MATB43: Introduction to Analysis
Lecture Notes

Anmol Bhullar

Undergraduate at the University of Toronto, Scarborough
E-mail address: anmol.bhullar@mail.utoronto.ca

Abstract. Shamelessly Stolen from the course description of MATB43 on the 2016/2017 UTSC Calendar:
Generalities of sets and functions, countability. Topology and analysis on the real line: sequences, compactness, completeness, continuity, uniform continuity. Topics from topology and analysis in metric and Euclidean spaces. Sequences and series of functions, uniform convergence.

Contents

Chapter 1

Administration Details

1.1. Pre-requisites

- MATA37: Calculus for Mathematical Sciencies II
- MATB24: Linear Algebra II
- Core-requisite - MATB42: Multivariable Calculus II

1.2. Professor

- Name: Raymond Grinnell
- Office: IC 466
- Office Phone Number: (416) 287 - 5655 (Please do not call his home phone)

1.3. Resources

- Real Analysis and Foundations (3^{rd} ed.) by Steven G. Kratz
- Class Website: http://www.math.utsc.utoronto.ca/b43/

1.4. Grading Scheme

- There are five tutorial quizzes with the lowest one dropped. This accounts for 18% of your grade.
- There are weekly practice problems posted on the class website. These are just to prepare for the quizzes and/or practice your skills. These account for 0% of your grade.
- There are two lecture assignments worth a total of 7% each. In two random lectures for this course, the professor will post problems

that are to be handed in for marking in the next lecture. This is to ensure that your mathematical writing is up to what is expected of this course.

- There is one midterm worth 30%.
- There is a final exam worth 45%.

Part 1

Sets and Number Systems

Chapter 2

Sets and their Operations

2.1. What is a Set?

Perhaps the simplest, most common and most useful object in mathematics is a set. Thus, one of the first thing a mathematician must learn - that is, if they ever hopes to learn math in a rigorous fashion - is some basic set theory. Unfortunately, despite a set being one of the simplest objects in mathematics, talking about it in a very rigorous fashion is a very challenging endeavour that is most commonly taken up at the graduate or senior undergraduate level. Fortunately, most of mathematics is content with taking the intuitive definition of a set. We introduce it now:

Definition 2.1. A **set** is a collection of arbitrary elements. If α is a set, and $\alpha_1, \alpha_2, \alpha_3$ are its elements, then we say $\alpha = \{\alpha_1, \alpha_2, \alpha_3\}$.

2.2. Listing a Set and Some Useful Sets

A set can contain finite or infinite elements. One example of a contains infinite elements is the set of all natural numbers or counting numbers. We denote this as:

$$\mathbb{N} = \{1, 2, 3, \ldots\}$$

The integers also form a set. We denote them as:

$$\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$$

Sometimes, it is not possible to explicitly list the elements of a set, instead we write the condition that an arbitrary element has to fullfill in order to be a part of the said set. For example:

$$\mathbb{Q} = \{\frac{p}{q} : p \in \mathbb{Z},\ q \in \mathbb{Z}\}$$

is the set of the rational numbers. Another example is the set:

$$\mathbb{R} = \{\text{Set of Infinite Decimal Expansion with the Rule} \\ i.999\ldots = i + 1 \text{ where } i \in \mathbb{Z}\}$$

This is the set of real numbers. A final, and also much simpler to comprehend example is the set:

$$\{x : x \geqslant 0, x \in \mathbb{Z}\}$$

which is the set of *non-negative* integers.

Definition 2.2. We denote the set of no elements by $\varnothing$ and call it the **empty set**.

If you have trouble believing or simply refuse to accept that such a set can exist, it is still very useful to have some notation that can represent an object with the properties of an empty set because it will be widely used in this text.

2.3. Some Common Set Notation

Definition 2.3. If a set α contains an element α_1, we say that $\alpha_1 \in \alpha$, and if α_1 is not in α, we write $\alpha_1 \notin \alpha$.

Definition 2.4. If every element in α is also in β, and every element of β is also in α, we say that, $\alpha = \beta$. If every element of the set α is contained in a set β, we say that $\alpha \subset \beta$ as in "Alpha is a **subset** of Beta". Note that, $\alpha = \beta$, if and only if $\alpha \subset \beta$ and $\beta \subset \alpha$. Consider sets α, β where $\alpha \subset \beta$. If we need to explicitly state that every element of α is in β but not vice verse, then we may write $\alpha \subseteq \beta$ and say that α is a **proper subset** of β. That being said, $\subset$ does not necessairly mean two sets are not proper subsets, however, it may be different in other texts.

Some common subsets of $\mathbb{R}$ that we will see in the text are *closed intervals* or *open intervals.*

Definition 2.5. We say a set I is a **closed interval** if it can be written in the form $\{x : \alpha \leqslant x \leqslant \beta,\ x \in \mathbb{R}\}$ where α and β are two real numbers such that $\alpha \leqslant \beta$. These sets are denoted as:

$$[\alpha,\ \beta]$$

Definition 2.6. Similarly to the definition above, we say that a set I is an **open interval** if it can be written in the form $\{x : \alpha < x < \beta,\ x \in \mathbb{R}\}$. These sets are denoted as:

$$(\alpha,\ \beta)$$

Remark 2.7. The notation used for the above two sets is very malleable. For example, $\{x : \alpha < x \leqslant \beta\}$ can be denoted as $(\alpha, \beta]$. Similarly, $\{x : \alpha \leqslant x < \beta\}$ can be denoted as $[\alpha, \beta)$

Remark 2.8. The set of one element α (given that α is in $\mathbb{R}$) can be written as a closed interval since it is in the form $\{x : \alpha \leqslant x \leqslant \alpha,\ x \in \mathbb{R}\}$.

2.4. Set Operations

Perhaps the most common operations on sets are *unions*, *intersections* and *cartesian product*. We define them now.

Definition 2.9. Let $\alpha = \{\alpha_1, \alpha_2, \ldots\}$ and $\beta = \{\beta_1, \beta_2, \ldots\}$, then we say that their **union** is the set where

$$\alpha \cup \beta = \{x : x \in \alpha \text{ or } x \in \beta\}$$

Definition 2.10. Using the same sets as definition 1.5, we say that the **intersection** of α and β is the set

$$\alpha \cap \beta = \{x : x \in \alpha \text{ and } x \in \beta\}$$

Definition 2.11. Again, let use the same sets as definition 1.5. Then, we say that the **cross product** of α and β is the set:

$$\alpha \times \beta = \{(x, y) : x \in \alpha, y \in \beta\}$$

We will do some quick examples, then quickly generalize the two definitions above as that is where most of the interesting results and behaviours lie.

Example 2.12. The intersection of the null set and any set is the null set. Similarly, the union of a null set and any arbitrary set α is equal to α.

Example 2.13. $\mathbb{Z} \cap \mathbb{R} = \mathbb{Z}$. To generalize this, let us say that $\alpha \subset \beta$ where α, β are arbitrary sets. Then, $\alpha \cap \beta = \alpha$.

Example 2.14. $\mathbb{Z} \cup \mathbb{R} = \mathbb{R}$. To generalize this, let us say similarly to example 1.8, that $\alpha \subset \beta$, then $\alpha \cup \beta = \beta$.

Example 2.15. We can construct $\mathbb{R}^2$ via taking the cross product of $\mathbb{R}$ with itself. This is because any element in $\mathbb{R}^2$ can be explicitly written as (x, y) where $x \in \mathbb{R}$ and $y \in \mathbb{R}$. By the definition of cross product (1.8), we know that every element in $\mathbb{R} \times \mathbb{R}$ can be written in this form, thus $\mathbb{R} \times \mathbb{R} = \mathbb{R}^2$.

The reader is invited to prove any of the results above (except perhaps example 1.12 since there is nothing much to prove), it is fairly easy and just requires for one to apply the appropriate definitions.

2.5. Generalized Set Operations

Generalizing definitions 1.6 and 1.7 mostly involves just being able to take the intersections, unions or cross products of just more than two sets. To talk about this efficiently, we introduce the idea of a *collection of sets*.

Definition 2.16. We say $\mathcal{A}$ is a **collection of sets** if its elements are sets. That is, if $\alpha_1, \alpha_2, \alpha_3, \ldots$ are arbitrary sets, and $\mathcal{A} = \{\alpha_1, \alpha_2, \alpha_3, \ldots\}$, then we say that $\mathcal{A}$ is a collection of sets.

When we say something like $\alpha = \{\alpha_1, \alpha_2, \alpha_3, \ldots\}$, we are *labelling* arbitrary elements through the counting numbers, this allows us to very easily talk about a large amount of arbitrary elements. We generalize this process of labelling.

Definition 2.17. A set α whose elements can associated to another set β in that β's elements are *labelled* or *indexed* by the elements of α is called a **index set**.

Example 2.18. Consider, $\alpha = \{x : x > 0, x \in \mathbb{Z}\}$. Then, α seems like it is an index set which could easily replace the natural numbers. For example, let us suppose

$$\beta = \{\beta_1, \beta_2, \beta_3, \ldots\}$$

Then, if we associate -1 to be the same element 1 is associated to and so continue the same way for the rest of the numbers, then saying:

$$\beta = \{\beta_{-1}, \beta_{-2}, \beta_{-3}, \ldots\}$$

is completely accurate.

Definition 2.19. Let $\mathcal{A}$ be an arbitrary index set onto A which is an arbitrary collection of sets. Then, the **union of arbitrary sets** is defined to be:

$$\bigcup_{\alpha \in \mathcal{A}} A_\alpha = \{x : x \in A_\alpha\}$$

Definition 2.20. Using the same index set and collection as definition 1.16, we say that the **intersection of arbitrary sets** is:

$$\bigcap_{\alpha \in \mathcal{A}} A_\alpha = \{x : \forall \ \alpha \in \mathcal{A},\ x \in A_\alpha\}$$

While we can introduce the what it means to take the cross product of an arbitrarily amount of sets, writing it down with the language that we have now, it may become messy. Therefore, we will introduce it later on in the text.

Now we give some examples of some of the definitions above.

Example 2.21. Let $i \in \mathbb{Z}$, and $A_i = \{x : i \leqslant x \leqslant i + 1,\ x \in \mathbb{R}\}$. Then,

$$\bigcup_{i \in \mathbb{Z}} A_i = \mathbb{R}$$

Also,

$$\bigcap_{i \in \mathbb{Z}} A_i = \mathbb{Z}$$

Example 2.22. Let $I = \{x : x > 0,\ x \in \mathbb{Z}\}$ be an index set, then:

$$\bigcap_{i \in I} (-\frac{1}{i}, \frac{1}{i}) = \{0\} = [0]$$

Thus, we see that the *infinite* intersections of open intervals is not necessairly open.

Chapter 3

Fields

3.1. What is a Field?

Roughly put, a field is an abstract number system consisting of addition and multiplicative operations which work however one wants to and consisting of any elements which would otherwise represent numbers. The only limitation is that these rules and elements have to be consistent within itself, that is, we cannot do something which doesn't make sense i.e. obtaining $2 = 1$.

We now give this a rigorous definition.

Definition 3.1. A field F is a set on which two binary operations (denoted by $+$ and $\cdot$, called *addition* and *multiplication* respectively) are defined so that, for each pair of elements x, $y \in F$, there are unique elements $x + y$ and $x \circ y$ in F for which the following conditions hold for all elements a, b, c in F.

- $F1$ $a + b = b + a$ and $a \cdot b = b \cdot a$. This is known as *commutativity* of addition and multiplication.
- $F2$ $(a + b) + c = a + (b + c)$ and $(a \cdot b) \cdot c = a \cdot (b \cdot c)$.This is known as *associativity* of addition and multiplication.
- $F3$ There exists distinct elements denoted by 0 and 1 in F such that

$$0 + a = a \quad \text{and} \quad 1 \cdot a = a$$

  We call this the *existence of identity elements* for addition and multiplication.

$F4$ For each element a in F and each nonzero element b in F, there exist elements c and d in F such that

$$a + c = 0 \quad \text{and} \quad b \cdot d = 1$$

This is known as the *existence of inverses* for addition and multiplication.

$F5$ $a \cdot (b + c) = a \cdot b + a \cdot c$. This is known as the *distributivity of multiplication over addition.*

3.2. Examples of Fields

Example 3.2. The sets $\mathbb{Q}$ and $\mathbb{R}$ form a field over the normal rules of addition and multiplication.

Example 3.3. $\{a + \sqrt{2}b \in \mathbb{R} : a,\ b \in \mathbb{Q}\}$ forms a field over the "normal" rules of addition and multiplication operations that we see in $\mathbb{R}$ or $\mathbb{Q}$

Example 3.4. Not all fields have infinite elements, there are many examples of of what we call *finite fields*, that is fields which have a finite number of elements in them. Let $F = \{O,\ I,\ A,\ B\}$ with the operations

·	O	I	A	B
O	O	O	O	O
I	O	I	A	B
A	O	A	B	I
B	O	B	I	A

+	O	I	A	B
O	O	I	A	B
I	I	O	B	A
A	A	B	O	I
B	B	A	I	O

Then, we say F forms a field over the operations stated above. Since F is finite and contains 4 elements, we denote F by F_4.

Example 3.5. There are other examples of finite fields as well. For example, consider the set $F = \{x \in \mathbb{Z} : 0 < x < n\}$ (where $n \in \mathbb{N}$) but we define its operations differently than what is normally defined on $\mathbb{Z}$. We define addition as the sum of two numbers modulo n, or in other words, if $, x y \in F$, then their sum is defined as: $x + y \mod n$. Furthermore, multiplication is defined as, $x \cdot y \mod n$. Similarly, multiplication is defined as $x \cdot y \mod n$. Then F forms a field if n is prime. The reason why n has to be prime is because if n is not prime, then there will exist some elements which have no multiplicative inverse. For example, if $x \in F$ divides $y \in F$ or otherwise stated $(x = y \cdot m)$ for some $m \in F$ with $1 < y < x$, then y has no multiplicative inverse.

Example 3.6. If $q \in \mathbb{N}$, there exists a finite field with q elements if and only if $q = p^m$, $m \in \mathbb{N}$ where p is prime.

Below are some useful elements in a field, that we may need to refer to.

Definition 3.7. Let F be a field. Then,

(i) For $a \in F$, denote by $-a \in F$, the unique element such that $a + (-a) = 0$. For $a,\ b \in F$, define $a - b := a + (-b)$.

(ii) For $a \in F$, $a \neq 0$ denote $a^{-1} \in F$ the unique element such that $a \cdot a^{-1} = 1$. For $a,\ b \in F$, $b \neq 0$, define $\frac{a}{b} := a \cdot b^{-1}$.

3.3. Properties of Fields

We now state some properties which hold for any general field.

Theorem 3.8 (The Cancellation Law)**.** *Let F be a field. Then,*

$$\forall\, a,\ b,\ c \in \mathbb{F} : a + b = c + b \implies a = c \tag{3.1}$$

$$\forall\, a,\ b,\ c \in \mathbb{F},\ b \neq 0 : a \cdot b = c \cdot b \implies a = c \tag{3.2}$$

Proof. Let F be a field with $a, b, c \in F$. Then,

(1) By F4, there exists $-b \in F$, so consider: $a + b + (-b) = a = c + b + (-b) = c$.

(2) By F4, there exists $b^{-1} \in F$, so consider: $a \cdot b \cdot b^{-1} = a = c \cdot b \cdot b^{-1} = c$.

□

Theorem 3.9 (Uniqueness of Inverse Element)**.** *Let F be a field. Then,*

(1) $\forall\ a \in F$ *there is a* *unique* $b \in F : a + b = 0$

(2) $\forall\ a \in F,\ a \neq 0$, *there is a* *unique* $b \in F : a \cdot b = 1$

Proof. We leave the second part as an exercise due to similarity.

(i) Suppose $b,\ b' \in F$ with $a + b = 0 = a + b'$. By cancellation, we obtain $b = b'$

(ii) Exercise.

□

Theorem 3.10. *Let F be a field. Then if $a \in F$, $a \cdot 0 = 0$.*

Proof. Let $a \in F$. Consider:

$$\begin{aligned} a \cdot 0 &= a \cdot (0 + 0) && \text{By F3} \\ &= a \cdot 0 + a \cdot 0 && \text{By F5} \end{aligned}$$

So by F4 we have,

$$a \cdot 0 + (-a \cdot 0) = a \cdot 0 + a \cdot 0 + (-a \cdot 0)$$

which implies

$$0 = a \cdot 0$$

$\square$

Theorem 3.11. *Let F be a field. For any a, $b \in F$:*

$$a \cdot b = 0 \to a = 0 \text{ or } b = 0$$

Proof. Suppose $a \cdot b = 0$. If $a = 0$, we are done so assume $a \neq 0$. By $F4$, a^{-1} exists. Then consider,

$$\begin{aligned}
b &= 1 \cdot b \\
b &= (a^{-1} \cdot a) \cdot b \\
b &= a^{-1} \cdot (a \cdot b) \\
b &= a^{-1} \cdot 0 \\
b &= 0
\end{aligned}$$

$\square$

Theorem 3.12. *For $a, b, c, d \in F$ where F is a field, the following properties hold:*

(1) $-(-a) = a$
(2) $\frac{a}{b} = \frac{ac}{bd}$
(3) $(-a) \cdot b = -(a \cdot b)$
(4) $(a + b)(a - b) = a^2 - b^2$
(5) *If $a \neq 0$, then $(a^{-1})^{-1} = a$*

3.4. Ordered Fields

Definition 3.13. Let F be a field. Then F is an ordered field with order $\leqslant$ if and only if,

O1 For any $a, b \in F$: If $a \leqslant b$, then $a + c \leqslant b + c$.
O2 For any $a, b \in F$: If $0 \leqslant a$ and $0 \leqslant b$, then $0 \leqslant a \cdot b$.

Example 3.14. We define order on the real line using the concept of the number line. Similarly, order on the rational field is defined.

Remark 3.15. Consider a field where $\sum_{i=1}^{n} 1 = 0$ holds for some $n > 1$, then this field cannot be a ordered field. This follows from O1, specifically $1 < 1 + 1$.

We now describe some useful properties of ordered fields.

Theorem 3.16 (Properties of Ordered Fields)**.** *Suppose F is a ordered field with order $\leqslant$ and let $a, b, c, d \in F$. Then,*

1. *(Transitivity) If $a \leqslant b$ and $b \leqslant c$, then $a \leqslant c$.*
2. *(Summation of Inequalities) If $a \leqslant b$ and $c \leqslant d$, then $a + c \leqslant b + d$.*
3. *(Multiplication of an Inequality) If $a \leqslant b$, then $ac \leqslant bc$.*

Proof. $\square$

Chapter 4

The Real Numbers

4.1. Finding a Suitable Number System

In order to do advanced mathematics, we must have a solid foundations. This can mean a variety of things, but in this context, we want a "good" number system to perform math on. Let's consider the natural numbers to be a candidate for this.

One way number systems can be "nice" is that performing operations of arithmetics yields you an element within the set you started from. So first, we consider addition. Obviously, any two elements in the natural numbers added together will give you back a natural numbers. However, we are not so fortunate in the case for subtraction. For example, consider $1 - 2 = -1$.

This is not within $\mathbb{N}$. Since $1 - 2 \in \mathbb{Z}$, let us consider $\mathbb{Z}$ as the viable candidate. Addition, Subtraction seem to work as we want them to and we see that any two integers multiplied give you back an integer. This victory is shortlived however, when we consider the case of division such as $1 \div 2$.

Since $1 \div 2 \in \mathbb{Q}$, let us consider $\mathbb{Q}$ as our number system of choice. It seems to be that all four mathematical operations give results back in $\mathbb{Q}$, does that mean that we have concluded our search? One other "nice" property that our choice of number system should have is that it should be yield rational numbers when performing *limiting* operations on the rational numbers.

This is where we begin to see that we have a problem. For example, consider the set $\{x : x > 0 \text{ and } x^2 < 2\}$. It seems obvious that the upper limit of this set seems to be tending to some number x that fulfills the equation $x^2 = 2$ but as we will show, it turns out that this number is not rational, but instead *irrational* (not rational).

Theorem 4.1. *There is no $x \in \mathbb{Q}$ such that it fulfills the equation $x^2 = 2$.*

Proof. Suppose x is rational. Then there exists $a, b \in \mathbb{Z}$ where $b \neq 0$ such that $x = \frac{a}{b}$. Suppose that a and b have no common factors (If they do, then divide the factor). Then consider,

$$\begin{aligned} \frac{a}{b}^2 &= 2 \\ a^2 &= 2b^2 \end{aligned}$$

So that a^2 is even. But this can only be if a itself is even and that a^2 is divisble by 4. This in turn implies that b so that a and b are both even. This is a contradiction since we assumed they have no common factors. □

This shows that there "holes" within the rational number system so we then use a number system which includes "holes" (i.e. numbers which fulfill equations such as $x^2 = 3$, $y^2 = 5$, $z^2 = 7, \ldots$. This is not all of the holes in the rational numbers, but filling in all of the holes, we obtain the real number system $\mathbb{R}$. When we construct the real numbers, we will show that we have indeed filled in all of the holes of the rational numbers.

4.2. Least Upper Bound Property of the Reals

Chapter 5

Functions

Functions are yet another one of math's most common objects, they permeate through almost every inch of mathematics. In this chapter, we introduce them, and then talk about some of their properties, and then talk about some special functions which are useful.

In order to define what a function is, we must first define something a bit more primitive.

5.1. Relations

Definition 5.1. A **relation** on a set A is a subset C of the cartesian product $A \times A$. We can also define relations on sets formed through cartesian products of two different sets, in that a subset of $A \times B$ is still a relation.

We can also equivalently say that a relation is a collection of elements of the form (x, y) where $x \in A$ and $y \in A$. Elements of the form (x, y) are called **ordered pairs**.

The following is an example of a relation from James Munkres' textbook *Topology*.

Example 5.2. Let P denote the set of all people in the world, and define $D \subset P \times P$ by the equation

$$D = \{(x, y) : x \text{ is a descendant of } y\}$$

Then, D is a relation on the set P.

Example 5.3. Let $A = \{x : x = y^2,\ x \in \mathbb{R}\}$ and then let $D = \{(x, y) : y \in A,\ x \in \mathbb{R}\}$. It follows that $D \subset \mathbb{R} \times \mathbb{R}$, thus we obtain that D is a relation under A.

A function ends being just a special type of subset a cartesian product, in that a function is a relation but not all relations are functions.

Definition 5.4. A relation f of a set A and B is a **function** if and only if each element in the first component of an ordered pair of f once and only once.

This definition may be a bit hard to grasp, so consider the figure below.

`function.pdf`

This represents a relation $f = \{(1, T), (2, R), (3, Q)\}$

Since each element in the first component of every ordered pair in f (in this case is 1, 2 and 3) appear once and only once in the elements of f, we have that f is a function. Now consider,

`notFunction.pdf`

This represents a relation $g = \{(1, T), (1, Q), (2, R), (3, Q)\}$

This is *not* a function because 1 appears in two elements of f. Example 2.3 is another example (for the lack of a better term) of a relation which is not a function.

5.2. Properties of a Function

We define some terminology so that it becomes much easier for us talk about the different characteristics of a function.

Definition 5.5. Let f be a function. An equivalent way of saying that the ordered pair (x, y) is a part of f, we can say that $f(x) = y$. In this case, $f(x)$ is a **value** of f and y is the **image** of $f(x)$.

Definition 5.6. We write $f : A \to B$ to mean that $f \subset A \times B$. In this case, X is the **domain** of f, and Y is the **range** of f.

Definition 5.7. $f(X)$ is defined to be the **image set** of f. The **pre-image set** of f is defined to be:

$$f^{-1}(Y) = \{x \in X : \ f(x) \in Y\}$$

Remark 5.8. Let f be a function such that $f : A \to B$. By the definition of a relation, we know that f is supposed to be a subset of the set $A \times B$. If f is a proper subset of $A \times B$, then its pre-image is not the same as its domain. However, if it is not a proper subset, and instead $f = A \times B$, then the pre-image and the domain are the same sets. In this way, we can also always guarantee that the pre-image set is a subset of the domain.

Remark 5.9. Suppose f is a function where $f : \mathbb{R} \to \mathbb{R}$. Often times instead of describing every element in f, we may write an equation which gives a *rule* for every element in the pre-image to follow. For example,

$$f(x) = x^2$$

means that we take an arbitrary element from the pre-image of f and square it. So, $f = \{(x, x^2) : x \in \mathbb{R}\}$

Definition 5.10. Let f be a function $f : A \times B$. If $C \subset A$, then $f : C \to B$ is still a function and we say that f is **restricted** to the domain of C. This new relation is most commonly denoted by $f|_C$

5.3. Special Functions

Some very special types of functions that we will be working are now listed. Their usefullness is perhaps best realized via through the sheer number of times we are going to be talking about such functions in the upcoming chapters.

Definition 5.11. We say that a function f is **one-to-one** if whenever $x = y$ (and $x \in$ domain of f), then $f(x) = f(y)$. If a function exists between any two sets such that it is one-to-one, we write $A \sim B$ and say that they are *equinumerous* or *have the same cardinality.*

Conceptually, this means that given an element from the pre-image of f, it will only be associated with at most one element of the image of f.

Example 5.12. $(0, \infty)$ and $\mathbb{R}$ have the same cardinality. Consider the function $f(x) = \log x$ (base e). Then f is one-to-one so we have that $(0, \infty) \sim \mathbb{R}$. Note that, we could have picked any function from $\mathbb{R} \to \mathbb{R}$ which is one-to-one and whose image is $(0, \infty)$. Alternatively if we looked for a function $f : (0, \infty) \to \mathbb{R}$, then we would be looking for a function whose domain is instead on $(0, \infty)$ (or can be restricted to such a domain). In this case, $\log x$ works in both cases.

Example 5.13. Let $f : \mathbb{R} \to \mathbb{R}$ be a function. Then, $f(x) = x$ is a one-to-one function because if we are given that $x = y$, then it follows $f(x) = x = y = f(y)$.

Remark 5.14. One of the reasons one-to-one functions are interesting is because if a function is one-to-one, then we know that an *inverse* of that function exists. For example, let f be a one-to-one function. Then, since only one element from its pre-image is mapped to its image, we can create a another function f^{-1} which is the reverse mapping of f. This allows us to essentially *reverse* the effects of f on a set.

Now, we introduce the other special function, called the *surjective* function.

Definition 5.15. Let $f : A \to B$. If $f(A) = B$, then we say that f is **surjective** or **onto**.

Example 5.16. The function f in example 2.12 is also an example of a surjective function, in this case due to how the function is defined, it follows quite trivially but other functions may be more involved.

Definition 5.17. A function which is injective and surjective is said to be **bijective**.

Chapter 6

Countability and Cardinality

Countability is a way of "organizing" sets of infinite elements in a that makes them easier to work with. This is very vague so perhaps best of understanding such a concept is by diving straight into it.

6.1. Countability

Definition 6.1. Let $Z_n = \{1, 2, \ldots, n\}$. If there exists a one-to-one function between A and Z_n (for some $n \in \mathbb{Z}$) arbitrary set A, then we say that A is **finite**. If some one-to-one correspondence does exist, we write $A \sim Z_n$. If A is not finite, we say it is **infinite**.

Remark 6.2. From the definition above, it follows that if there is a one-to-one correspondence between two finite sets A and B, then A and B must contain the same number of elements. This follows from the fact that if they did not, then the "extra" element would have to be mapped to an element on the other set which is already being mapped to, thus the function loses its property of being one-to-one.

Definition 6.3. Suppose A is finite, then we can create a bijective function $f : A \to J_n$ for some $n \in \mathbb{N}$. In this case, we say that A **has cardinality** n.

Remark 6.4. It is customary to say that the empty set is finite.

Definition 6.5. Let A be an arbitrary infinite set. If there exists a bijective mapping between A and $\mathbb{N}$, we say that A is **countable**. If A is not countable, we say it is **uncountable**. If A is countable, then we write $A \sim \mathbb{N}$.

Thus notion of countability helps us talk about sets being the same "length" or "size" even when we're dealing with infinite sets. In a way, countable sets represent the "smallest" infinite sets if you believe that the set of natural numbers is the smallest infinite set. We list some examples in order to gain a better understanding about the notion of countability.

Example 6.6. Let A be the set of even numbers, then $A \sim \mathbb{N}$. Since we can explicitly write the elements of A, namely

$$A = \{2, 4, 6, 8, \ldots\}$$

Let $f : A \to \mathbb{N}$ such that 2 in A maps to 1 in N, 4 in A maps to 4 in $\mathbb{N}$, 6 in A to 3 in $\mathbb{N}$, and so on. This pattern seems simple enough that we may be able to capture it with an explicit formula. Thus, consider:

$$f(x) = \frac{x}{2}$$

Since f is bijective (an easy exercise), we have by definition that $A \sim \mathbb{N}$.

Example 6.7. The set of integers is countable. In order to find a suitable function $f : \mathbb{Z} \to \mathbb{N}$, consider:

$$\mathbb{Z} : 0, 1, -1, 2, -2, 3, \ldots$$
$$\mathbb{N} : 1, 2, 3, 4, 5, 6, \ldots$$

Intuitively, it should then be clear how the set of integers is countable. However, we must show this rigorously. Looking at the two sequences for patterns, we realize that every even number in $\mathbb{Z}$ is multiplied by 2 to obtain a corresponding number in $\mathbb{N}$. The other numbers in $\mathbb{Z}$ seemed to be obtained by multiplying by -1 to the even numbers in $\mathbb{Z}$ (the exception being 0). So, we take take the odd numbers in $\mathbb{N}$, subtract by 1, divide by 2, to obtain the same even number, then multiply by -1. Thus:

$$f(x) = \begin{cases} \frac{x}{2}, \ x \text{ even} \\ -\frac{x-1}{2}, \ \text{otherwise} \end{cases}$$

This is a bijective function, thus, we have that $\mathbb{Z} \sim \mathbb{N}$.

Example 6.8. The set of reals is not countable.
To prove this we will use the fact that there is a bijective correspondance between real numbers in base 2 (the binary numbers) real numbers in and base 10. So suppose A is the set of all real numbers represented in binary. Let E be a countable subset of A. We show that $E \subsetneq A$ so that $E \neq A$.

We can also define infinite sets without resorting to finite sets.

Remark 6.9. Let A be an arbitrary set. If there exists a set B such that $B \subsetneq A$, and $A \sim B$, then A must be infinite. This follows from the fact that if two finite sets have a one-to-one correspondence, then they must be equal. We note that a much stronger statement can be made - namely that A is infinite if and only if there exists a set $B \subsetneq A$ and $A \sim B$ but proving the other part of this statement requires us to be familiar with the axiom of choice, something which we probably not cover in this text.

Example 6.10. It turns out there is a nice relationship between a countable set and its subsets. Consider for example the set of natural numbers $\mathbb{N}$. Consider,

$$A = (n, \infty) \subset \mathbb{N} \qquad n \in \mathbb{N}$$

Then, if we have a function $f : A \to \mathbb{N}$ such that $f(x) = x - (n - 1)$, then f is one-to-one, so A is countable. Therefore, we see that every subset of $\mathbb{N}$ is countable. This is of course assuming that the subsets are not finite, for if it were not finite, then we could not create such a function f. We can generalize this relationship to any countable set.

Theorem 6.11. *For every infinite subset B of a countable set A, we have that $A \sim \mathbb{N}$.*

Proof. Suppose that $B \subseteq A$ where A is countable. By the definition of a countable set, we know there exists a function $f : A \to \mathbb{N}$ which is one-to-one. Then $f|_B$ is also one-to-one since we have not changed where the elements are being mapped, we have only changed the amount of elements being mapped. Thus there exists a function $f|_B : B \to \mathbb{N}$ which is one-to-one, so by definition B is countable. □

We give a term for sets that may be either countable or finite in case they come up again.

Definition 6.12. Let A be any arbitrary set such that it is finite or countable, then we say that A is **at most countable**.

6.2. Sequences

We make a detour to discuss the notion of a *sequence* - a very important tool for solving questions in mathematics, and they can do some pretty interesting mathematics in and of themselves.

Definition 6.13. Consider a function f such that $f \sim \mathbb{N}$. Then, we say that (a_n) is a **sequence** where:

$$(a_n) := (a_1, a_2, \ldots, a_n, \ldots)$$

and a_n is equal to $f(n)$. In general, we don't have to use the letter a, we can use any symbol we want as long as the term is clearly noted with a subscript of a natural number.

Remark 6.14. We can define a much more general notion of a sequence by requiring that a function have a one-to-one correspondence with just an index set and not necessairly the natural numbers. In this case, we would have to be explicit with the fact that we are using a different indexing set. For example, consider a sequence indexed by the set of even numbers I, then this sequence can be represented by $(a_n)_{n \in I}$.

6.3. Arithmetic of Countable and Uncountable Sets

In this section, we discuss the effects of arithmetics on countable and uncountable sets. By arithmetics, we do not mean addition or multiplication, but rather the operations of unions, intersections and cross products. We begin by talking *finite* operations on countable / uncountable sets, as in *what happens when you take the union of arbitrary finite amount of countable sets.*

Theorem 6.15. *Let S be a collection of sets such that its elements are countable. Let I be a non-empty finite index set (an index set which contains finite elements - an example being J_n.) Then,*

$$\bigcup_{i \in I} S_i \quad \textit{is countable}$$

Proof. Let (S_n) be a sequence of countable sets. Define:

$$S = \bigcup_{n \in \mathbb{N}} S_n$$

For all $n \in \mathbb{N}$, let $\mathcal{F}_n$ be the set of all surjections from $\mathbb{N}$ to S_n. Since S_n is countable and we have a surjection from the natural numbers, then S_n □

(intersections of countable sets)

(unions of countable sets)

(cartesian products of countable sets)

(rational numbers are countable)

(cardinality)