# MAT327: Introduction to Topology. Solutions to the Big List Problems

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Thank you to my instructor Ivan Khatchatourian for providing these wonderful problems.

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Chapter 1

# Preface

I attempt to answer and LaTeX all of the solutions to the big list of problems posted by my instructor Ivan Khatchatourian for MAT327: Introduction to Topology. The problems are separated into difficulties which are labelled via asterisks. One asterisk being the lowest difficulty and 3 being the highest. Especially hard problems are marked via a cross. This is the format my instructor uses and I'm merely copying it for consistency's sake.

Chapter 2

# Topologies

* **Ex. 1** — Fix $a < b \in \mathbb{R}$. Show explicitly that the open interval $(a, b)$ is open in $\mathbb{R}_{\text{usual}}$. Show explicitly that the interval $[a, b)$ is not open in $\mathbb{R}_{\text{usual}}$

**Answer (Ex. 1)** — First, we show that $(a, b)$ is open in $\mathbb{R}_{\text{usual}}$. To do this: we have to show that for any point $x \in (a, b)$, there exists a neigbourhood $N(x, \epsilon) \subseteq (a, b)$.
Thus, choose a point $x \in (a, b)$. By density of $\mathbb{R}$, there exists $x_1 \in (a, x)$ and $x_2 \in (x, b)$. Let $\epsilon = \min(x_1, x_2)$. Then $N(x, \epsilon) \subseteq (a, b)$.
Next, we show that $[a, b)$ is not open. To do this, we show that $\exists$ no $\epsilon > 0$ such that $N(a, \epsilon) \subseteq [a, b)$. Note that any $x \in (a - \epsilon, a]$ would not be in $(a, b)$ for any $\epsilon > 0$, so $N(a, \epsilon) \nsubseteq [a, b)$.

* **Ex. 2** — Let $X$ be a set and $\mathcal{B} = \{\{x\} : x \in X\}$. Show that the only topology on $X$ that contains $\mathcal{B}$ as a subset is the discrete topology.

**Answer (Ex. 2)** — To show that only $X_{\text{discrete}}$ has the property $\mathcal{B} \subseteq X_{\text{discrete}}$, consider the set:

$$\mathcal{T} := \{\{x\} : x \in X\}$$

For $\mathcal{T}$ to be a topology, it must be closed under finite intersections so we must put $\emptyset$ in $\mathcal{T}$. Furthermore, $\mathcal{T}$ must be closed under the union of an arbitrary collection elements of $X$. If this is to be true, then $\mathcal{T}$ must contain every subset of $X$ since any arbritrary subset can be written as the union of all of its elements. Thus an arbitrary subset $A \subseteq X$ must be in $\mathcal{T}$. This implies $\mathcal{T} = X_{\text{discrete}}$.

* **Ex. 4** — Let $(X, \mathcal{T}_{\text{co-countable}})$ be an infinite set with the co-countable topology. Show that $\mathcal{T}_{\text{co-countable}}$ is closed under countable intersections but not necessairly arbitrary ones.

**Answer (Ex. 4)** — Given a countable indexing set $I$, we have to show

$$\bigcap_{\alpha \in I} U_\alpha \in \mathcal{T}_{\text{co-countable}}$$

Equivalently, we can show $X \setminus (\cap U_\alpha)$ is countable. By DeMorgan's Law: $X \setminus (\cap_\alpha) = \cup(X \setminus U_\alpha)$. Since the right hand side is the countable union of countable sets, it is countable. Thus, $\cap U_\alpha \in \mathcal{T}_{\text{co-countable}}$. To show that an arbitrary intersection of elements is not open, it suffices to state that the intersection of arbitrary many elements does not necessarily form a countable set.

* **Ex. 5** — Let $(X, \mathcal{T})$ be a topological space, and let $A \subseteq X$ be a set with the property that for all $x \in A$, $\exists$ an open set $U_x \in \mathcal{T}$ such that $x \in U_x \subseteq A$. Show that $A$ is open.

**Answer (Ex. 5)** — Let $I$ be a set which indexes the elements of $A$. Then:

$$\bigcup_{\alpha \in I} \{x_\alpha\} = A$$

Similarly, since $x_\alpha \in U_{x_\alpha}$ and $U_{x_\alpha} \subseteq A$, then:

$$\bigcup_{\alpha \in I} U_{x_\alpha} = A$$

Since this is the union of arbitrary elements of $\mathcal{T}$, then $A \in \mathcal{T}$.

* **Ex. 6** — Let $(X, \mathcal{T})$ be a topological space, and let $f : X \to Y$ be injective. Is $\mathcal{T}_f := \{f(U) : U \in \mathcal{T}\}$ a topology on $Y$? Is it necessarily a topology on the range of $f$?

**Answer (Ex. 6)** — Lost solution : (

* **Ex. 7** — Let $X$ be a set and let $\mathcal{T}_1$ and $\mathcal{T}_2$ be two topologies on $X$. Is $\mathcal{T}_1 \cup \mathcal{T}_2$ a topology on $X$? What about $\mathcal{T}_1 \cap \mathcal{T}_2$? Is yes, prove it. If not, provide a counterexample.

**Answer (Ex. 7)** — Let $\mathcal{T}_1 := \mathcal{T}_{\text{usual}}$, $\mathcal{T}_2 := \mathcal{T}_7$ on $\mathbb{R}$. Consider:

$$(6,7) \cap [6.5, 7] \qquad \text{both in} \mathcal{T}_1 \cup \mathcal{T}_2)$$

but this intersection yields the interval: $[6.5, 7)$ which is not open in either $\mathcal{T}_1$ or $\mathcal{T}_2$. So $\mathcal{T}_1 \cup \mathcal{T}_2$ is not necessarily a topology. Now, we show that $\mathcal{T}_1 \cap \mathcal{T}_2$ is a topology:

$$\emptyset \in \mathcal{T}_1, \emptyset \in \mathcal{T}_2 \implies \emptyset \in \mathcal{T}_1 \cap \mathcal{T}_2$$
$$X \in \mathcal{T}_1, X \in \mathcal{T}_2 \implies X \in \mathcal{T}_1 \cap \mathcal{T}_2$$

Furthermore, given, $U, V \in \mathcal{T}_1, T_2$. Then $U \cap V$ is in both $\mathcal{T}_1$ and $\mathcal{T}_2$ so it follows $U, V \in \mathcal{T}_1 \cap \mathcal{T}_2$ and that since $U, V \in \mathcal{T}_1$, then $U \cap V \in \mathcal{T}_1$. Similarly for $\mathcal{T}_2$ so then $U \cap V \in \mathcal{T}_1 \cap \mathcal{T}_2$. Thus, the finite intersection of open sets in $\mathcal{T}_1 \cap \mathcal{T}_2$ is in $\mathcal{T}_1 \cap \mathcal{T}_2$. Now it is left to show that $\cup_{\alpha \in I} V_\alpha \in \mathcal{T}_1 \cap \mathcal{T}_2$ given that for all $\alpha \in I$, $V_\alpha \in \mathcal{T}_1 \cap \mathcal{T}_2$.

If every $V_\alpha \in \mathcal{T}_1 \cap \mathcal{T}_2$, then it follows $\cup V_\alpha \in T_1$ and similarly for $\mathcal{T}_2$. Thus $\mathcal{T}_1 \cap \mathcal{T}_2$ is a topology.

* **Ex. 8** — Let $X$ be an infinite set. Show that there are infinitely many topologies on $X$.

**Answer (Ex. 8)** — Define $\mathcal{T}_{\text{co-k}} := \{U \subseteq X : |U^c| \leq k\}$. Since $\mathcal{T}_{\text{co-k}}$ is a topology for all $k \in \mathbb{N}$, it follows that if $|X| = \infty$, then there are infinite topologies on $X$.

* **Ex. 9** — Let $\{\mathcal{T}_\alpha : \alpha \in I\}$ be a collection of topologies on a set $X$, where $I$ is some indexing set. Prove that there is a unique finest topology that is refined by all the $\mathcal{T}_\alpha' s$. That is, prove that there is a topology $\mathcal{T}$ on $X$ such that:

(1)$\mathcal{T}_\alpha$ refines $\mathcal{T}$ for every $\alpha \in I$

(2)If $\mathcal{T}^{'}$ is another topology that fulfills (a), then $\mathcal{T}$ is finer than $\mathcal{T}^{'}$

**Answer (Ex. 9)** — Claim: $\mathcal{T} = \cap \mathcal{T}_\alpha$. By Ex 7, $\mathcal{T}$ is a topology, and $\mathcal{T}_\alpha$ refines $\mathcal{T}$ by construction of $\mathcal{T}$. Suppose $T^{'}$ is another topology of $X$ such that all $\mathcal{T}_\alpha$ refine $\mathcal{T}^{'}$ and that $\mathcal{T}'$ refines $\mathcal{T}$. Then, there exists $x \in \mathcal{T}^{'}$ such that $x \notin \mathcal{T}$. But then $x \notin \cap \mathcal{T}_\alpha$ so there exists $T_{\alpha_0}$ such that $x \notin \mathcal{T}_{\alpha_0}$. Thus $\mathcal{T}$ is not refined by $\mathcal{T}^{'}$ and then $\mathcal{T}^{'} = \mathcal{T}$ which implies $\mathcal{T}$ is unique.

* **Ex. 10** — This extends exercise 6. Show with examples that the assumption that $f$ is injective is necessary. That is, give an example of a topological space $(X, \mathcal{T})$ and a non-injective function $f : X \to Y$ such that $\mathcal{T}_f$ is a topology and another example where $\mathcal{T}_f$ is not.

**Answer (Ex. 10)** — An example of a non-injective function which is not a topology is given by mapping all of the irrationals of $\mathbb{R}$ to themselves but mapping all rationals to 0. Since there is no open set in $\mathbb{R}$ which maps to 0, we arrive that $\mathcal{T}_f$ is not a topology. An example of a non injective function which is a topology is given by:

*** **Ex. 11** — Working in $\mathbb{R}_{\text{usual}}$:

(1)Show that every non empty open set contains a rational number

(2)Show that there is no uncountable collection of pairwise disjoint open subsets of $\mathbb{R}$.

**Answer (Ex. 11)** — Let $U$ be such a set. For any $x \in U$, $\exists \epsilon > 0$ such that $N(x, \epsilon) \subseteq U$, but this is impossible since by density of $\mathbb{Q}$, there exists a $y \in (x, x+\epsilon)$. Then, no such $U$ exists. Now, we prove (b). We know the set of all intervals $(a, b)$ where $a, b \in \mathbb{Q}$ is countable. Suppose $\{X_\alpha : \alpha \in I\}$ is such a set. If we show $\theta := \{X_\alpha : \alpha \in I\}$, $|\theta| \geq |\{(a, b) : a, b \in \mathbb{Q}\}|$ then we are done. Since we construct a set $\theta_2$ where for every $X_\alpha \in \theta$, $\theta_2$ contains two rational open subsets of $X_\alpha$. Then $|\theta| \leq |\theta_2|$ implying that $\theta_2$ is at most countable.

# Bases of Topologies

* **Ex. 12** — Show explicitly that $\mathcal{B} = \{(a, b) : a < b, a, b \in \mathbb{R}\}$ is a basis and that it generates the usual topology on $\mathbb{R}$.

**Answer (Ex. 12)** — To show that $\mathcal{B}$ generates $\mathbb{R}_{\text{usual}}$, it suffices to construct an arbitrary open set $U \subseteq \mathbb{R}_{\text{usual}}$ through the unions of elements of $\mathcal{B}$. If $U$ is connected, then:

$$U = (x - \epsilon, x + \epsilon)$$

so $U$ is simply an element of $\mathcal{B}$. If $U$ is not connected, then simply repeat the process for every disconnected subset of $U$. Thus $\mathcal{B}$ generates $\mathbb{R}_{\text{usual}}$.

* **Ex. 13** — Show that the collection $\mathcal{B}_Q := \{(a, b) : a, b \in \mathbb{Q}, a < b\}$ is a basis for the usual topology on $\mathbb{R}$.

**Answer (Ex. 13)** — It suffices to show we can construct an open interval from $B_Q$ where $U = (d, c)$ such that $c \in \mathbb{Q}^c$. Let $\overline{a} = \{a_1, a_2a_1, a_3a_2a_1, \ldots\}$ be the sequence of decimal expansion of $c$ where each $a_k \in \mathbb{Q}$. Then $\overline{a} \to c$ and if $c^{'}$ is the floor of $c$:

$$\bigcup_{k \in \mathbb{N}} (d, c^{'}.a_k a_{k-1} \ldots a_1) = (d, c)$$

* **Ex. 15** — If $X$ is a finite set, with $|X| = n$, then $\mathcal{B} = \{\{x\} : x \in X\}$ has $n$ elements. Is there a basis with fewer than $n$ elements that generates the discrete topology on $X$?

**Answer (Ex. 15)** — Let $|B| < n \in \mathbb{N}$. Then $|\mathcal{P}(\mathcal{B})| < 2^n$. This implies that if $\mathcal{B}$ has less than $n$ elments, then maximum amount of sets it can generate for a topology is less than $2^n$. Since $|X_{\text{discrete}}| = 2^n$, then $\mathcal{B}$ cannot generate $X_{\text{discrete}}$ if $|\mathcal{B}| < n$