MATD11: Functional Analysis
Assignment 3

Anmol Bhullar - 1002678140

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Preface.

We may say $x_n \to x$ to say that $(x_n)_{n=1}^{\infty}$ is a sequence which converges to x. Instead of writing $(x_n)_{n=1}^{\infty}$, we may just say $(x_n)_1^{\infty}$ or even just (x_n) where context is clear.

P1.

Suppose M is a dense subspace in a Banach space X (meaning that the closure of M is all of X) and suppose that $T : M \to Y$ is linear, where Y is a Banach space, with $\|Tm\|_Y \leq K \|m\|_X$ for some $K < \infty$ and all $m \in M$. Show that T extends, in a unique way, to a bounded linear operator from X into Y.

Solution.
Note $\overline{M} = X$. Thus, by definition, we have that for all $x \in X$, there exist some sequence $(x_n)_1^{\infty}$ in M such that $x_n \to x$. Define a mapping $T' : X \to X$ by $T'(x) = \lim_{n\to\infty} T(x_n)$. Supposing this mapping is well defined, it is clear that T' is then a mapping from X to itself. Note our definition of T' does not depend on our choice of (x_n). To see why choose sequences $x_n \to x$ and $y_n \to x$ such that both $(x_n)_1^{\infty}$ and $(y_n)_1^{\infty}$ lie in M. Using uniqueness of limits in a Banach space (every Banach space is a metric space which are always Hausdorff), we know:

$$\lim_{n\to\infty} T(x_n) = T'(x) \qquad \text{and} \qquad \lim_{n\to\infty} T(y_n) = T'(x)$$

is always true. Thus, we have that $\lim_{n\to\infty} T(x_n) = \lim_{n\to\infty} T(y_n)$ which implies the value of $T'(x)$ is independent of our choice of sequence and so T' is a well defined function.

Now, we want to show that T' is a *continuous* extension of T. In order to do this, it suffices to show $T'|M = T$ is true and T' is continuous on X. $T'|M$ is clearly a map from $M \to X$ where $T'|_M(x \in M) = \lim_{n\to\infty} T(x_n)$ for some sequence $x_n \to x$ in M. Simply, choose the constant sequence $(x)_1^{\infty}$ since $x \in M$. Clearly, this converges to x. Then, note $\lim_{n\to\infty} T(x) = T(x)$ so that

$T'|_M(x) = T(x)$ as wanted. It is left to show T' is continuous.

Fix any $x \in X$. We want to show:

$$\forall\, \epsilon > 0,\ \exists\, \delta > 0 \text{ such that if } 0 < |x-y| < \delta,\ \text{then } |f(x) - f(y)| < \epsilon$$

Thus, choose $\epsilon > 0$, then for $\delta = \epsilon/K$ (if $K = 0$, then T is identically 0 (recall T is bounded by K) so that T' is identically zero which we know is continuous since it is a constant function), if $0 < |x-y| < \delta$, then:

$$\begin{aligned}
\|T'(x) - T'(y)\|_Y &= \left\| \lim_{n\to\infty} T(x_n) - \lim_{n\to\infty} T(y_n) \right\|_Y && \text{where } x_n \to x,\ y_n \to y \\
&= \left\| \lim_{n\to\infty} [T(x_n) - T(y_n)] \right\|_Y && \text{linearity of the limit operator} \\
&= \left\| \lim_{n\to\infty} T(x_n - y_n) \right\|_Y && \text{linearity of } T \\
&= \lim_{n\to\infty} \|T(x_n - y_n)\|_Y && \text{continuity of } \|\cdot\| \\
&= \lim_{n\to\infty} K \|x_n - y_n\|_X < \lim_{n\to\infty} K \cdot \delta = \epsilon
\end{aligned}$$

so that T' is continuous at x and since x was arbitrarily chosen, we have that T' is continuous everywhere. Note this also implies that T' is a bounded mapping by Proposition 2.2.

In order to show T' is a bounded linear operator from $X \to Y$, it is left to show T' is linear.

Uniquness of T' follows from the Hausdorff property of Y.

P2.

Let $\Lambda : X \to \mathbb{C}$ is a bounded linear functional on a normed linear space X. Recall that $\|\Lambda\|$ is defined as $\sup\{|\Lambda(x)| : \|x\| \leq 1\}$. Show that

$$\begin{aligned} \|\Lambda\| &= \sup\{|\Lambda(x)| : \|x\| = 1\} \\ &= \sup\{|\Lambda(x)/\|x\|\,| : x \neq 0\} \end{aligned}$$

P3.

Let X, Y and V be normed linear spaces and let $A \in \mathcal{B}(X, Y)$ and $B \in \mathcal{B}(Y, V)$. Prove that $BA \in \mathcal{B}(X, V)$ and $\|BA\| \leq \|B\|\,\|A\|$.

P4.

Let X be a Banach space. Let $\{A_n\}$ be a sequence in $\mathcal{B}(X)$ such that $\sum_{n=1}^{\infty} \|A_n\|$ converges. Prove that the series $\sum_{n=1}^{\infty} A_n$ converges to an operator $A \in \mathcal{B}(X)$ and $\|A\| \leq \sum_{n=1}^{\infty} \|A_n\|$.

P5.

Let X be a Banach space and let $A \in \mathcal{B}(X)$. Explain how to define e^A and prove that $e^A \in \mathcal{B}(X)$.

P6.

A sequence $\{h_n\}$ in a Hilbert space $\mathcal{H}$ is said to **converge weakly** to $h \in \mathcal{H}$ if

$$\lim_{n \to \infty} \langle h_n, g \rangle = \langle h, g \rangle$$

for every $g \in \mathcal{H}$.

(a) If $\{e_n\}$ is an orthonormal sequence in $\mathcal{H}$, show that $e_n \to 0$ weakly.

(b) Show that if $h_n \to h$ in norm, then $h_n \to h$ weakly. Show that the converse is false, but that if $h_n \to h$ weakly and $\|h_n\| \to \|h\|$, then $h_n \to h$ in norm.