MAT1847 OVERVIEW

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February 16, 2018

1 Riemann Surfaces

1.1 Simply Connected Surfaces

Definition 1. If $V \subset \mathbb{C}$ is an open set of complex numbers, a function $f : V \to \mathbb{C}$ is called **holomorphic** (or "complex analytic") if the first derivative

$$z \mapsto f'(z) = \lim_{h \to 0} \frac{(f(z+h) - f(z))}{h}$$

is defined and continuous as a function from V to $\mathbb{C}$, or equivalently if f has a power series expansion about any point $z_0 \in V$ which converges to f in some neighborhood of z_0. Such a function is **conformal** if the derivative $f'(z)$ never vanishes (for all $z \in U$).

Definition 2. By a **Riemann surface** S we mean a connected complex analytic manifold of complex dimension 1. The surface S is **simply connected** if every map from a circle *into* S can be continuously deforemed to a constant map. By definition, two Riemann surfaces S and S' are **conformally isomorphic** (or **biholomorphic**) if and only if there is a homeomorphism from S onto S' which is holomorphic in terms of the respective local charts.

Theorem 1 (Uniformization Theorem)**.** Any simply connecnted Riemann surface is conformally isomorphic either

1. to the plane $\mathbb{C}$ consisting of all complex nubmers $z = x + iy$
2. to the open disk $\mathbb{D} \subset \mathbb{C}$ consisting of all z with $|z|^2 = x^2 + y^2 < 1$, or
3. to the Riemann sphere $\hat{\mathbb{C}}$ consisting of $\mathbb{C}$ together with a point at infinity, using $\zeta = {}^1\!/_z$ as a chart in a neighborhood of the point at infinity.

These three cases are referred to as the **Euclidean**, **hyperbolic**, and **spherical** cases, respectively.

1.1.1 The unit disk $\mathbb{D}$

Some nice results about the surface $\mathbb{D}$ are as follows:

Lemma 1 (Schwarz Lemma)**.** If $f : \mathbb{D} \to \mathbb{D}$ is holomorphic map with $f(0) = 0$, then the derivative at the origin satisfies $|f'(0)| \leq 1$. If equality holds, $|f'(0)| = 1$, then f is a rotation about the origin. That is, $f(z) = cz$ for some constant $c = f'(0)$ on the unit circle. On the other hand, if $|f'(0)| < 1$, then $|f(z)| < |z|$ for all $z \neq 0$.

Lemma 2 (Maximum Modulus Principle)**.** A nonconstant holomorphic function cannot attain its maximum absolute value at any interior point of its region of definition.

Lemma 3 (Cauchy's Derivative Estimate)**.** If f maps the disk of radius r about z_0 into some disk of radius s, then

$$|f'(z_0)| \leq {}^{s}/_{r}$$

A useful corollary of this is as follows:

Theorem 2 (Liouville's Theorem)**.** A bounded function f which is defined and holomorphic everywhere on $\mathbb{C}$ must be constant.

Theorem 3 (Weierstrass Uniform Convergence Theorem)**.** If a sequence of holomorphic functions $f_n : U \to \mathbb{C}$ converges uniformily to the limit function f, then f itself is holomorphic. Furthermore, the sequence of derivatives f'_n converges, uniformily on any compact set of U, to the derivatives f'.

1.1.2 Conformal Automorphism Groups

For any Riemann surface S, the notation $\mathcal{G}(S)$ will be used for the group consisting of all conformal automorphisms of S. The identity map will be denoted by $I = I_S \in \mathcal{G}(S)$.

Lemma 4 (Möbius Transformations)**.** The group $\mathcal{G}(\hat{\mathbb{C}})$ of all conformal automorphisms of the Riemann sphere is equal to the group of all **fractional linear transformations** (also called **Möbius transformations**)

$$g(z) = \frac{az + b}{cz + d}$$

where the coefficients are complex numbers with $ad - bc \neq 0$.

The identification to the complex Lie group $\mathbb{PSL}_2(\mathbb{C})$ is as follows:
Multiply the numerator and denominator by a common factor, then it is always possible to normalize so that the determinant $ad - bc$ is equal to $+1$. The resulting coefficients are well defined up to a simultaneous change of sign. Thus it follows that the group $\mathcal{G}(\mathbb{C})$ of conformal transformations can be identified with the complex 3-dimensional Lie group $\mathbb{PSL}_2(\mathbb{C})$ consisting of all 2×2 complex

matrices with determinant $+1$ modulo the subgroup $\{\pm I\}$.

Next, it is shown that both $\mathcal{G}(\mathbb{C})$ and $\mathcal{G}(\mathbb{D})$ can be considered as Lie subgroups of $\mathcal{G}(\hat{\mathbb{C}})$.

Corollary 1 (The Affine Group)**.** The group $\mathcal{G}(\mathbb{C})$ of all conformal automorphisms consists of all affine transformations

$$f(z) = \lambda z + c$$

with complex coefficients $\lambda \neq 0$ and c.

Note every conformal automorphism f of $\mathbb{C}$ extends uniquely to a conformal automorphism of $\hat{\mathbb{C}}$ with $\lim_{z\to\infty} f(z) = \infty$.

Theorem 4 (Automorphisms of $\mathbb{D}$)**.** The group $\mathcal{G}(\mathbb{D})$ of all conformal automorphisms of the unit disk can be identified with the subgroup of $\mathcal{G}(\hat{\mathbb{C}})$ consisting of all maps

$$f(z) = e^{i\theta}\frac{z-a}{1-\overline{a}z}$$

where a ranges over the open disk $\mathbb{D}$ and where $e^{i\theta}$ ranges over the unit circle $\partial\mathbb{D}$.

This is no longer a *complex* Lie group. $\mathcal{G}(\mathbb{D})$ is a *real* 3-dimensional Lie group, having the topology of a "solid torus" $\mathbb{D} \times \partial\mathbb{D}$.

It is often more convenient to work with the **upper half-plane** $\mathbb{H}$, consisting of all complex numbers $w = u + iv$ with $v > 0$.

Lemma 5 ($\mathbb{D} \cong \mathbb{H}$)**.** The half-plane $\mathbb{H}$ is conformally isomorphic to the disk $\mathbb{D}$ under the holomorphic mapping

$$w \mapsto \frac{i-w}{i+w}$$

with inverse

$$z \mapsto \frac{i(1-z)}{1+z}$$

where $z \in \mathbb{D}$ and $w \in \mathbb{H}$.

Corollary 2 (Automorphisms of $\mathbb{H}$)**.** The group $\mathcal{G}(\mathbb{H})$ consiting of all conformal automorphisms of the upper half-plane can be identified with the group of all fractional linear transformations $w \mapsto {}^{aw+b}/_{cw+d}$, where the coefficients a, b, c, d are real with determinant $ad - bc > 0$.

If we normalize so that $ad - bc = 1$, then the coefficients are well defined up to a simultaneous change of sign. Thus $\mathcal{G}(\mathbb{H})$ is isomorphic to the group $\mathbb{PSL}_2(\mathbb{R})$, consisting of all 2×2 real matrices with determinant $+1$ modulo the subgroup $\{\pm I\}$.

To conclude this section, we will try to say something more about the structure of these three groups. For any map $f : X \to X$, it will be convenient to use the notation $\mathrm{Fix}(f) \subset X$ for the set of all fixed points $x = f(x)$. If f and g are commuting maps from X to itself, $f \circ g = g \circ f$, note that

$$f(\mathrm{Fix}(g)) \subset \mathrm{Fix}(g)$$

Now, we find the commuting elements of $\mathcal{G}(\mathbb{C})$, $\mathcal{G}(\mathbb{D})$, $\mathcal{G}(\hat{\mathbb{C}})$.

Lemma 6 (Commuting elements of $\mathcal{G}(\mathbb{C})$)**.** Two non-identity affine transformations of $\mathbb{C}$ commute if and only if they have the same fixed point set.

An affine transformation with two fixed points must be the identity map.

Now consider the group $\mathcal{G}(\hat{\mathbb{C}})$ of automorphisms of the Riemann sphere. By definition, an automorphism g is called an **involution** if $g \circ g = I$, but $g \neq I$.

Theorem 5 (Commuting elements of $\mathcal{G}(\hat{\mathbb{C}})$)**.** For every $f \neq I$ in $\mathcal{G}(\hat{\mathbb{C}})$, the set $\mathrm{Fix}(f) \subset \hat{\mathbb{C}}$ contains either one point or two points (if f has more than 2 fixed points, it must be the identity map). In general, two nonidentity elements $f, g \in \mathcal{G}(\hat{\mathbb{C}})$ commute if and only if $\mathrm{Fix}(f) = \mathrm{Fix}(g)$. The only exceptions to this statement are provided by pairs of commuting involutions each of which interchanges the two fixed points of each other.

As an example, the involution $f(z) = -z$ with $\mathrm{Fix}(f) = \{0, \infty\}$ commutes with the involution $g(z) = {}^1\!/\!{}_z$ with $\mathrm{Fix}(g) = \{\pm 1\}$.

We want a corresponding statement for the open disk $\mathbb{D}$. However, it is better to work with the closed disk $\overline{\mathbb{D}}$, in order to obtain a richer set of fixed points. Using Theorem 4, we see that every automorphism of the open disk extends uniquely to an automorphism of the closed disk so that $\mathcal{G}(\mathbb{D}) \cong \mathcal{G}(\overline{\mathbb{D}})$.

Theorem 6. For every $f \neq I$ in $\mathcal{G}(\mathbb{D}) \cong \mathcal{G}(\overline{\mathbb{D}})$, the set $\mathrm{Fix}(f) \subset \overline{\mathbb{D}}$ consists of either a single point of the boundary circle $\partial\mathbb{D}$, a single point of the open disk $\mathbb{D}$, or two points of $\partial\mathbb{D}$. Two nonidentity automorphisms $f, g \in \mathcal{G}(\mathbb{D})$ commute if and only if they have the same fixed point set in $\overline{\mathbb{D}}$.

1.2 Universal Coverings and the Poincaré Metric

Definition 3. A map $p : M \to N$ between connected manifolds is called a **covering map** if every point of N has a connected open neighborhood U within N which is **evenly covered**; that is, each component $p^{-1}(U)$ must map onto U by a homeomorphism. The manifold N is **simply connected** if it has no nontrivial coverings, that is, if every such covering map $M \to N$ is a homeomorphism. For any connected manifold N, there exists a covering map $\tilde{N} \to N$ such that $\tilde{N}$ is simply connected. This is called the **universal covering** of N and is unique up to homeomorphism.

Definition 4. By a **deck transformation** associated with a covering map $p : M \to N$ we mean a continuous map $\gamma : M \to M$ which satisfies the identity $p \circ \gamma = p$. For our purposes, the **fundamental group** $\pi_1(N)$ can be defined as the group Γ consisting of all deck transformations for the universal covering $\tilde{N} \to N$. Note that this universal covering is always a **normal covering** of N. That is, given two points $x, x' \in M = \tilde{N}$ with $p(x) = p(x')$, there exists one and only one deck transformation which maps x to x'. It follows that N can be identified with the quotient $\tilde{N}/\Gamma$ of $\tilde{N}$ by this action of Γ.

In particular, we have that a given group Γ of homeomorphisms of a connected manifold M gives rise in this way to a normal covering $M \to M/\Gamma$ if and only if

1. Γ acts **properly discontinuously**; that is, any compact set $K \subset M$ intersects only finitely many of its translates $\gamma(K)$ under the action of Γ; and

2. Γ acts **freely**; that is, every nonidentity element of Γ acts without fixed points on M

Theorem 7 (Uniformization for Arbitrary Riemann Surfaces)**.** Every Riemann surface S is conformally isomorphic to a quotient of the form $\tilde{S}/\Gamma$, where $\tilde{S}$ is a simply connected Riemann surface, which is necessairly isomorphic to either $\mathbb{D}$, $\mathbb{C}$, or $\hat{\mathbb{C}}$,and where $\Gamma \cong \pi_1(S)$ is a group of conformal automorphisms which acts freely and properly discontinuously on $\tilde{S}$.

Since the action of Γ on $\tilde{S}$ is properly discontinuous, Γ must be a **discrete** subgroup of $\mathcal{G}(\tilde{S})$; that is, there exists a neighborhood of the identity element in $\mathcal{G}(\tilde{S})$ which intersects Γ only in the identity element.

Corollary 3 (σ-Compactness)**.** Every Riemann surface can be expressed as a countable union of compact subsets.

This follows from Theorem 7 and from the fact that this is true for all the three simply connected surfaces.