \in J}) = x_\beta;$$

it is called the **projection mapping** associated with the index β.

Definition 0.18. Let $\mathcal{S}_\beta$ denote the collection

$$\mathcal{S}_\beta = \{\pi_\beta^{-1}(U_\beta) : U_\beta \text{ open in } X_\beta\}$$

and let $\mathcal{S}$ denote the union of these collections,

$$\mathcal{S} = \bigcup_{\beta \in J} \mathcal{S}_\beta$$

The topology generated by the subbasis $\mathcal{S}$ is called the **product topology**. In this topology $\prod_{\alpha \in J} X_\alpha$ is called a **product space**.

The following two facts may be important:

$$(\prod_{\alpha \in J} U_\alpha) \cap (\prod_{\alpha \in J} V_\alpha) = \prod_{\alpha \in J} (U_\alpha \cap V_\alpha)$$
$$\pi_\beta^{-1}(U_\beta) \cap \pi_\beta^{-1}(V_\beta) = \pi_\beta^{-1}(U_\beta \cap V_\beta)$$

Theorem 23 (Theorem 19.1)**.** *The box topology on $\prod X_\alpha$ has as basis all sets of the form $\prod U_\alpha$, where U_α is open in X_α for each α. The product topology on $\prod X_\alpha$ has as basis all sets of the form $\prod U_\alpha$, where U_α is open in X_α for each α and U_α equals X_α except for finitely many values of α.*

Theorem 24 (Theorem 19.2)**.** *Suppose the topology on each space X_α is given by a basis $\mathcal{B}_\alpha$. The collection of all sets of the form*

$$\prod_{\alpha \in J} B_\alpha$$

where $B_\alpha \in \mathcal{B}_\alpha$ for each α, will serve as a basis for the box topology on $\prod_{\alpha \in J} X_\alpha$. The collection of all sets of the same form, where $B_\alpha \in \mathcal{B}_\alpha$ for finitely many indices α and $B_\alpha = X_\alpha$ for all the remaining indices, will serve as a basis for the product topology $\prod_{\alpha \in J} X_\alpha$.

Theorem 25 (Theorem 19.3)**.** *Let A_α be a subspace of X_α, for each $\alpha \in J$. Then $\prod A_\alpha$ is a subspace of $\prod X_\alpha$ if both products are given the box topology, or if both products are given the same product topology.*

Theorem 26 (Theorem 19.4)**.** *If each space X_α is a Hausdorff space, then $\prod X_\alpha$ is a Hausdorff space in both the box and product topologies.*

Theorem 27 (Theorem 19.5)**.** *Let $\{X_\alpha\}$ be an indexed family of spaces; let $A_\alpha \subseteq X_\alpha$ for each α. If $\prod X_\alpha$ is given either the product or the box topology, then*

$$\prod \overline{A_\alpha} = \overline{\prod A_\alpha}$$

Theorem 28 (Theorem 19.6)**.** *Let $f : A \to \prod_{\alpha \in J} X_\alpha$ be given by the equation*

$$f(a) = (f_\alpha(a))_{\alpha \in J}$$

where $f_\alpha : A \to X_\alpha$ for each α. Let $\prod X_\alpha$ have the product topology. Then the function f is continuous if and only if each function f_α is continuous.

Chapter 2: Section 20.

Definition 0.19. A **metric** on a set X is a function

$$d : X \times X \to \mathbb{R}$$

having the following properties:

1. $d(x, y) \geq 0$ for all $x, y \in X$; equality holds if and only if $x = y$
2. $d(x, y) = d(y, x)$ for all $x, y \in X$.
3. (Triangle Inequality) $d(x, y) + d(y, z) \geq d(x, z)$ for all $x, y, z \in X$.

Given a metric d on X, the number $d(x, y)$ is often called the **distance** between x and y in the metric d. Given $\epsilon > 0$, consider the set

$$B_d(x, \epsilon) = \{y : d(x, y) < \epsilon\}$$

of all points y whose distance from x is less than ϵ. It is called the **ϵ-ball centered at** x.

Definition 0.20. If d is a metric on the set X, then the collection of all ϵ-balls $B_d(x, \epsilon)$, for $x \in X$ and $\epsilon > 0$, is a basis for a topology on X, called the **metric topology** induced by d.

Definition 0.21. If X is a topological space, X is said to be **metrizable** if there exists a metric d on the set X that induces the topology of X. A **metric space** is a metrizable space X together with a specific metric d that gives the topology of X.

Definition 0.22. Let X be a metric space with metric d. A subset A of X is said to be **bounded** if there is some number M such that

$$d(a_1, a_2) \leq M$$

for every pair a_1, a_2 of points of A. If A is bounded and non-empty, the **diameter** of A is defined to be the number

$$\text{diam } A = \sup\{d(a_1, a_2) : a_1, a_2 \in A\}.$$

Theorem 29 (Theorem 20.1)**.** *Let X be a metric space with metric d. Define $\overline{\mathrm{d}} : X \times X \to \mathbb{R}$ by the equation*

$$\overline{\mathrm{d}}(x, y) = \ min\{d(x, y), 1\}$$

Then $\overline{\mathrm{d}}$ is a metric that induces the same topology as d. The metric $\overline{\mathrm{d}}$ is called the ***standard bounded metric*** *corresponding to d.*

Definition 0.23. Given $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ in $\mathbb{R}^n$, we define the **norm** of $\mathbf{x}$ by the equation

$$||x|| = (x_1^2 + \ldots + x_n^2)^{\frac{1}{2}}$$

and we define the **euclidean metric** d on $\mathbb{R}^n$ by the equation

$$d(\mathbf{x}, \mathbf{y}) = ||\mathbf{x} - \mathbf{y}|| = [(x_1 - y_1)^2 + \ldots + (x_n - y_n)^2]^{\frac{1}{2}}.$$

We define the **square metric** ρ by the equation

$$\rho(\mathbf{x}, \mathbf{y}) = \max\{|x_1 - y_1|, \ldots, |x_n - y_n|\}.$$

Lemma 30 (Lemma 20.2). Let d and d' be two metrics on the set X; let τ and τ' be the topologies they induce, respectively. Then τ' is finer than τ if and only if for each x in X and each $\epsilon > 0$, there exists $\delta > 0$ such that

$$B_{d'}(x, \delta) \subseteq B_d(x, \epsilon).$$

Chapter 2: Section 21

Theorem 31 (Theorem 21.1)**.** *Let $f : X \to Y$; let X and Y be metrizable with metrics d_X and d_Y, respectively. Then continuity of f is equivalent to the requirement that given $x \in X$ and given $\epsilon > 0$, there exists $\delta > 0$ such that*

$$d_X(x, y) < \delta \implies d_Y(f(x), f(y)) < \epsilon.$$

Lemma 32 (Lemma 21.2). Let X be a topological space; let $A \subseteq X$. If there is a sequence of points of A converging to x, then $x \in \overline{A}$; the converse holds if X is metrizable.

Theorem 33 (Theorem 21.3)**.** *Let $f : X \to Y$. If the function f is continuous, then for every convergent sequence $x_n \to x$ in X, the sequence $f(x_n)$ converges to $f(x)$. The converse holds if X is metrizable.*

Lemma 34 (Lemma 21.4). The addition, subtraction, and multiplication operation are continous functions from $\mathbb{R} \times \mathbb{R}$ *into* $\mathbb{R}$; and the quotient operation is continous function from $(\mathbb{R} \times (\mathbb{R} - \{0\})$ into $\mathbb{R}$.

Theorem 35 (Theorem 21.5)**.** *If X is a topological space, and if $f, g : X \to \mathbb{R}$ are continuous functions, then $f + g, f - g$ and $f \cdot g$ are continuous. If $g(x) \neq 0$ for all x, then ${}^f/_g$ is continuous.*

Chapter 2: Section 22

Definition 0.24. Let X and Y be topological spaces; let $p : X \to Y$ be a surjective map. The map p is said to be a **quotient map** provided a subset U of Y is open in Y if and only if $p^{-1}(U)$ is open in X.

Recall that a map $f : X \to Y$ is said to be an **open map** if for each open set U of X, the set $f(U)$ is open in Y. It is said to be a closed map if for each closed set A of X, the set $f(A)$ is closed in Y. Continuous and surjective open and closed maps are one type of quotient maps but not all quotient maps are open or closed.

Definition 0.25. If X is a space and A is a set and if $p : X \to A$ is a surjective map, then there exists exactly one topology τ on A relative to which p is a quotient map, it is called the **quotient topology** induced by p.

Example.

$$D^n \setminus S^{n-1} \cong S^n$$

The square in $\mathbb{R}^2$ quotients to the taurus.

An example of a quotient map which is neither open or closed.

Example. *Let $\pi_1 : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be a projection on the first coordinate. Let A be the subspace of $\mathbb{R} \times \mathbb{R}$ consisting of all points $x \times y$ for which either $x \geq 0$ or $y = 0$ (or both); let $q : A \to \mathbb{R}$ be obtained by restricting π_1.*

Chapter 3: Section 23

Definition 0.26. Let X be a topological space. A **separation** of X is a pair U, V of disjoint nonempty open subsets of X whose union is X. The space X is said to be **connected** if there does not exist a separation of X.

A space X is connected if and only if the only subsets of X that are clopen in X are the empty set and X itself.

Lemma 36 (Lemma 23.1). If Y is a subspace of X, a separation of Y is a pair of disjoint nonempty sets A and B whose union is Y, neither of which contains a limit point of the other (i.e. $\overline{A} \cap B = \emptyset$ and $A \cap \overline{B} = \emptyset$). The space Y is connected if there exists no separation of Y.

The following is an example of a subspace of $\mathbb{R}^2$ which is not connected.

Example (Page 149). *Consider the following subset of the plane $\mathbb{R}^2$:*

$$X = \{x \times y : y = 0\} \cup \{x \times y : x > 0 \; and \, y = 1/x\}$$

Then X is not connected; indeed, the two indicated sets form a separation of X because neither contains a limit point of the other. Refer to figure 23.1 on page 149.

Lemma 37 (Lemma 23.2). If the sets C and D form a separation of X, and if Y is a connected subspace of X, then Y lies entirely within C or D.

Theorem 38 (Theorem 23.3)**.** *The union of a collection of connected subspaces of X that a point in common is connected.*

Theorem 39 (Theorem 23.4)**.** *Let A be a connected subspace of X. If $A \subseteq B \subseteq \overline{A}$, then B is also connected.*

Proof. Let A be connected and let $A \subseteq B \subseteq \overline{A}$. Suppose $B = C \cup D$ is a separation of B. By Lemma 23.2, the set A must lie entirely within in C or D. Suppose it lies within C, so $A \subseteq C$. Then, $\overline{A} \subseteq \overline{C}$. By Lemma 23.1, we know $\overline{C} \cap D = \emptyset$ and more specifically, since $B \subseteq \overline{A}$, we know that $B \cap D = \emptyset$ implying D is empty which is a contradiction. $\square$

Theorem 40 (Theorem 23.5)**.** *The image of a connected space under a continuous map is connected.*

Theorem 41. *A finite cartesian product of connected spaces is connected.*

Proof. We prove the theorem first for the product of two conncted spaces X and Y. Choose a base point $a \times b$ in the product $X \times Y$. Note that the horizontal slice $X \times b$ is connected (homeomorphic to X) and similarly $x \times Y$ is connected. As a result, each T-shaped space

$$T_x = (X \times b) \cup (x \times Y)$$

is connected since they have the point $x \times b$ in common. Now form union $\bigcup_{x \in X} T_x$ of all these T-shaped spaces. This union is connected because each T-shaped space contains $a \times b$ so that the union has $a \times b$ in common. Since this union equals $X \times Y$, the space $X \times Y$ is connected. $\square$

Chapter 3: Section 24

An arbitrary product of connected spaces is connected in the product topology but not in the box topology.

Definition 0.27. Given points x and y of the space X, a **path** in X from x to y is a continuous map $f : [a, b] \to X$ of some closed intervals in the real line into X, such that $f(a) = x$ and $f(b) = y$. A space X is **path-connected** if every pair of points of X can be joined by a path in X.

Path connectedness implies connectedness.
Continuous image of a path-connected space is path connected.

Example. *Let S denote the following subset of the plane.*

$$S = \{x \times \sin(1/x) : 0 < x \leq 1\}.$$

Because S is the image of the connected set $(0, 1]$ under a continuous map, S is connected. Therefore, its closure $\overline{S}$ in $\mathbb{R}^2$ is also connected. The set $\overline{S}$ is a classical example in topology called ***topologist's sine curve****. It equals the union of S and the vertical interval $0 \times [-1, 1]$.*

Chapter 3: Section 26

Definition 0.28. A collection $\mathcal{A}$ of subsets of a space X is said to **cover** X, or to be a **covering** of X, if the union of the elements of $\mathcal{A}$ is equal to X. It is called an **open covering** of X if its elements are open subsets of X.

Definition 0.29. A space X is said to be **compact** if every open covering $\mathcal{A}$ of X contains a finite subcollection that also covers X.

If Y is a subspace of X, a collection $\mathcal{A}$ of subsets of X is said to **cover** Y if the union of its elements *contains* Y.

Lemma 42 (Lemma 26.1). Let Y be a subspace of X. Then Y is compact if and only if every covering of Y by sets open in X contains a finite subcollection covering Y.

Theorem 43 (Theorem 26.2)**.** *Every closed subspace of a compact space is compact.*

Proof. Let Y be a closed subspace of the compact space X. Given a covering $\mathcal{A}$ of Y by sets open in X, let us form an open covering $\mathcal{B}$ of X adjoining to $\mathcal{A}$ the single open set $X - Y$, that is,

$$\mathcal{B} = \mathcal{A} \cup \{X - Y\}.$$

Some finite subcollection of $\mathcal{B}$ covers X. If this subcollection contains the set $X - Y$, discard $X - Y$; otherwise, leave the subcollection alone. The resulting collection is a finite subcollection of $\mathcal{A}$ that covers Y. □

Theorem 44 (Theorem 26.3)**.** *Every compact subspace of Hausdorff space is closed.*

Lemma 45 (Lemma 26.4). If Y is a compact subspace of the Hausdorff space X and x_0 is not in Y, then there exist disjoint open sets U and V of X containing x_0 and Y, respectively.

Theorem 46 (Theorem 26.5)**.** *The image of a compact space under a continuous map is compact.*

Theorem 47 (Theorem 26.6)**.** *Let $f : X \to Y$ be a bijective continuous function. If X is compact and Y is Hausdorff, then f is a homeomorphism.*

Theorem 48 (Theorem 26.7)**.** *The product of finitely many compact space is compact.*

Theorem 49 (Theorem 27.3)**.** *A subspace A of $\mathbb{R}^n$ is compact if and only if it is closed and is bounded in the euclidean metric d or the square metric ρ.*

Proof. It will suffice to consider only the metric ρ; the inequalities

$$\rho(x, y) \leq d(x, y) \leq \sqrt{n}\rho(x, y)$$

imply that A is bounded under d if and only if it is bounded under ρ.
Suppose that A is compact. Then by Theorem 26.3, it is closed ($\mathbb{R}^n$ Hausdorff).
Consider the collection of open sets

$$\{B_\rho(\mathbf{0}, m) : m \in \mathbb{Z}_+\},$$

whose union covers all of $\mathbb{R}^n$. Some finite subcollection covers A (A compact). It follows that $A \subseteq B_\rho(\mathbf{0}, M)$ for some M. Therefore, for any two points x and y of A, we have $\rho(x, y) \leq 2M$. Thus A is bounded under ρ so that A is cloesd and bounded.
Conversely, suppose that A is closed and bounded under ρ; suppose that $\rho(x, y) \leq N$ for every pair x, y of points of A. choose a point x_0 of A, and let $\rho(x_0, \mathbf{0}) = b$. The triangle inequality implies that $\rho(x, \mathbf{0}) \leq N + b$ for every $x \in A$. If $P = N + b$, then A is a subset of the cube $[-P, P]^n$, which is compact. Being closed, A is also compact. $\square$

Definition 0.30. Let (X, d) be a metric space; let A be a nonempty subset of X. For each $x \in X$, we define the **distance from x_0 to A** by the equation

$$d(x, A) = \inf\{d(x, a) : a \in A\}$$

Chapter 3: Section 30

Definition 0.31. A space X is said to have a **countable basis at** x if there is a countable collection **B** of neighborhood of x such that each neighborhood of x contains at least one of the elements of $\mathcal{B}$ or equivalently, for every open neighborhood U of x, there exists an element of the countable collection contained in U. A space that has a countable basis at each of its points is said to satisfy the **first countability axiom**, or to be **first-countable**.

Theorem 50 (Theorem 30.1)**.** *Let X be a topological space.*

1. *Let A be a subset of X. If there is a sequence of points of A, converging to x, then $x \in \overline{A}$; the converse holds if X is first-countable.*

2. *Let $f : X \to Y$. If f is continuous, then for every convergent sequence $x_n \to x$ in X, the sequence $f(x_n)$ converges to $f(x)$. The converse holds if X is first-countable.*

Definition 0.32. IF a space X has a countable basis for its topology, then X is said to satisfy the **second countability axiom**, or to be **second-countable**.

Note not every metrizable space is second-countable. Every second countable space is Lindelöf. The product of two Lindelöf spaces need not be Lindelöf. The counter-example to this is the Sorgenfrey plane i.e. the topology generated by the basis which has sets of the form $[a, b) \times [c, d)$. The lower limit topology is first-countable but not second.

Theorem 51 (Theorem 30.2)**.** *A subspace of a first-countable space is first-countable, and a countable product of first-countable spaces is first-countable. A subspace of a second-countable space is second-countable, and a countable product of second-countable spaces is second-countable.*

Chapter 3: Section 31

Definition 0.33. Suppose that one point sets are closed in X. Then X is said to be **regular** if for each pair consisting of a point x and a closed set B disjoint from x, there exist disjoint open sets containing x and B, respectively. The space X is said to be **normal** if for each pair A, B of disjoint closed sets of X, there exist disjoint open sets containing A and B, respectively.

The space $\mathbb{R}_K$ is Hausdorff but not regular. The space $\mathbb{R}_l$ is normal. The Sorgenfrey plane is not normal but is regular (this space is the product of two regular spaces namely $\mathbb{R}_l \times \mathbb{R}_l$). Thus, the cartesian product of two normal spaces need not be normal.

Lemma 52 (Lemma 31.1). Let X be a topological space. Let one points sets in X be closed.

1. X is regular if and only if given a point x of X and a neighborhood U of x, there is a neighborhood V of x such that $\overline{V} \subseteq U$.

2. X is normal if and only if given a closed set A and an open set U containing A, there is an open set V containing A such that $\overline{V} \subseteq U$.

Theorem 53 (Theorem 31.2)**.** *1. A subspace of a Hausdorff space is Hausdorff; a product of Hausdorff spaces is Hausdorff.*

2. A subspace of a regular space is regular; a product of regular spaces is regular.

Definition 0.34. A $\mathbf{T}_0$ or **Kolmogrov** space is defined to be a space where if $x, y \in X$, $x \neq y$, then there exists a neighborhood of x such that $y \notin U$ or vice versa.
A $\mathbf{T}_1$ or **Frechét** space is defined to be a space where if $x, y \in X$, then there exists U, V open sets such that $x \in U$ and $y \in V$ but $x \notin V$ and $y \notin U$.

Chapter 3: Section 32

Arbitrary products of normal spaces is not normal.

Theorem 54 (Theorem 32.1)**.** *Every regular space with a countable basis is normal.*

Theorem 55 (Theorem 32.2)**.** *Every metrizable space is normal.*

Proof. Let X be a metrizable space with metric d. Let A and B be disjoint closed subsets of X. For each $a \in A$, choose ϵ_a so that the ball $B(a, \epsilon_a)$ does not intersect B. Similarly, for each $b \in B$, choose ϵ_b so that the ball $B(b, \epsilon_b)$ does not intersect A. Define

$$U = \bigcup_{a \in A} B(a, \epsilon_a/2) \qquad \text{and} \qquad V = \bigcup_{b \in B} B(b, \epsilon_b/2)$$

Then U and V are disjoint open sets containing A and B, respectively; we assert they are disjoint. For if, $z \in U \cap V$, then

$$z \in B(a, \epsilon_a/2) \cap B(b, \epsilon_b/2)$$

for some $a \in A$ and $b \in B$. The triangle inequality applies to show that $d(a, b) < (\epsilon_a + \epsilon_b)/2$. If $\epsilon_a \leq \epsilon_b$, then $d(a, b) < \epsilon_b$ so that the ball $B(b, \epsilon_b)$ contains a or vice versa, if $\epsilon_b \leq \epsilon_a$, then $d(a, b) < \epsilon_a$, so that the ball $B(a, \epsilon_a)$ contains the point b. Neither situation is possible. $\square$

Theorem 56 (Theorem 32.3)**.** *Every compact Hausdorff space is normal.*