MATD11: Functional Analysis
Assignment 3

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Preface.

We may say $x_n \to x$ to say that $(x_n)_{n=1}^{\infty}$ is a sequence which converges to x. Instead of writing $(x_n)_{n=1}^{\infty}$, we may just say $(x_n)_1^{\infty}$ or even just (x_n) where context is clear. We may say fx instead of $f(x)$ for any operator f.

P1.

Suppose M is a dense subspace in a Banach space X (meaning that the closure of M is all of X) and suppose that $T : M \to Y$ is linear, where Y is a Banach space, with $\|Tm\|_Y \leq K \|m\|_X$ for some $K < \infty$ and all $m \in M$. Show that T extends, in a unique way, to a bounded linear operator from X into Y.

Solution.

Note $\overline{M} = X$. Thus, by definition, we have that for all $x \in X$, there exist some sequence $(x_n)_1^{\infty}$ in M such that $x_n \to x$. Using this, define a mapping $T' : X \to Y$ by $T'(x) = \lim_{n\to\infty} T(x_n)$. Supposing this mapping is well defined, it is clear that T' is then a mapping from X to Y because Y is a Banach space so if $\lim_n(T(x_n))$ converges, it converges to a point in Y and $\lim_n(T(x_n))$ converges since T is a continuous map (prop. 2.2). Note our definition of T' does not depend on our choice of (x_n). To see why choose sequences $x_n \to x$ and $y_n \to x$ such that both $(x_n)_1^{\infty}$ and $(y_n)_1^{\infty}$ lie in M. Using uniqueness of limits in a Banach space (every Banach space is a metric space which are always Hausdorff), we know:

$$\lim_{n\to\infty} T(x_n)$$

only converges to one point, which is $T'(x)$. Thus $x \mapsto T'(x)$ is actually a function. Furthermore,

$$\lim_{n\to\infty} T(x_n) = T'(x) \qquad \text{and} \qquad \lim_{n\to\infty} T(y_n) = T'(x)$$

is always true. Thus, we have that $\lim_{n\to\infty} T(x_n) = \lim_{n\to\infty} T(y_n)$ which implies the value of $T'(x)$ is independent of our choice of sequence and so T' is a well defined function (i.e. does not depend on our choice of sequence).

Now, we want to show that T' is a *continuous* extension of T. In order to do this, it suffices to show $T'|_M = T$ is true and T' is continuous on $X \backslash M$. $T'|_M$ is clearly a map from $M \to Y$ (recall $T' : X \to Y$) where $T'|_M(x \in M) = \lim_{n\to\infty} T(x_n)$ for some sequence $x_n \to x$ in M. Simply, choose the constant sequence $(x)_1^\infty$ since $x \in M$. Clearly, this converges to x. Then, note $\lim_{n\to\infty} T(x) = T(x)$ so that $T'|_M(x) = T(x)$ as wanted. It is left to show T' is continuous on $X \backslash M$.

Fix any $x \in X \backslash M$. We want to show:

$$\forall\, \epsilon > 0,\ \exists\, \delta > 0 \text{ such that if } 0 < |x - y| < \delta, \text{ then } |f(x) - f(y)| < \epsilon$$

Thus, choose $\epsilon > 0$, then for $\delta = \epsilon/K$ (if $K = 0$, then T is identically 0 (recall T is bounded by K) so that T' is identically zero because each $T'(x)$ would then be a limit of zero's. Since T' is then identically 0, it is constant, therefore continuous), if $0 < |x - y| < \delta$, then:

$$\begin{aligned}
\|T'(x) - T'(y)\|_Y &= \left\| \lim_{n\to\infty} T(x_n) - \lim_{n\to\infty} T(y_n) \right\|_Y && \text{where } x_n \to x,\ y_n \to y \\
&= \left\| \lim_{n\to\infty} [T(x_n) - T(y_n)] \right\|_Y && \text{linearity of the limit operator} \\
&= \left\| \lim_{n\to\infty} T(x_n - y_n) \right\|_Y && \text{linearity of } T \\
&= \lim_{n\to\infty} \|T(x_n - y_n)\|_Y && \text{continuity of } \|\cdot\| \\
&= \lim_{n\to\infty} K \|x_n - y_n\|_X < \lim_{n\to\infty} K \cdot \delta = \epsilon
\end{aligned}$$

so that T' is continuous at x and since x was arbitrarily chosen, we have that T' is continuous on $X \backslash M$.

In order to show T' is a bounded linear operator from $X \to Y$, we first to show T' is linear. Choose a, b scalars and $x, y \in X$. Then, we show $T'(ax+by) = aT'(x)+bT'(y)$. We know since $x \in X$ and $\overline{M} = X$, then there exists a sequence $x_n \to x$ and similarly $y_n \to y$, so that $ax_n \to ax$ and $by_n \to b$. In particular, $ax_n + by_n \to ax + by$ by elementary sequence properties. Therefore, we write:

$$T'(ax_n + by_n) = \lim_{n\to\infty} T(ax_n + by_n)$$

By linearity of T, $T(ax_n + by_n) = aT(x_n) + bT(y_n)$. Since $x_n \to x$, then $T'(x) = \lim_n T(x_n)$ and similarly, $\lim_n T(y_n) = T(y)$. Thus by linearity of the limit operator:

$$\begin{aligned}
T'(ax_n + by_n) &= \lim_{n\to\infty} [aT(x_n) + bT(y_n)] \\
&= a \lim_{n\to\infty} T(x_n) + b \lim_{n\to\infty} T(y_n) \\
&= aT'(x) + bT'(y)
\end{aligned}$$

so that T' is linear. Furthermore, since $T'|_M = T$, $T'|_M$ is bounded and so it is continuous (prop. 2.2). Also, since if x not in M, then it is already shown that T' is continuous on $X \backslash M$. Therefore, T' is continuous on both M and $X \backslash M$ implying that T' is continuous everywhere on X and so it is bounded on X (prop. 2.2).

Therefore, $T' \in \mathcal{B}(X, Y)$. Finally, T' extends T in a unique manner. Consider two distinct (don't agree on at least 1 point) maps T' and $T^{(2)}$ which extend T in a continuous manner such that it is linear and bounded. Then, since $T^{(2)}$ is continuous, we have that

$$\lim_{n\to\infty} x_n = x \implies \lim_{n\to\infty} T^{(2)}(x_n) = T^{(2)}(x)$$

So, choose some sequence $x_n \to x$ lying in M ($x \in M$ not required). Then since $T^{(2)}$ extends T, we have that $T(x_n) = T^{(2)}(x_n)$, so we have that

$$x_n \to x \implies \lim_{n\to\infty} T(x_n) = T^{(2)}(x)$$

but this is percisely the definition of T', so we obtain: $T'(x) = T^{(2)}(x)$. Since x is arbitrary, but T' and $T^{(2)}$ must differ on at least one point, we have a contradiction. Thus, $T^{(2)}$ does not exist and T' is then a uniquely defined function.$\blacksquare$

P2.

Let $\Lambda : X \to \mathbb{C}$ is a bounded linear functional on a normed linear space X. Recall that $\|\Lambda\|$ is defined as $\sup\{|\Lambda(x)| : \|x\| \leq 1\}$. Show that

$$\begin{aligned} \|\Lambda\| &= \sup\{|\Lambda(x)| : \|x\| = 1\} \\ &= \sup\{|\Lambda(x)/\left\|x\right\||: x \neq 0\} \end{aligned}$$

Solution.

Let $\Lambda_1 = \sup\{|\Lambda(x)| : \|x\| = 1\}$ and $\Lambda_2 = \sup\{|\Lambda(x)/\left\|x\right\|| : x \neq 0\}$. First note, $\{|\Lambda x| : \|x\| = 1\} \subseteq \{|\Lambda x| : \|x\| \leq 1\}$ and since the supremum of both sets exist, we obtain that $\Lambda \geq \Lambda_1$. Now, note for all $x \in X$ (not 0), $x/\left\|x\right\|$ has norm 1 and by linearity of Λ:

$$|\Lambda(\frac{x}{\|x\|})| = |\frac{1}{\|x\|}\Lambda(x)| = \frac{|\Lambda(x)|}{\|x\|}$$

Thus, we see that $\{|\Lambda(x)/\left\|x\right\|| : x \neq 0\}$ consists of vectors $|\Lambda(y)|$ where $\|y\| = 1$. Thus, $\{|\Lambda(x)/\left\|x\right\|| : x \neq 0\} \subseteq \{|\Lambda(x)| : \|x\| = 1\}$. As before, we get that $\Lambda_2 \leq \Lambda_1$. So far, we have that $\Lambda \geq \Lambda_1 \geq \Lambda_2$. To obtain that $\Lambda = \Lambda_1 = \Lambda_2$, it suffices to show that $\Lambda_2 \geq \Lambda$. Note, for $x \neq 0$, we have $|\Lambda(x)/\left\|x\right\|| \leq \Lambda_2$ and so $|\Lambda(x)| \leq \Lambda_2 \left\|x\right\|$ which implies Λ_2 is an upper bound of $\{|\Lambda(x) : \|x\| \leq 1\}$ and since Λ is the *least* upper bound, it follows that $\Lambda_2 \geq \Lambda$. Thus, we obtain

$$\Lambda = \Lambda_1 = \Lambda_2$$

as wanted for $x \neq 0$. If $x = 0$, we still have $\Lambda_2 \geq \Lambda$ since for $x = 0$, $\Lambda(x) = 0$ (recall $0 \mapsto 0$ in linear maps) so if the least upper bound is 0 and $\Lambda \geq \Lambda_2$, then it must be that $\Lambda_2 = 0$ so we still obtain the equality stated above regardless.$\blacksquare$

P3.

Let X, Y and V be normed linear spaces and let $A \in \mathcal{B}(X, Y)$ and $B \in \mathcal{B}(Y, V)$. Prove that $BA \in \mathcal{B}(X, V)$ and $\|BA\| \leq \|B\| \, \|A\|$.

Solution.

Choose any $x \in X$. Then $Ax \in Y$. Since $Ax \in Y$, then $B(Ax) \in V$. Since x arbitrary, we obtain that BA is a mapping $X \to V$. Choose $x, y \in X$ and scalars a, b. Then $BA(ax + by) = B(aA(x) + bA(y))$ by linearity of A and applying by linearity of B, we obtain: $BA(ax + by) = B(aA(x)) + B(bA(y)) = aB(A(x)) + bB(A(y)) = aBA(x) + bBA(y)$. Therefore, BA is a linear map. Now, note that $\|BAx\| = \|B(Ax)\| \leq \|B\| \, \|Ax\|$ and we know $\|B\|$ exists since $B \in \mathcal{B}(Y, V)$. Furthermore, $\|B\| \, \|Ax\| \leq \|B\| \, (\|A\| \, \|x\|)$ for similar reasons. So we obtain that:

$$\|BAx\| \leq \|B\| \, \|A\| \, \|x\|$$

which implies $\|BA\| \leq \|B\| \, \|A\|$ (recall x arbitrary). In particular, this implies $\|BA\|$ exists so that BA is a bounded linear operator. Thus, $BA \in \mathcal{B}(Y, V)$. $\blacksquare$

P4.

Let X be a Banach space. Let $\{A_n\}$ be a sequence in $\mathcal{B}(X)$ such that $\sum_{n=1}^{\infty} \|A_n\|$ converges. Prove that the series $\sum_{n=1}^{\infty} A_n$ converges to an operator $A \in \mathcal{B}(X)$ and $\|A\| \leq \sum_{n=1}^{\infty} \|A_n\|$.

Solution.

Define $Ax = \sum_{n=1}^{\infty} A_n(x)$. Claim: $A_n \to A \in \mathcal{B}(X)$. Therefore, the first property we show is that $\sum_{n=1}^{\infty} A_n(x)$ converges, i.e. Ax exists. Recall Cauchy Criterion for series, the sequence of partial sums of $\sum A_n x$ must be Cauchy. We prove this. Fix $x \in X$. $\sum_{n=1}^{\infty} \|A_n\|$ converges so, for any choice of $\epsilon > 0$, there exists N such that if $n' > n > N$, then $|\sum_{m=n}^{n'} \|A_m\| \, | < \epsilon$. From this, we get our choice of N to prove $\sum A_n x$ is Cauchy. Observe: Pick an $\epsilon' > 0$, then pick an $N > 0$ so that $|\sum_{m=N}^{N'} \|A_m\| \, | < \epsilon' / \|x\|$ for $N' > N$. Then for any $n' > n > N$:

$$\left\| \sum_{m=n}^{n'} A_m x \right\| \leq \sum_{m=n}^{n'} \|A_m(x)\|$$

by the triangle inequality. Also, $\|A_m(x)\| \leq \|A_m\| \, \|x\|$ since $A_m \in \mathcal{B}(X)$. Thus:

$$\left\| \sum_{m=n}^{n'} A_m x \right\| \leq \sum_{m=n}^{n'} \|A_m\| \, \|x\| = \|x\| \, [\sum_{m=n}^{n'} \|A_m\|] \tag{1}$$

Since the sum in the square bracket is less than $\epsilon' / \|x\|$, we have that:

$$\left\| \sum_{m=n}^{n'} A_m(x) \right\| < \epsilon'$$

so that the sequence of partial sums Cauchy converges. X is a Banach space so this sum converges to an element of X, thus Ax exists. It is left to show $A \in \mathcal{B}(X)$. To show $\|A\|$ exists, recall each $A_n \in \mathcal{B}(X)$ and consider:

$$\begin{aligned} \|A(x)\| = \left\| \lim_{n\to\infty} \sum_{i=1}^{n} A_n(x) \right\| &= \lim_{n\to\infty} \left\| \sum_{i=1}^{n} A_n(x) \right\| \\ &\leq \lim_{n\to\infty} \sum_{i=1}^{n} \|A_n(x)\| \\ &\leq \lim_{n\to\infty} \sum_{i=1}^{n} \|A_n\| \, \|x\| < \infty \end{aligned}$$

since $\sum_{i=1}^{\infty} \|A_i\|$ converges so a constant $\|x\|$ multiplied by the sum should also converge, thus, $\|x\| \, (\sum_{i=1}^{\infty} \|A_i\|) = \sum_{i=1}^{\infty} \|A_i\| \, \|x\|$ should also converge. (note we actually used absolute convergence here). Thus,

$$\|Ax\| \leq \sum_{i=1}^{\infty} \|A_i\| \, \|x\| < \infty$$

which implies that $\|A\|$ exists so that A is bounded. In particular, it shows $\|A\| \leq \sum_{i=1}^{\infty} \|A_i\|$. Now, we show that A is linear. This is done simply through definition, using the fact that each A_n is linear. Consider for scalars a, b and vectors $x, y \in X$:

$$\begin{aligned} A(ax+by) &= \sum_{i=1}^{\infty} A_i(ax+by) \\ &= \lim_{n\to\infty} \sum_{i=1}^{n} A_i(ax+by) \\ &= \lim_{n\to\infty} \sum_{i=1}^{n} [aA_i(x) + bA_i(y)] \\ &= a[\lim_{n\to\infty} \sum_{i=1}^{n} A_i(x)] + b[\lim_{n\to\infty} \sum_{i=1}^{n} A_i(y)] \\ &= aA(x) + bA(y) \end{aligned}$$

as wanted. Thus, $A \in \mathcal{B}(X)$. ∎

P5.

Let X be a Banach space and let $A \in \mathcal{B}(X)$. Explain how to define e^A and prove that $e^A \in \mathcal{B}(X)$.

Solution.

Define $e^A : X \to X$ by $e^A = I + A + 1/2!A^2 + 1/3!A^3 + \ldots = \sum_{n=0}^{\infty} A^n/n!$ where I (or A^0) is the identity mapping and A^n is the composition of A with itself n times. As is usually the case, we first show e^A exists, i.e. the infinite sum converges to some element of $\mathcal{B}(X)$. Define the sequence $(e_n^A)_{n=1}^{\infty}$ to be the nth term in the sum e^A. By problem 3 and induction, we obtain that A^n is an operator in $\mathcal{B}(X)$. Since I is the identity map, it is clearly bounded and linear so $I \in \mathcal{B}(X)$. Note since $\mathcal{B}(X)$ is a vector space, we have that for any scalar a, and an operator A, $aA \in \mathcal{B}(X)$ and also that $\sum_{i=1}^{k} A^i \in \mathcal{B}(X)$ for all k (VS. closed over scalar mult. and vector addition). Combining all these facts, we get that for every k, $\sum_{i=1}^{k} e_i^A \in \mathcal{B}(X)$. Before going further, We show $\sum_{n=1}^{\infty} \left\|e_n^A\right\|$ converges. We prove this via the technique of Cauchy Criterion as was done in the last problem. Choose $\epsilon > 0$. Recall that the factorial function $n \mapsto n!$ increases much faster than a exponential function $x \mapsto e^x$, thus $\lim_{n\to\infty} \|A\|^n/n! = 0$. In fact, we can say for $N > 0$ such that $N! > 2^N$ (this implies $n! > 2^n$ for subsequent $n > N$), there exists some $N' > N$ such that for all $n > N'$: $\|A\|^n/n! < \epsilon/2^n$. Now, consider for all $n' > n > N'$:

$$|\sum_{i=n}^{n'} \left\|e_i^A\right\| | = \sum_{i=n}^{n'} \left\|e_i^A\right\| \tag{2}$$

which is true since the norm function always outputs non-negative values. Now, By problem 3, if $A \in \mathcal{B}(X)$, then $A^2 \in \mathcal{B}(X)$ and $\left\|A^2\right\| \leq \|A\|^2$. Thus $\left\|e_n^A\right\| = \|A^n/n!\| \leq \|A\|^n/n!$. Thus:

$$\begin{aligned} |\sum_{i=n}^{n'} \left\|e_i^A\right\| | &= \sum_{i=n}^{n'} \left\|A^i/i!\right\| \\ &= \sum_{i=n}^{n'} \|A\|/i! \\ &< \sum_{i=n}^{n'} \epsilon/2^i = \epsilon[\sum_{i=n}^{n'} 1/2^i] \end{aligned}$$

By our geometric series knowledge, we know $\epsilon[\sum_{i=0}^{\infty} 1/2^i] = 2\epsilon$. Since this infinite sum is clearly larger than the finite sum from n to n'. We have that: $|\sum_{i=n}^{n'} \left\|e_i^A\right\| | < 2\epsilon$. This is sufficient to show that the sequence of partial sums Cauchy converge (if one wishes, they can easily make the desired sum less than ϵ but it is quite trivial as it amounts to re choosing our original ϵ). Thus, $\sum_{n=1}^{\infty} \left\|e_i^A\right\|$ converges as wanted.

We then have that for all k, $\sum_{i=1}^{k} e_i^A \in \mathcal{B}(X)$, and the sequence $(s_n)_1^\infty = (\sum_{i=1}^{k} e_i^A)_{k=1}^\infty$ of elements in $\mathcal{B}(X)$ such that $\sum_{i=1}^{\infty} \left\|e_i^A\right\|$ converges. Thus, by problem 4, (s_n) converges to an operator in $\mathcal{B}(X)$. Since s_n consists of the partial sums of e^A, it is clear $s_n \to e^A$ and by uniqueness of limits in Banach spaces (which $\mathcal{B}(X)$ is), we have that $e^A \in \mathcal{B}(X)$ as wanted. ■

P6.

A sequence $\{h_n\}$ in a Hilbert space $\mathcal{H}$ is said to **converge weakly** to $h \in \mathcal{H}$ if

$$\lim_{n\to\infty} \langle h_n, g\rangle = \langle h, g\rangle$$

for every $g \in \mathcal{H}$.

(a) If $\{e_n\}$ is an orthonormal sequence in $\mathcal{H}$, show that $e_n \to 0$ weakly.

(b) Show that if $h_n \to h$ in norm, then $h_n \to h$ weakly. Show that the converse is false, but that if $h_n \to h$ weakly and $\|h_n\| \to \|h\|$, then $h_n \to h$ in norm.

Solution.

The claim we want to prove is: $\lim_{n\to\infty}\langle e_n, g\rangle = \langle 0, g\rangle$. But $\langle 0, g\rangle = 0$, so we have to show $\lim_{n\to\infty}\langle e_n, g\rangle = 0$. Choose any $\epsilon > 0$ and fix any $g \in \mathcal{H}$. By Bessell's inequality, we have that

$$\sum_{n=1}^{\infty} \langle e_n, g\rangle \leq \|g\|^2$$

which implies that the sum on the left side converges. By the zero test for series, we have that the sequence where the nth term is the nth term in the sum above goes to zero i.e. $\lim_{n\to\infty}\langle e_n, g\rangle = 0$. This is what we wanted to show.

To show that if $h_n \to h$ in norm, then $h_n \to h$ converges weakly, first fix $g \in \mathcal{H}$. Since $h_n \to h$ in norm, then for any choice of $\epsilon > 0$, there exists an $N > 0$ such that if $n > N$, then $\|h_n - h\| < \epsilon$. Now, consider, for any $n > N$:

$$|\langle h_n, g\rangle - \langle h, g\rangle|^2 = |\langle h_n - h, g\rangle|^2$$

By Cauchy-Schwarz: $|\langle h_n - h, g\rangle|^2 = \|h_n - h\| \cdot \|g\| < \epsilon \cdot \|g\|$. Note, this $\|g\|$ is simply a constant so we can re choose our N so that if $n > N$, then $\|h_n - h\| < \epsilon^2 / \|g\|$. Thus, we have that $|\langle h_n, g\rangle - \langle h, g\rangle|^2 < \epsilon^2$ which of course, implies that:

$$|\langle h_n, g\rangle - \langle h, g\rangle| < \epsilon$$

so that $\langle h_n, g\rangle$ converges to $\langle h, g\rangle$ for all $g \in \mathcal{H}$ proving that $h_n \to h$ weakly.

To show the converse does not hold, examine any orthonormal sequence in a given Hilbert space[1] (e.g. l^2). We know that for any $n \geq 1$, $\|e_n - 0\| = \|e_n\| = 1$.

[1] It is a fact that an orthnormal sequence exists for every non-trivial Hilbert space, but the reader may choose a space for which they know one exists

This shows that for $0 < \epsilon < 1$, for all choices of $N > 0$, if $n > N$, then $\|e_n\| \geq \epsilon$ implying that e_n does *not* converge to 0 in norm. Thus, the converse fails. In fact, using the idea of Cauchy sequences and the Parallelogram law, one can prove that $\{e_n\}$ does not converge to any point of the given Hilbert space.

Recall, $h_n \to h$ in norm means that $\|h_n - h\| \to 0$. To do this, we need to establish a few small results. First, recall by Proposition 1.22 that $\|h_n - h\|^2 = \|h_n\|^2 - 2\mathrm{Re}\langle h_n, h\rangle + \|h\|^2$. Next, we are given $\|h_n\| \to h$ so it follows by elementary properties of sequences that $\|h_n\|^2 \to \|h\|^2$. Finally, we are given $h_n \to h$ weakly. So, choose $g = h$, then $\langle h_n, h\rangle \to \langle h, h\rangle$. Finally, we are ready to prove (b). Note we use (the just now) established results without stating them:

$$\begin{aligned} \|h_n - h\|^2 &= \|h_n\|^2 - 2\mathrm{Re}\langle h_n, h\rangle + \|h\|^2 \\ \implies \lim_{n\to\infty} \|h_n - h\|^2 &= \|h\|^2 + 2\mathrm{Re}\langle h, h\rangle + \|h\|^2 \\ &= 2\|h\|^2 - 2\|h\|^2 = 0 \end{aligned}$$

Thus, $\|h_n - h\|^2 \to 0$ and by taking the square root of both sides (each term in $\|h_n - h\|^2$ is clearly positive), we obtain the desired result. ■

P7.

Let S be the forward shift on l^2. Verify that S^* is the backward shift on l^2.

Solution.
Let $x, y \in l^2$. By Theorem 2.12, since $S \in \mathcal{B}(l^2)$, there exists S^* so that $\langle Sx, y\rangle = \langle x, S^*y\rangle$. We prove this S^* is the backward shift. Let B denote the backward shift on l^2.

If $x = (x_1, x_2, \ldots)$, then $Sx = (0, x_1, x_2, \ldots)$ so if $\langle x, y\rangle = \sum_{i=1}^{\infty} x_j \overline{y_j}$ (definition of inner product in l^2), then $\langle Sx, y\rangle = \sum_{i=1} x_{j-1}\overline{y_j}$ where $x_0 := 0$. Thus, $\langle Sx, y\rangle = \sum_{i=2}^{\infty} x_{j-1}\overline{y_j}$.

Similar to before, if $y = (y_1, y_2, \ldots)$, then $By = (y_2, y_3, \ldots)$. So, $\langle x, By\rangle = \sum_{i=1}^{\infty} x_j \overline{y_{j+1}}$. This is the same as saying, $\sum_{i=2}^{\infty} x_{j-1}\overline{y_j}$.

Thus, $\langle Sx, y\rangle = \sum_{i=2}^{\infty} x_{j-1}\overline{y_j} = \langle x, By\rangle$ so that the adjoint of the forward shift is the backward shift as wanted. ■

P8.

Let $\mathcal{H}$ be a Hilbert space and let $S, T \in \mathcal{B}(\mathcal{H})$. Determine whether $\langle Tx, x\rangle = \langle Sx, x\rangle$ for all $x \in \mathcal{H}$, implies $S = T$.

Solution.
S and T are in $\mathcal{B}(\mathcal{H})$ so by Theorem 2.12, there exists adjoints S^* and T^* in $\mathcal{B}(\mathcal{H})$. We first show $S^* = T^*$. Note for all $x, k \in \mathcal{H}$, we have:

$$\langle Sx, h\rangle = \langle x, S^*h\rangle \qquad \text{and} \qquad \langle Tx, h\rangle = \langle x, T^*h\rangle$$

In particular, if we choose $x = h$, we have that $\langle Sx, x\rangle = \langle Tx, x\rangle$. In particular, this equality shows that for all $x = h$, we have $S^*(x) = T^*(x)$. Since $S^{**} = S$, we obtain:

$$\text{For all } x = h,\ S(x) = T(x)$$

Since this holds for all $x \in \mathcal{H}$, we have that $S = T$ as wanted. ■

P9.

Let $\mathcal{H}$ be a Hilbert space and let $P : \mathcal{H} \to M$ be the orthogonal projection of $\mathcal{H}$ onto a closed subspace M of $\mathcal{H}$. Verify that P is self-adjoint and that $P^2 = P$.

Solution.
We know that the projection map $P \in \mathcal{B}(\mathcal{H})$. Thus, by theorem 2.12, there exists some map $P^* \in \mathcal{B}(\mathcal{H})$ such that $\langle Px, y\rangle = \langle x, P^*y\rangle$. We show $P^* = P$. By the projection theorem, there exists another projection $Q : \mathcal{H} \to M^\perp$ such that, $x = Px + Qx$ for every $x \in \mathcal{H}$. Choose $x, y \in \mathcal{H}$. Then:

$$\langle Px, y\rangle = \langle Px, Py + Qy\rangle = \langle Px, Py\rangle + \langle Px, Qy\rangle$$

and,

$$\langle x, Py\rangle = \langle Px + Qx, Py\rangle = \langle Px, Py\rangle + \langle Qx, Py\rangle$$

Note that $Qx \in M^\perp$ so that by definition since $Py \in M$, we have $\langle Qx, Py\rangle = 0$ and similarly, $\langle Px, Qy\rangle = 0$. Thus, we have that $\langle Px, y\rangle = \langle Px, Py\rangle = \langle x, Py\rangle$ as wanted. Thus, P is self-adjoint.

To show $P^2 = P$, it suffices to note that if $x \in M$, then $Px = x$. Thus, $P^2(x \in \mathcal{H}) = P(Px)$ where Px is then in M since $P : \mathcal{H} \to M$. Thus, $P(Px) = Px$. Since x is arbitrary, we have $P^2 = P$. ■

P10.

For A and B in $\mathcal{B}(\mathcal{H})$ we have (c) $(\alpha A)^* = \overline{\alpha} A^*$ for $\alpha \in \mathbb{C}$ and (d) $(AB)^* = B^*A^*$.

Solution.
(c) By theorem 2.12, there exists $(\alpha A)^* \in \mathcal{B}(\mathcal{H})$ such that for $h, k \in \mathcal{H}$: $\langle \alpha Ah, k\rangle = \langle h, (\alpha A)^*k\rangle$. Recall the property of inner product that: $\langle x, \alpha y\rangle = \overline{\alpha}\langle x, y\rangle$ and $\alpha\langle x, y\rangle = \langle \alpha x, y\rangle$. From this, consider:

$$\alpha\langle Ah, k\rangle = \langle \alpha(Ah), k\rangle = \langle(\alpha A)k, \rangle$$

where we can say $\alpha(Ah) = (\alpha A)h$ since A lies in a linear space and

$$\alpha\langle Ah, k\rangle = \alpha\langle h, A^*k\rangle = \langle h, \overline{\alpha}(A^*(k))\rangle = \langle h, (\overline{\alpha}A^*)k\rangle$$

Thus, we have that $\langle(\alpha A)h, k\rangle = \langle h, (\overline{\alpha}A^*)k\rangle$ implying that the since adjoints are unique, the adjoint of αA is $\overline{\alpha}A^*$ i.e. $(\alpha A)^* = \overline{\alpha}A^*$ as wanted.

(d) By problem 3, we know that $AB \in \mathcal{B}(\mathcal{H})$, thus by theorem 2.12, it follows that there exists a mapping $(AB)^*$ such that it is the adjoint of AB. Note since $Bh \in \mathcal{H}$ for arbitrary $h \in \mathcal{H}$, we have that for any $k \in \mathcal{H}$:

$$\langle A(Bh), k\rangle = \langle Bh, A^*k\rangle$$

and as noted before, we have

$$\langle (AB)h, k\rangle = \langle h, (AB)^*k\rangle$$

Since $A(Bh) = (AB)h$ just by definition of AB, we obtain from the two equalities above,

$$\langle Bh, A^*k\rangle = \langle h, (AB)^*k\rangle$$

Furthermore, note since B^* and A^* are in $\mathcal{B}(\mathcal{H})$, so there exists an adjoint for the map B^*A^*. Thus, we can say:

$$\langle B^*(A*k), h\rangle = \langle A^*k, (B^*)^*h\rangle = \langle A^*k, Bh\rangle$$

where $(B^*)^*$ by Prop. 2.13. Note for the above equality, we have:

$$\begin{aligned} \overline{\langle h, B^*(A^*k)\rangle} &= \overline{\langle Bh, A^*k\rangle} \\ \implies \langle h, B^*(A^*k)\rangle &= \langle Bh, A^*k\rangle \end{aligned}$$

because the conjugate of a conjugate of a complex number is the complex number itself. Thus, we obtain:

$$\langle h, B^*(A^*k)\rangle = \langle h, (B^*A^*)k\rangle$$

so that $\langle ABh, k\rangle = \langle h, (AB)^*k\rangle = \langle h, B^*A^*k\rangle$ so that $(AB)^* = B^*A^*$ by uniquness of adjoints. ■

P11.

Show that T^{-1} is linear given a linear bijective mapping T between vector spaces X and Y.

Solution.

T is bijective so T^{-1} exists. We want to show for a, b scalars, $x, y \in Y$, that $T^{-1}(ax + by) = aT^{-1}(x) + T^{-1}(y)$. Since T is surjective, we have that there exist some $x' \in X$ such that $T(x') = x$ and similarly, $T(y' \in X) = y$. In particular, $T(ax') = aT(x') = ax$ and $T(by') = bT(y') = by$ since T is linear. Thus, we can write $T^{-1}(ax+by)$ as $T^{-1}(aT(x')+bT(y'))$. Since T is linear, we have that,

$$T^{-1}(ax + by) = T^{-1}(T(ax') + T(by')) = T^{-1}(T(ax' + by'))$$

since T and T^{-1} are inverses, we have that $T^{-1}(T(ax' + by')) = ax' + by'$. However, $T^{-1}(x) = x'$ and similarly, $T^{-1}(y) = y'$ so,

$$T^{-1}(ax + by) = aT^{-1}(x) + bT^{-1}(y)$$

as wanted. ■