MATC34 Exam Study

Anmol Bhullar

December 13, 2017

Part 1 Preliminaries to Complex Analysis
1.1 Basic Properties

Definition 0.1. A **complex number** takes the form $z = x + iy$ where x and y are real, and i the imaginary number that satisfies $i^2 = -1$. x and y are said to be the **real part** and the **imaginary part** of z, respectively, and we write

$$x = \text{Re}(z) \qquad \text{and} \qquad y = \text{Im}(z)$$

A complex number with zero real part is said to be **purely imaginary**. The set of all numbers of the form $z = x + iy$ is denoted by $\mathbb{C}$.

The natural rules for adding and multiplying complex numbers are:

$$z_1 + z_2 = (x_1 + x_2) + i(y_1 + y_2)$$

and

$$z_1 z_2 = (x_1 x_2 - y_1 y_2) + i(x_1 y_2 + y_1 x_2)$$

$\mathbb{C}$ is a field with these operations so 0 is the additive identity, 1 is the multiplicative identity, and $\mathbb{C}$ has the properties of commutativity, associativity and distributivity.
Addition of complex numbers can be seen as the addition of two vectors in $\mathbb{R}^2$. Multiplication, however consists of a rotation composed with a dilation, where multiplication by i specifically, corresponds to a rotation by an angle of $\pi/2$.

Definition 0.2. The notion of length, or absolute value of a complex number is identical to the notion of Euclidean length in $\mathbb{R}^2$. Define the **absolute value** of a complex number $z = x + iy$ by

$$|z| = (x^2 + y^2)^{\frac{1}{2}}$$

so that $|z|$ is precisely the distance from the origin to the point (x, y). Note this is a norm, so the triangle inequality holds as well for all points in $\mathbb{C}$.

Definition 0.3. The **complex conjugate** of $z = x + iy$ is defined by

$$\overline{z} = x - iy,$$

and it is obtained by a reflection across the real axis in the plane. One can show that a complex number is real if and only if $z = \overline{z}$, and it is purely imaginary if and only if $z = -\overline{z}$.

Lemma 1 (Results of conjugation). One has,

$$\mathrm{Re}(z) = \frac{z + \overline{z}}{2} \qquad \text{and} \qquad \mathrm{Im}(z) = \frac{z - \overline{z}}{2i}$$

Furthermore,

$$|z|^2 = z\overline{z} \qquad \text{and as a consequence} \quad \frac{1}{z} = \frac{\overline{z}}{|z|^2}$$

whenever $z \neq 0$.

Definition 0.4. Any non-zero complex number z can be written in **polar form**

$$z = re^{i\theta}$$

where $r > 0$; also $\theta \in \mathbb{R}$ is called the **argument** of z (defined uniquely up a multiple of 2π) and is often denoted by $\arg z$, and

$$e^{i\theta} = \cos\theta + i\sin\theta.$$

Since $\left|e^{i\theta}\right| = 1$ we observe that $r = |z|$, and θ is simply the angle (with positive counterclockwise orientation) between the positive real axis and half-line starting at the origin and passing through z.

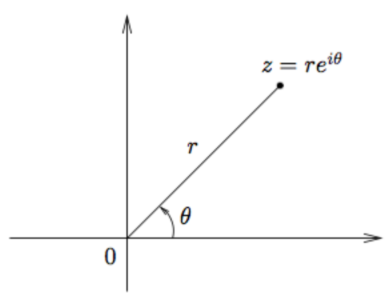

Figure 2. The polar form of a complex number

Finally, note that if $z = re^{i\theta}$ and $w = se^{i\varphi}$, then

$$zw = rse^{i(\theta+\varphi)},$$

so multiplication by a complex number corresponds to a homothety in $\mathbb{R}^2$ (that is, a rotation composed with a dilation).

2.1 Functions on the complex plane

Definition 0.5. Let f be a function defined on a set Ω of complex numbers. We say that f is **continuous** at the point $z_0 \in \Omega$ if for every $\epsilon > 0$ there exists $\delta > 0$ such that whenever $z \in \Omega$ and $|z - z_0| < \Omega$ then $\left|f(z) - f(z_0)\right| < \epsilon$. Equivalently, if for every sequencec $\{z_1, z_2, \ldots\} \subseteq \Omega$ such that $\lim z_n = z_0$, then $\lim f(z_n) = f(z_0)$. The function f is said to be continuous on Ω if it is continuous at every point of Ω.

Definition 0.6. We say that f attains a **maximum** at the point $z_0 \in \Omega$ if

$$\left|f(z)\right| \leq \left|f(z_0)\right| \qquad \text{for all } z \in \Omega$$

with the inequality reversed for the definition of a **minimum**.

2.2 Holomorphic functions

Definition 0.7. Let Ω be an open set in $\mathbb{C}$ and f a complex-valued function on Ω. The function f is **holomorphic at the point** $z_0 \in \Omega$ if the quotient

$$\frac{f(z_0 + h) - f(z_0)}{h} \tag{1}$$

converges to a limit when $h \to 0$. Here $h \in \mathbb{C}$ and $h \neq 0$ with $z_0 + h \in \Omega$, so that the quotient is well defined. The limit of the quotient, when it exists, is denoted by $f'(z_0)$, and is called the **derivative of f at z_0**:

$$f'(z_0) = \lim_{h \to 0} \frac{f(z_0 + h) - f(z_0)}{h}$$

The function f is said to be **holomorphic on** Ω if f is holomorphic at every point of Ω. If f is holomorphic in all of $\mathbb{C}$ we say that f is **entire**.

Example. *The function $f(z) = \overline{z}$ is not holomorphic. Indeed, we have*

$$\frac{f(z_0 + h) - f(z_0)}{h} = \frac{\overline{h}}{h}$$

which has no limit as $h \to 0$, as one can see by first taking h real and then h purely imaginary.

Example. *In terms of real variables, the function $f(z) = \overline{z}$ corresponds to the map $F : (x, y) \mapsto (x, -y)$ which is differentiable in the real sense. Its derivative at a point is the linear map given by its Jacobian, the 2×2 matrix of partial derivatives of the coordinate functions. In fact, F is linear and is therefore equal to its derivative. This implies F is actually indefinitely differentiable. In particular, the existence of the real derivative need not guarantee that f is holomorphic.*

Cauchy-Riemann Equations

Associate to each complex-valued function $f = u+iv$ the mapping $F(x, y) = (u(x, y), v(x, y))$ from $\mathbb{R}^2$ to $\mathbb{R}^2$.
Recall that a function $F(x, y) = (u(x, y), v(x, y))$ is said to be differentiable at a point $P_0 = (x_0, y_0)$ if there exists a linear transformation $J : \mathbb{R}^2 \to \mathbb{R}^2$ such that

$$\frac{\big|F(P_0 + H) - F(P_0) - J(H)\big|}{|H|} \to 0 \qquad \text{as } H \to 0, H \in \mathbb{R}^2 \tag{2}$$

Equivalently, we can write

$$F(P_0 + H) - F(P_0) = J(H) + |H|\,\Psi(H),$$

with $\big|\Psi(H)\big| \to 0$ as $|H| \to 0$. The linear transformation J is unique and is called the derivative of F at P_0. If F is differentiable of u and v exist, and the linear transformation J is described in the standard basis of $\mathbb{R}^2$ by the Jacobian matrix of F

$$J = J_F(x, y) = \begin{pmatrix} \partial u/\partial x & \partial u/\partial y \\ \partial v/\partial x & \partial v/\partial y \end{pmatrix}$$

Consider the limit in (2) when h is first real, say $h = h_1 + ih_2$ with $h_2 = 0$. Then, if we write $z = x + iy$, $z_0 = x_0 + iy_0$, and $f(z) = f(x, y)$, we find that

$$\begin{aligned} f'(z_0) &= \lim_{h_1 \to 0} \frac{f(x_0 + h_1, y_0) - f(x_0, y_0)}{h_1} \\ &= \frac{\partial f}{\partial x}(z_0), \end{aligned}$$

where $\partial/\partial x$ denotes the usual partial derivative in the x variable. Now taking h purely imaginary, say $h = ih_2$, a similar argument yields

$$\begin{aligned} f'(z_0) &= \lim_{h_2 \to 0} \frac{f(x_0, y_0 + h_2) - f(x_0, y_0)}{ih_2} \\ &= \frac{1}{i}\frac{\partial f}{\partial y}(z_0), \end{aligned}$$

where $\partial/\partial y$ is partial differentiation in the y variable. Therefore, if f is holomorphic we have shown that

$$\frac{\partial f}{\partial x} = \frac{1}{i}\frac{\partial f}{\partial y}.$$

Writing $f = u + iv$, we find that separating real and imaginary parts and using $1/i = -i$, that the partials of u and v exist, and they satisfy the following non-trivial relations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad \text{and} \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

These are the **Cauchy-Riemann** equations. We can also say:

$$\frac{\partial}{\partial z} = \frac{1}{2}\big(\frac{\partial}{\partial x} + \frac{1}{i}\frac{\partial}{\partial y}\big) \qquad \text{and} \qquad \frac{\partial}{\partial \overline{z}} = \frac{1}{2}\big(\frac{\partial}{\partial x} - \frac{1}{i}\frac{\partial}{\partial y}\big).$$

Proposition 1. If f is holomorphic at z_0, then

$$\frac{\partial f}{\partial \overline{z}}(z_0) = 0 \qquad \text{and} \qquad f'(z_0) = \frac{\partial f}{\partial z}(z_0) = 2\frac{\partial u}{\partial z}(z_0).$$

Also, if we write $F(x, y) = f(z)$, then F is differentiable in the sense of real variables, and

$$\det J_F(x_0, y_0) = \left|f'(z_0)\right|^2$$

Theorem 2. *Suppose $f = u + iv$ is a complex-valued function defined on an open set Ω. If u and v are continuously differentiable and satisfy the Cauchy-Riemann equations on Ω, then f is holomorphic on Ω and $f'(z) = {}^{\partial f}/_{\partial z}$.*

2.3 Power Series

Example. *A prime example of a power series is the complex* ***exponential*** *function, which is defined for $z \in \mathbb{C}$ by*

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}.$$

Definition 0.8. In general, a **power series** is an expansion of the form

$$\sum_{n=0}^{\infty} a_n z^n, \tag{3}$$

where $a_n \in \mathbb{C}$.

Definition 0.9. Given a power series $\sum_{n=0}^{\infty} a_n z^n$, there exists $0 \leq R \leq \infty$ such that:

1. If $|z| < R$ the series absolutely converges.
2. If $|z| < R$ the series diverges.

Moreover, if we use the convention that $1/0 = \infty$ and $1/\infty = 0$, then R is given by Hadamard's formula

$$1/R = \limsup |a_n|^{1/n}$$

The number R is called the **radius of convergence** of the power series, and the region $|z| < R$ the **disc of convergence**.

Example. *In particular, we have $R = \infty$ in the case of the exponential function. In contrast, the geometric series*

$$\sum_{n=0}^{\infty} z^n$$

converges absolutely only in the disc $|z| < 1$ (since its sum is equal to the function $1/(1-z)$ which is holomorphic in the open set $\mathbb{C} - 1$) so $R = 1$ for the geometric series.

Remark. On the boundary of the disk of the convergence, $|z| = R$, the situation is more delicate as one can have either convergence or divergence. For example, the power series $\sum nz^n$ does not converge on any point of the unit circle but the power series $\sum z^n/n^2$ converges at every point of the unit circle.

Example. *The power series of the* ***standard trigonometric functions*** *are defined by*

$$\cos z = \sum_{n=0}^{\infty} (-1)^n \frac{z^n}{(2n)!}, \quad \textit{and} \quad \sin z = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!},$$

and they agree with the usual cosine and sine of a real argument whenever $z \in \mathbb{R}$. Furthermore, we have the **Euler formulas** for sine and cosine functions,

$$\cos z = \frac{e^{iz} + e^{-iz}}{2} \qquad \text{and} \qquad \sin z = \frac{e^{iz} - e^{-iz}}{2i}$$

Theorem 3. *The power series $f(z) = \sum_{n=0}^{\infty} a_n z^n$ defines a holomorphic function in its disc of convergence. The derivative of f is also a power series obtained by differentiating term by term the series of f, that is,*

$$f'(z) = \sum_{n=0}^{\infty} n a_n z^{n-1}$$

Moreover, f' has the same radius of convergence as f'.

Corollary 3.1. A power series is infinitely complex differentiable in its disc of convergence, and the higher derivatives are also power series obtained by termwise differentation.

Definition 0.10. A function f is defined on an open set Ω is said to be **analytic** (or have a **power series expansion**) at a point $z_0 \in \Omega$ if there exists a power series $\sum a_n(z - z_0)^n$ centered at z_0, with a positive radius of convergence, such that

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n \qquad \text{for all } z \text{ in a neighborhood of } z_0.$$

If f has a power series expansion at every point in Ω, we say that f is **analytic on** Ω.

All holomorphic functions are analytic and vice versa.

3 Integration along curves

Definition 0.11. A **parameterized curve** is a function $z(t)$ which maps a closed interval $[a, b] \subseteq \mathbb{R}$ to the complex plane. We say that the parameterized curve is **smooth** if $z'(t)$ exists and is continuous on $[a, b]$ and $z'(t) \neq 0$ for $t \notin [a, b]$. At the points $t = a$ and $t = b$, the quantities $z'(a)$ and $z'(b)$ are interpreted as one-sided limits

$$z'(a) = \lim_{h\to 0, h>0} \frac{z(a+h) - z(a)}{h} \qquad \text{and} \qquad z'(b) = \lim_{h\to 0, h<0} \frac{z(b+h) - z(b)}{h}$$

Similarly, we say that the parameterized curve is **piecewise smooth** if z is continuous on $[a, b]$ and if there exist points

$$a = a_0 < a_1 < \ldots < a_n = b$$

where $z(t)$ is smooth in the intervals $[a_k, a_{k+1}]$. Two parameterizations,

$$z : [a, b] \to \mathbb{C} \qquad \text{and} \qquad \tilde{z} : [c, d] \to \mathbb{C},$$

are **equivalent** if there exists a continuously differentiable bijection $s \mapsto t(s)$ from $[c, d]$ to $[a, b]$ so that $t'(s) > 0$ and

$$\tilde{z}(s) = z(t(s)).$$

A smooth or piecewise-smooth curve is **closed** if $z(a) = z(b)$ for any of its parameterizations. Finally, a smooth or piecewise-smooth curve is **simple** if it is not self-intersecting, that is, $z(t) \neq z(s)$ unless $s = t$. If a curve is closed to begin with, then we say that it is simple whenever $z(t) \neq z(s)$ unless $s = t$, or $s = a$ and $t = b$.

Example. *Consider the circle $C_r(z_0)$ centered at z_0 and of radius r, which by definition is the set*

$$C_r(z_0) = \{z \in \mathbb{C} : |z - z_0| = r\}.$$

*The **positive orientation** (counterclockwise) is the one that is given by the standard parameterization*

$$z(t) = z_0 + re^{it}, \qquad \textit{whenever}\, t \in [0, 2\pi],$$

*while the **negative orientation** (clockwise) is given by*

$$z(t) = z_0 + re^{-it}, \qquad \textit{whenever}\, t \in [0, 2\pi].$$

In general, we shall consider a circle with the positive orientation.

Definition 0.12. Given a smooth curve γ in $\mathbb{C}$ parameterized by $z : [a, b] \to \mathbb{C}$, and f a continuous function on γ, we define the **integral of f along γ** by

$$\int_\gamma f(z)dz = \int_a^b f(z(t))z'(t)dt.$$

In order for this to be meaninful, we have to show this integral is *independent* of the parameterization we choose. Say that $\tilde{z}$ is an equivalent parameterization as above. Then the change of variables of formula and the chain rule imply that

$$\int_a^b f(z(t))z'(t)dt = \int_c^d f(z(t(s)))z'(t(s))t'(s)ds = \int_c^d f(\tilde{z}(s))\tilde{z}'(s)ds.$$

This proves that the integral of f over γ is well defined.

Definition 0.13. By definition, the **length** of the smooth curve γ is

$$\text{lenght}(\gamma) = \int_a^b \left|z'(t)\right| dt.$$

Proposition 2. Integration of continuous functions over curves satisfies the following properties:

1. It is linear, that is, if $\alpha, \beta \in \mathbb{C}$, then

$$\int_\gamma (\alpha f(z) + \beta g(z))dz = \alpha \int_\gamma f(z)dz + \beta \int_\gamma g(z)dz.$$

2. If γ^- is γ with the reverse orientation, then

$$\int_\gamma f(z)dz = -\int_{\gamma^-} f(z)dz.$$

3. One has the inequality

$$\left|\int_\gamma f(z)dz\right| \leq \sup_{z\in\gamma}\left|f(z)\right| \cdot \text{lenght}(\gamma).$$

Theorem 4. *If a continuous function f has a primitive F in Ω, and γ is a curve in Ω that begins at ω_1 and ends at ω_2, then*

$$\int_\gamma f(z)dz = F(w_2) - F(w_1)$$

Corollary 4.1. If γ is a closed curve in an open set Ω, and f is continuous and has a primitive in Ω, then

$$\int_\gamma f(z)dz = 0.$$

Corollary 4.2. If f is holomorphic in a region Ω and $f' = 0$, then f is constant.

Part 2 Cauchy's Theorem and Its Applications

1 Goursat's theorem

Theorem 5. *If Ω is an open set in $\mathbb{C}$, and $T \subseteq \Omega$ a triangle whose interior is also contained in Ω, then*

$$\int_T f(z)dz = 0$$

whenever f is holomorphic in Ω.

Corollary 5.1. If f is holomorphic in an open set Ω that contains a rectangle R and its interior, then

$$\int_R f(z)dz = 0.$$

2 Local existence of primitives and Cauchy's theorem in a disc

Theorem 6. *A holomorphic function in an open disc has a primitive in that disc.*

Theorem 7 (Cauchy's theorem for a disc)**.** *If f is holomorphic in a disc, then*

$$\int_\gamma f(z)dz = 0$$

for any closed curve γ in that disc.

Corollary 7.1. Suppose f is holomorphic in an open set containing the circle C and its interior. Then

$$\int_C f(z)dz = 0$$

3 Evaluation of some integrals

Example. *We show that if $\xi \in \mathbb{R}$, then*

$$e^{-\pi\xi^2} = \int_{-\infty}^{\infty} e^{-\pi x^2} e^{-2\pi i x\xi} dx \tag{4}$$

If $\xi = 0$, the formula is precisely the known integral

$$1 = \int_{-\infty}^{\infty} e^{-\pi x^2} dx.$$

Now suppose that $\xi > 0$, and consider the function $f(z) = e^{-\pi z^2}$, which is entire, and in particular holomorphic in the interior of the toy contour

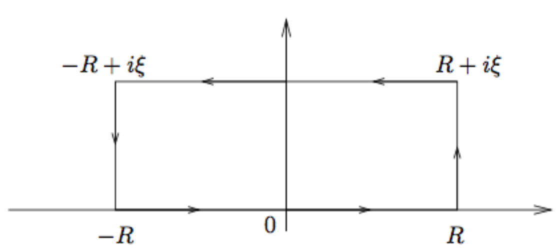

Figure 8. The contour γ_R in Example 1

The contour γ_R consists of a rectangle with vertices $R, R+i\xi, -R+i\xi, -R$ and the positive counterclockwise orientation. By Cauchy's theorem,

$$\int_{\gamma_R} f(z)dz = 0 \tag{5}$$

The integral over the real segment is simply

$$\int_{-R}^{R} e^{-\pi x^2} dx,$$

which converges to 1 as $R \to \infty$. The integral on the vertical side on the right is

$$I(R) = \int_0^{\xi} f(R+iy)idy = \int_0^{\xi} e^{-\pi(R^2+2iRy-y^2)} idy.$$

This integral goes to 0 as $R \to \infty$ since ξ is fixed and we may estimate it by

$$\big|I(R)\big| \leq Ce^{-\pi R^2}.$$

The integral over the vertical segment on the left also goes to 0 as $R \to \infty$ for the same reasons. Finally, the integral over the horizontal segment on top is

$$\int_{-R}^{R} e^{-\pi(x+i\xi)^2} dx = -e^{\pi\xi^2} \int_{-R}^{R} e^{-\pi x^2} e^{-2\pi i x\xi} dx.$$

Therefore, we find in the limit as $R \to \infty$ that (5) gives

$$0 = 1 - e^{\pi\xi^2} \int_{-\infty}^{\infty} e^{-\pi x^2} e^{-2\pi i x\xi} dx,$$

and our desired formula is established. In the case $\xi < 0$, we then consider the symmetric rectangle, in the lower half plane.

4 Cauchy's integral formulas

Theorem 8. *Suppose f is holomorphic in an open set that contains the closure of a disc D. If C denotes the boundary circle of this disc with the positive orientation, then*

$$f(z) = \frac{1}{2\pi i}\int_C \frac{f(\zeta)}{\zeta - z}d\zeta \quad \textit{for any point}\, z \in D.$$

Remark. Our earlier discussion of toy contours provides simple extensions of the Cauchy integral formula; for instance, if f is holomorphic in an open set that contains a (positively oriented) rectangle R and its interior, then

$$f(z) = \frac{1}{2\pi i}\int_R \frac{f(\zeta)}{\zeta - z}d\zeta,$$

whenever z belongs to the interior of R. To establisch this result, it suffices to repeat the proof of Theorem 4.1 replacing the "circular" keyhole with a "rectangular" keyhole.

Corollary 8.1. If f is holomorphic in an open set Ω, then f has infinitely many complex derivations in Ω. Moreover, if $C \subseteq \Omega$ is a circle whose interior is also contained in Ω, then

$$f^{(n)}(z) = \frac{n!}{2\pi i}\int_C \frac{f(\zeta)}{(\zeta - z)^{n+1}}d\zeta$$

for all z in the interior of C.

Definition 0.14. The formulas in the above theorem and corollary are said to be the **Cauchy integral formulas**.

Corollary 8.2 (Cauchy's inequalities). If f is holomorphic in an open set that contains the closure of a disc D centered at z_0 and of radius R, then

$$\left|f^{(n)}(z_0)\right| \leq \frac{n!\|f\|_C}{R^n},$$

where $\|f\|_C = \sup_{z\in C}\left|f(z)\right|$ denotes the supremum of $|f|$ on the boundary circle C.

Theorem 9. *Suppose f is holomorphic in an open set Ω. If D is a disc centered at z_0 and whose closure is contained in Ω. If D is a disc centered at z_0 and whose closure is contained in Ω, then f has a power series expansion at z_0*

$$f(z) = \sum_{n=0} a_n(z - z_0)^n$$

for all $z \in D$, and the coefficients are given by

$$a_n = \frac{f^{(n)}(z_0)}{n!} \qquad \textit{for all}\, n \geq 0$$

Corollary 9.1 (Liouville's theorem). If f is entire and bounded, then f is constant.

Corollary 9.2. Every non-constant polynomial $P(z) = a_n z^n + \ldots + a_0$ with complex coefficients has a root in $\mathbb{C}$.

Corollary 9.3. Every polynomial $P(z) = a_n z^n + \ldots + a_0$ of degree $n \geq 1$ has precisely n roots in $\mathbb{C}$. If these roots are denoted by $\omega_1, \ldots, \omega_n$, then P can be factored as

$$P(z) = a_n(z - \omega_1)(z - \omega_2) \cdots (z - \omega_n).$$

Theorem 10. *Suppose f is a holomorphic function in a region Ω that vanishes on a sequence of distinct points with a limit point in Ω. Then f is identically 0.*

In other words, if the zeros of a holomorphic function f in the connected set Ω accumulate *in* Ω, then $f = 0$.

Corollary 10.1. Suppose f and g are holomorphic in a region Ω and $f(z) = g(z)$ for all z in some non-empty open subset of Ω (or more generally for z in some sequence of distinct points with limit point in Ω). Then $f(z) = g(z)$ throughout Ω.

Suppose we are given a pair of functions f and F analytic in regions Ω and Ω', respectively, with $\Omega \subseteq \Omega'$. If the two functions agree on the smaller set Ω, we say that F is an **analytic continuation** of f into the region Ω'. The corollary then guarantees that there can only be one such analytic continuation, since F is uniquely determined by f.

5 Further applications

5.1 Morera's Theorem

Theorem 11. *Suppose f is a continuous function in the open disc D such that for any triangle T contained in D*

$$\int_T f(z)dz = 0,$$

then f is holomorphic.

5.4 Schwarz reflection principle

Let Ω be an open subset of $\mathbb{C}$ that is symmetric with respect to the real line, that is

$$z \in \Omega \qquad \text{if and only if} \qquad \overline{z} \in \Omega$$

Let Ω^+ denote the part of Ω that lies in the upper half plane and ω^- that part that lies in the lower half-plane.
Also, let $I = \Omega \cap \mathbb{R}$ so that I denotes the interior of the part of the boundary of Ω^+ and Ω^- that lies on the real axis. Then we have

$$\Omega^+ \cup I \cup \Omega^- = \Omega$$

and the only interesting case of the next theorem occurs, of cours, when I is non-empty.

Theorem 12 (Symmetry principle)**.** *If f^+ and f^- are holomorphic functions in Ω^+ and Ω^- respectively, that extend continuously to I and*

$$f^+(x) = f^-(x) \qquad \textit{for all}\, x \in I$$

then the function f defined on Ω by

$$\begin{cases} f^+(z) & \textit{if}\, z \in \Omega^+, \\ f^+(z) = f^-(z) & \textit{if}\, z \in I, \\ f^-(z) & \textit{if}\, z \in \Omega^- \end{cases}$$

is holomorphic on all of Ω.

Theorem 13 (Schwarz reflection principle)**.** *Suppose that f is a holomorphic function in Ω^+ that extends continuously to I and such that f is real-valued on I. Then there exists a function F holomorphic in all of Ω such that $F = f$ on Ω^+.*

Part 3 Meromorphic Functions and the Logarithm
1 Zeros and poles

Definition 0.15. A **point singularity** of a function f is a cmoplex number z_0 such that f is defined in a neighborhood of z_0 but not at the point z_0 itself. We shall also call such points **isolated singularities**.

Definition 0.16. A complex number z_0 is a **zero** of the holomorphic function f if $f(z_0) = 0$. In particular, analytic continuation shows that zeros of a non-trivial holomorphic function are isolated. In other word, if f is holomorphic in Ω and $f(z_0) = 0$ for some $z_0 \in \Omega$, then there exists an open neighborhood U of z_0 such that $f(z) \neq 0$ for all $z \in U - \{z_0\}$, unless f is identically zero.

Theorem 14. *Suppose that f is holomorphic in a connected open set Ω, has a zero at a point $z_0 \in \Omega$, and does not vanish identically in Ω. Then there exists a neighborhood $U \subseteq \Omega$ of z_0, a non-vanishing holomorphic function g on U, and a unique positive integer n such that*

$$f(z) = (z - z_0)^n g(z) \qquad \text{for all}\, z \in U.$$

Definition 0.17. In the case of the above theorem, we say that f has a **zero of order** n (or **multiplicity** n) at z_0. If a zero is of order 1, we say that it is **simple**. We observe that, quantitatively, the order describes the rate at which a function vanishes.

Definition 0.18. A **deleted neighborhood** of z_0 to be an open disc centered at z_0, minus the point z_0, that is, the set

$$\{z_0 : 0 < |z - z_0| < r\}$$

for some $r > 0$. Then, we say that a function f defined in a deleted neighborhood of z_0 has a **pole** at z_0, if the function $1/f$, defined to be zero at z_0 is holomorphic in a full neighborhood of z_0.

Theorem 15. *If f has a pole at $z_0 \in \Omega$, then in a neighborhood of that point there exist a non-vanishing holomorphic function h and a unique positive integere n such that*

$$f(z) = (z - z_0)^{-n} h(z)$$

Definition 0.19. The integer n is called the **order** (or **multiplicity**) of the pole, and describes the rate at which the function grows near z_0. If the pole is of order 1, we say that it is **simple**.

Theorem 16. *If f has a pole of order n at z_0, then*

$$f(z) = \frac{a_{-n}}{(z - z_0)^n} + \frac{a_{-n+1}}{(z - z_0)^{n-1}} + \ldots + \frac{a_{-1}}{(z - z_0)} + G(z) \tag{6}$$

where G is a holomorphic function in a neighborhood of z_0.

Definition 0.20. The sum

$$\frac{a_{-n}}{(z-z_0)^n} + \frac{a_{-n+1}}{(z-z_0)^{n-1}} + \ldots + \frac{a_{-1}}{(z-z_0)}$$

is called the **principal part** of f at the pole z_0, and the coefficient a_{-1} is called the **residue** of f at that pole. We write $\mathrm{res}_{z_0} f = a_{-1}$.

In the case when f has a simple pole at z_0, it is clear that

$$\mathrm{res}_{z_0} f = \lim_{z \to z_0} (z-z_0) f(z).$$

If the pole is of higher order,

Theorem 17. *If f has a pole of order n at z_0, then*

$$res_{z_0} f = \lim_{z \to z_0} \frac{1}{(n-1)!} \left(\frac{d}{dz}\right)^{n-1} (z-z_0)^n f(z).$$

The theorem is an immediate consequence of formula (6), which implies

$$(z-z_0)^n f(z) = a_{-n} + a_{-n+1}(z-z_0) + \ldots + a_{-1}(z-z_0)^{n-1} + G(z)(z-z_0)^n.$$

2 The residue formula

Theorem 18. *Suppose that f is holomorphic in an open set containing a circle C and its interior, except for a pole at z_0 inside C. Then*

$$\int_C f(z)dz = 2\pi i res_{z_0} f.$$

Corollary 18.1. Suppose that f is holomorphic in an open set containing a circle C and its interior, except for poles at the points $z_1, \ldots, z_N$ inside C. Then

$$\int_C f(z)dz = 2\pi i \sum_{k=1}^{N} \mathrm{res}_{z_k} f.$$

Corollary 18.2. Suppose that f is holomorphic in an open set containing a toy contour γ and its interior, except for poles at the points $z_1, \ldots, z_N$ inside γ. Then

$$\int_\gamma f(z)dz = 2\pi i \sum_{k=1}^{N} \mathrm{res}_{z_k} f.$$

Definition 0.21. The identity $\int_\gamma f(z)dz = 2\pi i \sum_{k=1}^{N} \mathrm{res}_{z_k} f$ is referred to as the **residue formula**.

2.1 Examples

3 Singularities and meromorphic functions

Definition 0.22. Let f be a function holomorphic in an open set Ω except possibly at one point z_0 in Ω. If we can define f at z_0 in such a way that f becomes holomorphic in all of Ω, we say that z_0 is a **removable** singularity for f.

Theorem 19. *Suppose that f is holomorphic in an open set Ω except possibly at a point z_0 in Ω. If f is bounded on $\Omega - \{z_0\}$, then z_0 is a removable singularity.*

Corollary 19.1. Suppose that f has an isolated singularity at the point z_0. Then z_0 is a pole of f if and only if $\left|f(z)\right| \to \infty$ as $z \to z_0$.

Isolated singularities can belong to one of three categories:

- Removable singularities (f bounded near z_0)
- Pole singularities ($\left|f(z)\right| \to \infty$ as $z \to z_0$)
- Essential singularities

Definition 0.23. By default, any singularity that is not removable or a pole is defined to be an **essential singularity**.

Theorem 20 (Casorati-Weierstrass)**.** *Suppose f is holomorphic in the punctured disc $D_r(z_0) - \{z_0\}$ and has an essential singularity at z_0. Then, the image of $D_r(z_0) - \{z_0\}$ under f is dense in the complex plane.*

Definition 0.24. A function f on an open set Ω is **meromorphic** if there exists a sequence of points $\{z_0, z_1, \ldots\}$ that has no limit points in Ω, and such that

1. the function f is holomorphic in $\Omega - \{z_0, z_1, \ldots\}$, and
2. f has poles at the points $\{z_0, z_1, z_2, \ldots\}$.

Definition 0.25. If f is holomorphic for all large values of z, we consider $F(z) = f({}^1\!/\!{}_z)$, which is now holomorphic in a deleted neighborhood of the origin. We say that f has a **pole at infinity** if F has a pole at the origin. Similarly, we can speak of f having an **essential singularity at infinity**, or a **removable singularity** (hence holomorphic) **at infinity** in terms of the corresponding behaviour of F at 0. A meromorphic function in the complex plane that is either holomorphic at infinity or has a pole at infinity is said to be **meromorphic in the extended complex plane**.

Theorem 21. *The meromorphic functions in the extended complex plane are the rational functions*

4 The argument principle and applications

Theorem 22 (Argument principle)**.** *Suppose f is meromorphic in an open set containing a circle C and its interior. If f has no poles and never vanishes on C, then*

$$\frac{1}{2\pi i}\int_C \frac{f'(z)}{f(z)}dz = (\textit{number of zeros of } f \textit{ inside } C) \textit{ minus}$$
$$(\textit{number of poles of } f \textit{ inside } C),$$

where the zeros and poles are counted with their multiplicities.

Corollary 22.1. The above theorem holds for any toy contours.

Theorem 23 (Rouché's theorem)**.** *Suppose that f and g are holomorphic in an open set containing a circle C and its interior. If*

$$|f(z)| > |g(z)| \qquad \textit{for all } z \in \mathbb{C}$$

then f and $f + g$ have the same number of zeros inside the circle C.

Definition 0.26. A mapping is said to be **open** if it maps open sets to open sets.

Theorem 24 (Open mapping theorem)**.** *If f is holomorphic and non-constant in a region Ω, then f is open.*

Definition 0.27. We shall refer to the **maximum** of a holomorphic function f in an open set Ω as the maximum of its absolute value $|f|$ in Ω.

Theorem 25 (Maximum modulas principle)**.** *If f is a non-constant holomorphic function in a region Ω, then f cannot attain a maximum in Ω.*

5 Homotopies and simply connected domains

Definition 0.28. Let γ_0 and γ_1 be two curves in an open set Ω with common end-points. So if $\gamma_0(t)$ and $\gamma_1(t)$ are two parameterizations defined on $[a, b]$, we have

$$\gamma_0(a) = \gamma_1(a) = \alpha \qquad \text{and} \qquad \gamma_0(b) = \gamma_1(b) = \beta$$

These two curves are said to be **homotopic in** Ω if for each $0 \leq s \leq 1$ there exists a curve $\gamma_s \subseteq \Omega$, parameterized by $\gamma_s(t)$ defined on $[a, b]$, such that for every s

$$\gamma_s(a) = \alpha \qquad \text{and} \qquad \gamma_s(b) = \beta,$$

and for all $t \in [a, b]$

$$\gamma_s(t)|_{s=0} = \gamma_0(t) \qquad \text{and} \qquad \gamma_s(t)|_{s=1} = \gamma_1(t)$$

Moreover, $\gamma_s(t)$ should be jointly continuous in $s \in [0, 1]$ and $t \in [a, b]$.

More informally, two curves are homotopic if one curve can be deformed into the other by a continuous transformation without ever leaving Ω.

Theorem 26. *If f is holomorphic in Ω, then*

$$\int_{\gamma_0} f(z)dz = \int_{\gamma_1} f(z)dz$$

whenever the two curves γ_0 and γ_1 are homotopic in Ω.

Theorem 27. *Any holomorphic function in a simply connected domain has a primitive.*

Corollary 27.1. If f is holomorphic in the simply connected region Ω, then

$$\int_{\gamma} f(z)dz = 0$$

for any closed curve $\gamma \in \Omega$.

6 The complex logarithm

If $z = re^{i\theta}$, and we want the logarithm to be the inverse to the exponential, then it is natural to get

$$\log z = \log r + i\theta$$

Here and below, we use the convention that $\log r$ denotes the standard (i.e. real) logarithm of the positive number r. The trouble with the above definition that θ is uniquely determined up to an integer multiple of 2π. One choice of this range is called a **branch** or **sheet** of the logarithm. An example of branch would be the choice $-\pi \leq \arg z \leq \pi$.

Theorem 28. *Suppose that Ω is simply connected with $1 \in \Omega$, and $0 \notin \Omega$. Then in Ω there is a branch of the logarithm $F(z) = \log_\Omega(z)$ so that*

1. *F is holomorphic in Ω,*
2. *$e^{F(z)} = z$ for all $z \in \Omega$,*
3. *$F(r) = \log(r)$ whenever r is a real number and near 1.*

In other words, each branch $\log_\Omega(z)$ is an extension of the standard logarithm defined for positive numbers.

Definition 0.29. In the slit plane, $\Omega = \mathbb{C} - \{(-\infty, 0]\}$ we have the **principal branch** of the logarithm

$$\log z = \log r + i\theta$$

where $z = re^{i\theta}$ with $|\theta| < \pi$. (Here we drop the subscript Ω, and write simply $\log z$.)

In general

$$\log(z_1 z_2) \neq \log(z_1) + \log(z_2).$$

For the principal branch of the logarithm the following Taylor expansion holds:

$$\log(1+z) = z - \frac{z^2}{2} + \frac{z^3}{3} - \ldots = -\sum_{n=1}^{\infty}(-1)^n \frac{z^n}{n} \quad \text{for} |z| < 1. \tag{7}$$