Algebraic Topology Companion Notes

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Filling in some details or trying some proofs myself from James Munkres' *Topology*.

§51 Homotopy of Paths

Lemma 51.1. *The relations $\simeq$ and $\simeq_p$ are equivalence relations.*

Proof. First, we show $\simeq$ is an equivalence relation between homotopic continuous functions from $X \to Y$.
We will prove reflexivity first. Let $f : X \to Y$ be continuous. We show f is homotopic with itself. To do this, we have to show there exists a continuous mapping $F : X \times I \to Y$ such that $F(x, 0) = f(x)$ and $F(x, 1) = f(x)$ for all $x \in X$. Thus, define F to be given by $(x, t) \mapsto f(x)$. Since f is continuous, it follows that F is then also continuous.
Now, we show symmetry. Let f and g be any homotopic functions from X to Y. We show $g \simeq f$. To do this, we have to show there exists a continuous function $F : X \times I \to Y$ (not to be confused with the one above, we are no longer using it) such that $F(x, 0) = g(x)$ and $F(x, 1) = f(x)$. Thus, define F to be given by the mapping $(x, t) \mapsto G(x, 1 - t)$ where G is the homotopy between f and g. Thus, we immediately obtain that $F(x, 0) = G(x, 1) = g(x)$ and $F(x, 1) = G(x, 0) = f(x)$ as wanted. To see this is continuous, it suffices to see that the component functions are continuous. It is clear that $\pi_1(F(x, t))$ is continuous since G is continuous. The fact that $\pi_2(F(x, t))$ is continuous follows from the fact that it is a composition of continuous functions (namely $(x, t) \mapsto (x, 1 - t)$ and $(x, t) \mapsto G(x, t)$).
Now, we prove $\simeq$ is transitive to conclude that $\simeq$ is an equivalence relation. Let $f \simeq g$ and $g \simeq h$ for homotopic functions $f, g : X \to Y$ and homotopic functions $g, h : X \to Y$. Let F' and F'' be the homotopy of f, g and g, h respectively. Now, define $H : X \times [0, 2] \to Y$ via $(x, t) \mapsto F'(x, t)$ if $t \leq 1$ and $(x, t) \mapsto F''(x, t-1)$ if $t \geq 1$. We see that H is well defined since $H(x, 1) = F'(x, 1) = g(x) = F''(x, 0)$. H is also continuous since H is continuous on $t < 1$ via F' and continuous on $t > 1$ via F''. Thus, by the pasting lemma, we have that H is continuous on $X \times [0, 2]$. Using this as motivation, we define the homotopy $\delta : X \times I \to Y$ between f, h via $(x, t) \mapsto F'(x, 2t)$ if $t \leq 1/2$ and $(x, t) \mapsto F''(x, 2t-1)$ if $t \geq 1/2$.

Now, we show $\simeq_p$ is an equivalence relation. Let $f : I \to X$ be a continuous path from x_0 to x_1 and define F to be the homotopy from the reflexive proof of $\simeq$. Then, we only need to show the additional condition that $F(0,t) = x_0$ and $F(1,t) = x_1$. Note $F(x,t) = f(x)$ so $F(0,t) = f(0) = x_0$ and $F(1,t) = f(1) = x_1$ as wanted. Thus, $f \simeq_p f$ showing reflexivity. Now, we show symmetry. Let $f, f' : I \to X$ be two homotopic paths from x_0 to x_1. Define $F : I^2 \to X$ via $(s,t) \mapsto F'(s, 1-t)$ where F' is the path homotopy between f and f'. Then from the symmetry proof of $\simeq$ above, we have only left to prove that $F(0,t) = x_0$ and $F(1,t) = x_1$. Note $F(0,t) = F'(0, 1-t) = x_0$ and $F(1,t) = F'(1, 1-t) = x_1$ as wanted. We now, only need show transitivity to conclude that $\simeq_p$ is an equivalence relation. Let $f, g : I \to X$ and $g, h : I \to X$ be path homotopies between x_0 and x_1. We show f, h are path homotopic. Define δ to be a path homotopy between f and h the same way it was done in the transitive proof above. Thus, it is left to prove $\delta(0,t) = x_0$ and $\delta(1,t) = x_1$. If $t < 1/2$, we have $\delta(0,t) = F'(0, 2t) = x_0$ since F' is a path homotopy between f and g which are paths between x_0 and x_1. Similarly, $\delta(1,t) = x_1$. We can repeat this for $t \geq 1$ to again, get the same result. Thus, δ is a path homotopy as wanted. Therefore, $\simeq_p$ is an equivalence relation as wanted. ☐

Example 51.2. *Let f and g be any two maps of a space X into $\mathbb{R}^2$. It is easy to see that f and g are homotopic; the map*

$$F(x,t) = (1-t)f(x) + tg(x)$$

is a homotopy between them called the ***straight line homotopy****.*

Proof. Note that $F(x,0) = (1-0)f(x) + 0g(x) = f(x)$ and $F(x,1) = (1-1)f(x) + 1g(x) = g(x)$ as wanted. F is continuous because f and g are continuous functions on X, thus, so are $(1-t)f(x)$ and $tg(x)$ for all $t \in \mathbb{R}$. $F(x,t)$ is the sum of these two functions so, we immediately obtain that F is a continuous function. Thus, F is a homotopy between f and g. ☐

Exercise 1

Show that if $h, h' : X \to Y$ are homotopic and $k, k' : Y \to Z$ are homotopic, then $k \circ h$ and $k' \circ h'$ are homotopic.

Proof. ☐