# 1 Smooth Manifolds

**Ex. 1** — Show that equivalent definitions of locally Euclidean spaces are obtained if, instead of requiring $U$ to be homeomorphic to an open subset of $\mathbb{R}^n$, we require it to be homeomorphic to an open ball in $\mathbb{R}^n$, or to $\mathbb{R}^n$ itself.

**Answer (Ex. 1)** — Let $\phi : U \to \tilde{U}$ where $p \in U \subset M$ and $\tilde{U} \subset \mathbb{R}^n$. Since $\tilde{U}$ is open, we can find an open ball around $\phi(p)$ (denoted $B_r(\phi(p))$ for some $r > 0$) such that $B_r(\phi(p)) \subset \tilde{U}$. We know $p \in \phi^{-1}(B_r(\phi(p)))$ and is open since $\phi$ and $\phi^{-1}$ is continuous. Therefore, the map $\phi' : \phi^{-1}(B_r(\phi(p))) \to B_r(\phi(p))$ is a homeomorphism. Using this along with the fact that open balls are of course open sets in $\mathbb{R}^n$, we obtain that requiring $U$ to be homeomorphic to open subsets of $\mathbb{R}^n$ or open balls of $\mathbb{R}^n$ makes no difference.

Next, consider a ball $B_r(x)$ centered at $x \neq 0$ of radius $r > 0$. The function $f : B_r(x) \to B_r(0)$ translates this ball to the origin via $z \mapsto z - x$. This map is clearly bijective and linear (so it is differentiable and so, continuous) with a continuous inverse, thus it is a homeomorphism. Therefore, it suffices to consider to balls centered at the origin. Now, consider the map $g : B_r(0) \to B_1(0)$ defined by $z \mapsto {}^{z}\!/_{r}$ which is clearly still a homeomorphism. Now, consider a map $\pi : B_1(0) \to \mathbb{R}^n$ defined by $x \mapsto (\tan({}^{\pi|x|}\!/_{2})x)$. ■

**Ex. 2** — Show that any topological subspace of a Hausdorff space is Hausdorff, and any finite product of Hausdorff spaces is Hausdorff.

**Answer (Ex. 2)** — Let $A$ be a subspace of a Hausdorff space $T$. Let $x \neq y \in A$, since $x, y \in T$, then there exist open sets $U, V$ in $T$ such that $x \in U$, $y \in V$ and $V \cap U = \emptyset$. Since $x \in U$ and $x \in A$, then $x \in U \cap A$ and similarly, for $y \in V \cap A$ and since $U \cap A \subset U \cap A$ and similarly for $V$, we have that $U \cap A \cap V \cap A = \emptyset$. Therefore, since $U \cap A$ and $V \cap A$ are open sets in $A$, we have the existence of two disjoint sets which contain $x$ and $y$ so that $A$ is also Hausdorff. A similar proof follows for finite products except we take the product of the disjoint sets i.e. if $U_1 \times \ldots \times U_n$ is our space, then take disjoint sets from each $U_i$ and product them together. ■

**Ex. 3** — Show that any topological subspace of a second countable space is second countable, and any finite product of second countable spaces is second countable.

**Answer (Ex. 3)** — Let $A$ be a subspace of the second countable space $T$. Since a basis of $A$ can be given by $B_i = \{A \cap T_i : T_i \in \{T_i\}\}$, we can simply let $\{T_i\}$ be our countable basis of $T$. Thus, $A$ is second countable. Note if $U_1 \times \ldots \times U_n$ is a finite product of second countable space, its basis is given by $B = \{U_{1_i} \times \ldots \times U_{n_i} : i \in \mathbb{N},$ $U_{k_i}$ is the $i$th basis element of $U_k\}$. From this, choosing the appropriate basis yields the fact that $U_1 \times \ldots \times U_n$ is second countable. ■

**Ex. 4** — Show that $\mathbb{P}^n$ is Hausdorff and second countable, and is therefore a topological $n$-manifold.

**Answer (Ex. 4)** — Let $x \neq y \in \mathbb{P}^n$. We show there exists $U, V \in \mathbb{P}^n$ such that $U \cap V = \emptyset$ with $x \in U$ and $y \in V$. Since $x \neq y$, we have that $\varphi_i[x] \neq \varphi_i[y]$ for any $1 \leq i \leq (n+1)$. Note $\varphi_i[x]$ and $\varphi_i[y]$ are points in $\mathbb{R}^n$ and so we can abuse the fact that $\mathbb{R}^n$ is Hausdorff. There exist open $U' \subset \mathbb{R}^n$ and $V' \subset \mathbb{R}^n$ disjoint sets that contain $\varphi_i[x]$ and $\varphi_i[y]$ respectively. Using the fact that $\varphi_i$ is a homeomorphism, we get that $U := \phi_i^{-1}(U')$ and $V := \varphi_i^{-1}(V')$ are disjoint and they contain $x$ and $y$ respectively. Furthermore, $U$ and $V$ are open since their images are open. Thus, $\mathbb{P}^n$ is Hausdorff.

It is left to show that $\mathbb{P}^n$ is second countable. We know that all of the $U_i$'s for $i = 1, \ldots, n+1$ cover $\mathbb{P}^n$. Let $\{X_i\}_{i=1}^{n+1} = \{\phi_i(U_i)\}_{i=1}^{n+1}$. Each $X_i \subset \mathbb{R}^d$ which is second countable, so each $X_i$ as a subspace is second countable. Let $\{Y_{i_j}\}_{j=1}^{\infty}$ be a countable basis for $X_i$. We claim,

$$M := \bigcup_{i=1}^{n+1} \{\phi_i^{-1}(Y_{i_j}) : j = 1, 2, 3 \ldots\}$$

is a countable basis of $\mathbb{P}^n$. Clearly $M$ is countable because each $\cup_{j=1}^{\infty} \phi_i^{-1}(Y_{i_j})$ is countable and the finite union of countable sets is countable. Let $O$ be an open set of $\mathbb{P}^n$. Then note $O = \cup_{i=1}^{n+1}(U_i \cap O)$. Note, each $\phi_i(U_i \cap O)$ can be written as a countable union of elements of $\{Y_{i_j}\}$ and so $U_i \cap O$ can be written as a countable union of elements of $\{\phi_i^{-1}(Y_{i_j}) : j = 1, 2, 3, \ldots\}$. Thus, we can write each $U_i \cap O$ as a countable union of elements of $M$ and so we can write $\cup_{i=1}^{n+1}(U_i \cap O)$ as a countable union of elements of $M$. Since $M$ only contains open sets of $\mathbb{P}^n$ (i.e. cannot generate a topology bigger than $M$), we have that $M$ is a countable basis of $\mathbb{P}^n$ as wanted. ■

**Ex. 5** — Let $M$ be a topological manifold. Two smooth atlases for $M$ determine the same maximal smooth atlas if and only if their union is a smooth atlas.

**Answer (Ex. 5)** — Let $\mathcal{X}, \mathcal{Y}$ be two smooth maximal smooth atlases containing $\mathcal{A}$ and $\mathcal{B}$ respectively. Let $\mathcal{A}$ and $\mathcal{B}$ be two smooth atlases for $M$ such that $\mathcal{A} \cup \mathcal{B}$ is a smooth atlas for $M$. Since $\mathcal{A} \subset \mathcal{X}$, it follows $(\mathcal{A} \cup \mathcal{B}) \cap \mathcal{X} \neq \emptyset$. Thus, if $\mathcal{X}$ is to be the unique maximal smooth atlas determined by $\mathcal{A}$, we must have that $\mathcal{A} \cup \mathcal{B} \subset \mathcal{X}$. For if this were not the case, we would have the existence of a chart smoothly compatible with $\mathcal{A}$ which is not in $\mathcal{X}$. Similarly, we obtain $\mathcal{A} \cup \mathcal{B} \subset \mathcal{Y}$. From the uniqueness of both $\mathcal{X}$ and $\mathcal{Y}$, we obtain $\mathcal{X} = \mathcal{Y}$ i.e. $\mathcal{A}$ and $\mathcal{B}$ determine the same smooth atlas. Now, suppose instead $\mathcal{A}$ and $\mathcal{B}$ determine the same maximal smooth atlas $\mathcal{X}$. Let $(U, \phi) \in \mathcal{A}$ and $(V, \psi) \in \mathcal{B}$. Since $(V, \psi) \in \mathcal{X}$, it follows $(U, \phi)$ and $(V, \psi)$ are smoothly compatible. Similarly, the converse holds i.e. every chart in $\mathcal{A}$ is smoothly compatible with charts in $\mathcal{B}$. Thus, $\mathcal{A} \cup \mathcal{B}$ is a smooth atlas. ■

**Ex. 6** — If $k$ is an integer between 0 and $\min(m, n)$, show that the set of $m \times n$ matrices whose rank is at least $k$ is an open submanifold of $\mathrm{M}(m \times n, \mathbb{R})$.

**Ex. 7** — By identifying $\mathbb{R}^2$ with $\mathbb{C}$ in the usual way, we can think of the unit circle $\mathbb{S}^1$ as a subset of the complex plane. An *angle function* on a subset $U \subset \mathbb{S}^1$ is a continuous function $\theta : U \to \mathbb{R}$ such that $e^{i\theta(p)} = p$ for all $p \in U$. Show that there exists an angle function on an open subset $U \subset \mathbb{S}^1$ if and only if $U \neq \mathbb{S}^1$. For any such angle function, show that $(U, \theta)$ is a smooth coordinate chart for $\mathbb{S}^1$ with its standard smooth structure.

**Ex. 8** — Let $0 < k < n$ be integers, and let $P, Q \subset \mathbb{R}^n$ be the subspaces spanned by $(e_1, \ldots, e_k)$ and $(e_{k+1}, \ldots, e_n)$, respectively, where $e_i$ is the $i$th standard basis vector. For any $k$-dimensional subspace $S \subset \mathbb{R}^n$ that has the trivial intersection with $Q$, show that the coordinate representation $\phi(S)$ constructed in the example on Grassmannian manifolds is the unique $(n - k) \times k$ matrix $B$ such that $S$ is spanned by the columns of the matrix $\left(\begin{smallmatrix} I_k \\ B \end{smallmatrix}\right)$, where $I_k$ denotes the $k \times k$ identity matrix.