MAT1847 OVERVIEW

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1 Riemann Surfaces

1.1 Simply Connected Surfaces

Definition. If $V \subset \mathbb{C}$ is an open set of complex numbers, a function $f : V \to \mathbb{C}$ is called **holomorphic** (or "complex analytic") if the first derivative

$$z \mapsto f'(z) = \lim_{h \to 0} \frac{(f(z+h) - f(z))}{h}$$

is defined and continuous as a function from V to $\mathbb{C}$, or equivalently if f has a power series expansion about any point $z_0 \in V$ which converges to f in some neighborhood of z_0. Such a function is **conformal** if the derivative $f'(z)$ never vanishes (for all $z \in U$).

Definition. By a **Riemann surface** S we mean a connected complex analytic manifold of complex dimension 1. The surface S is **simply connected** if every map from a circle *into* S can be continuously deforemed to a constant map. By definition, two Riemann surfaces S and S' are **conformally isomorphic** (or **biholomorphic**) if and only if there is a homeomorphism from S onto S' which is holomorphic in terms of the respective local charts.

Theorem 1 (Uniformization Theorem)**.** Any simply connecnted Riemann surface is conformally isomorphic either

1. to the plane $\mathbb{C}$ consisting of all complex nubmers $z = x + iy$
2. to the open disk $\mathbb{D} \subset \mathbb{C}$ consisting of all z with $|z|^2 = x^2 + y^2 < 1$, or
3. to the Riemann sphere $\hat{\mathbb{C}}$ consisting of $\mathbb{C}$ together with a point at infinity, using $\zeta = 1/z$ as a chart in a neighborhood of the point at infinity.

These three cases are referred to as the **Euclidean**, **hyperbolic**, and **spherical** cases, respectively.

1.1.1 The unit disk $\mathbb{D}$

Some nice results about the surface $\mathbb{D}$ are as follows:

Lemma 1 (Schwarz Lemma)**.** If $f : \mathbb{D} \to \mathbb{D}$ is holomorphic map with $f(0) = 0$, then the derivative at the origin satisfies $|f'(0)| \leq 1$. If equality holds, $|f'(0)| = 1$, then f is a rotation about the origin. That is, $f(z) = cz$ for some constant $c = f'(0)$ on the unit circle. On the other hand, if $|f'(0)| < 1$, then $|f(z)| < |z|$ for all $z \neq 0$.

Lemma 2 (Maximum Modulus Principle)**.** A nonconstant holomorphic function cannot attain its maximum absolute value at any interior point of its region of definition.

Lemma 3 (Cauchy's Derivative Estimate)**.** If f maps the disk of radius r about z_0 into some disk of radius s, then

$$|f'(z_0)| \leq {}^{s}\!/_{r}$$

A useful corollary of this is as follows:

Theorem 2 (Liouville's Theorem)**.** A bounded function f which is defined and holomorphic everywhere on $\mathbb{C}$ must be constant.

Theorem 3 (Weierstrass Uniform Convergence Theorem)**.** If a sequence of holomorphic functions $f_n : U \to \mathbb{C}$ converges uniformily to the limit function f, then f itself is holomorphic. Furthermore, the sequence of derivatives f'_n converges, uniformily on any compact set of U, to the derivatives f'.

1.1.2 Conformal Automorphism Groups

For any Riemann surface S, the notation $\mathcal{G}(S)$ will be used for the group consisting of all conformal automorphisms of S. The identity map will be denoted by $I = I_S \in \mathcal{G}(S)$.

Lemma 4 (Möbius Transformations)**.** The group $\mathcal{G}(\hat{\mathbb{C}})$ of all conformal automorphisms of the Riemann sphere is equal to the group of all **fractional linear transformations** (also called **Möbius transformations**)

$$g(z) = \frac{az + b}{cz + d}$$

where the coefficients are complex numbers with $ad - bc \neq 0$.

The identification to the complex Lie group $\mathbb{PSL}_2(\mathbb{C})$ is as follows:
Multiply the numerator and denominator by a common factor, then it is always possible to normalize so that the determinant $ad - bc$ is equal to $+1$. The resulting coefficients are well defined up to a simultaneous change of sign. Thus it follows that the group $\mathcal{G}(\mathbb{C})$ of conformal transformations can be identified with the complex 3-dimensional Lie group $\mathbb{PSL}_2(\mathbb{C})$ consisting of all 2×2 complex

matrices with determinant $+1$ modulo the subgroup $\{\pm I\}$.

Next, it is shown that both $\mathcal{G}(\mathbb{C})$ and $\mathcal{G}(\mathbb{D})$ can be considered as Lie subgroups of $\mathcal{G}(\hat{\mathbb{C}})$.

Corollary 1 (The Affine Group)**.** The group $\mathcal{G}(\mathbb{C})$ of all conformal automorphisms consists of all affine transformations

$$f(z) = \lambda z + c$$

with complex coefficients $\lambda \neq 0$ and c.

Note every conformal automorphism f of $\mathbb{C}$ extends uniquely to a conformal automorphism of $\hat{\mathbb{C}}$ with $\lim_{z\to\infty} f(z) = \infty$.

Theorem 4 (Automorphisms of $\mathbb{D}$)**.** The group $\mathcal{G}(\mathbb{D})$ of all conformal automorphisms of the unit disk can be identified with the subgroup of $\mathcal{G}(\hat{\mathbb{C}})$ consisting of all maps

$$f(z) = e^{i\theta}\frac{z-a}{1-\overline{a}z}$$

where a ranges over the open disk $\mathbb{D}$ and where $e^{i\theta}$ ranges over the unit circle $\partial\mathbb{D}$.

This is no longer a *complex* Lie group. $\mathcal{G}(\mathbb{D})$ is a *real* 3-dimensional Lie group, having the topology of a "solid torus" $\mathbb{D} \times \partial\mathbb{D}$.

It is often more convenient to work with the **upper half-plane** $\mathbb{H}$, consisting of all complex numbers $w = u + iv$ with $v > 0$.

Lemma 5 ($\mathbb{D} \cong \mathbb{H}$)**.** The half-plane $\mathbb{H}$ is conformally isomorphic to the disk $\mathbb{D}$ under the holomorphic mapping

$$w \mapsto \frac{i-w}{i+w}$$

with inverse

$$z \mapsto \frac{i(1-z)}{1+z}$$

where $z \in \mathbb{D}$ and $w \in \mathbb{H}$.

Corollary 2 (Automorphisms of $\mathbb{H}$)**.** The group $\mathcal{G}(\mathbb{H})$ consiting of all conformal automorphisms of the upper half-plane can be identified with the group of all fractional linear transformations $w \mapsto {}^{aw+b}/_{cw+d}$, where the coefficients a, b, c, d are real with determinant $ad - bc > 0$.

If we normalize so that $ad - bc = 1$, then the coefficients are well defined up to a simultaneous change of sign. Thus $\mathcal{G}(\mathbb{H})$ is isomorphic to the group $\mathbb{PSL}_2(\mathbb{R})$, consisting of all 2×2 real matrices with determinant $+1$ modulo the subgroup $\{\pm I\}$.

To conclude this section, we will try to say something more about the structure of these three groups. For any map $f : X \to X$, it will be convenient to use the notation $\mathrm{Fix}(f) \subset X$ for the set of all fixed points $x = f(x)$. If f and g are commuting maps from X to itself, $f \circ g = g \circ f$, note that

$$f(\mathrm{Fix}(g)) \subset \mathrm{Fix}(g)$$

Now, we find the commuting elements of $\mathcal{G}(\mathbb{C})$, $\mathcal{G}(\mathbb{D})$, $\mathcal{G}(\hat{\mathbb{C}})$.

Lemma 6 (Commuting elements of $\mathcal{G}(\mathbb{C})$)**.** Two non-identity affine transformations of $\mathbb{C}$ commute if and only if they have the same fixed point set.

An affine transformation with two fixed points must be the identity map.

Now consider the group $\mathcal{G}(\hat{\mathbb{C}})$ of automorphisms of the Riemann sphere. By definition, an automorphism g is called an **involution** if $g \circ g = I$, but $g \neq I$.

Theorem 5 (Commuting elements of $\mathcal{G}(\hat{\mathbb{C}})$)**.** For every $f \neq I$ in $\mathcal{G}(\hat{\mathbb{C}})$, the set $\mathrm{Fix}(f) \subset \hat{\mathbb{C}}$ contains either one point or two points (if f has more than 2 fixed points, it must be the identity map). In general, two nonidentity elements $f, g \in \mathcal{G}(\hat{\mathbb{C}})$ commute if and only if $\mathrm{Fix}(f) = \mathrm{Fix}(g)$. The only exceptions to this statement are provided by pairs of commuting involutions each of which interchanges the two fixed points of each other.

As an example, the involution $f(z) = -z$ with $\mathrm{Fix}(f) = \{0, \infty\}$ commutes with the involution $g(z) = {}^1\!/z$ with $\mathrm{Fix}(g) = \{\pm 1\}$.

We want a corresponding statement for the open disk $\mathbb{D}$. However, it is better to work with the closed disk $\overline{\mathbb{D}}$, in order to obtain a richer set of fixed points. Using Theorem 4, we see that every automorphism of the open disk extends uniquely to an automorphism of the closed disk so that $\mathcal{G}(\mathbb{D}) \cong \mathcal{G}(\overline{\mathbb{D}})$.

Theorem 6. For every $f \neq I$ in $\mathcal{G}(\mathbb{D}) \cong \mathcal{G}(\overline{\mathbb{D}})$, the set $\mathrm{Fix}(f) \subset \overline{\mathbb{D}}$ consists of either a single point of the boundary circle $\partial\mathbb{D}$, a single point of the open disk $\mathbb{D}$, or two points of $\partial\mathbb{D}$. Two nonidentity automorphisms $f, g \in \mathcal{G}(\mathbb{D})$ commute if and only if they have the same fixed point set in $\overline{\mathbb{D}}$.

1.2 Universal Coverings and the Poincaré Metric

Definition. A map $p : M \to N$ between connected manifolds is called a **covering map** if every point of N has a connected open neighborhood U within N which is **evenly covered**; that is, each component $p^{-1}(U)$ must map onto U by a homeomorphism. The manifold N is **simply connected** if it has no nontrivial coverings, that is, if every such covering map $M \to N$ is a homeomorphism. For any connected manifold N, there exists a covering map $\tilde{N} \to N$ such that $\tilde{N}$ is simply connected. This is called the **universal covering** of N and is unique up to homeomorphism.

Definition. By a **deck transformation** associated with a covering map $p : M \to N$ we mean a continuous map $\gamma : M \to M$ which satisfies the identity $p \circ \gamma = p$. For our purposes, the **fundamental group** $\pi_1(N)$ can be defined as the group Γ consisting of all deck transformations for the universal covering $\tilde{N} \to N$. Note that this universal covering is always a **normal covering** of N. That is, given two points $x, x' \in M = \tilde{N}$ with $p(x) = p(x')$, there exists one and only one deck transformation which maps x to x'. It follows that N can be identified with the quotient $^{\tilde{N}}/_{\Gamma}$ of $\tilde{N}$ by this action of Γ.

In particular, we have that a given group Γ of homeomorphisms of a connected manifold M gives rise in this way to a normal covering $M \to {}^{M}/_{\Gamma}$ if and only if

1. Γ acts **properly discontinuously**; that is, any compact set $K \subset M$ intersects only finitely many of its translates $\gamma(K)$ under the action of Γ; and

2. Γ acts **freely**; that is, every nonidentity element of Γ acts without fixed points on M

Theorem 7 (Uniformization for Arbitrary Riemann Surfaces)**.** Every Riemann surface S is conformally isomorphic to a quotient of the form $^{\tilde{S}}/_{\Gamma}$, where $\tilde{S}$ is a simply connected Riemann surface, which is necessarily isomorphic to either $\mathbb{D}$, $\mathbb{C}$, or $\hat{\mathbb{C}}$,and where $\Gamma \cong \pi_1(S)$ is a group of conformal automorphisms which acts freely and properly discontinuously on $\tilde{S}$.

Since the action of Γ on $\tilde{S}$ is properly discontinuous, Γ must be a **discrete** subgroup of $\mathcal{G}(\tilde{S})$; that is, there exists a neighborhood of the identity element in $\mathcal{G}(\tilde{S})$ which intersects Γ only in the identity element.

Corollary 3 (σ-Compactness)**.** Every Riemann surface can be expressed as a countable union of compact subsets.

This follows from Theorem 7 and from the fact that this is true for all the three simply connected surfaces.

Now, we can give a rough catalogue *for all Riemann* surfaces (not just simply connected ones!)

Spherical Case. According to Theorem 5, every conformal automorphisms of the Riemann sphere $\hat{\mathbb{C}}$ has at least one fixed point. Therefore if $S \cong \hat{\mathbb{C}}/\Gamma$ is a Riemann surface with universal covering with $\tilde{S} \cong \hat{\mathbb{C}}$, then the group $\Gamma \subset \mathcal{G}(\hat{\mathbb{C}})$ must be trivial, and hence S itself must be conformally isomorphic to $\hat{\mathbb{C}}$.

Euclidean Case. By Corollary 1, the group $\mathcal{G}(\mathbb{C})$ of conformal automorphisms of the complex plane consists of all affine transformations $z \mapsto \lambda z + c$ with $\lambda \neq 0$. Every such transformation with $\lambda \neq 1$ has a fixed point. Hence, if $S \cong \mathbb{C}/\Gamma$ is a surface with universal covering $\tilde{S} \cong \mathbb{C}$, then Γ must be a discrete group of translations $z \mapsto z + c$ of the complex plane $\mathbb{C}$. There are three subcases:

1. If Γ is trivial, then S itself is isomorphic to $\mathbb{C}$.
2. If Γ has just one generator, then S is isomorphic to the infinite **cylinder** $\mathbb{C}/\mathbb{Z}$, where $\mathbb{Z} \subset \mathbb{C}$ is the additive subgroup of integers. Note that this cylinder is isomorphic to the **punctured plane** $\mathbb{C}\backslash\{0\}$. under the isomorphism
$$z \mapsto \exp(2\pi i z) \in \mathbb{C}\backslash\{0\}$$
3. If Γ has two generators, then it can be described as a 2-dimensional **lattice** $\Lambda \subset \mathbb{C}$, that is, an additive group generated by two complex numbers which are linearly independent over $\mathbb{R}$. The quotient $\mathbb{T} = \mathbb{C}/\Lambda$ is called a **torus**.

So far, we have that Riemann surfaces can be conformally isomorphic to $\hat{\mathbb{C}}$, $\mathbb{C}$, $\mathbb{C}/\mathbb{Z}$ and the various tori, $\mathbb{C}/(\mathbb{Z} \oplus \tau\mathbb{Z})$. So, we have the final case:

Hyperbolic Case. In all other cases, the universal covering $\widetilde{S}$ must be conformally isomorphic to the unit disk. Such Riemann surfaces are said to be **hyperbolic**. So the Riemann surfaces above are the only Riemann surfaces are the only surfaces which are non-hyperbolic. Therefore, every Riemann surface not homeomorphic to the sphere or torus (in the compact case), or homeomorphic to the plane or cylinder (in the noncompact case), must necessarily be hyperbolic, with universal covering surface conformally isomorphic to the unit disk.

Remark. Recall the inclusions $\mathbb{D} \to \mathbb{C} \to \hat{\mathbb{C}}$.

Lemma 7 (Maps between Surfaces of Different Type)**.** Every holomorphic map from a Euclidean Riemann surface to a hyperbolic one is necessarily constant. Similarly, every holomorphic map from the Riemann sphere to a Euclidean or hyperbolic surface is necessairly constant.

Every hyperbolic surface has a non-abelian fundamental group so it must follow that all non-hyperbolic (e.g. Euclidean) Riemann surfaces have abelian fundamental groups. In particular, these can be trivial or isomorphic to $\mathbb{Z}$ or $\mathbb{Z} \oplus \mathbb{Z}$. However, there also exist hyperbolic surfaces with the fundamental group $\mathbb{Z}$. An example of this is the **punctured disk** $\mathbb{D}\backslash\{0\}$ or an **annulus** $\mathbb{A}_r = \{z \in \mathbb{C} : 1 < |z| < r\}$.

Lemma 8 (The Triply Punctured Sphere)**.** If we remove three or more points from the Riemann sphere, then the resulting Riemann sphere S is hyperbolic, with the universal covering $\widetilde{S}$ conformally isomorphic to the disk.

Theorem 8 (Picard's Theorem)**.** Every holomorphic map $f : \mathbb{C} \to \mathbb{C}$ which omits two different values must necessairly be constant.

The Poincaré Metric on $\mathbb{D}$. Every hyperbolic surface has a preferred Riemannian metric, constructed via the following discussion. We first consider the simply connected case:

Lemma 9 (The Poincaré Metric on $\mathbb{D}$)**.** There exists one and, up to multiplication by a positive constant, only one Riemannian metric on the disk $\mathbb{D}$ which is invariant under every conformal automorphism of $\mathbb{D}$.

As an immediate corollary, we get exactly the same statement for the upper half-plane $\mathbb{H}$, or for any other surface which is conformally isomorphic to $\mathbb{D}$.

Definition. This **Poincaré metric** on $\mathbb{H}$ is defined to be $ds = |dw|/v$ and for $\mathbb{D}$:

$$ds = 2|dz|/(1 - |z|^2) \text{ for } z \in \mathbb{D}$$

Remark. The most basic invariant for a Riemannian metric on a surface S is the **Gaussian curvature** function $K : S \to \mathbb{R}$. The metric defined above has Gaussian curvature $K = -1$.

Thus, there is a preferred Riemannian metric ds on $\mathbb{D}$ or on $\mathbb{H}$. In particular, if S is any hyperbolic surface, it follows that there is one and only one Riemannian metric on S so that the projection $\widetilde{S} \to S$ is a **local isometry** mapping any sufficiently small open subset of $\widetilde{S}$ isometrically onto its image in S. By definition, the metric ds constructed in this way is called the **Poincaré metric** on the hyperbolic surface S.

Definition. Let S be a hyperbolic surface with Poincaré metric ds. The integral $\int_P ds$ along any pieceweise smooth path $P : [0,1] \to S$ is called the **Poincaré length** of this path. For any two points z_1 and z_2 in S, the **Poincaré distance** $\mathrm{dist}(z_1, z_2) = \mathrm{dist}_S(z_1, z_2)$ is defined to be the infimum, over all piecewise smooth paths P joining z_1 to z_2, of the Poincaré length $\int_P ds$. In fact we will see that there always exists a path of minimum length.

Lemma 10 (Completeness Lemma)**.** Every hyperbolic surface S is **complete** with respect to its Poincaré metric. That is:

1. every Cauchy sequence with respect to the metric dist_S converges, or equivalently:

2. every closed neighborhood

$$N_r(z_0, \mathrm{dist}_S) = \{z \in S; \mathrm{dist}_S(z, z_0) \le r\}$$

   is a compact subset of S. Furthermore:

3. any two points of S are joined by at least one minimal geodesic.

Corollary 4 (Constant Curvature Metrics)**.** Every Riemann surface admits a complete conformal metric with constant curvature which is either positive, negative or zero according to whether the surface is spherical, hyperbolic, or Euclidean.

In the hyperbolic case, it is just the Poincaré metric with constant curvature $K = -1$, in the Euclidean case, it is the standard metric with $K = 0$ and in the spherica case, we obtain the **spherical metric**:

$$ds = 2|dz|/(1 + |z|^2)$$

with constant Gaussian curvature $+1$.

Theorem 9 (Pick Theorem)**.** If $f : S \to S'$ is a holomorphic map between hyperbolic surfaces, then exactly one of the following three statements is valid:

1. f is a conformal isomorphism from S onto S', and it maps S with its Poincaré metric isometrically onto S' with its Poincaré metric.

2. f is a covering map but it is not one-to-one. In this case, it is locally but not globally a Poincaré isometry. Every smooth path $P : [0, 1] \to S$ of arclength l in S maps to a smooth path $f \circ P$ of the same length l is S', and it follows that

$$\text{dist}_{S'}(f(p), f(q)) \leq \ \text{dist}_S(p, q)$$

for every $p, q \in S$. Here equality holds whenever p is sufficiently close to q, but strict inequality will hold, for example, if $f(p) = f(q)$ with $p \neq q$.

3. In all other cases, f strictly decreases all nonzero distances. In fact, for any compact set $K \subset S$ there is a constant $c_K < 1$ so that

$$\text{dist}_{S'}(f(p), f(q)) \leq c_k \text{dist}_S(p, q)$$

for every $p, q \in K$ and so that every smooth path in K with arclength l (using the Poincaré metric for S) maps to a path Poincaré arclength $\leq c_K l$ in S'.

One important application of Theorem 9 is to the inclusion map $\iota : S \to S'$ where S' is a hyperbolic Riemann surface and S is a connected open subset. If $S \neq S'$ then it follows from Theorem 9 that

$$\text{dist}_{S'}(p, q) \leq \ \text{dist}_S(p, q)$$

for every $p \neq q$ in S. *Thus distances measured realtive to a larger Riemann surface are always smaller.*

1.3 Normal Families: Montel's Theorem

Let S and T be Riemann surfaces. We will study the compactness of the function space $\mathrm{Hol}(S,T)$ consisting of all holomorphic maps with source S and target T. To define a topology on this space, we give it the subspace topology of $\mathrm{Map}(S,T)$ which is a set consisting of all continuous maps (C^0) from S to T. The topology on $\mathrm{Map}(S,T)$ is given by:

Definition. Let X be a locally compact space, recall that this means that every point of X has a compact neighbourhood. Let Y be a metric space. For any f in the space $\mathrm{Map}(X,Y)$ of continuous maps from X to Y, we define a family $N_{K,\epsilon}(f)$ of **basic neighborhoods** of f as follows. For any compact subset $K \subset Y$ and any $\epsilon > 0$, let $N_{K,\epsilon}(f)$ be the set of all $g \in \mathrm{Map}(X,Y)$ satisfying the condition that

$$\mathrm{dist}(f(x), f(y)) < \epsilon \qquad \text{for all} \quad x \in K$$

A subset $U \subset \mathrm{Map}(X,Y)$ is defined to be **open** if and only if, for every $f \in U$, there exists K and ϵ as above so that the basic neighborhood $N_{K,\epsilon}(f)$ is contained in U.

The topology on $\mathrm{Map}(X,Y)$ is known as the **topology of uniform convergence on compact subsets** or even **compact- open topology**.

Lemma 11 (The Topology of Locally Uniform Convergence)**.** With these definitions, $\mathrm{Map}(X,Y)$ is a well defined Hausdorff space. A sequence of maps $f_i \in \mathrm{Map}(X,Y)$ converges to the limit g in this topology if and only if