# (Selected) Exercise Solutions From Guillemin's Differential Topology

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**1.** [1.1.4] Let $B_a$ be the open ball $\{x : |x|^2 < a\}$ in $\mathbb{R}^k$. Show that the map:

$$x \to \frac{ax}{\sqrt{a^2 - |x|^2}}$$

is a diffeomorphism of $B_a$ onto $\mathbb{R}^k$.
For the second part of this problem, suppose that $X$ is a $k$-dimensional manifold. Show that every point in $X$ has a neighbourhood diffeomorphic to all of $\mathbb{R}^k$. Deduce that local parameterizations may be written with all of $\mathbb{R}^k$ for their domains.

**Solution:**
It is sufficient to find an inverse of the mapping $f : \mathbb{R}^k \to B_a$ (with the element mapping given above). One may compute directly to find that:

$$f^{-1}(x) = \frac{ax}{\sqrt{a^2 + |x|^2}}$$

Therefore, $f$ is a diffeomorphism.
(b) By definition, for any point $x \in X$, we have that there exists a diffeomorphism $\phi : V \to U$ where $U \subseteq \mathbb{R}^k$ is open and $V$ is some neighbourhood around $x$. Restrict $V$ so that $f(V) = B_a(f(x))$ for some $a$. Then, the mapping $f^{-1} \circ \phi : V \to \mathbb{R}^k$ is a diffeomorphism which maps a neighbhourhood around $x$ to all of $\mathbb{R}^k$. Furthermore, the inverse of the composition mapping is a diffeomorphism going from $\mathbb{R}^k \to V$. Thus, it is a local parameterization written with all of $\mathbb{R}^k$ for its domain.

**2.** [1.1.5] Show that every $k$-dimensional vector subspace of $\mathbb{R}^n$ is a manifold diffeomorphic to $\mathbb{R}^k$, and that all linear maps on $V$ are smooth. If $\phi : \mathbb{R}^k \to V$ is a linear isomorphism, then the corresponding coordinate functions are linear functionals on $V$ called *linear coordinates.*

**Solution:**
First, we will show $V$ is isomorphic to $\mathbb{R}^k$. Let $\{e_1, e_2, \cdots, e_k\}$ be some basis of $\mathbb{R}^k$ and $\{\varphi_1, \varphi_2, \cdots, \varphi_k\}$ be the basis of $V$. Then, there exists a linear map $L$ such that $L(\varphi_1) = e_1, L(\varphi_2) = e_2, \cdots, L(\varphi_k) = e_k$. By a similar process, $L^{-1}$ is defined. Thus, we obtain that $L$ is an isomorphism which implies $V$ and $\mathbb{R}^k$ are isomorphic.
Next, we will show $L$ is differentiable. Consider, for any $a \in \mathbb{R}^k$ and (let $A_{m \times n}$

be the matrix from $L(x) = Ax$). Then:

$$\begin{aligned}
&= \lim_{h\to 0} \frac{|L(a+h) - L(a) - Ah|}{|h|} \\
&= \lim_{h\to 0} \frac{|L(a) + L(h) - L(a) - Ah|}{|h|} \\
&= \lim_{h\to 0} \frac{|L(h) - Ah|}{|h|} \\
&= \lim_{h\to 0} \frac{|Ah - Ah|}{|h|} \\
&= 0
\end{aligned}$$

Since $DL(a) := A$ is stil a linear map (it is composed of $m$ linear maps), then via induction, we can obtain the result that $L$ is a smooth function. Thus $L$ is actually diffeomorphic implying that $V$ and $\mathbb{R}^k$ are diffeomorphic. Furthermore, since $L$ is arbitrary, any linear map on $V$ is smooth.

**3.** [1.1.8] Prove that the hyperboloid in $\mathbb{R}^3$, defined by $x^2 + y^2 - z^2 = a$, is a manifold if $a > 0$. Why doesn't $x^2 + y^2 - z^2 = 0$ define a manifold?

**Solution:**
Suppose $a > 0$. Then we show that $x^2 + y^2 - z^2 = a$ defines a manifold.
**Case I:** $y^2 - z^2 > a$ Then:

$$x^2 + y^2 - z^2 = a \implies x^2 + y^2 - a = z \implies z = \pm\sqrt{x^2 + y^2 - a}$$

So we get the covers:

$$\phi(x,y) = (x, y, \pm\sqrt{x^2 + y^2 - a})$$

**Case II:** One can check that for the other two cases $x^2 - z^2 > a$ and $x^2 + y^2 - z^2 > a$, we arrive at the same covers. Thus the diffeomorphisms:

$$\phi(x,y) = (x, y, \pm\sqrt{x^2 + y^2 - a})$$

cover the hyperboloid so it is a manifold.
Now suppose that $a = 0$. We show that $x^2 + y^2 = z^2$ does not define a manifold. Note that:

$$z = \pm\sqrt{x^2 + y^2}$$

Thus, if the hyperboloid is a manifold, then the maps

$$\phi(x,y) = (x, y, \pm\sqrt{x^2 + y^2})$$

are smooth on their domains. However, we prove that $\phi'(0,0)$ does not exist. First note that:

$$\phi'(0,0) = \begin{pmatrix} 1 & 0 & \pm\frac{x}{\sqrt{x^2+y^2}} \\ 0 & 1 & \pm\frac{y}{\sqrt{x^2+y^2}} \end{pmatrix}$$

It suffices to show that $\lim_{(x,y)\to(0,0)} \frac{x}{\sqrt{x^2+y^2}}$ does not exist. So, take $\|(x,y)\| < 1$ and consider approaching the function at $y = mx$:

$$\begin{aligned}
&\lim_{(x,y)\to(0,0)} \frac{x}{\sqrt{x^2+y^2}} \\
&= \lim_{(x,y)\to(0,0)} \frac{x}{\sqrt{x^2+(mx)^2}} \\
&= \lim_{(x,y)\to(0,0)} \frac{x}{\sqrt{x^2(1+m^2)}} \\
&= \lim_{(x,y)\to(0,0)} \frac{1}{\sqrt{1+m^2}} \\
&\neq (0,0)
\end{aligned}$$

so $\phi$ is not smooth at $(0,0)$, thus $x^2 + y^2 = z^2$ does not define a manifold.

**4.** [1.1.9] Explicitly exhibit enough parameterizations to cover $S^1 \times S^1 \subset \mathbb{R}$.
**Solution:**
Let $a, b \in \{x, -x, \sqrt{1-x^2}, -\sqrt{1-x^2}\}$ and $c, d \in \{y, -y, -\sqrt{1-y^2}, \sqrt{1-y^2}\}$. Then the set: $\{(a,b,c,d)\}$ holds the parameterizations which cover $S^1 \times S^1$.

**5.** [1.1.10] The *torus* is the set of points in $\mathbb{R}^3$ at distance $b$ from the circle of radius $a$ in the $xy$ plane where $0 < b < a$. Prove that these tori are all diffeomorphic to $S^1 \times S^1$. Also, draw the cases $b = a$ and $b > a$; why are these not manifolds?

**6.** [1.1.11] Show that one cannot parameterize the $k$ sphere $S^k$ by a single parameterization.

**Solution:**
Suppose such a parameterization exists. Then let $\phi : U \subseteq \mathbb{R}^k \to S^n$ be the parameterization. We know that $\phi^{-1} : S^n \to U$ is continuous. Therefore, since $S^k$ is compact, $U$ must be as well. But note that by definition of $\phi$, $U$ must be open. Thus $U$ is open and closed and the only non-empty set which is open and closed is $\mathbb{R}^k$ so $U = \mathbb{R}^k$. But then $U$ cannot be compact (Heine-Borel) even though the mapping $\phi^{-1}$ implies that it should be. Thus we have a contradiction, implying that there exists no *single* parameterization which can cover $S^k$.

**7.** [1.1.12] A stereographic projection is a map $\pi$ from the punctured sphere $S^2 - \{N\}$ onto $\mathbb{R}^2$, where $N$ is the north pole. For any $p \in S^2 - \{N\}$, $\pi(p)$ is defined to be the point at which the line through $N$ and $p$ intersects the $xy$ plane. Prove that $\pi : S^2 - \{N\} \to \mathbb{R}^2$ is a diffeomorphism. Note, that if $p$ is near $N$, then $|\pi(p)|$ is large. Thus $\pi$ allows us to think of $S^2$ as a copy of $\mathbb{R}^2$ compactified by the addition of one point infinity. Since we can define stereographic projection by using the south pole instead of the north, $S^2$ may be covered by two local parameterizations.

**Solution:**
Let $(p_1, p_2, p_3)$ be a point in $S^2 - \{N\}$. Then the line starting from $N$ going through $(p_1, p_2, p_3)$ is given by:

$$t \mapsto (0,0,1) + t(p_1, p_2, p_3 - 1)$$

Let this function be reffered to as $\phi$. We know $\phi(t) = 0$ when $t(p_3 - 1) + 1 = 0$ or when $t = \frac{-1}{p_3-1}$. So,

$$\pi(p) = (\frac{-p_1}{p_3 - 1}, \frac{-p_2}{p_3 - 1}) \tag{1}$$

Now, to find $\pi^{-1}$, take any $(x,y) \in \mathbb{R}^2$. Then, consider the line $t \mapsto (0,0,1) + t(x,y,-1)$ (furthermore, let us refer to this function as $\theta$. To find when $\theta(t)$ intersects $S^2 - \{N\}$, consider:

$$\sqrt{(tx)^2 + (ty)^2 + (-t+1)^2} = 1$$
$$\Leftrightarrow (tx)^2 + (ty)^2 + t^2 - t = 0$$

which holds when $t = 0$ or when $t = \frac{1}{1+x^2+y^2}$. This implies $\pi^{-1}(x,y) = (0,0,1) + t(x,y,-1)$ where $t = \frac{1}{1+x^2+y^2}$. Note that if $x^2 + y^2 = 1$, then $\pi^{-1}(x,y) := (x,y,0)$. It is easy to see that both $\pi$ and $\pi^{-1}$ are both smooth. Thus, $\pi : S^2 - \{N\} \to \mathbb{R}^2$ is diffeomorphic.

**8.** [1.1.13] By generalizing stereographic projection define a diffeomorphism $S^k - \{N\} \to \mathbb{R}^k$.

**Solution:**
Define $\pi(p)$ to the point formed at the intersection between the line $\overline{Np}$ (where $N$ is the north pole) and the hyperplane. Specifically:

$$\pi(p) = (0, \cdots, 0, 1) + (\frac{p_1}{1 - p_{k+1}}, \frac{p_2}{1 - p_{k+1}}, \cdots, \frac{p_k}{1 - p_{k+1}})$$

and

$$\pi^{-1}(x) = (0, \cdots, 0, 1) + (1 - p_1^2, -p_2^2, \cdots, -p_k^2)(p_1, \cdots, p_k, -1)$$

**9.** [1.1.17] The *graph* of a map $f : X \to Y$ is the subset of $X \times Y$ defined by

$$\text{graph}(f) = \{(x, f(x)) : x \in X\}$$

Define $F : X \to \text{graph}(f)$ by $F(x) = (x, f(x))$. Show that if $f$ is smooth, then $F$ is a diffeomorphism; thus graph$(f)$ is a manifold if $X$ is.

**Solution:**
Note $F'(x) = (1, f(x))$. Since $f(x)$ is smooth, $f'$ exists and is continuous. Also note that $F$ is then differentiable ($\in C^1$ to be more specific) since both of the

components of $F$ are $C^1$. We can apply induction and use the same arguements to obtain that $F$ is smooth.

Now suppose that $F(x) = F(y)$. Then $(x, f(x)) = (y, f(y))$ which implies that in the first component, $x = y$. Thus $F$ is injective. The fact that $F$ is surjective follows from the definition of graph$(f)$.

Smoothness of inverse is given by the fact that it is a projection mapping $(x, f(x)) \mapsto x$ and all such mappings are smooth.

Thus $F$ is a diffeomorphism and $F$ or $F^{-1}$ would be the parameterization used to cover $X$ or graph$(f)$ if they were manifolds.

Problem[1.1.18(a)] An extermely useful function $f : \mathbb{R} \to \mathbb{R}$ is:

$$f(x) = \{e^{\frac{-1}{x^2}} \text{ if } x > 0,$$
$$0 \text{ if } x \leq 0\}$$