

**MAT327: Introduction to Topology. Solutions to  
the Big List Problems**

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Thank you to my instructor Ivan Khatchatourian for providing these wonderful problems.



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## Preface

I attempt to answer and  $\text{\LaTeX}$  all of the solutions to the big list of problems posted by my instructor Ivan Khatchatourian for MAT327: Introduction to Topology. The problems are separated into difficulties which are labelled via asterisks. One asterisk being the lowest difficulty and 3 being the highest. Especially hard problems are marked via a cross. This is the format my instructor uses and I'm merely copying it for consistency's sake.



# Topologies

\* **Ex. 1** — Fix  $a < b \in \mathbb{R}$ . Show explicitly that the open interval  $(a, b)$  is open in  $\mathbb{R}_{\text{usual}}$ . Show explicitly that the interval  $[a, b)$  is not open in  $\mathbb{R}_{\text{usual}}$ .

**Answer (Ex. 1)** — First, we show that  $(a, b)$  is open in  $\mathbb{R}_{\text{usual}}$ . To do this: we have to show that for any point  $x \in (a, b)$ , there exists a neighbourhood  $N(x, \epsilon) \subseteq (a, b)$ .

Thus, choose a point  $x \in (a, b)$ . By density of  $\mathbb{R}$ , there exists  $x_1 \in (a, x)$  and  $x_2 \in (x, b)$ . Let  $\epsilon = \min(x_1, x_2)$ . Then  $N(x, \epsilon) \subseteq (a, b)$ .

Next, we show that  $[a, b)$  is not open. To do this, we show that  $\exists$  no  $\epsilon > 0$  such that  $N(a, \epsilon) \subseteq [a, b)$ . Note that any  $x \in (a - \epsilon, a]$  would not be in  $(a, b)$  for any  $\epsilon > 0$ , so  $N(a, \epsilon) \not\subseteq [a, b)$ .

\* **Ex. 2** — Let  $X$  be a set and  $\mathcal{B} = \{\{x\} : x \in X\}$ . Show that the only topology on  $X$  that contains  $\mathcal{B}$  as a subset is the discrete topology.

**Answer (Ex. 2)** — To show that only  $X_{\text{discrete}}$  has the property  $\mathcal{B} \subseteq X_{\text{discrete}}$ , consider the set:

$$\mathcal{T} := \{\{x\} : x \in X\}$$

For  $\mathcal{T}$  to be a topology, it must be closed under finite intersections so we must put  $\emptyset$  in  $\mathcal{T}$ . Furthermore,  $\mathcal{T}$  must be closed under the union of an arbitrary collection elements of  $X$ . If this is to be true, then  $\mathcal{T}$  must contain every subset of  $X$  since any arbitrary subset can be written as the union of all of its elements. Thus an arbitrary subset  $A \subseteq X$  must be in  $\mathcal{T}$ . This implies  $\mathcal{T} = X_{\text{discrete}}$ .

\* **Ex. 4** — Let  $(X, \mathcal{T}_{\text{co-countable}})$  be an infinite set with the co-countable topology. Show that  $\mathcal{T}_{\text{co-countable}}$  is closed under countable intersections but not necessarily arbitrary ones.



**Answer (Ex. 4)** — Given a countable indexing set  $I$ , we have to show

$$\bigcap_{\alpha \in I} U_\alpha \in \mathcal{T}_{\text{co-countable}}$$

Equivalently, we can show  $X \setminus (\bigcap U_\alpha)$  is countable. By DeMorgan's Law:  $X \setminus (\bigcap U_\alpha) = \bigcup (X \setminus U_\alpha)$ . Since the right hand side is the countable union of countable sets, it is countable. Thus,  $\bigcap U_\alpha \in \mathcal{T}_{\text{co-countable}}$ . To show that an arbitrary intersection of elements is not open, it suffices to state that the intersection of arbitrary many elements does not necessarily form a countable set.

\* **Ex. 5** — Let  $(X, \mathcal{T})$  be a topological space, and let  $A \subseteq X$  be a set with the property that for all  $x \in A$ ,  $\exists$  an open set  $U_x \in \mathcal{T}$  such that  $x \in U_x \subseteq A$ . Show that  $A$  is open.

**Answer (Ex. 5)** — Let  $I$  be a set which indexes the elements of  $A$ . Then:

$$\bigcup_{\alpha \in I} \{x_\alpha\} = A$$

Similarly, since  $x_\alpha \in U_{x_\alpha}$  and  $U_{x_\alpha} \subseteq A$ , then:

$$\bigcup_{\alpha \in I} U_{x_\alpha} = A$$

Since this is the union of arbitrary elements of  $\mathcal{T}$ , then  $A \in \mathcal{T}$ .

\* **Ex. 6** — Let  $(X, \mathcal{T})$  be a topological space, and let  $f : X \rightarrow Y$  be injective. Is  $\mathcal{T}_f := \{f(U) : U \in \mathcal{T}\}$  a topology on  $Y$ ? Is it necessarily a topology on the range of  $f$ ?

**Answer (Ex. 6)** — Lost solution : (

\* **Ex. 7** — Let  $X$  be a set and let  $\mathcal{T}_1$  and  $\mathcal{T}_2$  be two topologies on  $X$ . Is  $\mathcal{T}_1 \cup \mathcal{T}_2$  a topology on  $X$ ? What about  $\mathcal{T}_1 \cap \mathcal{T}_2$ ? Is yes, prove it. If not, provide a counterexample.

**Answer (Ex. 7)** — Let  $\mathcal{T}_1 := \mathcal{T}_{\text{usual}}$ ,  $\mathcal{T}_2 := \mathcal{T}_7$  on  $\mathbb{R}$ . Consider:

$$(6, 7) \cap [6.5, 7] \quad \text{both in } \mathcal{T}_1 \cup \mathcal{T}_2$$

but this intersection yields the interval:  $[6.5, 7)$  which is not open in either  $\mathcal{T}_1$  or  $\mathcal{T}_2$ . So  $\mathcal{T}_1 \cup \mathcal{T}_2$  is not necessarily a topology. Now, we show that  $\mathcal{T}_1 \cap \mathcal{T}_2$  is a topology:

$$\begin{aligned} \emptyset \in \mathcal{T}_1, \emptyset \in \mathcal{T}_2 &\implies \emptyset \in \mathcal{T}_1 \cap \mathcal{T}_2 \\ X \in \mathcal{T}_1, X \in \mathcal{T}_2 &\implies X \in \mathcal{T}_1 \cap \mathcal{T}_2 \end{aligned}$$

Furthermore, given,  $U, V \in \mathcal{T}_1, \mathcal{T}_2$ . Then  $U \cap V$  is in both  $\mathcal{T}_1$  and  $\mathcal{T}_2$  so it follows  $U, V \in \mathcal{T}_1 \cap \mathcal{T}_2$  and that since  $U, V \in \mathcal{T}_1$ , then  $U \cap V \in \mathcal{T}_1$ . Similarly for  $\mathcal{T}_2$  so then  $U \cap V \in \mathcal{T}_1 \cap \mathcal{T}_2$ . Thus, the finite intersection of open sets in  $\mathcal{T}_1 \cap \mathcal{T}_2$  is in  $\mathcal{T}_1 \cap \mathcal{T}_2$ . Now it is left to show that  $\bigcup_{\alpha \in I} V_\alpha \in \mathcal{T}_1 \cap \mathcal{T}_2$  given that for all  $\alpha \in I$ ,  $V_\alpha \in \mathcal{T}_1 \cap \mathcal{T}_2$ .

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If every  $V_\alpha \in \mathcal{T}_1 \cap \mathcal{T}_2$ , then it follows  $\cup V_\alpha \in \mathcal{T}_1$  and similarly for  $\mathcal{T}_2$ . Thus  $\mathcal{T}_1 \cap \mathcal{T}_2$  is a topology.

\* **Ex. 8** — Let  $X$  be an infinite set. Show that there are infinitely many topologies on  $X$ .

**Answer (Ex. 8)** — Define  $\mathcal{T}_{\text{co-}k} := \{U \subseteq X : |U^c| \leq k\}$ . Since  $\mathcal{T}_{\text{co-}k}$  is a topology for all  $k \in \mathbb{N}$ , it follows that if  $|X| = \infty$ , then there are infinite topologies on  $X$ .

