CSF Notes

Anmol Bhullar

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1 Basics

Suppose $\{\Gamma_t \subset \mathbb{R}^2\}$ is one-parameter family of embedded (i.e. simple) curves. If this family moves by curve shortening flow (CSF), by definition, it satisfies:

$$\partial_t(p) = \vec{\kappa}(p) \tag{1.1}$$

where p is some point on Γ_t. In order to compute $\partial_t(p)$ and $\vec{\kappa}(p)$, first, we parameterize Γ_t by some parameteric equation $\phi_t : [0, 2\pi] \to \mathbb{R}^2$. Suppose this is in arc-length parameterization. Then, $\partial_t(p)$ is just computed by differentiating $\phi_t(x)$ with respect to t (x some element of domain) and $\vec{\kappa}(p)$ is computed by $\partial_x^2[\phi_t(x)]$. Suppose ϕ_t is not given in arc-length parameterization. Then, if $\phi_t(s) = (x(s), y(s))$, we have that the signed curvature is given by the formula:

$$k(s) = \frac{x'y'' - x''y'}{((x')^2 + (y')^2)^{3/2}}$$

so that we can then compute $\vec{k}$ by evaluating $k(s)N(s)$ where N denotes the normal vector at of α at s.

1.1 Round shrinking circles

We show that if $\Gamma_t = \partial B^2_{r(t)} \subset \mathbb{R}^2$, then (1.1) reduces to the ODE:

$$\dot{r} = -1/r$$

and if we give it the initial value $r(0) = R$, then $r(t) = \sqrt{R^2 - 2t}$ for $t \in (-\infty, R^2/2)$.

Proof. Let $r(t)$ be a C^1 function that gives a radius dependent on the parameter t. Assume $B^2_{r(t)} = \{x \in \mathbb{R}^2 : |x| < r(t)\}$. Then $\partial B^2_{r(t)} = \{x \in \mathbb{R}^2 : |x| = r(t)\}$. Fix t. We let $\partial B^2_{r(t)}$ be given by the parameterization $\phi_t : [0, 2\pi] \to \mathbb{R}^2$,

$$\phi_t(s) = r(t)(\cos(s), \sin(s))$$

Now, we compute $\partial_t(\phi_t(s))$. Consider:

$$\partial_t(\phi_t(s)) = \partial_t(r(t)(\cos(s), \sin(s))) = \dot{r}(t)(\cos(s), \sin(s))$$

and also, we can compute the signed curvature:

$$\begin{aligned} k(s) &= \frac{r^2(t)[\cos'(s)\sin''(s) - \cos''(s)\sin'(s)]}{((r(t)\cos'(s))^2 + (r(t)\sin'(s))^2)^{3/2}} \\ &= \frac{r^2(t)[\sin^2(s) + \cos^2(s)]}{(r^2(t)[\sin^2(s) + \cos^2(s)])^{3/2}} = \frac{r^2(t)}{r^3(t)} = \frac{1}{r(t)} \end{aligned}$$

Now, we compute the unit normal. Note $v(s) = |\partial_s(\phi_t(s))| = r(t)$. So, the unit tangent vector $\mathbf{T}(s)$ is given by,

$$\mathbf{T}(s) = \partial_s(\phi_t(s))/v(s) = r(t)(-\sin(s), \cos(s))/r(t) = (-\sin(s), \cos(s))$$

from this, we can compute the unit normal:

$$\kappa \mathbf{N} = \frac{\partial_s(T(s))}{\partial_s(v(s))} = (-\cos(s), -\sin(s)) \cdot 1/r(t)$$

From this, we see that $\mathbf{N} = (-\cos(s), -\sin(s))$ so that (1.1) reduces down to:

$$\dot{r}(t)(\cos(s), \sin(s)) = [1/r(t)](-\cos(s), -\sin(s))$$

which of course, yields:

$$\dot{r}(t) = -1/r(t)$$

as wanted. Now, set $R > 0$ and give the initial value data $r(0) = R$. We have $dr/dt = -1/r$ which we can rearrange to get $rdr = -1dt$. Integrating both sides, we get $r^2/2 = -t + c$ for some constant c. If $r(0) = R$, then $c = R$. Thus, we get:

$$r(t) = \sqrt{2}\sqrt{R - t} \text{ for } t \in (-\infty, R)$$

$\square$

1.2 Grim Reaper Curve

A self similar solution to the curve shortening flow is given by the grim reaper curve. This is given by the family $\Gamma_t = \text{graph}(\log \cos p) + t$ where $p \in (-\pi/2, \pi/2)$ and $t \in \mathbb{R}$.

Now, instead consider the family:

$$\Phi_t = \text{ graph}(u_t(p)) = \{(p, u_t(p)) : p \in \text{ Dom}(u_t : U \subset \mathbb{R} \to \mathbb{R})\}.$$

We ask which equation does u_t satisfy. We parameterize Φ_t via the map $\phi_t : U \to \mathbb{R}^2$ given by the mapping $p \in U \mapsto (p, u_t(p))$. Immediately, we see that

$$\partial_t(\phi_t(p)) = (\partial_t(p), \partial_t(u_t(p))) = (0, \partial_t(u_t(p)))$$

Now, we calculate $\vec{k}$ by calculating $k\mathbf{N}$. Note,

$$\nu(p) = |\partial_p(\phi_t(p))| = |(\partial_p(p), \partial_p(u_t(p)))| = \sqrt{1^2 + (\partial_p u_t(p))^2}$$

so that,

$$\mathbf{T}(p) = \frac{\partial_p u_t(p)}{\nu(p)} = \frac{(1, \partial_p(u_t(p)))}{\sqrt{1^2 + (\partial_p u_t(p))^2}}$$

and so,

$$\begin{aligned}
\partial_p \mathbf{T}(p) &= \partial_p \Big(\frac{(1, \partial_p u_t(p))}{\sqrt{1 + (\partial_p u_t(p))^2}} \Big) \\
&= \frac{\partial_p(1, \partial_p u_t(p))}{\sqrt{1 + (\partial_p u_t(p))^2}} - (1, \partial_p u_t(p)) \partial_p [1 + (\partial_p u_t(p))^2]^{-1/2} \\
&= \frac{(0, \partial_p^2 u_t(p))}{\sqrt{1 + (\partial_p u_t(p))^2}} - \frac{\partial_p u_t(p)[\partial_p^2 u_t(p)](1, \partial_p u_t(p))}{[1 + (\partial_p u_t(p))^2]^{3/2}} \\
&= \frac{[1 + (\partial_p u_t(p))^2](0, \partial_p^2 u_t(p))}{[1 + (\partial_p u_t(p))^2]^{3/2}} - \frac{\partial_p u_t(p)[\partial_p^2 u_t(p)](1, \partial_p u_t(p))}{[1 + (\partial_p u_t(p))^2]^{3/2}} \\
&= \frac{[\partial_p^2 u_t(p)](0, 1 + (\partial_p u_t(p))^2) - [\partial_p^2 u_t(p)](\partial_p u_t(p), (\partial_p u_t(p))^2}{[1 + (\partial_p u_t(p))^2]^{3/2}} \\
&= [\partial_p^2 u_t(p)] \frac{(0, 1 + (\partial_p u_t(p))^2) - (\partial_p u_t(p), (\partial_p u_t(p))^2}{[1 + (\partial_p u_t(p))^2]^{3/2}}) \\
&= [\partial_p^2 u_t(p)] \frac{(-\partial_p u_t(p), 1)}{[1 + (\partial_p u_t(p))^2]^{3/2}}
\end{aligned}$$

Thus $\vec{\kappa}$ can be computed by:

$$\begin{aligned}
\kappa \mathbf{N} &= \frac{\partial_p T(p)}{\nu(p)} = \frac{1}{\nu(p)} \partial_p T(p) \\
&= \frac{1}{\sqrt{1^2 + (\partial_p u_t(p))^2}} \frac{[\partial_p^2 u_t(p)](-\partial_p u_t(p), 1)}{[1 + (\partial_p u_t(p))^2]^{3/2}} \\
&= \frac{[\partial_p^2 u_t(p)](-\partial_p u_t(p), 1)}{[1 + (\partial_p u_t(p))^2]^2}
\end{aligned}$$

Note, that by computing the norm of $\vec{\kappa}$, we get:

$$\begin{aligned}
|\kappa \mathbf{N}| &= \frac{|\partial_p^2 u_t(p)}{(1 + (\partial_p u_t(p))^2)^2} \big| (-\partial_p u_t(p), 1) \big| \\
&= \frac{|\partial_p^2 u_t(p)|}{(1 + (\partial_p u_t(p))^2)^{3/2}}
\end{aligned}$$

Thus, we obtain the following:

$$\kappa = \frac{\partial_p^2 u_t(p)}{(1 + (\partial_p u_t(p))^2)^{3/2}}, \qquad \mathbf{N} = \frac{(-\partial_p u_t(p), 1)}{\sqrt{1 + (\partial_p u_t(p))^2}}$$

Now, we use the equation:

$$\partial_t \phi_t(p) \cdot \mathbf{N} = \kappa$$

which after substitution, reduces down to:

$$\frac{\partial_t u_t(p)}{(1 + (\partial_p u_t(p))^2)^{1/2}} = \frac{\partial_p^2 u_t(p)}{(1 + (\partial_p u_t(p))^2)^{3/2}}$$

Thus, we get:

$$\partial_t u_t(p) = \frac{\partial_p^2 u_t(p)}{1 + (\partial_p u_t(p))^2}, \text{ or } \quad \partial_p^2 u_t(p) = \partial_t u_t(p)[1 + (\partial_p u_t(p))^2]$$

If $\partial_t u_t(p) = 1$, we get the ODE:

$$\partial_p^2 u_t(p) = 1 + (\partial_p u_t(p))^2$$

We attempt to solve this. First, we rewrite it as:

$$y''(x) = 1 + (y'(x))^2$$

1.3 Evolution equation of length

We derive the evolution equation of $L(t) = \int_{\Gamma_t} ds$. Note if Γ_t is parameterized by $\gamma(x,t) : S^1 \times [0,T) \to \mathbb{R}^2$, then:

$$L(t) = \int_{S^1} \langle \partial_x \gamma, \partial_x \gamma \rangle^{1/2} dx$$

We compute,

$$\begin{aligned}
\partial_t L(t) &= \partial_t \Big(\int_{S^1} \langle \partial_x \gamma, \partial_x \gamma \rangle^{1/2} dx \Big) \\
&= \int_{S^1} \partial_t \langle \partial_x \gamma, \partial_x \gamma \rangle^{1/2} dx \\
&= \int_{S^1} \frac{1}{2} \langle \partial_x \gamma, \partial_x \gamma \rangle^{-1/2} [\langle \partial_t \partial_x \gamma, \partial_x \gamma \rangle + \langle \partial_x \gamma, \partial_t \partial_x \gamma \rangle] dx \\
&= \int_{S^1} \frac{1}{2} |\partial_x \gamma|^{-1} 2 \langle \partial_t \partial_x \gamma, \partial_x \gamma \rangle dx \\
&= \int_{S^1} \langle \partial_t \partial_x \gamma, \frac{\partial_x \gamma}{|\partial_x \gamma|} \rangle dx \\
&= \int_{S^1} \langle \partial_x \partial_t \gamma, T \rangle dx
\end{aligned}$$

By definition of the curve shortening flow, we know that $\partial_t \gamma = \kappa N$. Thus,

$$\partial_t L(t) = \int_{S^1} \langle \partial_x (\kappa N), T \rangle dx$$

By the Frenet-Serret formulas, we have $\partial_s N = -\kappa T$ and so by chain rule, we obtain $\partial_x N = -(\partial_x s)\kappa T$. Thus,

$$\langle \partial_x(\kappa N), T\rangle = \langle (\partial_x \kappa)N, T\rangle + \langle \kappa \partial_x N, T\rangle = 0 + \kappa[(\partial_x s)\kappa T]\cdot T = -\kappa^2 \partial_x s$$

Writing instead $\langle \partial_x(\kappa N), T\rangle = -\kappa^2 \frac{ds}{dx}$, we easily obtain the final equation:

$$\partial_t L(t) = \int_{\Gamma_t} -\kappa^2 ds$$

1.4 Evolution equation of curvature

We show that $\kappa_t = \kappa_{ss} + \kappa^3$. For convenience sake, set $|\partial_x \gamma| = 1$ and $\langle \partial_x^2 \gamma, T\rangle = 0$ at the point (x,t). Note $\langle \partial_x^2\gamma, N\rangle = \langle \kappa N, N\rangle = \kappa$. Since $|\partial_x\gamma| = 1$, we can also say:

$$\kappa = \frac{\langle \partial_x^2 \gamma, N\rangle}{|\partial_x \gamma|^2}$$

We evaluate κ_t i.e. $\partial_t \kappa$.

$$\begin{aligned}
\partial_t \kappa &= \frac{\partial_t(\langle \partial_x^2\gamma, N\rangle)}{|\partial_x\gamma|^2} + \frac{\langle \partial_x^2\gamma, N\rangle}{\partial_t|\partial_x\gamma|^2} \\
&= \partial_t(\langle \partial_x^2\gamma, N\rangle) + \partial_t(\langle \partial_x\gamma, \partial_x\gamma\rangle^{-1})\langle \partial_x^2\gamma, N\rangle \\
&= \partial_t(\langle \partial_x^2\gamma, N\rangle) - \langle \partial_x\gamma, \partial_x\gamma\rangle^{-2}\partial_t(\langle \partial_x\gamma, \partial_x\gamma\rangle)\langle \partial_x^2\gamma, N\rangle \\
&= \partial_t(\langle \partial_x^2\gamma, N\rangle) - 2|\partial_x\gamma|^{-4}\langle \partial_t\partial_x\gamma, \partial_x\gamma\rangle\langle \partial_x^2\gamma, N\rangle \\
&= \partial_t(\langle \partial_x^2\gamma, N\rangle) - 2\langle \partial_t\partial_x\gamma, T\rangle\langle \partial_x^2\gamma, N\rangle \\
&= \langle \partial_t\partial_x^2\gamma, N\rangle + \langle \partial_x^2\gamma, \partial_t N\rangle - 2\langle \partial_t\partial_x\gamma, T\rangle\langle \partial_x^2\gamma, N\rangle \\
&= \langle \partial_x^2\partial_t\gamma, N\rangle + \langle \partial_x^2\gamma, \partial_t N\rangle - 2\langle \partial_x\partial_t\gamma, T\rangle\langle \partial_x^2\gamma, N\rangle
\end{aligned}$$

Now, substitute κN for $\partial_t\gamma$ and κN for $\partial_x^2\gamma$ to get:

$$\begin{aligned}
\partial_t\kappa &= \langle \partial_x^2(\kappa N), N\rangle + \langle \kappa N, \partial_t N\rangle - 2\langle \partial_x(\kappa N), T\rangle\langle \kappa N, N\rangle \\
&= \partial_x^2(\kappa)\langle N, N\rangle + \kappa\langle \partial_x^2 N, N\rangle + \kappa\langle N, \partial_t N\rangle - 2[\partial_x(\kappa)\langle N, T\rangle + \kappa\langle \partial_x N, T\rangle]\kappa\langle N, N\rangle \\
&= \partial_x^2(\kappa) + \kappa\langle \partial_x(\partial_x N), N\rangle - 2\kappa^2\langle \partial_x N, T\rangle
\end{aligned}$$

By the Frenet formulas, we know $\partial_x N = -\kappa T$. So we can continue our computation:

$$\begin{aligned}
\partial_t\kappa &= \partial_x^2(\kappa) + \kappa\langle \partial_x(-\kappa T), N\rangle - 2\kappa^2\langle -\kappa T, T\rangle \\
&= \partial_x^2(\kappa) + \kappa[\langle \partial_x(-\kappa)T, N\rangle - \langle \kappa\partial_x T, N\rangle] + 2\kappa^3 \\
&= \partial_x^2(\kappa) - \kappa^2[\langle \partial_x T, N\rangle] + 2\kappa^3 \\
&= \partial_x^2(\kappa) - \kappa^3 + 2\kappa^3 \\
&= \partial_x^2(\kappa) + \kappa^3
\end{aligned}$$

From our assumption in the beginning, we can regard $\partial_x^2(\kappa)$ as $\partial_{ss}(\kappa)$. From this, we get:

$$\kappa_t = \partial_{ss}\kappa + \kappa^3$$

as wanted.

1.5 Evolution equation of area

Let $A(t)$ denote the area enclosed by Γ_t. We equate $\partial_t A(t)$.

Let Γ_t be given by $(x, t) = F(u)$. By Green's theorem, we immediately obtain:

$$A(t) = \frac{1}{2}\int_0^{2\pi} \left(x\frac{\partial y}{\partial u} - y\frac{\partial y}{\partial u}\right)du$$

We have $\partial F/\partial u = \partial(x, y)/\partial u = (\partial x/\partial u, \partial y/\partial u)$. Note

$$\langle(\partial x/\partial u, \partial y/\partial u), -(\partial y/\partial u, \partial x/\partial u)\rangle = 0$$

so these are orthogonal. Since $\partial F/\partial u$ is the tangent, we then obtain that $n := -(\partial y/\partial u, \partial x/\partial u)$ is proportional to the inward pointing unit normal. In particular, $N = n/|n|$. Note $|n|$ is simply equal to the norm of $\partial_u F$ so we denote it by v. Furthermore,

$$\langle n, F\rangle = -x\frac{\partial y}{\partial u} + y\frac{\partial x}{\partial u}$$

but $n = vN$ so, we simply write:

$$\langle vN, F\rangle = -x\frac{\partial y}{\partial u} + y\frac{\partial x}{\partial u}$$

Thus,

$$A(t) = \frac{-1}{2}\int_0^{2\pi} \langle F, vN\rangle du$$

Now, we can compute $\partial_t A(t)$. Consider:

$$\begin{aligned}
\partial_t A(t) &= \frac{-1}{2}\int_0^{2\pi} \langle \partial_t F, vN\rangle + \langle F, \partial_t vN\rangle du \\
&= \frac{-1}{2}\int_0^{2\pi} \langle \partial_t F, vN\rangle + \langle F, \partial_t(v)N\rangle + \langle F, v\partial_t N\rangle du
\end{aligned}$$