MATD11: Functional Analysis
Assignment 1

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P1.

Let $(X, \|\cdot\|)$ be a normed linear space over $\mathbb{R}$.

1. Prove that for $x, y \in X$, $|\,\|x\| - \|y\|\,| \geq \|x - y\|$.
2. Let $\{x_n\}$ be a convergent sequence in X and $\{a_n\}$ a convergent sequence in $\mathbb{R}$. Prove that the sequence $\{a_n x_n\}$ is convergent.
3. Let $a \in X$ and $r > 0$. Prove that $B(a, r) = B(0, r) + a$.
4. Prove that the open ball in X is a convex set.

Solution.

1. First, note that $\|x - y\| = \|y - x\|$ since

$$\|y - x\| = \|-1(x - y)\| = |-1|\,\|x - y\| = \|x - y\|\,.$$

Now, note that:

$$\begin{aligned}
\|x\| = \|(x - y) + y\| \leq \|x - y\| + \|y\| &\implies \|x\| - \|y\| \leq \|x - y\| \\
\|y\| = \|(y - x) + x\| \leq \|y - x\| + \|x\| &\implies \|y\| - \|x\| \leq \|y - x\| \\
&\implies \|x\| - \|y\| \leq -\|x - y\|
\end{aligned}$$

Thus:

$$-\|x - y\| \leq \|x\| - \|y\| \leq \|x - y\|$$

which implies,

$$|\,\|x\| - \|y\|\,| \leq |\,\|x - y\|\,| = \|x - y\| \qquad \text{(positive definiteness)}$$

as wanted. ■

2. Claim: $\{a_n x_n\} \to ax$.

$$\|a_n x_n - ax\| = \|a_n x_n - x_n a + x_n a - ax\| \leq \|x_n\|\,|a_n - a| + |a|\,\|x_n - x\|$$

We are given that $x_n \to x \in X$. Since this is a convergent series, then x_n is bounded i.e. $\|x_n\| \leq M \in \mathbb{R}^+$. Now, choose some $\epsilon > 0$ and write $\epsilon = \epsilon_1 + \epsilon_2$ for some $\epsilon_1 > 0$ and $\epsilon_2 > 0$. Now, we choose ${\epsilon_1}/{M} > 0$, then there must exist some $N \in \mathbb{N}$ such that if $n > N$, then $|a_n - a| < {\epsilon_1}/{M}$. Next, we choose ${\epsilon_2}/{|a|} > 0$, then there must exist some $M \in \mathbb{N}$ such that if $m > M$, then $\|x_m - x\| < {\epsilon_2}/{|a|}$. Choose $J = max\{N, M\}$. Then, consider for all $j > J$:

$$\begin{aligned}
\|a_j x_j - ax\| &= \|a_j x_j - x_j a + x_j a - ax\| \\
&\leq \|x_j\| \, |a_j - a| + |a| \, \|x_j - x\| \\
&< M \frac{\epsilon_1}{M} + |a| \frac{\epsilon_2}{|a|} \\
&= \epsilon_1 + \epsilon_2 \\
&= \epsilon
\end{aligned}$$

so that $a_n x_n \to ax$ as wanted. Note that if $a = 0$, then

$$\|a_j x_j - ax\| \leq \|x_j\| \, |a_j - a| < \epsilon_1 < \epsilon.$$

as wanted. ∎

3. To solve this, we show set equality i.e. a) show $B(a, r) \subseteq B(0, r) + a$ and b) show $B(0, r) + a \subseteq B(a, r)$.
a) First, note that $d(0, x-a) = \|0 - (x - a)\| = \|a - x\| = d(a, x)$. Now, choose $x \in B(a, r)$, then $d(a, x) < r \implies d(0, x - a) < r$. Then, $x - a \in B(0, r)$ which furthermore implies $x \in B(0, r) + a$.
b) Choose $y \in B(0, r)$. Then $y + a \in B(0, r) + a$. Note, $d(a, y + a) = \|a - (y + a)\| = \|y\| < r$. Thus, $y + a \in B(a, r)$ so that $B(0, r) + a \subseteq B(a, r)$ as wanted. ∎

4. Choose $a \in X$ and $r > 0$. Then $B(a, r)$ is said to be convex if for all $x, y \in B(a, r)$, the straight line connecting them is in $B(a, r)$.
Choose $x, y \in B(a, r)$. Let the line connecting them be given by,

$$f(t) = ty + (1 - t)x \qquad \forall\, t \in (0, 1)$$

We want to show $\|f(t) - a\| < r$. Consider,

$$\|f(t) - a\| = \|ty + (1 - t)x - a\| = \|t(y - a) + (1 - t)(x - a)\|$$

By using the triangle inequality, we get:

$$\|f(t) - a\| \leq \|(1 - t)(x - a)\| + \|t(y - a)\|$$

so that, we get:

$$|f(t) - a| \leq |1 - t| \, \|x - a\| + |t| \, \|y - a\| < |1 - t|r + |t|r = r$$

Thus, $B(a, r)$ is convex as wanted. ∎

P2.

Show the map $x \mapsto \|x\|$ is continuous. Is it uniformily continuous?

Solution.

Choose $x_0 \in X$. We want to show $\lim_{x \to x_0} \|x\| = \|x_0\|$ i.e.

$$\forall\, \epsilon > 0,\ \exists\, \delta > 0 \text{ such that if } \|x - x_0\| < \delta \implies |\,\|x\| - \|x_0\|\,| < \epsilon$$

Thus, choose a $\epsilon > 0$. Let $\epsilon = \delta$. Then, by the reverse triangle inequality:

$$|\,\|x\| - \|x_0\|\,| \le \|x - x_0\| < \delta = \epsilon$$

Therefore, $\|\cdot\|$ is a continuous mapping. Note, even if we did not fix x_0 in the beginning, our proof would not have changed. Therefore, our proof does not depend on our choice of x_0 implying that $\|\cdot\|$ is also uniformily continuous. ■

P3.

Let c_0 be the set of real sequences that converge to 0. Prove that c_0 is complete with respect to the sup norm.

Solution.

Let $\{\xi_n\}$ be a Cauchy sequence of elements of c_0. We want to show that $\{\xi_n\}$ converges and it converges to some sequence $\{a_n\}$ in c_0.
Choose a $\epsilon > 0$, then there exists $N \in \mathbb{N}$ such that if $n, m > N$, then $\|\xi_n - \xi_m\| < \epsilon$. By the definition of the sup norm, we know that for all $j \in \mathbb{N}$,

$$|\xi_n(j) - \xi_m(j)| \le \ \sup\{|\xi_n(j) - \xi_m(j)| : j \in \mathbb{N}\} = \|\xi_n - \xi_m\| < \epsilon$$

Thus $\{\xi_n(j)\}_{j=1}^{\infty}$ is a sequence of real numbers and is a Cauchy sequence from the line above. Since this is a sequence of real numbers, by the completeness of the reals, we know this converges to some real number m_j. Therefore, we can write

$$\lim_{n \to \infty} \xi_n(j) = m_j$$

so we can define a sequence $\{m_j\}_{j=1}^{\infty}$ using the same process for different j's and then claim that

$$\{m_j\} \in c_0 \qquad \text{and} \qquad \lim_{n \to \infty} \xi_n = \{m_j\}.$$

First, we want to show that $\lim_{n \to \infty} \xi_n = \{m_j\}$.
Choose $\epsilon > 0$. We know for all $n \in \mathbb{N}$, $\{\xi_m(n)\}$ is a convergent sequence (of $\mathbb{R}$), thus, there exists some $M \in \mathbb{N}$ such that if $m \ge M$, then:

$$|\xi_m(n) - m_n| < \epsilon \tag{1}$$

Now, to show $\lim_{n\to\infty} \xi_n = \{m_j\}$, choose the same M that makes (1) hold and consider:

$$\|\xi_m - \{m_j\}\| = \max_{n\in\mathbb{N}} |\xi_m(n) - m_n| < \epsilon$$

which is enough to show that $\lim_{n\to\infty} \xi_n = \{m_j\}$ as wanted.

Claim: $m_j \to 0$ as $j \to \infty$. Choose a $\epsilon > 0$. Then: We know ξ_n is an element of c_0. Thus, for our choice of ϵ, there exists a $J \in \mathbb{N}$ such that if $j \geq J$, then $|\xi_n(j)| < \epsilon$. Furthermore, since $\{\xi_n(j)\}_{j=1}^{\infty} \to m_j$ as discussed earlier, then for our choice of ϵ, there exists $J' \in \mathbb{N}$, then if $j' \geq J'$, then $|\xi_n(j') - m_{j'}| < \epsilon$. Then, define $J := \max\{J, J'\}$, and for all $j \geq J$, consider:

$$\begin{aligned} |\xi_n(j) - m_j| = |m_j - \xi_n(j)| &< \epsilon \\ |m_j| - |\xi_n(j)| \leq ||m_j| - |\xi_n(j)|| &< \epsilon \qquad \text{(reverse triangle inequality)} \\ |m_j| &< \epsilon + |\xi_n(j)| < 2\epsilon \end{aligned}$$

Although $\epsilon \leq 2\epsilon$, this does not matter as we can just go back and choose $\epsilon/2$ so that in the end, we get $|m_j - 0| < \epsilon/2 + \epsilon/2 = \epsilon$ which is sufficient to show that $\{m_j\} \to 0$ as $j \to \infty$ as wanted.

Therefore, the Cauchy sequence $\{\xi_n\} \subseteq c_0$ converges to a sequence in c_0 showing that c_0 is complete. ■

P4.

Let c_{00} be the set of all real sequences that have at most finitely many non-zero entries.

1. Prove that c_{00} is a proper subspace of c_0.

2. Find the Hamel basis for c_{00} that is a generalization of the standard basis for $\mathbb{R}^n$.

3. Prove that if $\mathcal{A}$ is a Hamel basis for c_0, then $\mathcal{A}$ is not a subset of c_{00}.

4. Prove that the closure of c_{00} (with respect to the sup norm) is c_0.

Solution.

1. $\{1/n\}_{n=1}^{\infty}$ is an example of a sequence of real numbers which clearly does not have finitely many non-zero terms but still converges to 0. Therefore, if $c_{00} \subseteq c_0$, then $c_{00} \subsetneq c_0$. Now, choose an arbitrary $\{x_n\} \in c_{00}$. Since this has finitely many non-zero terms, there exists some $K \in \mathbb{N}$ such that for all $k \geq K$, then $x_k = 0$. Thus, for all $\epsilon > 0$, we choose this choice of K, and obtain for all $k \geq K$ that $|x_k| = 0 < \epsilon$. Therefore, $\{x_n\} \to 0$ implying that $\{x_n\} \in c_0$ which further implies that $c_{00} \subseteq c_0$. From our discussion earlier, we then obtain that

$$c_{00} \subsetneq c_0$$

2. Let e_k be the kth standard basis vector of $\mathbb{R}^n$ (obviously $n \geq k$). This has only one non-zero term which is located at the kth component of the n-dimensional vector and is equal to 1 there. To generalize this, let $e_k^{'} := (0, \ldots, 0, 1, 0, \ldots)$ which has only one non-zero term located at the kth component of the sequence (or equivalently, an ∞-dimensional vector). Then, the set $\{e_k^{'}\}_{k=1}^{\infty}$, we claim gives the basis of c_{00}. It is easy to see this is linearly independent as it is just a generalization of the standard basis from $\mathbb{R}^n$. To show that it spans c_{00}, we pick a sequence $\{x_n\}_{n=1}^{\infty} \in c_{00}$. Then, there exist finitely many $n_1, \ldots, n_m$ for some $m \in \mathbb{N}$ such that $\{x_{n_i}\}_{i=1}^{m}$ contains the only non-zero terms of $\{x_n\}$. $\{x_n\}$ can then be written as:
$$\{x_n\} = (x_{n_1})e_{n_1}^{'} + \ldots + (x_{n_m})e_{n_k}^{'}$$
implying that the basis $\{e_k^{'}\}_{k=1}^{\infty}$ clearly spans the space c_{00} as wanted and is clearly a generalization of the standard basis of $\mathbb{R}^n$.

3. Suppose such a set $\mathcal{A}$ exists i.e. $\mathcal{A}$ is a Hamel basis of c_0 and also a subset of c_{00}. Then each element $\{x_n\}$ of $\mathcal{A}$ only has finitely many non-zero terms and the rest are zero. The key to recognizing the contradiction is to realize that any element $\{y_n\}$ of c_0 must be able to be written as a *finite* combination of elements of $\mathcal{A}$. But if each element of $\mathcal{A}$ only has finitely many non-zero terms, then there exist no *finite* combination of elements of $\mathcal{A}$ such that the combination produces a sequence which has infinitely many non-zero terms. More mathematically put, pick a element $\{y_n\}$ of c_0 such that it has infinitely many non-zero terms (i.e. $\{1/n\}$). Then, we can write $\{y_n\}$ as a finite combintation of elements in $\mathcal{A}$ since it is a basis of c_0. Therefore, we write:
$$\{y_n\} = a_1\{b_1(n)\} + a_2\{b_2(n)\} + \ldots + a_m\{b_m(n)\}$$
where each $a_i \in \mathbb{R}$ and each $\{b_i(n)\} \in \mathcal{A}$ for $1 \leq i \leq m$ (also not all a_i's are 0). However, since each sequence $\{b_i(n)\}$ only has finitely many non-zero terms, then their sum produces a sequence which again, only has a finite number of non-zero terms. However, this is impossible since we picked $\{y_n\}$ to be a sequence which has an infinite number of non-zero terms. Therefore, there is a contradiction and no such $\mathcal{A}$ exists as wanted.

4. We claim that c_0 is the smallest closed set which contains c_{00} (def'n of closure). In order to prove this, we show that every element of c_0 is a limit point of c_{00}, and if there existed a smaller (closed) set than c_0, then it would not contain all limit points of c_{00} implying the set is not closed which is a contradiction. Therefore, choose some arbitrary element $\{x_n\}_{n=1}^{\infty}$ of c_0 and define a sequence by terms:
$$\alpha_n := \{x_1, x_2, \ldots, x_n, 0, 0, \ldots\}$$
and we claim that $\{\alpha_n\} \to \{x_j\}_{j=1}^{\infty}$ as $n \to \infty$. Clearly, each $\alpha_n \in c_{00}$ and so it is left to show the convergence. Choose a $\epsilon > 0$. We want to show

there exists some $N \in \mathbb{N}$ such that if $n \geq N$, then $\|\alpha_n - \{x_j\}\| < \epsilon$. Note $\{x_j\}$ goes to 0 since it is in c_0. Thus, for our choice of ϵ, there exists some $N_1 \in \mathbb{N}$ such that if $n' \geq N_1$, then $|x_{n'}| < \epsilon$. Let $N = N_1$. Then for all $n \geq N$, note:

$$\begin{aligned}\|\alpha_n - \{x_j\}\| &= \|(0, \ldots, 0, x_{n+1}, x_{n+2}, x_{n+3}, \ldots)\| \\ &= \ \sup_{k>n}|x_k| < \epsilon\end{aligned}$$

so that $\{\alpha_n\} \to \{x_j\}$ as $n \to \infty$ as wanted. Therefore, we have that for every point $\mathbf{x}$ of c_0, there exists some sequence $\{\alpha_n\}$ of elements of c_{00} such that this sequence converges to $\mathbf{x}$, thus implying that every point of c_0 is a limit point of c_{00}. Therefore, c_0 is the closure of c_{00}. ■

P5.

Let H be a Hilbert space.

1. Verify the second equation in Proposition 1.22 on Page 11.
2. Let $x, y \in H$. Prove that $x \perp y$ if and only if $\|x + \alpha y\| = \|x - \alpha y\|$ for all $\alpha \in \mathbb{C}$.

Solution.

1. We want to prove that for any vectors f, g in a Hilbert space $\mathcal{H}$, we have the equality:
$$\|f + g\|^2 = \|f\|^2 + 2\mathrm{Re}\langle f, g\rangle + \|g\|^2$$
Recall the definition of the induced norm by a given inner product, then:
$$\|f + g\|^2 = \langle f + g, f + g\rangle = \langle f, f\rangle + \langle g, f\rangle + \langle f, g\rangle + \langle g, g\rangle$$
Recall that $\langle f, g\rangle = \overline{\langle g, f\rangle}$ and since $\langle f, g\rangle$ is a complex number, we can write $\langle f, g\rangle = a + ib$ for $a, b \in \mathbb{R}$ and:
$$\langle f, g\rangle + \overline{\langle g, f\rangle} = (a + ib) + (a - ib) = 2a$$
i.e.
$$\|f + g\|^2 = \langle f, f\rangle + 2\mathrm{Re}\langle f, g\rangle + \langle g, g\rangle$$
which is just equal to:
$$\|f\|^2 + 2\mathrm{Re}\langle f, g\rangle + \|g\|^2$$
as wanted. ■

2. First, assume $x \perp y$ for $x, y \in \mathcal{H}$ where $\mathcal{H}$ is a Hilbert space. Then for any $\alpha \in \mathbb{C}$:
$$\begin{aligned}\|x + \alpha y\|^2 &= \langle x + \alpha y, x + \alpha y\rangle \\ &= \langle x, x + \alpha y\rangle + \langle \alpha y, x + \alpha y\rangle \\ &= \langle x, x\rangle + \bar{\alpha}\langle x, y\rangle + \alpha\langle y, x\rangle + \alpha\bar{\alpha}\langle y, y\rangle\end{aligned}$$

By recalling that for all $\alpha \in \mathbb{C}$, we have that $\alpha\bar{\alpha} = |\alpha|^2$, we obtain:

$$\begin{aligned} \|x + \alpha y\|^2 &= \langle x, x\rangle + \alpha\langle x, y\rangle + \bar{\alpha}\langle y, x\rangle + |\alpha|^2\langle g, g\rangle \\ &= \langle x, x\rangle + \alpha(0) + \bar{\alpha}(0) + |\alpha|^2\langle g, g\rangle = \langle x, x\rangle + |\alpha|^2\langle g, g\rangle \end{aligned}$$

where the last line follows from the fact that if $\langle x, y\rangle = 0$, then $\langle y, x\rangle = 0$. Furthermore, consider:

$$\begin{aligned} \|x - \alpha y\|^2 &= \langle x, x\rangle + \langle x, -\alpha y\rangle + \langle -\alpha y, x\rangle + \langle -\alpha y, -\alpha y\rangle \\ &= \langle x, x\rangle + \overline{-\alpha}\langle x, y\rangle - \alpha\langle y, x\rangle + |\alpha|^2\langle y, y\rangle \\ &= \langle x, x\rangle + |\alpha|^2\langle y, y\rangle \end{aligned}$$

implying that $\|x + \alpha y\| = \|x - \alpha y\|$ as wanted. Now, we prove the reverse direction. So, assume $\|x + \alpha y\| = \|x - \alpha y\|$, and we show $x \perp y$ i.e. $\langle x, y\rangle = 0$. Consider:

$$\begin{aligned} \|x + \alpha y\| &= \|x - \alpha y\| \\ \|x + \alpha y\|^2 &= \|x - \alpha y\| \\ \langle x + \alpha y, x + \alpha y\rangle &= \langle x - \alpha y, x - \alpha y\rangle \\ \langle x, x\rangle + \langle x, \alpha y\rangle + \langle \alpha y, x\rangle + \langle \alpha y, \alpha y\rangle &= \langle x, x\rangle + \langle x, -\alpha y\rangle + \langle -\alpha y, x\rangle + \langle -\alpha y, -\alpha y\rangle \\ \bar{\alpha}\langle x, y\rangle + \alpha\langle y, x\rangle &= -\bar{\alpha}\langle x, y\rangle - \alpha\langle y, x\rangle \\ 2\bar{\alpha}\langle x, y\rangle &= -2\alpha\langle y, x\rangle \\ \bar{\alpha}\langle x, y\rangle &= -\alpha\langle y, x\rangle \end{aligned}$$

Let $\langle x, y\rangle = a + ib$ for some $a, b \in \mathbb{R}$ and choose $\alpha = 1$. Then $\bar{\alpha} = -1$ and:

$$-1(a + ib) = 1(a - ib) \implies ib - ib = a + a \implies a = 0$$

i.e. $\mathrm{Re}\langle x, y\rangle = 0$. Now, choose $\alpha = i$. Then $\bar{\alpha} = -i$ and:

$$-i(a+ib) = i(a-ib) \implies -ia+b = -i(a-ib) \implies -ia+b+ia+b = 0 \implies b = 0$$

which implies $\mathrm{Im}\langle x, y\rangle = 0$. This is enough to imply that $\langle x, y\rangle = 0$ i.e. $x \perp y$ as wanted. ∎

P6.

Let A be a subset of a Hilbert space $\mathcal{H}$. Prove that $A^{\perp}$ is a closed subspace of $\mathcal{H}$.

Solution.

First, we prove that $A^{\perp}$ is closed. To do this, we use this fact provided in the book:

$$A^{\perp} = \cap_{a \in A} a^{\perp}$$

Define a function $f : \{a\} \times X \to \mathbb{C}$ defined by $f(a, x) = \langle a, x \rangle$. Note the inner product function is a continuous map, so when it is restricted like as in f, it is still continuous. Since f is continuous, then the pre-image of every closed set in $\mathbb{C}$ is mapped to a closed set in $\{a\} \times X$. Therefore, choose $\{0\} \subseteq \mathbb{C}$ which is clearly closed. The pre-image of this set is the set of all points $x, \in X$ such that $\langle a, x \rangle = 0$ i.e. $\pi_2(f^{-1}(0)) = \{a\}^{\perp}$. Therefore, $\{a\}^{\perp}$ is closed. Since the arbitrary intersection of closed sets is closed, we obtain that $A^{\perp}$ is closed.

To show that $A^{\perp}$ is a subspace of $\mathcal{H}$, it suffices to show that $A^{\perp}$ is closed under vector addition and scalar multiplication. Choose $x, y \in A^{\perp}$. We want to show $x+y \in A^{\perp}$ i.e. $\langle a, x+y \rangle = 0$. Note: $\langle a, x+y \rangle = \langle a, x \rangle + \langle a, y \rangle = 0+0 = 0$ so that $x + y \in A^{\perp}$. Similarly, choose $c \in \mathbb{C}$, we want to show $cx \in A^{\perp}$ i.e. $\langle a, cx \rangle = 0$. Note, $\langle a, cx \rangle = \bar{c}\langle a, x \rangle = \bar{c}(0) = 0$ so that $cx \in A^{\perp}$. Therefore, $A^{\perp}$ is a linear closed subspace of $\mathcal{H}$. ■

P7.

Verify the proper inclusions of real sequences $l^1 \subset l^p \subset l^q \subset c_0 \subset l^{\infty}$ where $1 < p < q < \infty$.

Solution.

Let $1 \leq p < q$. We want to show $l^p \subseteq l^q$. Note, this will show $l^1 \subseteq l^p \subseteq l^q$. Choose an element $\{x_n\} \in l^p$, we want to show $\{x_n\} \in l^q$. We know $\sum_{n=1}^{\infty} |x_n|^p < \infty$.

$$L = \lim_{n \to \infty} |\frac{x_{n+1}}{x_n}|^p \implies L^{q/p} = \lim_{n \to \infty} |\frac{x_{n+1}}{x_n}|^q$$

Since $L < 1$ (series converges in l^p norm) and $q/p > 1$, then $L^{q/p} < 1$ implying $\sum_{n=1}^{\infty} |x_n|^q$ converges by our elementary series tests. Therefore, $\{x_n\} \in l^q$ as wanted. By the zero test for series, we know that if $\sum_{n=1}^{\infty} |a_n|^p < \infty$, then $\lim_{n \to \infty} a_n = 0$. This is enough to imply that $\{a_n\} \in c_0$. Furthermore, every element of c_0 is a convergent sequence (in $\mathbb{R}$) and therefore it is bounded implying that this sequence is also in l^{∞}. More mathematically put, if $\{a_n\}_{n=1}^{\infty}$ is an element of c_0. Then by definition of c_0, we know $\{a_n\} \to 0$ as $n \to \infty$

i.e. it converges and therefore all the terms $|a_n|$ for every n is bounded by some $M \in \mathbb{R}$. Therefore, $\sup\{|a_n| : n \in \mathbb{N}\} < \infty$ implying that $\{a_n\}$ is also in l^∞. We have so far proven that:

$$l^1 \subseteq l^p \subseteq l^q \subseteq c_0 \subseteq l^\infty$$

We will now show *proper* inclusions. We know $\{1, 1, \ldots\}$ is in l^∞ but clearly this does not converge to 0, therefore, it is not an element of c_0. We know by the p-series test that $\{{}^1/_{n^{1/p}}\}$ is an element of l^q since:

$$\sum_{n=1}^{\infty}(\frac{1}{n^{1/p}})^q = \sum \frac{1}{n^{q/p}}$$

since ${}^q/_p > 1$, we know this series converges (p-series test), and therefore is in l^q but we claim that this is not in l^q since:

$$\sum_{n=1}^{\infty}(\frac{1}{n^{1/p}})^p = \sum(\frac{1}{n^1})$$

which clearly does not converge as it is the Harmonic series. Therefore, $\{{}^1/_{n^{1/p}}\}$ is in l^q but not in l^p. Since our proof did not rely on $p > 1$, only on $p \geq 1$, we have shown in total that:

$$l^1 \subsetneq l^p \subsetneq l^q \subsetneq c_0 \subsetneq l^\infty$$

as wanted. ■

P8.

Show that $C[0, 1]$ in the supremum norm is not an inner product space; that is, the norm cannot be derived from an inner product.

Solution

We know that the parallelogram law only holds in inner product spaces, therefore, we assume that $C[0, 1]$ in the supremum norm is an inner product space, and show that the parallelogram law does not hold thereby showing a contradiction and obtaining the result we want.

The parallelogram law states that for all $f, g \in C[0, 1]$:

$$\|f + g\|^2 + \|f - g\|^2 = 2\|f\|^2 + 2\|g\|^2$$

Choose $f(x) = x$ and $g(x) = 1 + x$. Then $\|f + g\| = \|x + 1\| = 2$, $\|f - g\| = \|1\| = 1$, $\|f\| = 1$, $\|g\| = 2$. Therefore,

$$3^2 + 1^2 = 2(3 + 1) \implies 9 + 1 = 2(4) \implies 10 = 8$$

which is clearly not true, therefore no such inner product space structure exists on $C[0, 1]$ such that it induces the sup norm on $C[0, 1]$. ■