Problem Set 12

MAT237: Advanced Calculus
January 29, 2017 MESSENGER BOY

Problem 0.1. *For each of the following manifolds, determine can be written as the graph of the function, the zero-locus of a function, and parameterically. Give the appropriate functions in each case.*

(a) *The ellipse* $ax^2 + by^2 = c^2$.

(b) *The set* $\{(t^2 + t, 2t - 1) : t \in \mathbb{R}\}$

(c) *The plane* $3x - 4y + 3z - 10 = 0$.

(d) *The sphere of the radius* r: $S_r = \{x \in \mathbb{R} : |x| = r\}$

(e) *The cylinder* $\{(x, y, z) : x^2 + y^2 = 4\}$

(f) *The intersection of the plane* $x + z = 1$ *with the sphere* $x^2 + y^2 + z^2 = 1$.

(g) *If* $f : [a, b] \to \mathbb{R}$ *let* S *be the space defined by revolving the graph about the* $x - axis$.

Solution.

(a) An ellipse in $\mathbb{R}^2$ is not injective so it cannot be written as the graph of a function. The zero locus is $ax^2 + by^2 - c^2 = 0$. This can be written parameterically as $\{(\frac{c}{\sqrt{b}} \cos\theta, \frac{c}{\sqrt{b}} \sin\theta\}$.

(b) Parameterically, this can be written as the function $f : t \to (t^2 + t, 2t - 1)$. This cannot be written as the graph of a function $\mathbb{R} \to \mathbb{R}$. Finally, its zero locus is given by an equation of the form $y^2 - x = 0$.

(c) The plane $3x - 4y + 3z = 10$. It's zero locus is $3x + 4y + 3z - 10 = 0$. By definition a plane is not injective, we cannot write it as the graph of a function $\mathbb{R}^2 \to \mathbb{R}^2$. Finally, its parameterization. Consider: $3x = 10 + 4y - 3z \to x = \frac{10+4y-3z}{3}$ so it is given by the function $f : (y, z) \mapsto (\frac{10+4y-3z}{3}, y, z)$

(d) The zero locus is the set: $\{x \in \mathbb{R}^3 : |x| - r = 0\}$. There is no function $f : \mathbb{R}^3 \to \mathbb{R}^3$ such that it is the graph of S_r since it is a sphere. Finally, the parameteric function is given by $f : (\theta, \phi) \to (r \sin\theta \cos\phi, r \sin\theta \sin\phi, r \cos\theta)$

(e) The zero locus is the set: $\{(x, y, z) : x^2 + y^2 - 4 = 0\}$. There is no function $f : \mathbb{R}^3 \to \mathbb{R}^3$ that can graph this set. The parameteric function is given by $f : (\theta, z) \mapsto (2\cos\theta, 2\sin\theta, z)$

(f) Subsituting $x + z = 1$ into $x^2 + y^2 + z^2 = 1$, we obtain that: $2x^2 + y^2 - 2 = 0$ is the zero locus.

(g) Assume we are in $\mathbb{R}^3$, and our graph is an $\mathbb{R}^2$ object. First, of all we cannot describe a function $\mathbb{R}^3 \to \mathbb{R}^3$ such that it is the intersection of those two equations. Parameterically, we are using x, θ so the parameterization is given by: $f : (x, \theta) \to (x, f(x)\cos\theta, f(x)\sin\theta)$. Letting $y = f(x)\cos\theta$ and $z = f(x)\sin\theta$, then $f(x)^2 = y^2 + z^2$ which gives us our zero locus.

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Problem 0.2. *Determine which of the following spaces are smooth.*

(a) The set $C = \{(\cos t, \sin 2t) : t \in [0, 2\pi)\}$

(b) Let S *be the surface defined by the image of* $g : \mathbb{R}^2 \to \mathbb{R}^2$*:*

$$g(s,t) = (3s, s^2 - 2t, s^3 + t^2)$$

(c) Let $S = F^{-1}(0)$ *where* $F(x, y, z) = 3xy + x^2 + z$*.*

(d) Let $S = F^{-1}(0)$*, where* $F(x, y, z) = \cos xy + e^z$*.*

Solution.

(a) Let $f : t \mapsto (\cos t, \sin 2t)$ with $t \in [0, 2\pi)$. Then, we have that $f^{'}(t) = (-\sin t, 2\cos 2t)$. which has no solutions for $(0,0)$ in $[0, 2\pi)$ and all remains to check is, if the fucntion is injective. One easy way to tell is that the graph of the function is the leminscate implying that the function is not injective at 0. Therefore, we can never find a neighborhood N around 0 such that $C \cap N$ is the graph of a function.

(b) Consider,

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