Problem Set 8

MAT237: Advanced Calculus
November 30, 2016 ANMOL BHULLAR

Problem 0.1. *Define $f : \mathbb{R}^2 \to \mathbb{R}$ by*

$$f(x,y) = \begin{cases} \frac{x^3y - xy^3}{x^2+y^2} & (x,\ y) \neq (0,\ 0) \\ 0 & (x,\ y) = (0,0) \end{cases}$$

Compute the mixed partial derivatives $\partial_{xy} f$, $\partial_{yx} f$. Why does this not contradict Clairut's theorem?

Solution. First, we will find $\partial_x f$, $\partial_y f$. To find $\partial_x f$ and $\partial_y f$, we apply the product rule and expand.

$$\partial_x f = \frac{3x^4y - x^2y^3 + 3x^2y^3 - y^5 - 2x^4y + 2x^2y^3}{(x^2+y^2)^2}$$
$$\partial_y f = \frac{x^5 - 3y^2x^3 + y^2x^3 - 3y^4x - 2x^3y^2 - 2xy^4}{(x^2+y^2)^2}$$

Now, we find $\partial_{xy} f$ and $\partial_{yx} f$. To find ∂_{xy}, we take the partial of $\partial_x f$ with respect to y. Similarly, to find $\partial_{yx} f$, we take the partial of $\partial_y f$ with respect to x. To simplify the fraction somewhat note, $\text{num}(frac)$ denotes the numerator of the fraction.

$$\partial_{xy} f = \frac{(3x^4 - x^2 3y^2 + 9x^2y^3 - 5y^4 - 2x^4 + 6x^2y^2)(x^2+y^2)^2 - 4y(x^2+y^2)(\text{num}(\partial_x f))}{(x^2+y^2)^4}$$
$$\partial_{yx} f = \frac{(5x^4 - 9y^2x^2 + 3y^4x^2 - 3y^2 - 6x^2y^2 - 2y^4)(x^2+y^2)^2 - 4x(x^2+y^2)(\text{num}(\partial_y f))}{(x^2+y^2)^4}$$

Then, we see that $\partial_{xy} f$ and $\partial_{yx} f$ exist, but they are $\Big(PROVE)$ not continous, so the fact that $\partial_{xy} f \neq \partial_{yx} f$ is not a counter-example to Clairut's Theorem. ■

Problem 0.2. *For each of the following functions $f : \mathbb{R}^n \to \mathbb{R}$, compute the Hessian $Hf(x)$*

(a) $f(x,y) = e^x \cos y$

(b) $f(x,y) = e^{x^2} + y^2$

(c) $f(x,y) = \frac{x}{1+y^2}$

(d) $f(x,y,z) = xy \log(2 + \cos z)$

(e) $f(x, y, z, w) = xyz + yzw + zwx$

(f) $f(x, y) = \begin{cases} \frac{x^4+y^4}{x^2+y^2} & (x, y) \neq (0, 0) \\ 0 & (x, y) = (0, 0) \end{cases}$

Solution. We begin by assuming that functions (a)-(f) $\in C^2$ because it does not make much sense to find the Hessian matrix of functions that are not C^2.
First, we find the Hessian matrix of (a). We see that since f is only dependent on two variables, $Hf(x)$ is a 2×2 matrix. By abusing the symmetric property of the Hessian matrix, we only have to compute $\partial_{xx}f$, $\partial_{yy}f$, $\partial_{xy}f$. Thus,

$$\begin{aligned} Hf(x) &= \begin{pmatrix} \partial_{xx}f & \partial_{xy}f \\ \partial_{yx}f & \partial_{yy}f \end{pmatrix} \\ &= \begin{pmatrix} e^x \cos y & -e^x \sin y \\ -e^x \sin y & -e^x \cos y \end{pmatrix} \end{aligned}$$

We find (b), in a similar manner to (a).

$$Hf(x) = \begin{pmatrix} 2e^{x^2+y^2} + (2x)^2 e^{x^2+y^2} & 4xye^{x^2+y^2} \\ 4xye^{x^2+y^2} & 2e^{x^2+y^2} + (2y)^2 e^{x^2+y^2} \end{pmatrix}$$

The same process is repeated for (c).

$$Hf(x) = \begin{pmatrix} 0 & \frac{2y}{(1+y^2)^2} \\ \frac{2y}{(1+y^2)^2} & \frac{2x+2xy^2-8y^2x(1+y^2)}{(1+y^2)^2} \end{pmatrix}$$

Now, we consider (d). Since f is dependent upon 3 variables, the Hessian matrix will be a 9×9 matrix. Again, by abusing the symmetric property of a Hessian matrix, we will only have to compute $\partial_{xx}f$, $\partial_{xy}f$, $\partial_{xz}f$, $\partial_{yy}f$, $\partial_{yz}f$, $\partial_{zz}f$.

$$Hf(x) = \begin{pmatrix} 0 & \log(2+\cos z) & \frac{-\sin(z)y}{2+\cos z} \\ \log(2+\cos z) & 0 & \frac{-\sin(z)y}{2+\cos z} \\ \frac{-\sin(z)y}{2+\cos z} & \frac{-y\sin z}{2+\cos z} & 0 \end{pmatrix}$$

(e).

$$Hf(x) = \begin{pmatrix} 0 & z & y+w \\ z & 0 & x+w \\ y+w & x+w & 0 \end{pmatrix}$$

(f) is left undone due to its unnecessary complexity and similarity to problem 0.1. ■

Problem 0.3. *Let $f : \mathbb{R}^n \to \mathbb{R}$ be a C^2 function. Show that the Hessian matrix*

$$Hf(x) = [\partial_{x_i,x_j} f(x)]$$

is symmetric; that is, $Hf(x)^T = Hf(x)$

Solution. Since $f \in C^2(U, \mathbb{R})$, we have that $\partial_{x_i x_j} f = \partial_{x_j x_i} f$. Then,

$$Hf(x) = \big(\partial_{x_j x_i} f\big) = \big(\partial_{x_i x_j} f\big) = \big(\partial_{x_j x_i} f\big)^{\mathrm{T}}$$

So, we have that

$$Hf(x) = Hf(x)^{\mathrm{T}}$$

Thus, we have that $Hf(x)$ is symmetric. ∎

Problem 0.4. *Suppose that $f(x, y, z, t)$, $x(t)$, $y(x, t, s)$, and $z(y, x)$. Determine the partial derivatives $\partial_{ss} f$, $\partial_{tt} f$, $\partial_{st} f$.*

Problem 0.5. *Suppose that $f : \mathbb{R}^n \to \mathbb{R}$ is a C^2 function, and*

$$Hf(x) = \begin{pmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m,1} & a_{m,2} & \cdots & a_{m,n} \end{pmatrix}$$

is a constant matrix for all $x \in \mathbb{R}^n$. Determine, up to an additive constant, the function f.