MAT327: Introduction to Topology. Solutions to the Big List Problems

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Thank you to my instructor Ivan Khatchatourian for providing these wonderful problems.

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Preface

I attempt to answer and LaTeXall of the solutions to the big list of problems posted by my instructor Ivan Khatchatourian for MAT327: Introduction to Topology. The problems are separated into difficulties which are labelled via asterisks. One asterisk being the lowest difficulty and 3 being the highest. Especially hard problems are marked via a cross. This is the format my instructor uses and I'm merely copying it for consistency's sake.

Topologies

* **Ex. 1** — Fix $a < b \in \mathbb{R}$. Show explicitly that the open interval (a, b) is open in $\mathbb{R}_{\text{usual}}$. Show explicitly that the interval $[a, b)$ is not open in $\mathbb{R}_{\text{usual}}$

Answer (Ex. 1) — First, we show that (a, b) is open in $\mathbb{R}_{\text{usual}}$. To do this: we have to show that for any point $x \in (a, b)$, there exists a neigbourhood $N(x, \epsilon) \subseteq (a, b)$.
Thus, choose a point $x \in (a, b)$. By density of $\mathbb{R}$, there exists $x_1 \in (a, x)$ and $x_2 \in (x, b)$. Let $\epsilon = \min(x_1, x_2)$. Then $N(x, \epsilon) \subseteq (a, b)$.
Next, we show that $[a, b)$ is not open. To do this, we show that $\exists$ no $\epsilon > 0$ such that $N(a, \epsilon) \subseteq [a, b)$. Note that any $x \in (a - \epsilon, a]$ would not be in (a, b) for any $\epsilon > 0$, so $N(a, \epsilon) \not\subseteq [a, b)$.

* **Ex. 2** — Let X be a set and $\mathcal{B} = \{\{x\} : x \in X\}$. Show that the only topology on X that contains $\mathcal{B}$ as a subset is the discrete topology.

Answer (Ex. 2) — To show that only X_{discrete} has the property $\mathcal{B} \subseteq X_{\text{discrete}}$, consider the set:

$$\mathcal{T} := \{\{x\} : x \in X\}$$

For $\mathcal{T}$ to be a topology, it must be closed under finite intersections so we must put $\emptyset$ in $\mathcal{T}$. Furthermore, $\mathcal{T}$ must be closed under the union of an arbitrary collection elements of X. If this is to be true, then $\mathcal{T}$ must contain every subset of X since any arbritrary subset can be written as the union of all of its elements. Thus an arbitrary subset $A \subseteq X$ must be in $\mathcal{T}$. This implies $\mathcal{T} = X_{\text{discrete}}$.

* **Ex. 4** — Let $(X, \mathcal{T}_{\text{co-countable}})$ be an infinite set with the co-countable topology. Show that $\mathcal{T}_{\text{co-countable}}$ is closed under countable intersections but not necessairly arbitrary ones.

Answer (Ex. 4) — Given a countable indexing set I, we have to show

$$\bigcap_{\alpha\in I} U_\alpha \in \mathcal{T}_{\text{co-countable}}$$

Equivalently, we can show $X\setminus(\cap U_\alpha)$ is countable. By DeMorgan's Law: $X\setminus(\cap_\alpha) = \cup(X\setminus U_\alpha)$. Since the right hand side is the countable union of countable sets, it is countable. Thus, $\cap U_\alpha \in \mathcal{T}_{\text{co-countable}}$. To show that an arbitrary intersection of elements is not open, it suffices to state that the intersection of arbitrary many elements does not necessarily form a countable set.

* **Ex. 5** — Let $(X, \mathcal{T})$ be a topological space, and let $A \subseteq X$ be a set with the property that for all $x \in A$, $\exists$ an open set $U_x \in \mathcal{T}$ such that $x \in U_x \subseteq A$. Show that A is open.

Answer (Ex. 5) — Let I be a set which indexes the elements of A. Then:

$$\bigcup_{\alpha\in I}\{x_\alpha\} = A$$

Similarly, since $x_\alpha \in U_{x_\alpha}$ and $U_{x_\alpha} \subseteq A$, then:

$$\bigcup_{\alpha\in I} U_{x_\alpha} = A$$

Since this is the union of arbitrary elements of $\mathcal{T}$, then $A \in \mathcal{T}$.