Problem Set 17

MAT237: Advanced Calculus
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Problem 0.1. *Determine the Jacobian of the following transformations. Whenever, possible, write the infinitesimal area/volume element in terms of one another.*

(a) $(x, y) = (e^{\xi}, \eta^3)$

(b) $(x, y) = (5u - 2v, u + v)$

(c) $(x, y) = (sin(u^2 v), cos(v^2 u))$

(d) $(x, y, z) = (v + w^2, w + u^2, u + v^2)$

(e) $(x, y, z) = (u^3 - v^2, u^3 + v^2, u^3 + v^2 + w)$

Solution. (a). Let $(x, y) = G(\xi, \eta) = (e^{\xi}, \eta^3)$. Then:

$$|DG(\xi, \eta)| = \left| \begin{pmatrix} e^{\xi} & 0 \\ 0 & 3\eta^2 \end{pmatrix} \right| = e^{\xi} 3\eta^2$$

so that $dxdy = 3e^{\xi}\eta^2 d\xi d\eta$. To calculate $d\xi d\eta$ in terms of $dxdy$, note that:

$$x = e^{\xi}, \ y = \eta^3$$

so that $\log x = \xi$ and $\sqrt[3]{y} = \eta$ so that $(\xi, \eta) = (\log x, \sqrt[3]{y})$. Let $(\xi, \eta) = G^{-1}(x, y) = (\log x, \sqrt[3]{y})$, then:

$$\left|DG^{-1}(x, y)\right| = \left| \begin{pmatrix} x^{-1} & 0 \\ 0 & \frac{1}{3} y^{\frac{-2}{3}} \end{pmatrix} \right| = \frac{1}{3} x^{-1} y^{\frac{-2}{3}}$$

so that $d\xi d\eta = \frac{1}{3} x^{-1} y^{\frac{-2}{3}} dxdy$.

(b). Let $(x, y) = G(u, v) = (5u - 2v, u + v)$, then:

$$|DG(u, v)| = \left| \begin{pmatrix} 5 & -2 \\ 1 & 1 \end{pmatrix} \right| = |5 + 2| = 7$$

so that $dxdy = 7dudv$ which implies that: $dudv = \frac{1}{7} dxdy$.

(c) Let $(x,y) = G(u,v) = (\sin(u^2v), \cos(v^2u))$, then:

$$\begin{aligned}|DG(u,v)| &= \left|\begin{pmatrix} 2uv\cos(u^2v) & u^2\cos(u^2v) \\ -v^2\sin(v^2u) & -2vu\sin(v^2u)\end{pmatrix}\right| \\ &= (uv)^2[-4\cos u^2v\sin(v^2u) + \cos u^2v\sin(v^2u)] \\ &= -3(uv)^2\sin(u^2v)\cos(u^2v)\end{aligned}$$

so that $dxdy = -3(uv)^2\sin(u^2v)\cos(u^2v)dudv$. Next, we try to express $dudv$ in terms of $dxdv$. First, note that:

$$x = \sin(u^2v),\ y = \cos(v^2u),\ (uv)^2 = ???$$

so that $dudv = \frac{-1}{3}xy(???)$.

(d) Let $(x,y,z) = G(v,w,u) = (v+w^2, w+u^2, u+v^2)$, then:

$$\begin{aligned}|DG(u,v)| &= \left|\begin{pmatrix} 1 & 2w & 0 \\ 0 & 1 & 2u \\ 2v & 0 & 1\end{pmatrix}\right| \\ &= \left|\begin{pmatrix} 1 & 2u \\ 0 & 1\end{pmatrix} - \begin{pmatrix} 0 & 2u \\ 2v & 1\end{pmatrix}\right| \\ &= 1 + 8uvw\end{aligned}$$

so that $dxdydz = (1+8uvw)dudvdw$. ■

Problem 0.2. *Let R be the region bounded by the curves $y = x^2$, $4y = x^2$, $xy = 1$, $xy = 2$. Compute the integral:*

$$\iint_R x^2y^2dxdy$$

Solution. We are given that $1 \le xy \le 2$ and $y \le x^2 \le 4y$. Note, that $y \le x^2 \le 4y$ is the same as saying $1 \le \frac{x^2}{y} \le 4$. Let $u = xy$, $v = \frac{x^2}{y}$ so that:

$$1 \le u \le 2 \ \text{ and } \ 1 \le v \le 4$$

Since $y \neq 0$, $(u,v) = G(x,y) = (xy, \frac{x^2}{y})$ is a diffeomorphism. Note,

$$|DG(x,y)| = \left|\begin{pmatrix} y & x \\ \frac{2x}{y} & -\frac{x^2}{y^2}\end{pmatrix}\right| = \left|-\frac{x^2}{y} - \frac{2x^2}{y}\right| = \frac{3x^2}{y}$$

so that $dudv = \frac{3x^2}{y}dxdy$ which implies $\frac{y}{3x^2}dudv = dxdy$ so that we need to write $\frac{y}{3x^2}$ in terms of u, v. Note:

$$\frac{1}{3}v^{-1} = \frac{y}{3x^2}$$

so $dxdy = (3v)^{-1}dudv$ so that by the change of variables theorem, we have that:

$$\begin{aligned}\iint_R x^2 dxdy &= \int_1^4 \int_1^2 (u^2)(3v)^{-1} dudv \\ &= \frac{1}{3}\int_1^4 v^{-1}dv \cdot \int_1^2 u^2 du \\ &= \frac{1}{9}[\log v]_{v=1}^{v=4} \cdot [u^3]_{u=1}^{u=2} \\ &= \frac{7}{9}\log 4\end{aligned}$$

■

Problem 0.3. *Determine* $\iint_S \frac{(x+y)^4}{(x-y)^5} dA$ *where* $S = \{-1 \le x+y \le 1, 1 \le x-y \le 3\}$.

Solution. We are given that:

$$-1 \le x+y \le 1,\ 1 \le x-y \le 3$$

so let $u = x+y$ and $v = x-y$ which is a diffeomorphism. Then, note:

$$|DG(x,y)| = \left|\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\right| = 1$$

so that $dudv = dxdy$. Then by the change of variables theorem:

$$\iint_S \frac{(x+y)^4}{(x-y)^5} dA = \int_1^3 \int_{-1}^1 \frac{u^4}{v^5} dudv = \int_1^3 v^{-5} dv \cdot \int_{-1}^1 u^4 du$$

which can be computed easily. ■

Problem 0.4. *Compute* $\iint_R (4x+8y) dA$ *where R is the quadrilateral with endpoints:* $(-1,3)$, $(1,-3)$, $(3,-1)$, $(-3,1)$.

Problem 0.5. *Compute* $\iint_R \sin(9x^2+4y^2) dA$ *where R is the circle* $9x^2+4y^2 = 36$.

Solution. We use elliptical coordinates, specifically, let:

$$x = 3^{-1} r\cos(\theta),\ y = 2^{-1} r \sin(\theta)$$

We note that $0 \le \theta \le 2\pi$ and to bound r, consider the following:

$$\begin{aligned} 9x^2 + 4y^2 &= 36 \\ 9(3^{-1}r\sin(\theta))^2 + 4(2^{-1}r\cos(\theta))^2 &= 36 \\ r^2\sin^2(\theta) + r^2\cos^2(\theta) &= 36 \\ r^2(\sin^2(\theta)+\cos^2(\theta)) &= 36 \\ r &= \pm 4 \end{aligned}$$

so that $-4 \le r \le 4$. Since we have already computed the determinant of the Jacobian of a polar coordinates change of coordinates, and an elliptical change of coordinates only differ by constants, then: $dxdy = 2^{-1}3^{-1}rdrd\theta$. Thus, by the change of variables theorem, we have that:

$$\begin{aligned}\iint_R \sin(9x^2+4y^2)dA &= \int_0^{2\pi}\int_{-4}^{4}\sin(r^2)2^{-1}3^{-1}rdrd\theta \\ &= \frac{1}{6}\int_0^{2\pi}d\theta\cdot\int_{-4}^{4}r\sin(r^2)dr\end{aligned}$$

which can be computed easily by making the subsitution (in $\mathbb{R}^1$) $u = r^2$ to the right-most integral. $\blacksquare$