MAT257: Analysis II Lecture Notes

Anmol Bhullar

Caven Library

E-mail address: `anmol.bhullar@mail.utoronto.ca`

Thank you to Dr. Dror Bar Natan for providing the lecture notes.

Contents

About This Class

1.1. Crucial Information

The agenda is "Adult-Level" Calculus; especially:

$$\int_M d\omega = \int_{\partial M} \omega$$

1.1.1. Instructor. Dror Bar-Natan, drorbn@math.toronto.edu (no math over email!), Bahen 6178, 416-946-5438. Office hours: Wednesdays 10:30 - 11:30 at Bahen 6178 (on teaching weeks), or, if truly impossible, by appointment.

1.1.2. Classes. Mondays, Wednesdays, and Fridays 2-3PM at MC 254 (fall semester) and at RS 211 (winter/spring). The class website is: http://drorbn.net/1617-257.

1.1.3. Teaching Assistants. Ahmed J. Zerouali (ahmed.zerouali@mail.utoronto.ca) and Jeff Im (jim@math.toronto.edu). Jeff's Office hours: Fridays 11:30 - 12:30 at the lounge on the 10th floor of 215 Huron.

1.1.4. Tutorials. Wednesdays 5-6PM with Ahmed J. Zerouali at AB 107 and Thursdays 4-5PM with Jeff Im at BA 2185.

1.2. Text Book(s)

Our main textbook will be *Analysis on Manifolds* by James R. Munkres; it is a required reading. You may also wish to look at Spivak's *Calculus on Manifolds*.

1.3. Abstract

Taken from the Faculty of Arts and Science Calendar:

Topology of $\mathbb{R}^n$; compactness, functions and continuity, extreme value theorem. Derivatives; inverse and implicit function theorems, maxima and minima, Lagrange multipliers. Integeration; Fubini's theorem, partitions of unity, change of variables. Differential forms. Manifolds in $\mathbb{R}^n$; integration on manifolds; Stokes' theorem for differential forms and classical versions.

- Prerequisites: MAT157Y1, MAT240H1, MAT247H1
- Distribution Requirement Status: Science
- Breadth Requirement: The Physical and Mathematical Universes (5)

1.4. Plan

If we don't do $\int_M d\omega = \int_{\partial M} \omega$, we've failed. So we'll do $\int_M d\omega = \int_{\partial M} \omega$ no matter what it takes. The base plan is to go over the textbook skipping almost nothing at a pace of a little less than two sections a week (and then just add a bit). If we lag, we'll skip and take shortcuts. Yet $\int_M d\omega = \int_{\partial M} \omega$ we will do.

1.5. Warning and Recommendation

This will be a very tough class, not for the faint of heart. It's about "adult-level calculus": I will make every effort to make it understandable, but certain parts of the material require very high level of mathematical sophistication. If you are not ready to put in tremendous effort it takes to reach this level of sophistication, do yourself a favour and take another class. Every bit of this class absolutely makes sense. But you'll have to think hard at times, and be ready to repeatedly adjust your perspective, to see that it is so. Don't let go! If you'll fall behind, you'll find it nearly impossible to catch up. This actually does not mean "do your homework in time" (highly recommended anyway). It means "do your deep thinking in time".

1.6. Marking Scheme

There will be three term tests (each worth 15% of your total grade) and a final exam (40%), as well as about 19 homework assignments (15%).

1.6.1. Term Tests. The term tests will take place outside of class time at 5-7PM on Tuesday November 1st at BI 131, on Tuesday January 17 at EX 300, and on Tuesday March 14 at EX 300. There will be no make-up term tests. If you miss a term test without providing a valid reason (for example, a doctors note) within one week of the test you will receive a mark of 0 on that term test. If you miss one term test for a valid reason, the the weight of the other two term tests will be increased to 22.5% each. If you miss two term tests for valid reasons, the the weight of the remaining term test will be increased to 30%, HW to 20%, and the final to 50%. You cannot miss all terms tests and receive a mark for this class.

1.6.2. Homework. About 19 assignments will be posted on the course web page approximately on the weeks shown in the class timeline, usually on Wednesdays. They will usually be due a week later and they will be (at least partially) marked by the TAs. All students (including those who join the course late) will receive a mark of 0 on each assignment not handed in; though to account for sick days and other unusual circumstances, your worst 4 assignments will not count towards your HW grade. I encourage you to discuss the assignments with other students or browse the web, so long as you do at least some of the thinking on your own and you write up your own solutions. Remember that cheating is always possible and may increase your homework grade a bit. But it will hurt your appreciation of yourself, your knowledge, and your exam grades a lot more.

1.7. Accessibility Needs

The University of Toronto is especially commited to accessibility. If you require accomodations for a disability, or have any accessibility concerns about the course, the classroom, or course materials, please contact Accessibility Services as soon as possbile: disability.services@utoronto.ca or aoda.utoronto.ca

1.8. Academic Integrity

I have been asked to include with the course syllabus a link to the Office of Academic Integrity. Here it is: artsci.utoronto.ca/osai/students

1.9. How to succeed in this class

- **Keep up!** Don't fall behind on reading, listening, and doing assignments! University goes at a different pace than high school. New material is covered once and just once. There will be no going over the same thing again and again - if you fall behind, you stay behind. Unless you are an Einstein, there is no way to do well in this class merely by attending lectures - you must think about the material more than 3 or 5 hours a week if you want it to sink in. And if you are planning on not attending lectures, well, think again. Most people find it very hard to pace their own studies without a human contact; if you'll try, you are likely to discover the hard way that you belong to the majority.
- Take your own class notes, in your own handwriting, and strive to make them as complete as possible. Writing "burns" things into your brain and forces you to keep from daydreaming. And nothing beats reading your own notes when you review the material later on.
- If in high school you were the best in your class in math, now remember that everybody around you was the same. You may find that what was enough then simply doesn't cut it any more. Try to catch that early in the year!
- Math is about **understanding**, not about memorizing. To understand is to internalize; it is to come to the point where whatever the professor does on the blackboard or whatever is printed in the books becomes **yours**; it is to come to the point where you appreciate why everything is done the way it is

done, what does it mean, what are the reasons and motivations and what is it all good for. Don't settle for less!

- Keep asking yourself questions; many of them will be answered in class, but not all. Remember the old Chinese proverb: "**Teachers open the door, but you must enter by yourself!**"

Chapter 2

Linear Algebra: Revision of Previous Materials

Proofs of theorems will not be covered in this chapter as it is review.

2.1. Vector Spaces

Definition 2.1. V is a vector space if for all $x, y, z \in V$, c, d scalars:

i) $x + y = y + x$
ii) $x + (y + z) = (x + y) + z$
iii) there is a unique vector $0 \in V$
iv) $x + (-1)x = 0$ where -1 is a scalar.
v) $1 \cdot x = x$ where 1 is a scalar.
vi) c(dx) = (cd)x
vii) (c+d)x = cx + dx
viii) c(x+y) = cx+cy

2.1.1. Component-Wise Addition and Scalar Multiplication.

Often times we have vectors $x, y \in V$ which are written as the components of vectors in other vector spaces. Suppose:

$$x = (x_1, x_2, \dots, x_n) \text{ and } y = (y_1, y_2, \dots, y_n)$$

then *component wise addition* is defined as:

$$x + y = (x_1 + y_1, x_2 + y_2, \dots, x_n + y_n)$$

and *component wise scalar multiplication* is defined as:

$$cx = (cx_1, cx_2, \dots, cx_n)$$

2.2. Independence

Definition 2.2 (span, linear combination, independence). Let V be a vector space. A set $a_1, \ldots, a_m$ of vectors in V is said to **span** V if for all $x \in V$, there exist scalars $c_1, \ldots, c_m$ such that $\sum_{i=1}^{m} c_i a_i$. In this case, we say that x can be written as a **linear combination** of the vectors $a_1, \ldots, a_m$. The set of vectors is said to be **independent** if to each x in V, there exist at most one m-tuple of scalars $c_1, \ldots, c_m$ such that $x = c_1 a_1 + \ldots + c_m a_m$ or equivalently $\sum_{i=1}^{m} c_i a_i = 0 \leftrightarrow c_1 = c_2 = \ldots = c_m = 0$. A **basis** of V is a linearly independent subset of V that spans V.

Theorem 2.3. *Suppose V has a basis consisting of m vectors. Then any set of vectors that span V has at least m vectors; any set of vectors of V that is independent has at most m vectors. In particular, any basis of V has m vectors.*

Definition 2.4. If V has a basis consisting of m vectors, we say that m is the **dimension** of V. The vector space consisting of the zero vector alone has dimension zero.

Remark 2.5. The *standard basis* of $\mathbb{R}^n$ is:

$$e_1 = (1, 0, \ldots, 0), e_2 = (0, 1, \ldots, 0), \ldots, e_n = (0, \ldots, 1)$$

2.3. Subspaces

Definition 2.6. A **subspace** W of a vector space V is a vector space with the same structure as V such that $W \subseteq V$. Specifically, if $W \subset V$, and W is a vector space itself under the operations of vector addition and scalar multiplication, is a subspace of V.

Theorem 2.7. *Let V be a vector space of dimension m. If W is a linear subspace of V (different to V), then the dimension of the subspace is less than the dimension of V. Particularly, if we refer to the subspace as W, then we write: $dim(W) < dim(V)$. Furthermore, any basis $a_1, \ldots, a_k$ of W can be extended to a basis $a_1, \ldots, a_k, a_{k+1}, \ldots, a_m$ for V.*

2.4. Inner Product and Norm

Definition 2.8 (Inner Product). $\langle , \rangle$ is an **inner product** for a vector space V if for all $x, y, z \in V$, c scalar:

i) $\langle x, y \rangle = \langle y, x \rangle$
ii) $\langle x + y, z \rangle = \langle x, z \rangle + \langle y, z \rangle$
iii) $\langle cx, y \rangle = c\langle x, y \rangle = \langle x, cy \rangle$
iv) $\langle x, x \rangle > 0$ if $x \neq 0$

Remark 2.9. Suppose $x, y \in \mathbb{R}^n$, then: $\langle x, y \rangle^2 = x_1 y_1 + \ldots + x_n y_n$. Furthermore, $\|x\| = \langle x, x \rangle^{\frac{1}{2}}$

Definition 2.10 (Norm). $\|\cdot\|$ denotes a **norm** in V if for all $x \in V$:

i) $\|x\| > 0$ if $x \neq 0$

ii) $\|cx\| = |c| \|x\|$

iii) $\|x + y\| \leq \|x\| + \|y\|$

Remark 2.11. Take the hypothesis of the definition above. Then, $\|x - y\| \geq \|x\| - \|y\|$.

Example 2.12. The sup norm, $\|x\| = max\{\|x_1\|, \ldots, \|x_n\|\}$. sup S where S is a set, is the smallest number that is larger than any other upper bound of S. Supremum may not in S is S is infinite but sup is equal to max is S is finite. Note, that the sup norm of a matrix: $A = (a_{ij})$ then $\|A\| = max|a_{ij}|$. In

2.5. Matrix Multiplication

Definition 2.13 (Matrix Multiplication)**.** If A is a matrix of size $n \times m$, B is a matrix of size $m \times p$, then the **product** of $A \cdot B$ is defined to be the matrix C of size $n \times p$ where

$$C_{ip} = \sum_{i=1}^{m} a_{ik} b_{kj}$$

Remark 2.14. Suppose x and y are vectors in $\mathbb{R}^n$ where x is a row vector and y is a column vector, then x and y can be thought of as matrices where $x \cdot y$ is simply the result of sum of all i where $x_i y_i$. Then, matrix multiplication can be thought of as the combination of all such column and row vectors.

Theorem 2.15. *Suppose A, B, C are matrices such that they are the same size and c is a scalar. Then:*

i) $A \cdot (B \cdot C)$

ii) $A \cdot (B + C) = A \cdot B + A \cdot C$

iii) $(A + B) \cdot C = A \cdot C + B \cdot C$

iv) $(cA) \cdot B = c(A \cdot B) = A \cdot (c \cdot B)$

Remark 2.16. Note that $AB \neq BA$.

Definition 2.17 (Identity Matrix)**.** For all $k \in \mathbb{N}^*$, there exists $k \times k$ matrix I_k such that if A is any $n \times m$ matrix, $I_n \cdot A = A$ and $A \cdot I_m = A$. I_k is called the identity matrix.

Theorem 2.18. *If A has size $n \times m$, B has size $m \times p$, then $A \cdot B \leq m|A||B|$*

Proof. Let $C = A \cdot B$. Then,

$$|C_{ij}| = |\sum_{k=1}^{m} a_{ik} b_{kj}| \leq \sum_{k=1}^{m} |a_{ik}||b_{kj}| \leq \sum_{k=1}^{m} |A| \cdot |B| = m|A| \cdot |B|$$

□

2.6. Linear Transformations

Definition 2.19. Let V, W be vector spaces, a **linear transformation** $T : V \to W$ is a mapping such that: for $x, y \in V$, $T(x+y) = T(x) + T(y)$ and if c scalar, then $T(cx) = cT(x)$.

Theorem 2.20. *Let V be a vector space with basis $a_1, \ldots, a_m$. Let W be a vector space. Given any m vectors $b_1, \ldots, b_m \in W$, there is exactly one linear transformation $T : V \to W$ such that for all i, $1 \leq i \leq m$, $T(a_i) = b_i$.*

Remark 2.21. Let $T : V \to W$ be a linear transformation where V is n-dimensional and W is m-dimensional. Then consider:

$$\begin{tikzcd} A \arrow[r, "T"] \arrow[d, "\phi"] & B \arrow[d, "\eta"] \\ \mathbb{R}^n \arrow[r, "A"] & \mathbb{R}^m \end{tikzcd}$$

where ϕ and η are bijective, then we obtain that: $T(x) = A \cdot x$. This allows us to consider linear transformations as matrices.

Theorem 2.22. *For any matrix A, the row rank of A equals the column rank of A. In particular,*

$$\textit{rank}\, T = \textit{dim Im}\, T = \textit{dim colspace}\, T = \textit{dim rowspace}\, T$$

Definition 2.23. Let A be a matrix. Then the following are **elementary row operations**:

i) Exchange rows i and j of A $(i \neq j)$

ii) Replace row i of A by itself plus the scalar c times row j $(i \neq j)$.

iii) Multiply row i of A by a non zero scalar λ.

Theorem 2.24. *If B is a matrix obtained from applying an elementary row operations to A, then rank B = rank A.*

Theorem 2.25. *Let A be an $n \times m$ matrix. Any elementary row operation on A may be carried out by premultiplying A by the corresponding elementary matrix.*

2.7. Invertible Matrices

Definition 2.26. Let A be an $n \times m$ matrix, and B, C be $m \times n$ matrices. Then, B is a **left inverse** of A if $B \cdot A = I_m$. C is a **right inverse** for A if $A \cdot C = I_n$.

Theorem 2.27. *If A has both a left inverse B and a right inverse C, then they are unique and equal.*

Definition 2.28. A is **invertible** if A has both a right inverse and a left inverse. The unique matrix that is both a right inverse and a left inverse of A is called the **inverse** of A and is denoted A^{-1}.

Theorem 2.29. *Let A be the matrix of size $n \times m$. If A is invertible, then $n = m =$rank A.*

Theorem 2.30. *Let A be a matrix of size $m \times n$. If $n = m =$rank A, then A is invertible. Furthermore, A is a product of elementary matrices.*

Theorem 2.31. *If A is a square matrix, B is a left matrix of A implies B is also the right inverse.*

2.8. Determinants

Definition 2.32. Let A be an $n \times n$ matrix, a function det: $M_{n \times n} \to \mathbb{R}$ is called the **determinant function** if it satisfies the following axioms:

i) If B is the matrix obtained by exchanging any two rows of A, then det $B =$ det A.
ii) Given i, the function det A is linear as a function of the i^{th} row alone. Otherwise stated, holding all else constant $det(x_0, \ldots, x_i, \ldots, x_n)$ is a linear function.
iii) det $I_n = 1$

Using this definition, we can construct the det function.

Remark 2.33. Let $A = (a_{ij})$ where $1 \leq i, j \leq n$. Then, det $A = \sum_{\sigma \in S}$ sign $\sigma a_{1\sigma(1)} a_{2\sigma(2)} \ldots a_{n\sigma(n)}$ where S is the set of bijective functions $\sigma : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, n\}$.

Theorem 2.34. *Let A be an $n \times n$ matrix.*

- *If E is the elementary matrix corresponding to the operation that exchanges rows i and j then det $(E \cdot A)$ = -det A.*
- *If $E^{'}$ is the elementary matrix corresponding to the operation that replaces row i of A by itself plus c times row j, then det $(E^{'} \cdot A)$ = det A.*
- *If $E^{''}$ is the elementary matrix corresponding to the operation that multiplies row i of A by the non zero scalar λ, then det $(E^{''} \cdot A) = \lambda$ det (A)*
- *If A is the identity matrix I_n, then det A = 1.*

Theorem 2.35. *Let A be a square matrix. If the rows of A are independent, then det $A \neq 0$. If the rows are dependent, then det A = 0. Thus, an $n \times n$ matrix A has rank n if and only if det $A \neq 0$.*

Theorem 2.36. *Given a square matrix A, reducing it to echelon form B by elementary row operations of E and $E^{'}$. If B has a zero row, then det A = 0. Otherwise, let k be the number of row exchanges involved in the reduction process, then det A = $(-1)^k$ times the product of the diagonal entries of B.*

Theorem 2.37. *Let A and B be $n \times n$ matrices. Then det $(A \cdot B)$ = det $(A) \cdot$ det (B).*

Theorem 2.38. *det (A^{tr}) = det A.*

Definition 2.39. Let A be an $n \times n$ matrix. The **(i,j) minor** of A is a $(n-1) \times (n-1)$ matrix obtained from A by deleting the i^{th} row and j^{th} column of A, denoted A_{ij}.

Theorem 2.40. *A is an $n \times n$ matrix of rank n, $B = A^{-1}$, then*

$$b_{ij} = (-1)^{j+1} \frac{det\,(A_{ji})}{det\,(A)}$$

Theorem 2.41. *Let A be an $n \times n$ matrix, i fixed, then:*

$$det\, A = \sum_{k=1}^{n} (-1)^{k+1} a_{ik} \cdot det\,(A_{ik})$$

Topology of $\mathbb{R}^n$

3.1. Metric Spaces

Definition 3.1. Let X be a set. A **metric** on X is a function $d : X \times X \to \mathbb{R}$ such that:

i) $d(x, y) = d(y, x)$ for $x, y \in X$
ii) $d(x, y) \geq 0$ with $d(x, y) = 0$ if and only if $x = y$
iii) For all $x, y, z \in X$, $d(x, y) + d(y, z) \geq d(x, z)$. This is known as the **triangle inequality**

Example 3.2. Let $X = \mathbb{R}$, and define $d(x, y) = |x - y|$. Then X is a metric on $\mathbb{R}$.

Example 3.3. Let $X = \mathbb{R}^n$, and define $d_1(x, y) = \|x - y\| = (\sum_{k=1}^{n}(x_k - y_k)^2)^{\frac{1}{2}}$. Furthermore, let $d_2(x, y) = |x - y| = \max |x_i - y_i|$. d_2 is known as the **sup metric** and d_1 is known as the **Euclidean metric**.

Example 3.4. Let X be any set and define $d(x, y)$ be 0 if $x = y$ and 1 if $x \neq y$. Then d is a metric space and is called the **discrete metric**.

Example 3.5. Let $X = \mathcal{C}([0, 1])$ be the set of all continous functions $f : [0, 1] \to \mathbb{R}$ with $f, g \in X$. Define $d(f, g) = max_{x \in [0,1]}|f(x) - g(x)|$. Note that, f, g continous functions which implies $f - g$ is continous which implies $|f - g|$ is continous. Since $[0, 1]$ is bounded and $|f - g|$ is continous, max exists over $x \in [0, 1]$.

Definition 3.6. A metric space X is a set X along with a choice of a metric on it. For example, the Euclidean metric is a metric space.

3.2. Open and Closed Sets

Definition 3.7. Given a metric space X, and $x_0 \in X$ and $\epsilon > 0$; $U(x_0, \epsilon) = \{x \in X : d(x_0, x) < \epsilon\}$ is the **epsilon-neighbourhood** (ϵ-nbd) or the $\epsilon-$ball around x_0.

Example 3.8. Consider the sup metric on $\mathbb{R}^2$. Then $U(0, 1)$ is as shown in Figure 1. This shows that just because it is called the ϵ-ball, it doesn't have to look like

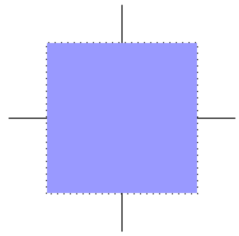

Figure 1. A "ball" of radius 1 centered at (0,0) in the sup metric

one! Consider the Euclidean metric on $\mathbb{R}^2$ and $U(0,1)$. Then, it looks like an actual "ball" as shown in Figure 2.

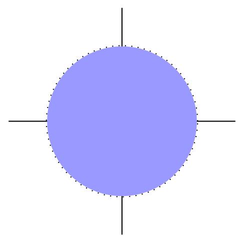

Figure 2. A "ball" of radius 1 centered at (0,0) in the Euclidean metric

Definition 3.9. A set $U \subseteq X$ is called **open** if for all $x_0 \in U$, there exists $\epsilon > 0$ such that $U(x_0, \epsilon) \subseteq U$. In other words, for every point in U, we can draw an ϵ-nbd around the point such that it is contained in U. A set $F \subseteq X$ is **closed** if $F^{\mathrm{c}} = X \setminus C$.

Remark 3.10. There exist sets which are closed and open. It follows by vacous logic that the empty set is open, and since X contains every point in X, then any ϵ-neighbhourhood around a point in X will also be in X. Since $X^{\mathrm{c}} = \emptyset$, then it follows that the empty set is open and closed! By similar logic, we also arrive at the statement that X is both closed and open.

Theorem 3.11. *Let X be a metric space and for any $x \in X$, $U(x,r)$ for $r > 0$ is open.*

Proof. Let $y \in U(x,r)$, and take $\epsilon = r - d(x,y)$. Then, we have that $\epsilon > 0$ as $y \in U(x,r)$, thus $d(x,y) < r$. Thus it is left to show that $U(y,\epsilon) \subseteq U(x,r)$. Let $z \in U(y,\epsilon)$, implying:

$$d(x,z) < d(x,y) + d(y,z) < d(x,y) + \epsilon = d(x,y) + r - d(x,y) = r$$

so that $z \in U(x,r)$. $\square$

Theorem 3.12 (Properties of Open Sets)**.** *Let I be an indexing set and $U_{\alpha \in I}$ be open sets. Then, the following are true:*

i) $\emptyset, X$ *are open*

ii) *An arbitrary union of open sets is open.*

iii) *For all $1 \leq i \leq n$, U_i is open implies that $\cap_{i=1}^{n} U_i$ is open.*

Proof. Let X be a metric space, I an indexing set and $U_{\alpha \in I}$ be open sets. Then:

ii) Let x be some point in $\cup_{\alpha \in I} U_\alpha$. Without loss of generalization, let $x \in U_{\alpha_0}$. Since U_α itself is open, there exists $\epsilon > 0$ such that $U(x, \epsilon)$ is within $U_{\alpha_0} \subseteq \cup_{\alpha \in I} U_\alpha$.

iii) Let $x \in \cap_{i=1}^{n} U_i$. Then for all $i, x \in U_i$, choose $\epsilon_i > 0$ such that $U(x, \epsilon_i) \subseteq U_i$. Then let $\epsilon = min\{\epsilon_i : 1 \leq i \leq n\}$, then $\epsilon > 0$ and for all i, $U(x, \epsilon) \subseteq U(x, \epsilon_i) \subseteq U_i$ which implies $U(x, \epsilon) \subseteq U_i$.

□

Theorem 3.13 (Properties of Closed Sets)**.** *Let I be an indexing set and $U_{\alpha \in I}$ be open sets. Then, the following are true:*

i) *$\emptyset, X$ are closed.*

ii) *For all α, F_α is closed implies that $\cap_\alpha F_\alpha$ is closed.*

iii) *For all $1 \leq i \leq n$, F_i is closed so that $\cap_{i=1}^{n} F_i$ is closed.*

Proof. Related to the proof of theorem above. □

Theorem 3.14. *Let $\mathbb{R}^n$ be a metric space with inscribed metrics d_1 and d_2. Let $d_1(x, y) = \|x - y\|$ and $d_2(x, y) = |x - y|$:*

i) *U is open relative to $d_1 \leftrightarrow U$ is open relative to d_2.*

ii) *F is closed relative to $d_1 \leftrightarrow U$ is open relative to d_2.*

Proof. We will first show (i) is true. Assume U is d_1-open. Let $x \in U$. Then there exists $\epsilon > 0$ such that $U_{d_1}(x, \epsilon) \subseteq U$. We want to show $U_{d_2}(x, \frac{\epsilon}{\sqrt{n}}) \subseteq U_{d_1}(x, \epsilon) \subseteq U_{d_2}(x, \epsilon)$.
If $y \in U_{d_1}(x, \epsilon)$ then $d_2(x, y) \leq d_1(x, y) \leq \epsilon$, so $y \in U_{d_2}(x, \epsilon)$. If $y \in U_{d_2}(x, \frac{\epsilon}{\sqrt{n}})$, then $d_2(x, y) < \frac{\epsilon}{\sqrt{n}}$ and $d_1(x, y) \leq \sqrt{n} d_2(x, y) < \epsilon$, then $y \in U_{d_1}(x, \epsilon)$.
Now: $U_{d_2}(x, \frac{\epsilon}{\sqrt{n}} \subseteq U_{d_1}(x, \epsilon) \subseteq U$ which implies d_2 is open.
All is left to show that if d_2-open then d_1-open. ii) is similar to i). □

Theorem 3.15. *Let X be a metric space and $Y \subseteq X$ also a metric space. Then:*

(1) *$U \subseteq Y$ is open if and only if there exists $V \subseteq X$ such that V is open in X and $U = V \cap Y$.*

(2) *$F \subseteq Y$ is closed if and only if there exists $G \subseteq X$ such that G is closed in X and $F = Y \cap G$.*

Example 3.16. Let $X = \mathbb{R}^2, Y = \mathbb{R} \times \{0\} \subseteq \mathbb{R}^2$. Refer to Figure 3. This shows that $\mathbb{R}$ is closed in $\mathbb{R}^2$ and not open and that if we take a open subset of $\mathbb{R}$, it is not open in $\mathbb{R}^2$.

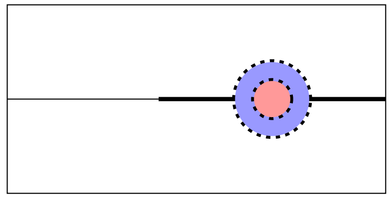

Figure 3. Where the rectangle is X, the thin line is Y, the red circle is U and the blue circle is V

3.3. Limit Points and Closure

Definition 3.17. Let $x_o \in X$. Then x_0 is a **limit point** of $A \subseteq X$ if for all $\epsilon > 0$, then: $(U(x_0, \epsilon) \cap A)n\{x_o\} \neq \emptyset$.

Remark 3.18. It is an easy exercise to show that the definition above can be reformulated to state that

$$\forall\ \epsilon > 0,\ |U(x_0, \epsilon) \cap A| = \infty$$

Definition 3.19. The closure of a set A denoted $\overline{A}$ is defined to be $A\ \cup\ lp(A)$ where $lp(A) := \{\text{limit points of } A\}$.

Example 3.20. In $\mathbb{R}$, the closure of $(0, 1)$ is $[0, 1]$ or using appropriate notation

$$\overline{(0, 1)} = [0, 1]$$

Example 3.21. The limit points of the set of sequence $\{\frac{1}{n}\}_{n=1}^{\infty}$ is $\{0\}$.

Theorem 3.22. *A is closed if and only if $A = \overline{A}$.*

Proof. Since $lp(A) \subseteq A \leftrightarrow \overline{A} = A$, proving the statement in the theorem is the same as showing that A^c is open if and only if $lp(A) \subseteq A$.
We will first show the $\implies$ direction.
Suppose by contradiction that $x_0 \in lp(A) \cap A^c$. If we try to find an $\epsilon > 0$ such that $U(x_0, \epsilon) \subseteq A^c$ but then $U(x_0, \epsilon) \cap A = \emptyset$, contradicting the statement that $x_0 \in lp(A)$.
The other direction is left as an exercise. □

Remark 3.23. We make the following remarks. It is left to the reader to prove them.

(i) The smallest closed containing A is $\overline{A}$.
(ii) The intersection of all sets containing A.

3.4. Continuity

Definition 3.24. Let X and Y be metric spaces with metrics d_X and d_Y respectively. Let $F : X \to Y$ $x_0 \in X$ we say that F is continous at x_0 if there exists a neighbourhood U of x_0 such that $F(U) \subseteq V$. This is the same as saying for all $\epsilon > 0$, there exists $\delta > 0$ such that $d_X(x_0, x) < \delta \implies d_Y(F(x_0), F(x)) < \epsilon$. Note that, if $X = Y = \mathbb{R}$, this definition is equivalent of that given in MAT157Y1.

Theorem 3.25. *The following are equivalent (for $F : X \to Y$):*

(i) *F is continous*
(ii) *For every open $V \subseteq Y$, $f^{-1}(V)$ is open*
(iii) *For every F closed in Y, $f^{-1}(F)$ is closed*

Before we prove this theorem, it is useful to keep a few facts about pre-images and images in mind.

Remark 3.26. Let $A \cup B \subseteq X$ and $C \cup D \subseteq Y$ and $f : X \to Y$. Then, the following is true:

(i) $f^{-1}(C \cup D) = f^{-1}(C) \cup f^{-1}(D)$
(ii) $f^{-1}(C \cap D) = f^{-1}(C) \cap f^{-1}(D)$
(iii) $f^{-1}(D^{\mathrm{c}}) = (f^{-1}(D))^{\mathrm{c}}$
(iv) $f(A \cup B) = f(A) \cup f(B)$
(v) $f(A \cap B) \subset f(A) \cap f(B)$
(vi) $f(A^{\mathrm{c}}) \supseteq (f(A))^{\mathrm{c}}$

Now, we give the proof of theorem 3.25.

Proof. First, we show $(i) \implies (ii)$: Let $V \subseteq Y$ be open. Let $x_0 \subseteq f^{-1}(V)$ or equivalently, $f(x_0) \in V$ so v_0 is a neighbourhood of $f(x_0)$ by assumption of continuity. There is a neighbourhood U of x_0 such that $f(U) \subseteq V$. But then $x_0 \in U \in f^{-1}(V)$ so that $f^{-1}(V)$ is open.
Second, we show $(ii) \implies (i)$: Let $x_0 \in X$, let V be a neighbourhood of $f(x_0)$. Let $U = f^{-1}(V)$. By assumption, U is open, clearly $x_0 \in U$ so U is a neighbourhood of x_0 and $f(U) = f(f^{-1}(V)) \subseteq V$ as required.
$(ii) \implies (iii)$ follows from the fact that $f^{-1}(f^{\mathrm{c}}) = (f^{-1}(f))^{\mathrm{c}}$ □

Theorem 3.27 (Properties of Continous Functions)**.** *The following are true:*

(i) *Constant functions are continous*
(ii) *Let $A \subseteq X$ and suppose $f : X \to Y$ be continous. Then $f\big|_A$ is continous.*
(iii) *Let both $f : X \to Y$ and $g : R \to S$ be continous. Then $f \circ g$ is continous*
(iv) *$F : X \to \mathbb{R}^n$ is continous if and only if each F_i is continous*
(v) $\pi_i : \mathbb{R}^n \to \mathbb{R}, \pi_i(X) = X_i$
(vi) *f, g continous. Then $f + g, f - g, \frac{f}{g}, f \cdot g$ are all continous.*

3.5. Exteriors, Interiors and Boundaries

Definition 3.28 (The Interior of a Set)**.** If $A \subset X$, then the **interior of** A is defined as the union of all open sets contained in A or otherwise stated:

$$\mathrm{int}(A) := \{x \in A : \exists\, \epsilon > 0,\ \cup(x, \epsilon) \subset A\}$$

One can also check that the interior of a set is the *maximal* open set contained in A.
The following two definitions borrow the set A from Definition 3.28.

Definition 3.29. The **exterior of** A is defined to be $int(A^c)$ and is denoted via $\text{Ext}(A)$.

Definition 3.30. The **boundary of** A is defined to be $X \backslash (\text{int}(A) \cup \text{ext}(A))$ and is denoted via $\text{Bd}(A)$.

We make the following propositions:

Proposition 3.31. *Let $A \subset X$. Then:*

(1) $ext(A) = X \backslash \overline{A}$
(2) $int(A) = X \backslash \overline{A^c}$
(3) $Bd(A) = \overline{A} \cap \overline{A^c}$

3.5.1. Compactness.

Definition 3.32. Let A be a set. An **open cover** of A is defined to be a collection of open sets such that if $x \in A$, then $x \in \mathcal{O}_\alpha$ where $\mathcal{O}_\alpha$ is an open set in the open cover. We say the open cover is **finite** (or a finite cover) if the collection contains only finite open sets.

Definition 3.33. Let $\{\mathcal{O}_{\alpha \in A}\}$ (A, an index set) be an arbitrary open cover of the set Y. Then an open cover $\{\mathcal{Y}_{\beta \in B}\}$ is said to be a **sub cover of** $\{\mathcal{O}_{\alpha \in A}\}$ if every open set in $\{\mathcal{Y}_{\beta \in B}\}$ is also in $\{\mathcal{O}_{\alpha \in A}\}$

Example 3.34. Let $\mathcal{O}$ be the union of all open sets such that $\mathcal{O} = (0,1)$. Then $\mathcal{O}$ is an *open cover* of $(0,1)$. It can be easily proven that such a cover $\mathcal{O}$ can be finite.

Example 3.35. Let $\mathcal{F}$ be the union of all intervals $(\alpha, \alpha+1)$ where $\alpha \in \mathbb{Z}$, then it can be proven that $\mathcal{F}$ is an open cover of $\mathbb{R}$ and that no $\mathcal{F}$ can exist such that $\mathcal{F}$ is finite.

Definition 3.36. A space X is called **compact** if for every open cover of X, there exists a finite sub-cover such that it covers X.

Example 3.37. We now give some common examples of compact sets. A set with finitely many points is compact. The set $[0,1]$ is compact but $(0,1]$ is not compact since $(0,1] = \bigcup_{n \in \mathbb{N}} (\frac{1}{n}, 1]$ which is a cover for which we cannot find a finite number of subsets.

Theorem 3.38. *A continuous function on a compact space is bounded*

Proof. Let f be continuous and X be the bounded set. Let $\mathcal{X}$ be an open cover for $f(X)$ where $\mathcal{X} = \cup_{\alpha \in I} \mathcal{X}_\alpha$. Since f is continuous and each $\mathcal{X}_\alpha$ is open, then $f^{-1}(\mathcal{X}_\alpha)$ is open. Furthermore the union of all such pre-images forms an open cover of X. Since X is compact, a finite subcover of this cover exists. Take the image of this open cover to obtain a finite cover of $f(X)$. □

Definition 3.39. $A \subset X$ is compact if θ is a covering of X where finitely many of the sets from θ already cover A.

Theorem 3.40. *The two definitions of compactness are equivalent.*

Proof. The main idea is that $V \subset A$ is open if and only if there $\exists\, U \subset X$ such that $U \cap A = V$. □