MATC37 Lecture Notes

Lecture notes on a course in real analysis

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Chapter 1

Shortcomings of the Riemann integral

In the turn of the 19th century many mathematicians were starting to discover that the Riemann integral had many shortcomings and were starting to look for a new formulation. Thus, was born the Lebesgue integral. Here we will discuss some of the problems that motivated the idea for the Lebesgue measure and the Lebesgue integral.

1.1 Limits of continuous functions

Let $f : [0,1] \to \mathbb{R}$ be continuous. Let $f_n : [0,1] \to \mathbb{R}$ be a continuous function so that $\{f_n\}_{n\in\mathbb{N}}$ is a sequence of continuous functions on the interval $[0,1]$. Assume that

$$\lim_{n\to\infty} f_n(x) = f(x)$$

exists for every $x \in [0,1]$. We know from MATB43 that if each f_n is uniformily continuous, then f is continuous but suppose each f_n were not uniformily continuous, then there are issues that may arise which may seem subtle but have radical effects. For example, we can construct a sequence of continuous functions $\{f_n\}$ converging everywhere to f so that

(a) $0 \le f_n(x) \le 1$ for all x

(b) $f_n \ge f_{n+1}$ i.e. the sequence $\{f_n\}$ is monotonically decreasing as $n \to \infty$.

(c) $f(x) = \lim_{n\to\infty} f_n(x)$ exists for all x

However, f is not Riemann integrable!
Note that, in view of (b), we obtain $\int_0^1 f_n(x)dx \ge \int_0^1 f_{n+1}(x)dx$. Combining this with the fact that there is a lower bound (using (a)) on $\int_0^1 f_n(x)dx$, we know that

the sequence $\int_0^1 f_n(x)dx$ converges to a limit. Therefore, the question arises: What method of integration can be used to integrate f and obtain that for it

$$\int_0^1 \lim_{n\to\infty} f_n(x)dx = \lim_{n\to\infty} \int_0^1 f_n(x)dx$$

It is with Lebesgue integration that we can solve this problem.

1.2 Irregular Geometric Objects

We say a curve is **rectifiable** if its length L (defined in terms of the supremum of lengths of all polygonal lines joining successively finitely many points of the curve) is finite. Rectifiable curves, because they are endowed with length are one-dimensional in nature. However, it is worth asking if there are non-rectifiable curves which are two-dimensional or rather just any dimensions more than one. We shall see the answer to this question in terms of space-filling curves and generally have dimensions between 1 and 2.
Recall our area/length definitions for regular polygons. It is worth asking if we can extend our understanding of area and length so that we can measure geometric objects that are not as simple as regular polygons such as fractional objects such as Koch's snowflake. Note these objects are still parameterized by lines so this question and the previous are well connected.

1.3 Differentiation and Integration

The fundamental theorem of calculus expresses the notion that differentiation and integration are inverse operations and it does this via the two equations below

$$F(b) - F(a) = \int_a^b F'(x)dx, \tag{1.1}$$

$$d/dx \int_0^x f(y)dy = f(x). \tag{1.2}$$

It is natural to try and find a general class of functions which respect either of these equations. However, for the first equation, the existence of continuous functions F which are nowhere differentiable, or for which $F'(x)$ exists for every x, but F' is not integrable, leads to the problm of finding a general class of functions for which this equation is valid. For the second equation, the question is to formulate properly and establish this assertion for the general class of integrable functions f. These questions can be answered with the help of certain covering arguments, and the notion of absolute continuity.

1.4 Probability Theory

Define S to be a space of $\{0,1\}$-valued sequences (e.g. fair coin toss). Define $f : S \to [0,1]$ where

$$f(s) = \sum_{n=1}^{\infty} s_n/2^n$$

Note this implies $0 \leq f \leq 1$. Then, we can compute,

$$P(s_1 = 0, s_2 = 1) = l([1/4, 1/2]) = 1/4$$

However, what happens if we try to take an arbitrary set $A \subset [0,1]$ and try to compute $P(f^{-1}(A)) = l(A)$. This problem is answered (not completely, however) by the Lebesgue measure.

1.5 The measure problem (MP)

To put matters clearly, the fundamental issue that must be understood in order to try to answer all of the questions raised above is that of measure. Stated impercisely in one dimension, we are trying to assign to each subset E of $\mathbb{R}$ its one-dimensional measure $l(E)$, that is, its length, thereby extending the standard notion of length defined for closed intervals. For example, consider the sets below:

$$(i)\ [2,6] \quad (ii)\ (2,6] \quad (iii)\ \{2\} \quad (iv)\ \mathbb{R} \quad (v)\ \mathbb{Q} \quad (vi)\ \cup_{n=1}^{\infty} [n, n + 1/n^2] \quad (vii)\ \emptyset$$

It seems intuitive to assign 4 to (i) so that $l([2,6]) = 6 - 2 = 4$. We can assign (ii) the same measure as well. (iii) is just a single point so it is a 0 dimensional object, therefore it seems intuitive to assign it the measure 0 so that $l(\{2\}) = 0$. We assign (vii) the same measure since it contains *no* points whatsoever. There is no intuitive way to assign $\mathbb{R}$ a finite measure, therefore we write $l(\mathbb{R}) = \infty$. However, we cannot apply the same reasoning to (v). Recall from MATB43 that any countable set has Lebesgue measure zero, therefore we write $l(\mathbb{Q}) = 0$. (vi) is a bit tricky but easily understood if we remember that $l([n, n + 1/n^2]) = (n + 1/n^2) - n = 1/n^2$ so that

$$l(\cup_n [n, n + 1/n^2]) = \sum_n l([n, n + 1/n^2]) = \sum_n 1/n^2 = \pi^2/6$$

Moving from left-most part to the second-left most part is done through a equality that we have not yet proven but it does seem intuitive to want this to be true. Therefore, we should want l to have this property.
Before, we move on further, it is important to discuss the closely related integral problem as well.

1.5.1 The integral problem (IP)

This is concerned with the question: For which $f : [0,1] \to \mathbb{R}$ can we make sense of $\int_0^1 f(x)dx$? Thus, we are concerned with finding the class of integrable functions. Note that MP and IP are closely related.
Given a set $A \subset \mathbb{R}$, consider the characteristic function

$$\mathcal{X}(A) = \begin{cases} 1, & x \in A \\ 0 & x \notin A \end{cases}$$

Then $\int \mathcal{X}_A(x)dx = l(A)$ i.e. integrating the characteristic function over a set is the same as taking its measure.
Conversely, given $f : [0,1] \to \mathbb{R}$ we can try to write f as

$$f = \sum_n a_n \mathcal{X}_{A_n}$$

for some suitable $a_n \in \mathbb{R}$ and $A_n \subset \mathbb{R}$ and then we try to compute

$$\int f(x)dx = \int \sum_n a_n \mathcal{X}_{A_n} dx = \sum a_n \int \mathcal{X}_{A_n} dx = \sum a_n l(A_n)$$

Note, we actually used the equality mentioned in section 1.1.

1.6 Wishlist for MP

We are looking for a non-negative (length intuitively should be non-negative) function m defined on any given subset of $\mathbb{R}$. Therefore, we are looking for a function $l : \mathcal{P}(\mathbb{R}) \to \mathbb{R}^+$ such that:

1. If $A_1, A_2, \ldots$ are disjoint then

   $$l(\bigcup_n A_n) = \sum_n l(A_n)$$

   This is called the **countably additive** property.

2. $l(A + t) = l(A)$ for any $t \in \mathbb{R}$. This is called the **translation invariance** property.

3. $l([0,1]) = 1$ i.e. l is *normalized*

However, we shall see that this is an unrealistic wishlist.

Theorem 1 (Vitali, 1905). *There does not exist a measure $l : \mathcal{P}(\mathbb{R}) \to [0, \infty]$ such it satisfies all the properties above.*

Proof. Define an equivalence relation on the closed unit interval $\mathbb{I} = [0, 1]$ via

$$\forall\, x, y \in \mathbb{I} : \; x \sim y \Leftrightarrow x - y \in \mathbb{Q}$$

Then, we have a natural set of elements of each equivalence class given by $\sim$. Denote this set by A. Note, $M \subset \mathbb{I}$ and for each $x \in \mathbb{R}$ there exists a unique $y \in A$ and $r \in \mathbb{Q}$ such that $x = y + r$.
Then, define

$$A_r := \{y + r : y \in A\}$$

for each $r \in \mathbb{Q}$. Because of what we noted above, we obtain the result that $\mathbb{R}$ is partitioned into countably many disjoint sets, and specifically by the sets:

$$\mathbb{R} := \bigcup \{A_r : r \in \mathbb{Q}\}$$

Suppose that A is measurable by l. Then:

$$1 = l([0, 1]) = \sum_{r \in \mathbb{Q}} l(A_r) = l(A)$$

Then $l(A) \neq 0$ since by countable additivity, we would have $l([0, 1]) = 0$. Suppose $l(A) > 0$. Then

$$l([0, 1]) \geq l(\bigcup_{r \in \mathbb{Q} \cap [0,1]} A_r) = \sum_{r \in \mathbb{Q} \cap [0,1]} l(A_r) = \infty$$

since $l(A_r) = l(A)$ by translation invariance. But this is also a contradiction as $1 \geq \infty$. Therefore, l cannot satisfy *every* property above. □

It is because of this that we loosen our wishlist to allow the existence of non-measurable sets. The reader is warned that constructions of such sets are not trivial!

If we try to loosen countable additivity to just *finite* additivity, it is still too strong. Consider the Banach-Tarski paradox (very non trivial to prove so it will not be done here). This is possible by assuming all properties above but replacing countable additivity by finite additivity. It turns out that such a measure can exist in $\mathbb{R}^1$ but not any higher.

Theorem 2 (Banach-Tarski Paradox)**.** *The three-dimensional unit ball can be decomposed into a finite number of disjoint sets, which can then be reassembelled into two three-dimensional unit balls.*

While this is not a contradiction but it does allow for highly non intuitive results.

Chapter 2

Measure Theory

2.1 Preliminaries

This section is devoted to reviewing the basic notions that one has in $\mathbb{R}^d$ (positive integer d).

The rational numbers are denoted by $\mathbb{Q}$. The real numbers are denoted by $\mathbb{R}$ and can be written as $(-\infty, +\infty)$. The extended real numbers are written as $\mathbb{R} \cup \{\pm\infty\}$ or $[-\infty, +\infty]$.

Definition 2.1.1. Fix an integer $d \geq 1$. A **point** $x \in \mathbb{R}^d$ is given by an ordered d-tuple of real numbers ($\mathbb{R}^1$),

$$\mathbf{x} = (x_1, \dots, x_d) \qquad \text{where} \qquad x_1, \dots, x_d \in \mathbb{R}$$

$\mathbb{R}^d$ is called the **d-dimensional Euclidean space**.

Recall, that we can add vectors (points in a Euclidean space),

$$\mathbf{x} = (x_1, \dots, x_d),\ \mathbf{y} = (y_1, \dots, y_d) \implies \mathbf{x} + \mathbf{y} = (x_1 + y_1, \dots, x_d + y_d)$$

and multiply them by a scalar $\lambda \in \mathbb{R}$,

$$\lambda\mathbf{x} = (\lambda x_1, \dots, \lambda x_d)$$

Definition 2.1.2. The **norm** of $x \in \mathbb{R}^d$ is defined as $|x| = (\sum x_i^2)^{1/2}$. The norm operator has the following properties,

- $|x| \geq 0$ and $|x| = 0$ if and only if $x = 0$.
- $|\lambda x| = |\lambda||x|$
- $|x + y| \leq |x| + |y|$

In particular, we can *measure* the distance between $x, y \in \mathbb{R}^d$ via

$$d(x, y) := |x - y|$$

where d is then a function $d : \mathbb{R}^d \times \mathbb{R}^d \to [0, \infty)$ most comonly called the **distance function** with the following properties:

1. $d(x, y) \geq 0$ and $d(x, y) = 0$ if and only if $x = y$
2. $d(x, y) = d(y, x)$
3. $d(x, z) = d(x, y) + d(y, z)$ for $x, y, z \in \mathbb{R}^d$

Definition 2.1.3. For a set $E \subset \mathbb{R}^d$, the **complement** is

$$E^c := \{x \in \mathbb{R}^d : x \notin E\} = \mathbb{R}^d \backslash E$$

More generally, for two sets $E, F \subset \mathbb{R}^d$, we have the **difference of two sets** being given by

$$F \backslash E := \{x \in \mathbb{R}^d : x \in F \text{ and } x \notin E\}$$

Also, recall

$$E \cup F = \{x \in \mathbb{R}^d : x \in E \text{ and } x \in F\} \qquad (\textbf{union})$$

$$E \cap F = \{x \in \mathbb{R}^d : x \in E \text{ and } x \in F\} \qquad (\textbf{intersection})$$

Finally, for $E, F \subset \mathbb{R}^d$ define the **distance between sets** by

$$d(E, F) := \inf_{x \in E, y \in F} d(x, y)$$

This "d" is not to be confused with the one above because it does not have *all* of the properties that the distance function between points does. Namely, if $E = (0, 1)$ and $F = (1, 2)$, then $d(E, F) = 0$ but $E \neq F$, and in fact $E \cap F = \emptyset$.

2.1.1 Open, Closed and Compact sets

Definition 2.1.4. For $x \in \mathbb{R}^d$ and $r > 0$, define

$$B_r(x) := \{y \in \mathbb{R}^d : d(x, y) < r\}$$

$B_r(x)$ is said to be an **open ball** of radius r. A subset E of $\mathbb{R}^d$ is **open** if for all $x \in E$, there exists $r > 0$ such that $B_r(x) \subset E$.

Proposition 1. An open ball $B_r(x)$ is open.

Proof. Given any point $\widetilde{x} \in B_r(x)$, we want to find $\widetilde{r} > 0$ such that $B_{\widetilde{r}}(\widetilde{x}) \subset B_r(x)$.

Set $\widetilde{r} = r - d(\widetilde{x}, x)$. We know $\widetilde{r}$ is positive since $d(\widetilde{x}, x) < r$ (definition of an open ball), which implies $\widetilde{r} > 0$. Now if $z \in B_{\widetilde{r}}(\widetilde{x})$, then

$$d(x, z) \leq d(x, \widetilde{x}) + d(\widetilde{x}, z) \leq d(x, \widetilde{x}) + \widetilde{r} = r$$

Thus $z \in B_r(x)$. Hence $B_{\widetilde{r}}(\widetilde{x}) \subset B_r(x)$ as wanted. $\square$

Proposition 2.

1. $\emptyset$ and $\mathbb{R}^d$ are open.
2. If $\{\theta_\alpha\}$ is a collection of open sets, then $\cup_\alpha \theta_\alpha$ is open.
3. If $\theta_{\alpha_1}, \dots, \theta_{\alpha_n}$ are open, then $\cap_{i=1}^n \theta_{\alpha_i}$ is open.

Proof.

1. Clear.
2. Choose $x \in \cup_\alpha \theta_\alpha$. Then there exists α' such that $x \in \theta_{\alpha'}$. Because $\theta_{\alpha'}$ is an open set, there exists a $r > 0$ such that $B_r(x) \subset \theta_{\alpha'} \subset \cup_\alpha \theta_\alpha$ as wanted.
3. Let $x \in \cap_{i=1}^n \theta_{\alpha_i}$, then there exists $r_1, \dots, r_n$ such that $B_{r_1}(x) \subset \theta_{\alpha_1}, \dots, B_{r_n}(x) \subset \theta_{\alpha_n}$.

□

Remark. (3) only works for a finite collection. For example, $\cap_{n=1}^\infty (-1/n, 1/n) = \{0\}$ which is not open!

Definition 2.1.5. A set $E \subset \mathbb{R}^d$ is **closed** if E^c is open.

Example.

1. *$[0,1] \subset \mathbb{R}$ is closed. Indeed, $[0,1]^c = (-\infty, 0) \cup (1, \infty)$.*
2. *$\emptyset$ and $\mathbb{R}^d$ are both open and closed.*

Note that most sets are neither open nor closed e.g. $(0,1] \subset \mathbb{R}$ is not open or closed.

Definition 2.1.6. A set E is **bounded** if it is contained in some ball of finite radius. A bounded set is **compact** if it is also closed. Compact sets can also be defined using the Heine-Borel property:
If E is covered by some collection $\{Q_\alpha\}$ of open sets (called an **open cover**), then there are finitely many of the open sets $Q_{\alpha_1}, \dots, Q_{\alpha_n}$ such that E is also covered by this finite collection of open sets from $\{Q_\alpha\}$ (called a **finite subcover**).

Definition 2.1.7. A point $x \in \mathbb{R}^d$ is a **limit point** of the set E is for every $r > 0$, the ball $B_r(x)$ contains points of E. An **isolated point** of E is a point $x \in E$ such that there exists an $r > 0$ where $B_r(x) \cap E = \{x\}$.

Definition 2.1.8. A point $x \in E$ is an interior point of E if there exists $r > 0$ such that $B_r(x) \subset E$. The set of all interior points of E is called the **interior** of E. Also, the **closure** $\overline{E}$ of E consists of the union of E and all of its limit points. The **boundary** of a set E, denoted by ∂E, is the set of points which are in the closure of E but not in the interior of E. Note that the closure can be defined as the *smallest* closed set which *contains* E and similarly, the interior of a set can be defined as the *largest* open set which is *contained in* E.

2.1.2 Rectangles and Cubes

Definition 2.1.9. $R \subset \mathbb{R}^d$ is said to be a **closed rectangle** if it can be written in the form:

$$R = [a_1, b_1] \times \ldots \times [a_n, b_n] = \{(x_1, \ldots, x_n) \in \mathbb{R}^d \,:\, a_j \leq x \leq b_j, j = 1, \ldots, n\}$$

A **cube** is a rectangle for which $b_1 - a_1 = b_2 - a_2 = \ldots = b_d - a_d$.

Note each rectangle R has sides parallel to the coordinates axes and is closed.

Example. *In $\mathbb{R}$, rectangles are line segments, in $\mathbb{R}^2$, rectangles are the usual 2-d flat, 4 sided polygons we see in high school and rectangles in $\mathbb{R}^3$ are called parallelepipeds.*

Definition 2.1.10. We say that the lengths of the sides of the rectangle R are $b_1 - a_1, \ldots, b_d - a_d$. The **volume** of the rectangle R is denoted by $|R|$ and is defined to be

$$|R| = (b_1 - a_1) \cdots (b_d - a_d)$$

Continuing our most recent example, in $d = 1$, the volume equals the length of the line segment and when $d = 2$, it equals the area.

Definition 2.1.11. An open rectangle is the product of open intervals, and the interior of the rectangle R is then

$$(a_1, b_1) \times (a_2, b_2) \times \cdots \times (a_d, b_d)$$

Definition 2.1.12. A union of rectangles is said to be **almost disjoint** if the interiors of the rectangles are disjoint.

We shall make much use of coverings of sets with sets of rectangles and cubes. Some important results for these techniques is discussed:

Lemma 3. If a rectangle is the almost disjoint union of finitely many other rectangles, say $R = \cup_{k=1}^{N} R_k$, then

$$|R| = \sum_{k=1}^{N} |R_k|$$

Proof. We consider the grid formed by extending indefinitely the sides of all rectangles $R_1, \ldots, R_N$. This construction yields finitely many rectangles $\hat{R}_1, \ldots, \hat{R_M}$, and a partition $J_1, \ldots, J_N$ of the integers between 1 and M, such that the unions

$$R = \bigcup_{j=1}^{M} \hat{R}_j \quad \text{and} \quad R_k = \bigcup_{j \in J_k} \hat{R}_j, \qquad \text{for } k = 1, \ldots, N$$

are almost disjoint.

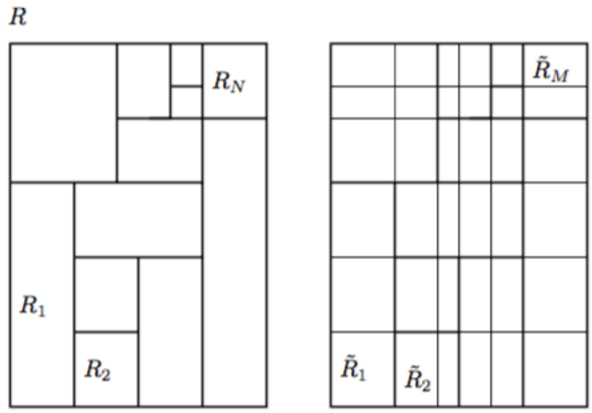

Figure 2. The grid formed by the rectangles R_k

For the rectangle R, for example, we see that $|R| = \sum_{j=1}^{M} |\hat{R}_j|$, since the grid actually partitions the sides of R and $\hat{R}_j$ consists of taking products of the intervals in these partitions. Thus when adding the volumes of the $\hat{R}_j$ we are summing the corresponding products of lengths of the intervals that arise. Since this also holds for other rectangles $R_1, \dots, R_N$, we conclude that

$$|R| = \sum_{j=1}^{M} |\hat{R}_j| = \sum_{k=1}^{N} \sum_{j \in J_k} |\hat{R}_j| = \sum_{k=1}^{N} |R_k|.$$

□

and a related claim:

Lemma 4. If $R, R_1, \dots, R_N$ are rectangles, and $R \subset \cup_{k=1}^{N} R_k$, then

$$|R| \leq \sum_{k=1}^{N} |R_k|.$$

Proof. Similar to the proof of Lemma 3. □

Theorem 5. *Every open set $O \subset \mathbb{R}$ can be written uniquely as a countable union of disjoint open intervals.*

Proof. For each $x \in O$, let I_x be the largest open interval that contains x and is contained in O, i.e. $I_x = (a_x, b_x)$, where

$$a_x = \inf\{a < x : (a, x) \subset O\}$$
$$b_x = \inf\{b > x : (x, b) \subset O\}$$

Then $x \in I_x$ and $I_x \subset O$, hence $O = \cup_{x \in O} I_x$.
If $I_x \cap I_y \neq \emptyset$ then $I_x \cup I_y \subseteq I_x$ by maximality of I_x. Similarly for $I_x \cup I_y \subset I_y$.

Thus, $I_x = I_y$. This proves the disjoint condition.
Finally, since every open interval contains a rational number, there are countably many distinct open intervals in our collection. □

Theorem 6. *Every open set $O \subset \mathbb{R}^d$ can be written as a countable collection Q of almost disjoint closed cubes such that $O = \cup_{q \in Q} q$.*

Proof. We must construct a countable collection Q of closed cubes whose interiors are disjoint, and so that $O = \cup_{Q \in Q} Q$.
As a first step, consider the grid in $\mathbb{R}^d$ formed by taking all closed cubes of side length 1 whose vertices have integer coordinates. We shall also use the grides formed by cubes of side length 2^{-N} obtained by successively bisecting the original grid.
we either accept of reject cubes in the initial grid as part of Q according to the following rule: if Q is entirely contained in O then we accept Q; if Q intersects both O and O^c then we tentatively accept it; and if Q is entirely contained in O^c then we reject it.
As a second step, we bisect the tentatively accepted cubes into 2^d cubes with side-length $1/2$. We repeat our procedure, by accepting the smaller cubes if they are completely contained in O, tentatively accepting them if they intersect both O and O^c, and rejecting them if they are contained in O^c.

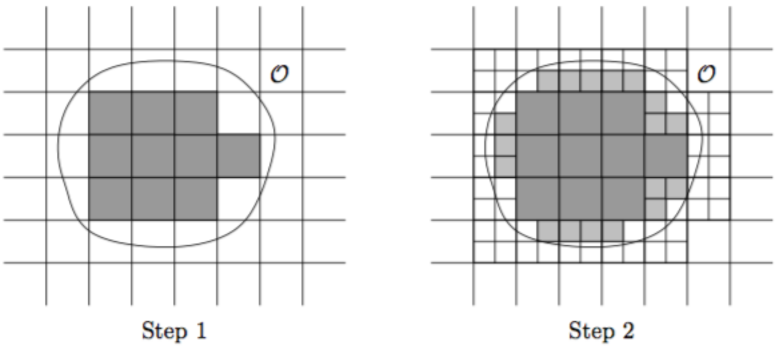

Figure 3. Decomposition of $\mathcal{O}$ into almost disjoint cubes

This procedure is then repeated indefinitely, and (by construction) the resulting collection Q of all accepted cubes is countable and consists of almost disjoint cubes. To see why their union is all of O, we note that given $x \in O$ there exists a cube of side length 2^{-N} that contains x and that is entirely contained in O. Either this cube has been accepted, or it is contained in a cube that has been previously accepted. This shows that the union of all cubes in Q is O. □

2.2 The exterior measure

In order to develop a suitable measure to work with in this course, we will first develop the Lebesgue *outer* measure which can measure any set of $\mathbb{R}^d$ (by Vitali, this should not have one of the other properties of measure) and then restrict the size of the number of sets we can measure to obtain all the desired properties of a measure.

Definition 2.2.1. If E is *any* subset of $\mathbb{R}^d$, the **exterior** or **outer** measure of E is defined as:

$$m_*(E) = \inf \sum_{j=1}^{\infty} |Q_j|,$$

where the infimum is taken over all countable coverings $E \subset \cup_{j=1}^{\infty} Q_j$ by closed cubes. The exterior measure is always non-negative but could be infinite so, in particular, we have $m_* : \mathcal{P}(\mathbb{R}^d) \to [0, \infty]$.

Remark. The infimum always exists since it is bounded and taken over a non-empty set. Suppose for some A, $m_*(A) = 42$. This means:

1. For all countable coverings by closed cubes, $A \subset \cup_{j=1}^{\infty} Q_j$, so $\sum_{j=1}^{\infty} |Q_j| \geq 42$.

2. For all $\epsilon > 0$, there exists a countable covering: $A \subseteq \cup_{j=1}^{\infty} Q_j$ by closed cubes such that $\sum_{j=1}^{\infty} |Q_j| \leq 42 + \epsilon$.

We can also replace the coverings by cubes with coverings by rectangles or even converings by balls.

Examples 1. *1.* $m_*(\emptyset) = 0$. *This follows from the fact that any cube of arbitrary volume covers* $\emptyset$.

2. $m_*(\{x \in \mathbb{R}^d\}) = 0$. *$x$ is covered by* $[p - 1/n, p + 1/n]$ *which are cubes of arbitrary length (by variying n).*

3. $m_*(\mathbb{Z}^+) = 0$ *where* $\mathbb{Z}^+$ *are the positive integers. Given* $\epsilon > 0$, *choose* $Q_j = [j - \epsilon/2^{j+1}, j + \epsilon/2^{j+1}]$. *Then* $\mathbb{Z}^+ \subset \cup_{j=1}^{\infty} Q_j$ *and*

$$\sum_{j=1}^{\infty} |Q_j| = \sum_{j=0}^{\infty} \epsilon/2^j = \epsilon$$

so that $m_*(\mathbb{Z}^+) = 0$.

4. *Other examples of outer measure being zero include* $\mathbb{Z}, \mathbb{Z}^d, \mathbb{Q}$.

5. *Let Q be a closed cube in $\mathbb{R}^d$. Simply cover it by itself to obtain $m_*(Q) \leq |Q|$. It is left to prove the reverse inequality. We consider an arbitrary cover $Q \subset \cup_{j=1}^{\infty} Q_j$ by cubes, and note that it suffices to prove that*

$$|Q| \leq \sum_{j=1}^{\infty} |Q_j|$$

Fix $\epsilon > 0$. Choose for each j and cube S_j which contains Q_j, and such that $|S_j| \leq (1+\epsilon)|Q_j|$. From the open covering $\cup_{j=1}^{\infty} S_j$ of the compact set Q, we may select a finite subcovering which, after possibly renumbering the rectangles ,we may write as $Q \subset \cup_{j=1}^{N} S_j$. Taking the closure of the cubes S_j, we may apply Lemma 4 to conclude $|Q| \leq \sum_{j=1}^{N} |S_j|$. Consequently,

$$|Q| \leq (1+\epsilon) \sum_{j=1}^{N} |Q_j| \leq (1+\epsilon) \sum_{j=1}^{\infty} |Q_j|$$

Since ϵ is aribtrary, we find that the inequality we wanted to prove holds; thus $|Q| \leq m_*(Q)$ as desired.

6. *If Q is an open cube, the result $m_*(Q) = |Q|$ still holds. Since Q is covered by its closure $|\overline{Q}|$, and $|\overline{Q}| = |Q|$, we immediately see that $m_*(Q) \leq |Q|$. Now, for the reverse inequality. Note that if Q_0 is a closed cube contained in Q, then $m_*(Q_0) \leq m_*(Q)$, since any covering of Q by a countable number of closed cubes is also a covering of Q_0. Hence $|Q_0| \leq m_*(Q)$, and since we can choose Q_0 with a volume of as close as we wish to $|Q|$, we musth have $|Q| \leq m_*(Q)$.*

7. $m_*(\mathbb{R}^d) = \infty$

8. $m_*(C) = 0$ *where C is the Cantor set*

9. $m_*(R) = |R|$ *where R is a rectangle.*

2.2.1 Properties of the exterior measure

Now, we cover some of the properties of this exterior measure.

Observation 1 (Monotonicity)**.** *If $A_1 \subseteq A_2$, then $m_*(A_1) \leq m_*(A_2)$.*

Proof. It suffices to note that any covering of A_2 covers A_1. □

Observation 2 (Countable subadditivity)**.** *If $A = \cup_{j=1}^{\infty} A_j$, then $m_*(A) \leq \sum_{j=1}^{\infty} m_*(A_j)$.*

Proof. Fix $\epsilon > 0$. For each j, there is a countable covering $A_j \subset \cup_{j=1}^{\infty} Q_{j_k}$ by closed cubes such that

$$\sum_{j=1}^{\infty} |Q_{j_k}| \leq m_*(A_j) + \epsilon/2^j$$

Then $A \subset \cup_{j,k=1}^{\infty} Q_{j,k}$ is a countable covering by closed cubes and:

$$\begin{aligned}
\sum_{j,k=1}^{\infty} |Q_{j,k}| &= \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} |Q_{j,k}| \\
&\leq \sum_{j=1}^{\infty} [m_*(A_j) + \epsilon/2^j] \\
&= \sum_{j=1}^{\infty} m_*(A_j) + \epsilon \\
&= m_*(A) \leq \sum_{j=1}^{\infty} m_*(A_j)
\end{aligned}$$

where hte last equality follows from the fact that ϵ is arbitrary. □

Observation 3 (Outer approximation)**.** $m_*(A) = \inf_{A \subset O\ open} m_*(A)$.

Proof. $A \subset O$ open implies $m_*(A) \leq m_*(O)$ which the implies $m_*(A) \leq \inf_{A \subset O \text{ open}} m_*(O)$. To show the other inequality, we have to show $\inf_{A \subset O \text{ open}} m_*(O) \leq m_*(A)$ i.e.

$$\forall\ \epsilon > 0,\ \exists\ A \subset O \text{ open, such that } m_*(O) \leq m_*(A) + \epsilon$$

□