Algebraic Topology Companion Notes

Anmol Bhullar

April 2018

Filling in some details or trying some proofs myself from James Munkres' *Topology*.

§51 Homotopy of Paths

Lemma 51.1. *The relations $\simeq$ and $\simeq_p$ are equivalence relations.*

Proof. First, we show $\simeq$ is an equivalence relation between homotopic continuous functions from $X \to Y$.
We will prove reflexivity first. Let $f : X \to Y$ be continuous. We show f is homotopic with itself. To do this, we have to show there exists a continuous mapping $F : X \times I \to Y$ such that $F(x, 0) = f(x)$ and $F(x, 1) = f(x)$ for all $x \in X$. Thus, define F to be given by $(x, t) \mapsto f(x)$. Since f is continuous, it follows that F is then also continuous.
Now, we show symmetry. Let f and g be any homotopic functions from X to Y. We show $g \simeq f$. To do this, we have to show there exists a continuous function $F : X \times I \to Y$ (not to be confused with the one above, we are no longer using it) such that $F(x, 0) = g(x)$ and $F(x, 1) = f(x)$. Thus, define F to be given by the mapping $(x, t) \mapsto G(x, 1 - t)$ where G is the homotopy between f and g. Thus, we immediately obtain that $F(x, 0) = G(x, 1) = g(x)$ and $F(x, 1) = G(x, 0) = f(x)$ as wanted. To see this is continuous, it suffices to see that the component functions are continuous. It is clear that $\pi_1(F(x, t))$ is continuous since G is continuous. The fact that $\pi_2(F(x, t))$ is continuous follows from the fact that it is a composition of continuous functions (namely $(x, t) \mapsto (x, 1 - t)$ and $(x, t) \mapsto G(x, t)$).
Now, we prove $\simeq$ is transitive to conclude that $\simeq$ is an equivalence relation. Let $f \simeq g$ and $g \simeq h$ for homotopic functions $f, g : X \to Y$ and homotopic functions $g, h : X \to Y$. Let F' and F'' be the homotopy of f, g and g, h respectively. Now, define $H : X \times [0, 2] \to Y$ via $(x, t) \mapsto F'(x, t)$ if $t \leq 1$ and $(x, t) \mapsto F''(x, t-1)$ if $t \geq 1$. We see that H is well defined since $H(x, 1) = F'(x, 1) = g(x) = F''(x, 0)$. H is also continuous since H is continuous on $t < 1$ via F' and continuous on $t > 1$ via F''. Thus, by the pasting lemma, we have that H is continuous on $X \times [0, 2]$. Using this as motivation, we define the homotopy $\delta : X \times I \to Y$ between f, h via $(x, t) \mapsto F'(x, 2t)$ if $t \leq 1/2$ and $(x, t) \mapsto F''(x, 2t-1)$ if $t \geq 1/2$.

Now, we show $\simeq_p$ is an equivalence relation. Let $f : I \to X$ be a continuous path from x_0 to x_1 and define F to be the homotopy from the reflexive proof of $\simeq$. Then, we only need to show the additional condition that $F(0,t) = x_0$ and $F(1,t) = x_1$. Note $F(x,t) = f(x)$ so $F(0,t) = f(0) = x_0$ and $F(1,t) = f(1) = x_1$ as wanted. Thus, $f \simeq_p f$ showing reflexivity. Now, we show symmetry. Let $f, f' : I \to X$ be two homotopic paths from x_0 to x_1. Define $F : I^2 \to X$ via $(s,t) \mapsto F'(s, 1-t)$ where F' is the path homotopy between f and f'. Then from the symmetry proof of $\simeq$ above, we have only left to prove that $F(0,t) = x_0$ and $F(1,t) = x_1$. Note $F(0,t) = F'(0, 1-t) = x_0$ and $F(1,t) = F'(1, 1-t) = x_1$ as wanted. We now, only need show transitivity to conclude that $\simeq_p$ is an equivalence relation. Let $f, g : I \to X$ and $g, h : I \to X$ be path homotopies between x_0 and x_1. We show f, h are path homotopic. Define δ to be a path homotopy between f and h the same way it was done in the transitive proof above. Thus, it is left to prove $\delta(0,t) = x_0$ and $\delta(1,t) = x_1$. If $t < 1/2$, we have $\delta(0,t) = F'(0, 2t) = x_0$ since F' is a path homotopy between f and g which are paths between x_0 and x_1. Similarly, $\delta(1,t) = x_1$. We can repeat this for $t \geq 1$ to again, get the same result. Thus, δ is a path homotopy as wanted. Therefore, $\simeq_p$ is an equivalence relation as wanted. $\square$

Example 51.2. *Let f and g be any two maps of a space X into $\mathbb{R}^2$. It is easy to see that f and g are homotopic; the map*

$$F(x,t) = (1-t)f(x) + tg(x)$$

is a homotopy between them called the ***straight line homotopy****.*

Proof. Note that $F(x,0) = (1-0)f(x) + 0g(x) = f(x)$ and $F(x,1) = (1-1)f(x) + 1g(x) = g(x)$ as wanted. F is continuous because f and g are continuous functions on X, thus, so are $(1-t)f(x)$ and $tg(x)$ for all $t \in \mathbb{R}$. $F(x,t)$ is the sum of these two functions so, we immediately obtain that F is a continuous function. Thus, F is a homotopy between f and g. $\square$

Exercise 1

Show that if $h, h' : X \to Y$ are homotopic and $k, k' : Y \to Z$ are homotopic, then $k \circ h$ and $k' \circ h'$ are homotopic.

Proof. We want to deform $k \circ h : X \to Z$ into $k' \circ h' : X \to Z$ in a continuous manner. Let H be the homotopy between h and h' and K be the homotopy between k and k'. Let,

$$G : (x,t) \mapsto K(H(x,t), t)$$

If $t = 0$, we have $K(H(x,0), 0) = k(H(x,0)) = k \circ h(x)$. If $t = 1$, we have $K(H(x,1), 1) = k'(H(x,1)) = k' \circ h'(x)$. G is a composition of continuous functions, so it is continuous. Thus, $k \circ h$ and $k' \circ h'$ are homotopic. $\square$

Exercise 2

Given spaces X and Y, let $[X, Y]$ denote the set of homotopy *classes* of maps of X into Y.

(a) Let $I = [0, 1]$. Show that for any X, the set $[X, I]$ has a single element.

(b) Show that if Y is path connected, the set $[I, Y]$ has a single element.

Proof. For (a), it suffices to show that any continuous maps $f, f' : X \to I$ is homotopic. Note I is convex so the straight line homotopy is enough to show that f and f' are homotopic. $\square$

Exercise 3

A space X is said to be **contractible** if the identity map $i_X : X \to X$ is nulhomotopic (able to be continuously deformed into a constant map).

(a) Show that I and $\mathbb{R}$ are contractible.

(b) Show that a contractible space is path-connected.

(c) Show that if Y is contractible, then for any X, the set $[X, Y]$ has a single element.

(d) Show that if X is contractible and Y is path connected, then $[X, Y]$ has a single element.

Proof. (a): Define $F : I \times I \to I$ to be given by $F(x, t) = (1 - t)x$. This is the straight line homotopy between the identity map i_I and the constant map $x \mapsto 0$. $\mathbb{R}$ is contractible for the same reason, simply apply the straight line homotopy between $i_{\mathbb{R}}$ and $x \mapsto 0$. Note, this is possible since both I and $\mathbb{R}$ are convex.

(b): Let X be a contractible space. Let F be the homotopy between i_X and some constant map f. Then, $\psi : t \mapsto F(x, t)$ is a continuous path with endpoints x and $f(x)$. Similarly, $\phi : t \mapsto F(y, t)$ is a continuous path with endpoints y and $f(y)$. Since f is a constant map, we have $f(x) = f(y)$ and so we can construct a continuous path $\Phi : I \to X$ using the pasting lemma with endpoints x and y.

(c): Let the identity map on Y be homotopic to the constant map, mapping everything to $y_0 \in Y$. Let $g : X \to Y$ be given by $x \mapsto y_0$ and $f : X \to Y$ be any continuous map. Let H be the homotopy between i_Y and the map $Y \mapsto y_0$. Define $F : X \times I \to Y$ be given by $F(x, t) = H(f(x), t)$. Then, F is a homotopy since H is and so $[X, Y]$ has a single element.

(d): Let H be the homotopy between the identity map on X and the map $X \mapsto x_0 \in X$. Then, $f \circ H : X \times I \to Y$ is a homotopy between f and $X \mapsto f(x_0)$ since $f \circ H(x, 0) = f(x)$ and $f \circ H(x, 1) = f \circ x_0 = f(x_0)$. Taking $g : X \to Y$

to be another continuous map, we can see $g \circ H$ is a homotopy between g and $X \mapsto g(x_0)$. Y is path connected so there exists a path between $g(x_0)$ and $f(x_0)$. Through this, we get a homotopy between f and g. Thus, $[X, Y]$ has a single element. □

Side note: One could think that you may not need contractibility to be able to answer (d) if we let our homotopy $H(x,t)$ be equal to the path that connects some continuous functions $f(x)$ and $g(x)$ in Y. However, this mapping is not continuous.

52 The Fundamental Group

Exercise 1

A subset A of $\mathbb{R}^n$ is said to be **star convex** if for some point a_0 of A, all the line segments joining a_0 to other points of A lie in A.

(a) Find a star convex set that is not convex

(b) Show that if A is star convex, A is simply connected.

Proof.

(a) Let A be a star in $\mathbb{R}^2$ centered at $(0,0)$. This is clearly not convex just by taking any two points on any (distinct) "tips" of the star.

(b) Let $a_0 \in A$ be the point which is connected to any other point of A via a straight line (lying entirely in A). From this property of a_0, we see that A is path connected. Now, we show that $|\pi_1(A, a_0)| = 1$ and so the fundamental group based at any other point also has order 1 (and so it must be the trivial fundamental group). To see this, let f and g be any two paths based at a_0. Define $F : I^2 \to A$ by $F(x,t) = [\phi_{f(x)} * \bar{\phi}_{g(x)}](t)$ where $\phi_a : I \to A$ is the straight line connecting a and a_0. Intuitively, F deforms f into g by first taking $f(x)$ to a_0, then takes a_0 to $g(x)$. Thus, $f \simeq_p g$ and so $|\pi_1(A, a_0)| = 1$ as wanted. □

Exercise 2

Let α be a path in X from x_0 to x_1; let β be a path in X from x_1 to x_2. Show that if $\gamma = \alpha * \beta$, then $\hat{\gamma} = \hat{\beta} \circ \hat{\alpha}$.

Proof.

$$\begin{aligned}
\hat{\beta} \circ \alpha(\hat{[f]}) &= [\overline{\beta}] * ([\overline{\alpha}] * [f] * [\alpha]) * [\beta] \\
&= ([\overline{\beta}] * [\overline{\alpha}]) * [f] * ([\alpha] * [\beta]) \\
&= [\overline{\beta} * \overline{\alpha}] * [f] * [\alpha * \beta]
\end{aligned}$$

Recall from group theory that $(a \cdot b)^{-1} = b^{-1} \cdot a^{-1}$ so:

$$\hat{\beta} \circ \hat{\alpha}([f]) = [\overline{\alpha * \beta}] * [f] * [\alpha * \beta] = \hat{\gamma}([f])$$

as wanted. $\square$

Exercise 3

Let x_0 and x_1 be points of the path-connected space X. Show that $\pi_1(X, x_0)$ is abelian if and only if for every pair α and β of paths from x_0 to x_1, we have $\hat{\alpha} = \hat{\beta}$.

Proof. Assume the fundamental group at x_0 is abelian. Consider:

$$\begin{aligned}
\hat{\alpha}([f]) &= [\overline{\alpha}] * [f] * [\alpha] \\
&= ([\overline{\beta}] * [\beta]) * [\overline{\alpha}] * [f] * [\alpha] * ([\overline{\beta}] * [\beta]) \\
&= [\overline{\beta}] * ([f] * [\alpha] * [\overline{\beta}]) * ([\beta]) * [\overline{\alpha}]) * [\beta] \\
&= [\overline{\beta}] * [f] * [\beta] = \hat{\beta}([f])
\end{aligned}$$

where we have used the fact that $([f] * [\alpha] * [\overline{\beta}])$ is a path from x_0 to x_0 and so is $([\beta]) * [\overline{\alpha}])$ and so we can switch them using our abelian property. Now, assume instead that $\hat{\alpha} = \hat{\beta}$.

We know that $\gamma := f * \alpha$ is a path from x_0 to x_1 and so $\hat{\gamma} = \hat{\alpha}$. Note $[\overline{\gamma}] = [\overline{f * \alpha}] = [\overline{\alpha}] * [\overline{f}]$ and so,

$$\begin{aligned}
\hat{\gamma}([f * g]) &= [\overline{\gamma}] * [f * g] * [\gamma] \\
&= [\overline{\alpha}] * [\overline{f}] * [f] * [g] * [f] * [\alpha] \\
&= [\overline{\alpha}] * [g] * [f] * [\alpha] \\
&= [\overline{\alpha}] * [g * f] * [\alpha] = \hat{\alpha}([g * f]) = \hat{\gamma}([g * f])
\end{aligned}$$

and so $\hat{\gamma}([f * g]) = \hat{\gamma}([g * f])$. Using cancellation, we get $[f * g] = [g * f]$ (multiply by $[\gamma]$ on the left and by $[\overline{\gamma}]$ on the right) so that $\pi_1(X, x_0)$ is abelian. $\square$

Exercise 4

Let $A \subset X$; suppose that $r : A \to X$ is a continuous map such that $r(a) = a$ for each $a \in A$. (The map r is called a retraction of X onto A). If $a_0 \in A$, show that

$$r_* : \pi_1(X, a_0) \to \pi_1(A, a_0)$$

is surjective.

Proof. Choose any $[f] \in \pi_1(A, a_0)$. Since f (and any map, path homotopic to it) lies in A, we know that $r([f]) = [f]$ by definition of retraction. Thus, $[f] \in \pi_1(X, x_0)$ as wanted. $\square$

53 Covering Spaces

A fiber over a point in a covering space is the discrete space

Let $p : E \to B$ be a covering map. Let $b \in B$. Equip $p^{-1}(b)$ with the subspace topology. Then $p^{-1}(b)$ is precisely the discrete space.

Proof. A subspace $p^{-1}(b)$ of E is discrete if and only if for each $h \in p^{-1}(b)$, there exists an open set V in E such that $V \cap p^{-1}(b) = \{h\}$. Note p is a covering map, so in particular, there exists some neighbourhood U of B that p evenly covers. So, we partition $p^{-1}(U)$ into open disjoint sets and denote them by V_α. Thus, $h \in p^{-1}(b)$ falls in some unique V_α and so $\{h\} \subset V_\alpha \cap p^{-1}(b)$. Note that p is a covering map so $p|_{V_\alpha} : V_\alpha \to B$ is a homeomorphism. In particular, this means that $\#(p|_{V_\alpha}^{-1}) = 1$ or equivalently, $\{h\} = V_\alpha \cap p^{-1}(b)$. Thus, $p^{-1}(b)$ is a discrete space. $\square$

54 Deformation Retracts and Homotopy Types

Exercise 2

For each of the following spaces, the fundamental group is either trivial, infinite cyclic, or isomorphic to the fundamental group of the figure eight. Determine for each space which of the three alternatives holds.

1. The 'solid torus', $B^2 \times S^1$.
2. The torus T with a point removed.
3. The cylinder $S^1 \times I$
4. The infinite cylinder $S^1 \times \mathbb{R}$
5. $\mathbb{R}^3$ with the non-negative x, y, and z axes deleted
6. $\{x : ||x|| > 1\}$
7. $\{x : ||x|| \geq 1\}$
8. $\{x : ||x|| < 1\}$
9. $S^1 \cup (\mathbb{R} \times 0)$
10. $S^1 \cup (\mathbb{R}_+ \times \mathbb{R})$
11. $S^1 \cup (\mathbb{R} \times 0)$
12. $\mathbb{R}^2 - (\mathbb{R}_+ \times 0$

Proof. 1. We can embed this solid torus in $\mathbb{R}^4$ via the natural parameterizations of S^1 and B^2. So, embed the solid torus in $\mathbb{R}^3$ so that its center is the origin (0,0,0) and so that horizontally, half of it lies above the xy-plane and half of it lies below the xy- plane. Then, we construct a homotopy $H : B^2 \times S^1 \times I \to \mathbb{R}^3$ where $H(x, y, z, t) = (x, y, (1 - t)z)$ so that we can flatten this donut into an annulus. Now, let $\phi : B^2 \times S^1 \to \mathbb{R}^3$ be a functions such that it returns the intersection of the line that cross through (x, y, z) and $(0, 0, 0)$ and the set $\{(x, y, z) : ||(x, y, z)|| = 1\}$. Using this, we can construct a homotopy which squishes the flat ring into a circle embedded in R^3. Thus, $S^1 \times 0$ is a deformation retract of $B^2 \times S^1$ implying $\pi_1(B^2 \times S^1) \simeq \pi_1(S^1)$.

2. (figure eight)

3. Embed $S^1 \times I$ so that its center is the origin and so that horizontally, half of it lies above the xy-plane and half of it lies below the xy- plane. Then, define $H : S^1 \times I \times I \to \mathbb{R}^3$ via $H(x, y, z, t) = (x, y, z, (1 - t)z)$ which flattens the cylinder into $S^1 \times 0$. Thus, giving us that $S^1 \times 0$ is a deformation retract of $S^1 \times I$ which implies $\pi_1(S^1 \times I) \simeq \pi_1(S^1)$.

4. We apply the same procedure from above to get $\pi_1(S^1 \times \mathbb{R}) \simeq \pi_1(S^1)$.

5. (figure eight)

6. This has $\partial B_2(0)$ as a deformation retract. Since $\pi_1(\partial B_2(0)) \simeq \pi_1(S^1)$, we are done.

7. Using the function ϕ from (1) but redefining it so that it maps $\{x : ||x|| > 1\} \to \mathbb{R}^2$, we get that $\{x : ||x|| > 1\}$ has S^1 as a deformation retract.

8. This is a star convex set so the fundamental group is trivial.

9. We can construct a homotopy which collapses $\mathbb{R}_+ \times 0$ into the point $(1, 0)$ but leaves S^1 the same. This shows S^1 is a deformation retract of this space.

10. Using a homotopy, we can collaps $\mathbb{R}_+ \times \mathbb{R}$ onto the positive (or right) side of S^1. Thus, S^1 is a deformation retract of this set.

11. We can retract $\mathbb{R} \times 0$ to $I \times 0$. Note, $S^1 \cup (I \times 0)$ is homeomorphic to $S^1 \cup (0 \times I)$ which is the theta space.

12. This set is star convex.

□

59 The Fundamental Group of S^n

Exercise 3

(a) Show that $\mathbb{R}^1$ and $\mathbb{R}^n$ are not homeomorphic if $n > 1$

(b) Show that $\mathbb{R}^2$ and $\mathbb{R}^n$ are not homeomorphic if $n > 2$

Proof. (a) Suppose $\mathbb{R}$ and $\mathbb{R}^n$ are homeomorphic for $n > 1$, then by definition, there exists some homeomorphism. $f : \mathbb{R}^1 \to \mathbb{R}^n$. Considering the restriction $\mathbb{R}^1 - \{0\}$ induces a homeomorphism

$$f_{\mathbb{R}-\{0\}} : \mathbb{R} - \{0\} \to \mathbb{R}^n - \{f(0)\}$$

We see that $\mathbb{R} - \{0\}$ is not connected but $\mathbb{R}^n - \{f(0)\}$ is (note $|\{f(0)\}| = 1$ since f is bijective). Since connectedness is a topological invariant, we have a contradiction and so such a mapping f cannot exist so that $\mathbb{R}$ and $\mathbb{R}^n$ are not homeomorphic.

(b) Suppose $\mathbb{R}^2$ and $\mathbb{R}^n$ are homeomorphic for $n > 2$, then by definition, there exists some homeomorphism $f : \mathbb{R}^2 \to \mathbb{R}^n$. Similarly, to before, we have another homeomorphism:

$$f_{\mathbb{R}^2-\{0\}} : \mathbb{R}^2 - \{0\} \to \mathbb{R}^n - \{f(0)\}$$

We know S^1 is a deformation retract of $\mathbb{R}^2 - \{0\}$ so that both spaces have isomorphic fundamental groups. Similarly, S^{n-1} is a deformation retract of $\mathbb{R}^n - \{0\}$. Since $n > 2$, we know $n - 1 > 1$ so that in particular, S^{n-1} has a trivial fundamental group. This implies that $\mathbb{R}^n - \{0\}$ is simply connected but $\mathbb{R}^2 - \{0\}$ is not. This is a contradiction as simple connectedness is a topological invariant. Therefore, we know that such a mapping f cannot exist and so $\mathbb{R}^2$ and $\mathbb{R}^n$ for $n > 2$ are not homeomorphic. $\square$

60 Fundamental Groups of Some Surfaces

Exercise 1

Compute the fundamental groups of the "solid torus" $S^1 \times B^2$ and the product space $S^1 \times S^2$:

Proof. Consider for the solid torus (and any point $x_0 \times y_0$ on the surface)

$$\begin{aligned} \pi_1(S^1 \times B^2, x_0 \times y_0) &= \pi_1(S^1, x_0) \times \pi_1(B^2, y_0) \qquad \text{[Theorem 60.1]} \\ &\simeq \mathbb{Z} \times \{e\} \qquad [B^2 \text{ is convex}] \\ &\simeq \mathbb{Z} \end{aligned}$$

so that the fundamental group of the solid torus is isomorphic to $\mathbb{Z}$. We can do a similar computation to this to get the same result for the surface $S^1 \times S^2$ taking advantage of the fact that S^2 is simply connected. $\square$