Practice Problems MATC37

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1. Review of some key definitions and theorems

Definition of the Lebesgue measure. If E is Lebesgue measurable, we say its *Lebesgue measure* is equal to its outer measure i.e. $m(E) = m_*(E)$ where m denotes the Lebesgue measure of E (m is a function $m : \mathscr{L} \to \mathbb{R} \cup \{\infty\}$ where $\mathscr{L}$ denotes the set of Lebesgue measurable sets) and $m_* : \mathcal{P}(\mathbb{R}^d) \to \mathbb{R} \cup \{\infty\}$ denotes the Lebesgue outer measure.

Definition of Lebesgue measurable. A subset E of $\mathbb{R}^d$ is said to be *Lebesuge measurable* if for all $\epsilon > 0$, we have the existence of some open set $\mathcal{O}$ such that $m_*(\mathcal{O}/E) \leq \epsilon$ where m_* denotes the Lebesgue outer measure. The Lebesgue outer measure of any $E \subset \mathbb{R}^d$ is the infimum taken over the sums of countable collections of closed cubes which cover E.

Definition of Lebesgue Integral. Let ϕ be a simple function, i.e. ϕ can be written as $\phi(x) = \sum_{k=1}^{N} a_k \chi_{E_k}$ where all a_k's are distinct and non-zero and all E_k's are disjoint with $m(E_k) < \infty$. We define the *Lebesgue integral* of ϕ to be $\int \phi := \sum_{k=1}^{N} a_k m(E_k)$. Now, let f be some bounded function with bounded support i.e. a bounded function such that $m(\{x : f(x) \neq 0\} < \infty$. Then, there exist a sequence of increasing simple functions $\{\phi_k\}_{k=1}^{\infty}$ which converge pointwise to f. We define its *Lebesgue integral* to be $\int f := \lim_{k\to\infty} \int \phi_k$. Now, take f to be any non-negative function. We define its *Lebesgue integral* to be $\int f := \sup_g \int g$ where the supremum is taken over all functions $0 \leq g \leq f$. If $\int f < \infty$, we say f is *Lebesgue integrable*. Finally, let f be any real valued function. We say f is *Lebesgue integrable* if $\int |f| < \infty$. Furthermore, we say its *Lebesgue integral* is given by $\int f = \int f^+ - \int f^-$ where $f^+(x) = \max\{f(x), 0\}$ and $f^-(x) = -\min\{0, f(x)\}$.

Dominated Convergence Theorem. Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of measurable functions converging pointwise to f for a.e. x. If $0 \leq |f| \leq |g|$ for some integrable function g, we have that $\int |f_n - f| \to 0$ as $n \to \infty$ and $\int f_n \to \int f$ as $n \to \infty$.

Monotone Convergence Theorem. Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of non-negative measurable functions. If $0 \leq |f_n| \leq |f_{n+1}|$ and $f_n \to f$ pointwise for a.e. x, then $\lim \int f_n = \int f$.

Fatou's Lemma. Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of non-negative measurable functions. Then, $\int \liminf_{n\to\infty} f_n \leq \liminf_{n\to\infty} \int f_n$.

Definition of L^1 and L^2. Let $\sim$ be a equivalence relation on Lebesgue integrable functions. We say that $f \sim g$ if and only if $f = g$ a.e. x on their domain. Then, we say that $L^1(E)$ is the set of all integrable functions quotiented over $\sim$ that are defined on the domain $E \subseteq \mathbb{R}^d$. This forms a vector space with the standard pointwise addition and pointwise scalar multiplication operations. In particular, it is a normed linear space with the norm $\|f\|_{L^1(E)} = \int_E |f|$ where the integral is taken over the domain of f. $L^2(E)$ as a vector space is defined similarly.

$$L^2(E) = \{f : E \subseteq \mathbb{R}^d \to \mathbb{R} \mid \int_E |f|^2 < \infty\}/\sim$$

The norm on $L^2(E)$ is given by $\|f\|_{L^2(E)} = (\int_E |f|^2)^{1/2}$.

Fundamental Theorem of Calculus. If $F : [a,b] \to \mathbb{R}$ is absolutely continuous, then F' exists for a.e. x, is integrable and $F(b) - F(a) = \int_a^b F'(t)dt$. Conversely, if $f \in L^1(\mathbb{R})$, then the function F defined by $F(x) = \int_a^x f(t)dt$ is absolutely continuous.

2. Space of integrable and square integrable functions

Is the function $f(x) = x/(1+x^2)$ in $L^1(\mathbb{R})$? To show $f(x) := x/(1+x^2)$ is not in $L^1(\mathbb{R})$, it suffices to show $\int_{\mathbb{R}} |x/(1+x^2)|dx = \infty$. This is a non-negative function clearly, so by definition, we have:

$$\int_{\mathbb{R}} |\frac{x}{1+x^2}|dx = \sup_g \int g$$

where the supremum is taken over all non-negative bounded functions with finite support. Since $\sup_n\{|1/(1+x^2)|\chi_{[-n,n]}\} \leq \sup_g \int g$, we have that

$$\sup_n\{|\frac{x}{1+x^2}|\chi_{[-n,n]}\} = \lim_{n\to\infty} \int_{[-n,n]} |\frac{x}{1+x^2}|dx \leq \int_{\mathbb{R}} |\frac{x}{1+x^2}|dx$$

Note $\lim_{n\to\infty} \int_{[-n,n]} |\frac{x}{1+x^2}|dx = \lim_{n\to\infty} \int_{[-n,n]}^{\mathcal{R}} |\frac{x}{1+x^2}|dx$ where $\mathcal{R}$ denotes the Riemann integral (since we are integrating over a compact interval). However, the limit then goes to infinity when we compute the integrals using the Riemann integral (since the limit is equal to the improper integral $\int_{\mathbb{R}}^{\mathcal{R}} |x/(1+x^2)|dx = \infty$). Thus, $\infty \leq \int_{\mathbb{R}} |\frac{x}{1+x^2}|dx$ and so $x/(1+x^2)$ is not in $L^1(\mathbb{R})$.

Give an example of a function that is in $L^1(\mathbb{R})$ but not in $L^2(\mathbb{R})$. We proved in one of the assignments that $|1/x^2|$ is integrable over the interval $[1, \infty)$ and not integrable over $[0, 1]$. Thus, consider the function $f(x) = (1/x)\chi_{[1,\infty)}(x)$. This is not clearly integrable over L^1 but over L^2 it is since $\int_1^\infty |1/x^2| dx < \infty$.

Prove that $L^2([a, b]) \subseteq L^1([a, b])$. s

3. Interchanging limits and integration

Compute the following limits (here and everywhere, you should justify your steps):

Compute $\lim_{n\to\infty} \int_0^1 \frac{1}{1+x^n} dx$. Consider the sequence of functions $f_n := 1/(1 + x^n)$ defined on the unit interval $[0, 1]$. Furthermore, $f_n \to f = 1_{\text{id}}$, each f_n is measurable and bounded by 1 with finite support. Thus, by Bounded Convergence theorem, we have that

$$\lim_{n\to\infty} \int_0^1 \frac{1}{1+x^n} dx = \int_0^1 \lim_{n\to\infty} \frac{1}{1+x^n} dx = \int_0^1 1dx = 1$$

Compute $\lim_{n\to\infty} \int_0^\infty \frac{1+\cos(x/n)}{2-\sin(x/n)} \frac{1}{1+x^2} dx$. Consider the sequence of (measurable) functions $f_n := \frac{1+\cos(x/n)}{2-\sin(x/n)} \frac{1}{1+x^2}$. Clearly, $|f_n| \le g := \frac{2}{1+x^2}$ which is clearly Lebesgue integrable and non-negative on $[0, \infty)$. Thus, by Dominated Convergence theorem, we have that:

$$\int_0^\infty \lim_{n\to\infty} \frac{1+\cos(x/n)}{2-\sin(x/n)} \frac{1}{1+x^2} dx = \lim_{n\to\infty} \int_0^\infty \frac{1+\cos(x/n)}{2-\sin(x/n)} \frac{1}{1+x^2} dx$$

Clearly the integrand in the left side of the equation above is just equal to $1/(1+x^2)$. This is non-negative on $[0, \infty)$. Thus, we compute this integral via Riemann integral techniques as explained in the answer below. Therefore, we obtain that the integrals above are equal to $\arctan(\infty) - \arctan(0) = \pi/2$ as wanted.

Compute $\lim_{n\to\infty} \int_0^\infty \frac{x^2+n^2}{x^2+2n^2} e^{-x} dx$. Consider the sequence $f_n := (x^2 + n^2)/(x^2 + 2n^2)e^{-x}$. Clearly $|f_n| \le g := e^{-x}$. It is left to show g is Lebesgue integrable on $[0, \infty)$. Recall $e^{-x} \le 1/x^2$ on $[1, \infty)$ and $1/x^2$ is Lebesgue integrable on this interval. Also, since e^{-x} is Riemann integrable on $[0, 1]$ it is also Lebesgue integrable. Thus, e^{-x} is Lebesgue integrable on $[0, \infty)$ as wanted. We can now use DCT to get:

$$\lim_{n\to\infty} \int_0^\infty \frac{x^2+n^2}{x^2+2n^2} e^{-x} dx = \int_0^\infty \lim_{n\to\infty} \frac{x^2+n^2}{x^2+2n^2} e^{-x} dx = \int_0^\infty \lim_{n\to\infty} \frac{x^2/n^2+1}{x^2/n^2+2} e^{-x} dx$$

Thus, we see that the integrand on the far right is equal to $(1/2)e^{-x}$. Thus, it is left to compute $(1/2) \cdot \int_0^\infty e^{-x} dx$. Note:

$$\int_{[0,\infty)} e^{-x} dx = \lim_{n\to\infty} \int_{[0,\infty)} e^{-x} \chi_{[0,n]} dx$$

by MCT and the non-negativity of e^{-x}. Since each $\int_{[0,\infty)} e^{-x}\chi_{[0,n]}dx = \int_0^n e^{-x} dx$. We can solve $\int_0^\infty e^{-x}$ using Riemann integral techniques. Thus, it is equal to $1/2(-e^{-\infty} + e^0) = 1/2$.

Compute $\lim_{n\to\infty} \int_{-n}^n \frac{\cos(x/n)}{1+x^2} dx$**.** Note:

$$\lim_{n\to\infty} \int_{-n}^n \frac{\cos(x/n)}{1+x^2} dx = \lim_{n\to\infty} \int_{\mathbb{R}} \frac{\cos(x/n)}{1+x^2} \chi_{[-n,n]} dx$$

We again apply DCT using $g := 1/(1+x^2)$ on $f_n := \cos(x/n)/(1+x^2)\chi_{[-n,n]}$ defined on $\mathbb{R}$. Clearly $f_n \to 1/(1+x^2)$. Thus:

$$\lim_{n\to\infty} \int_{-n}^n \frac{\cos(x/n)}{1+x^2} dx = \int_{\mathbb{R}} \frac{1}{1+x^2} dx$$

which is simply equal to 0 (odd function).

4. Lebesgue measurable and Lebesgue measure

Consider the set $E := \hat{C} \cup [7,9] \subseteq \mathbb{R}$, where $\hat{C}$ denotes the fat Cantor set where at each step a middle quarter is removed.

Prove that E is Lebesgue measurable. Since $[7,9]$ and $\hat{C}$ are disconnected with distance > 0, it suffices to show $\hat{C}$ is Lebesgue measurable. We claim $\hat{C}$ is closed which will show it is measurable. Note $\hat{C}$ is constructed via a countable intersection of closed sets. Thus, $\hat{C}$ is closed and so it is Lebesgue measurable which additionally implies that E is Lebesgue measurable as wanted.

Compute the Lebesgue measure of E. We are removing $(1/4)^{n+1}$ from each interval at each step n and doing this for 2^n intervals at the same step. Thus, we are removing: $\sum_{n=0}^\infty 2^n/4^{n+1} = (1/4)\sum_{n=0}^\infty (2/4)^n = 1/2$. Therefore, $m(E) = m(\hat{C}) + m([7,9]) = (1-1/2) + 2 = 5/2$.

5. A parameter dependent integral

The goal of this question is to compute the integral $\int_0^\infty \cos(tx) e^{-x^2/2} dx$.

Prove that $F(t) := \int_0^\infty \cos(tx)e^{-x^2/2}dx$ is differentiable. We have to compute the limit:

$$\lim_{h\to 0} \frac{F(t+h)-F(t)}{h}$$

and see if it exists. Substituting in the formula for F, we get:

$$\lim_{h\to 0} \int_0^\infty e^{-x^2/2} \frac{\cos((t+h)x)-\cos(tx)}{h} dx$$

Take an arbitrary sequence $(h_n)_1^\infty$ converging to 0. We get that the above is equal to:

$$\lim_{n\to\infty} \int_0^\infty e^{-x^2/2} \frac{\cos((t+h_n)x)-\cos(tx)}{h_n} dx$$

Note $\cos(tx)e^{-x^2/2}\chi_{[0,n]} \to \cos(tx)e^{-x^2/2}$ and $|\cos(tx)e^{-x^2/2}\chi_{[0,n]}| \le e^{-x^2/2}$ which is certainly integrable on $[0,\infty)$. Thus, we use DCT to get:

$$\int_0^\infty e^{-x^2/2} (\lim_{n\to\infty} \frac{\cos((t+h_n)x)-\cos(tx)}{h_n}) dx$$

which is equivalent to saying:

$$\int_0^\infty e^{-x^2/2} (\lim_{h\to 0} \frac{\cos((t+h)x)-\cos(tx)}{h}) dx$$

We know the derivative of $\cos(tx)$ taken with respect to t is simply $(-x)\sin(tx)$. Thus, we get:

$$F'(t) = \lim_{h\to 0} \frac{F(t+h)-F(t)}{h} = -\int_0^\infty x\sin(tx)e^{-x^2/2}dx$$

Thus $F'(t)$ existing comes down to showing that $t\sin(tx)e^{-x^2/2}$ is Lebesgue integrable on $[0,\infty)$. This is true so $F'(t)$ exists. Thus F is differentiable.

Compute $F(t)$ (Hint: Derive a differential equation for $F(t)$ and solve it). We know,

$$F'(t) = -\int_0^\infty x\sin(tx)e^{-x^2/2}dx$$

from the previous question. The integral on the right side is solvable. It is equal to $\sqrt{\pi/2}e^{-t^2/2}t$. Thus, we get,

$$F'(t) = \sqrt{\pi/2}e^{-t^2/2}t$$

We find the anti-derivative for the right side to solve for $F'(t)$:

$$F(t) = \sqrt{\pi/2}(-e^{-t^2/2})$$