Outer Measure

Definition: The *Lebesgue Outer Measure* of $E \subseteq \mathbb{R}^d$ is equal to $\inf(\sum_1^\infty |Q_j|)$ where the infimum is taken over all countable collections of closed cubes which cover E.
By definition, for every $\epsilon > 0$, there exists a covering $E \subseteq \cup_{j=1}^\infty Q_j$ with $\sum_{j=1}^\infty m_*(Q_j) \le m_*(E) + \epsilon$.
Observation 1: Monotonicity: If $E_1 \subseteq E_2$, then $m_*(E_1) \le m_*(E_2)$.
Observation 2: Countable sub-additivity If $E = \cup_{j=1}^\infty E_j$, then $m_*(E) \le \sum_{j=1}^\infty m_*(E_j)$.
Observation 3 If $E \subseteq \mathbb{R}^d$, then $m_*(E) = \inf m_*(\mathcal{O})$, where the infimum is taken over all open sets $\mathcal{O}$ containing E.
Observation 4 If $E = E_1 \cup E_2$, and $d(E_1, E_2) > 0$, then $m_*(E) = m_*(E_1) + m_*(E_2)$.
Observation 5 If a set E is the countable union of almost disjoint cubes $E = \cup_{j=1}^\infty Q_j$, then $m_*(E) = \sum_{j=1}^\infty |Q_j|$.

Measurable sets and the Lebesgue measure

Lebesgue Measurable A subset E of $\mathbb{R}^d$ is *Lebesgue measurable*, if for any $\epsilon > 0$ there exists an open set $\mathcal{O}$ with $E \subseteq \mathcal{O}$ and $m_*(\mathcal{O} - E) \le \epsilon$.
Lebesgue Measure If E is measurable, we define its *Lebesgue measure* $m(E)$ by $m(E) = m_*(E)$.
Property 1 Every open set in $\mathbb{R}^d$ is measurable.
Property 2 If $m_*(E) = 0$, then E is measurable. In particular, if F is a subset of a set of exterior measure 0, then F is measurable.
Property 3 A countable union of measurable sets is measurable.
Property 4 Closed sets are measurable.
Lemma 3.1 If F is closed, K is compact, and these sets are disjoint, then $d(F, K) > 0$.
Property 5 The complement of a measurable set is measurable.
Property 6 A countable intersection of measurable sets is measurable.
Theorem 3.2 If $E_1, E_2, \ldots$, are disjoint measurable sets, and $E = \cup_{j=1}^\infty E_j$, then $m(E) = \sum_{j=1}^\infty m(E_j)$.
Arrow Notation If $E_1, E_2, \ldots$ is a countable collection of subsets of $\mathbb{R}^d$ that increases to E in the sense that $E_k \subseteq E_{k+1}$ for all k, and $E = \cup_{k=1}^\infty E_k$, then we write $E_k \nearrow E$. Similarly, if $E_1, E_2, \ldots$ decreases to E in the sense that $E_k \supset E_{k+1}$ for all k, and $E = \cap_{k=1}^\infty E_k$, we write $E_k \searrow E$.
Corollary 3.3 Suppose $E_1, E_2, \ldots$ are measurable subsets of $\mathbb{R}^d$.

1. If $E_k \nearrow E$, then $m(E) = \lim_{N\to\infty} m(E_N)$.
2. If $E_k \searrow E$ and $m(E_k) < \infty$ for some k, then $m(E) = \lim_{N\to\infty} m(E_N)$.

Symmetic Difference The notation $E \triangle F$ stands for the symmetric difference between the sets E and F, defined by $E \triangle F = (E - F) \cup (F - E)$ which consists of those points that belong to only one of the two sets E or F.
Theorem 3.4 Suppose E is a measurable subset of $\mathbb{R}^d$. Then, for every $\epsilon > 0$:

1. There exists an open set $\mathcal{O}$ with $E \subseteq \mathcal{O}$ and $m(\mathcal{O} - E) \le \epsilon$.
2. There exists a closed set F with $F \subseteq E$ and $m(E - F) \le \epsilon$.
3. If $m(E)$ is finite, there exists a compact set K with $K \subseteq E$ and $m(E - K) \le \epsilon$.
4. If $m(E)$ is finite, there exists a finite union $F = \cup_{j=1}^N Q_j$ of closed cubes such that $m(E \triangle F) \le \epsilon$.

Invariance properties of Lebesgue measure If E is a measurable set and $h \in \mathbb{R}^d$, then the set $E_h := \{x + h : x \in E\}$ is also measurable, and $m(E_h) = m(E)$. Also, suppose $\delta > 0$, and denote δE the set $\{\delta x : x \in E\}$. We can then assert δE is measurable whenever E is, and $m(\delta E) = \delta^d m(E)$. Whenever E measurable, so is $-E = \{-x : x \in E\}$ and $m(-E) = m(E)$.

σ-algebras and Borel sets

Definition A σ-algebra of sets is a collection of subsets of $\mathbb{R}^d$ that is closed under countable unions, countable intersections, and complements.
The collection of all subsets of $\mathbb{R}^d$ is a σ-algebra. The collection of all Lebesgue measurable sets forms a σ-algebra.
Borel σ-algebra Denoted $\mathcal{B}_{\mathbb{R}^d}$, the Borel σ-algebra is the smallest σ-algebra which contains all open sets. We may also define $\mathcal{B}_{\mathbb{R}^d}$ as the intersection of all σ-algebras that contain the open sets.
Completion of the Borel algebra From the point of view of Borel sets, the Lebesgue sets arise as the *completion* of the σ-algebra, that is, by adjoining all subsets of Borel sets of measure zero. This is an immediate consequence of the corollary below.
The G_δ and F_σ sets G_δ is the set of all countable intersections of open sets. F_σ is the set of all countable union of closed sets. Both of these collections are in $\mathcal{B}_{\mathbb{R}^d}$.
Corollary 3.5 A subset E of $\mathbb{R}^d$ is measurable if and only if E differs from a G_δ by a set of measure zero. A subset E of $\mathbb{R}^d$ is measurable if and only if E differs from a F_σ by a set of measure zero.

Measurable functions

Characteristic function A characteristic function of a set E is defined by the function $\chi_E(x) = 1$ if $x \in E$ and $\chi_E(x) = 0$ if $x \notin E$.
Step functions A function f is a step function if it can be written in the form $f = \sum_{k=1}^N a_k \chi_{R_k}$ where a_k's are constants and R_k's are rectangles.
Simple functions A function f is a simple function if it can be written in the form $f = \sum_{k=1}^N a_k \chi_{E_k}$ where a_k's are constants and each $m(E_k) < \infty$.
Measurable functions A function f defined on a measurable set E of $\mathbb{R}^d$ is **measurable** if for all $a \in \mathbb{R}$, the set

$$f^{-1}((-\infty, a)) = \{x \in E : f(x) < a\}$$

is measurable. We denote the set above by $\{f < a\}$. This definition is equivalent to requiring $\{f < a\}$, $\{f \ge a\}$ or even $\{f \le a\}$ being measurable.
Property 1 f is measurable if and only if $f^{-1}(\mathcal{O})$ is measurable for every open set $\mathcal{O}$, and if and only if $f^{-1}(F)$ is measurable for every closed set F.
Property 2 If f is continuous on $\mathbb{R}^d$, then f is measurable. If f is measurable and finite-valued, and ϕ continuous, then $\phi \circ f$ is measurable.
Property 3 Suppose $\{f_n\}_{n=1}^\infty$ is a sequence of measurable functions. Then $\sup_n f_n(x)$, $\inf_n f_n(x)$, $\limsup_{n\to\infty} f_n(x)$ and $\liminf_{n\to\infty} f_n(x)$ are measurable.
Property 4 Suppose $\{f_n\}_{n=1}^\infty$ is a sequence of measurable functions and $\lim_{n\to\infty} f_n(x) = f(x)$, then f is measurable.
Property 5 If f and g are measurable, then the integer powers f^k, $k \ge 1$ are measurable. Additionally, $f + g$ and fg are also measurable.
Almost everywhere notation We shall say two functions f and g defined on a set are equal almost everywhere and write $f(x) = g(x)$ a.e. $x \in E$ if the set $m(\{x : f(x) \ne g(x)\}) = 0$.
Property 6 Suppose f is measurable, and $f(x) = g(x)$ for a.e. x. Then g is measurable.

Approximation by simple functions or step functions

Theorem 4.1 Suppose f is a non-negative measurable function on $\mathbb{R}^d$. Then there exist an increasing sequence of non-negative simple functions $\{\phi_k\}_{k=1}^\infty$ that converge pointwise to f, namely $\phi_k(x) \le \phi_{k+1}(x)$ and $\lim_{k\to\infty} \phi_k(x) = f(x)$ for all x.
Theorem 4.2 Suppose f is measurable on $\mathbb{R}^d$. Then there exist a sequence of simple functions $\{\phi_k\}_{k=1}^\infty$ that satisfies $|\phi_k(x)| \le |\phi_{k+1}(x)|$ and $\lim_{k\to\infty} \phi_k(x) = f(x)$ for all x.
Theorem 4.3 Suppose f is measurable on $\mathbb{R}^d$. Then there exist a sequence of step functions $\{\psi_k\}_{k=1}^\infty$ that converges pointwise to $f(x)$ for almost every x.

Littlewood's three principles

The three principles

1. Every set is nearly a finite union of intervals.
2. Every function is nearly continuous.
3. Every convergent sequence is nearly uniform convergent.

First principle Given by Theorem 3.4 (4).
Second principle (Lusin) Suppose f is measurable and finite valued on E with E of finite measure. Then for every $\epsilon > 0$ there exist a closed set F_ϵ, with $F_\epsilon \subseteq E$ and $m(E - F_\epsilon) \le \epsilon$ and such that $f|_{F_\epsilon}$ is continuous.
Third principle (Egorov) Suppose $\{f_k\}_{k=1}^\infty$ is a sequence of measurable functions defined on a measurable set with $m(E) < \infty$, and assume that $f_k \to f$ a.e. on E. Given $\epsilon > 0$, we can find a closed set $A_\epsilon \subseteq E$ such that $m(E - A_\epsilon) \le \epsilon$ and $f_k \to f$ uniformily on A_ϵ.

Integration Theory

The lebesgue integral is defined successively in four distinct stages: (1) Simple functions, (2) Bounded functions supported on a set of finite measure, (3) non-negative functions and (4) Integrable functions (the general case).

Stage one: simple functions

Canonical form The canonical form of a simple function ϕ is when we can write $\phi = \sum_{k=1}^N a_k \chi_{E_k}$ where all a_k's are distinct and non-zero. Furthermore, all E_k's are disjoint. This is a unique representation.
Lebesgue integral of simple functions Define the Lebesgue integral of simple functions to be the value $\int \phi := \sum_{k=1}^N a_k m(E_k)$.
Proposition 1.1 The integral of simple functions defined above satisfies the following: Independence of representation, linearity, additivity ($\int_{E\cup F} \phi = \int_E \phi + \int_F \phi$ whenever $E \cap F = \emptyset$), monotonicity (If $\phi \le \psi$, then $\int \phi \le \int \psi$) and the triangle inequality: If ϕ is a simple function, so is $|\phi|$ and also, $|\int \phi| \le \int |\phi|$.

Stage two: Bounded functions supported on a set of finite measure

Support The support of a measurable function is defined to be $\{x : f(x) \ne 0\}$. We shall say f is supported on E if $f(x) = 0$ whenever $x \notin E$. We say f is supported on a set of finite measure if $m(E) < \infty$.
Lemma 1.2 Let f be a bounded function supported on a set E of finite measure. If $\{\phi_k\}_{k=1}^\infty$ is any sequence of simple functions bounded by M, supported on E, and with $\phi_k(x) \to f(x)$ for a.e. x, then the limit $\lim_{n\to\infty} \int \phi_k$ exists and if $f = 0$, then the limit $\lim_{n\to\infty} \int \phi_k = 0$.
Lebesgue integral of bounded functions with finite support Let f be such a function. Then it's lebesgue integral is defined by $\int f = \lim_{n\to\infty} \int \phi_k$.
Proposition 1.3 Let f be a bounded function with finite support. Then the Lebesgue integral over such functions have the properties: Linearity, additivity, monotonicity and the triangle inequality.
Theorem 1.4 (Bounded convergence theorem) Suppose that $\{f_n\}$ is a sequence of measurable functions that are all bounded by M, are supported on a set of finite measure, and $f_n(x) \to f(x)$ for a.e. x. Then f is measurable, bounded and supported on a set of finite measure for a.e. x and $\int |f_n - f| \to 0$. In particular, $\int f_n \to \int f$.

Return to Riemann integrable functions

Theorem 1.5 Suppose f is Riemann integrable on the closed interval $[a, b]$. Then f is measurable, and $\int_{[a,b]}^{\mathcal{R}} f(x) = \int_{[a,b]}^{\mathcal{L}} f(x)dx$ where the integral denoted by $\mathcal{R}$ represents the standard Riemann integral and the integral on the right hand side represents the Lebesgue integral.

Stage three: non-negative functions

Lebesgue integral of non-negative functions Let f be a non-negative function. Then we define its Lebesgue integral by $\int f(x)dx = \sup_g \int g(x)dx$ where the supremum is taken over all $0 \le g \le f$, and where g is bounded and supported on a set of finite measure. We shall say that f is Lebesgue integral if $\int f(x)dx < \infty$.
Proposition 1.6 The integral of non-negative measurable functions enjoy the following properties: (1) Linearity, (2) Additivity, (3) Monotonicity (iv) If g integrable and $0 \le f \le g$, then f integrable (vi) If $\int f = 0$ then $f(x) = 0$ for a.e. x.
Is the limit of an integrable always the integral of a limit? No. Consider $f_n(x) = n$ if $0 < x < 1/n$ and 0 otherwise. Then $f_n(x) \to 0$ for all x, yet $\int f_n(x) = 1$ for all n.
Lemma 1.7 (Fatou) Suppose $\{f_n\}_{n=1}^\infty$ is a sequence of measurable functions with $f_n \ge 0$. If $\lim_{n\to\infty} f_n(x) = f(x)$ for a.e. x, then $\int f \le \liminf_{n\to\infty} \int f_n$.
Corollary 1.8 Suppose f is a non-negative measurable function, and $\{f_n\}$ a sequence of non-negative measurable functions with $f_n(x) \le f(x)$ and $f_n(x) \to f(x)$ for almost every x. Then $\lim_{n\to\infty} \int f_n = \int \lim_{n\to\infty} f_n$.
Arrow notation for sequences of functions We shall write $f_n \nearrow f$ whenever $\{f_n\}_{n=1}^\infty$ is a sequence of measurable functions that satisfies $f_n(x) \le f_{n+1}(x)$ a.e. x, all $n \ge 1$ and $\lim_{n\to\infty} f_n(x) = f(x)$ for a.e. x. Similarly, we shall write $f_n \searrow f$ whenever $f_n(x) \ge f_{n+1}(x)$ a.e. x, all $n \ge 1$ and $\lim_{n\to\infty} f_n(x) = f(x)$ a.e. x.
Corollary 1.9 (Monotone convergence theorem) Suppose $\{f_n\}_{n=1}^\infty$ is a sequence of non-negative measurable functions with $f_n \nearrow f$. Then $\lim_{n\to\infty} \int f_n = \int f$.
Corollary 1.10 Consider a series $\sum_{k=1}^\infty a_k(x)$, where $a_k \ge 0$ is measurable for every $k \ge 1$. Then $\int \sum_{k=1}^\infty a_k(x)dx = \sum_{k=1}^\infty \int a_k(x)dx$.

Stage four: general case
Lebesgue integral of measurable functions Given a measurable function f, we define its Lebesgue integral to be $\int f = \int f^+ - \int f^-$ where $f^+(x) := \max\{f(x), 0\}$ and $f^-(x) := \min\{f(x), 0\}$. We say f is Lebesgue integrable if $\int |f| < \infty$.
Proposition 1.11 The integral of Lebesgue integrable functions is linear, additive, monotonic and satisfies the triangle inequality.
Proposition 1.12 Suppose f is integrable on $\mathbb{R}^d$. Then for every $\epsilon > 0$, we have (1) There exists a set of finite measure B (a ball, for example) such that $\int_{B^c} |f| < \epsilon$ and (2) There is a $\delta > 0$ such that $\int_E |f| < \epsilon$ whenever $m(E) < \delta$.
Theorem 1.13 (Dominated Convergence Theorem) Suppose $\{f_n\}$ is a sequence of measurable functions such that $f_n(x) \to f(x)$ a.e. x. If $|f_n(x)| \le g(x)$, where g is integrable, then $\int |f_n - f| \to 0$ and consequently, $\int f_n \to \int f$.

Complex-valued functions

The space L^1 of integrable functions

Definition $L^1(E)$ is a vector space consisting of all Lebesgue integrable functions on $E \subseteq \mathbb{R}^d$ quotiented by the equivalence relation $\sim$. If $f, g \in L^1$, we say $f \sim g$ if $f = g$ a.e. x.
Norm on L^1 For any integrable function f on $E \subseteq \mathbb{R}^d$ we define the norm of f,

$$\|f\| = \|f\|_{L^1} = \|f\|_{L^1(E)} = \int_E |f(x)|dx$$

Proposition 2.1 Suppose f and g are two functions in $L^1(E)$. Then,

1. $\|af\|_{L^1(E)} = |a| \, \|f\|_{L^1(E)}$ for all $a \in \mathbb{C}$
2. $\|f + g\|_{L^1(E)} \le \|f\|_{L^1(E)} + \|g\|_{L^1(E)}$
3. $\|f\|_{L^1(E)} = 0$ if and only if $f \sim 0$.
4. $d(f, g) = \|f - g\|_{L^1(E)}$

Theorem 2.2 (Riesz-Fischer) The vector space $L^1(E)$ is complete in its metric.
Theorem 2.4 The following families of functions are dense in $L^1(E)$:

1. The simple functions
2. The step functions
3. The continuous functions of compact support

Invariance Properties If is a function defined on $\mathbb{R}^d$, the translation of f by a vector $h \in \mathbb{R}^d$ is the function f_h defined by $f_h(x) = f(x - h)$. We have translation invariance of the integral: $\int_{\mathbb{R}^d} f(x - h)dx = \int_{\mathbb{R}^d} f(x)dx$. We have relative invariance of the Lebesgue integral under dilation and reflection: $\delta^d \int_{\mathbb{R}^d} f(\delta x)dx = \int_{\mathbb{R}^d} f(x)dx$ and $\int_{\mathbb{R}^d} f(-x)dx = \int_{\mathbb{R}^d} f(x)dx$.
Convolution product The integral $\int_{\mathbb{R}^d} f(x - y)g(y)dy$ is denoted by $(f * g)(x)$ and is defined as the convolution of f and g.
Proposition 2.5 Suppose $f \in L^1(\mathbb{R}^d)$. Then $\|f_h - f\|_{L^1} \to 0$ as $h \to 0$.

Differentiation and Integration
Question 1: Suppose $f : [a, b] \to \mathbb{R}$ is integrable, $F(x) := \int_a^x f(t)dt$. Under what conditions is F differentiable (for a.e. x) with $F' = f$.
Question 2: What conditions on $F : [a, b] \to \mathbb{R}$ guarantee that F' exists (for a.e. x), F' is integrable and $F(b) - F(a) = \int_a^b F'(t)dt$.
Consider the Cantor-Lebesgue function F. $F'(x) = 0$ for a.e. x but $1 = F(1) - F(0) = \int_0^1 F'(x)dx = 0$. Thus, this does not hold for a general class of integrable functions.
Averaging problem Define $F(x) := \int_a^x f(t)dt$ for $x \in [a, b]$. We want to find if $\lim_{h\to 0}(F(x + h) - F(x))/h$ exists. This is equivalent to asking whether

$$\lim_{|I|\to 0, x\in I} \frac{1}{|I|} \int_{|I|} f(t)dt$$

is equal to $F'(x)$ or $f(x)$.
Maximal function For $f \in L^1(\mathbb{R})$, define its maximal function f^* by

$$f^*(x) = \sup_{0<r<\infty} \frac{1}{2r} \int_{x-r}^{x+r} |f|(t)dt$$

Theorem (Hardy-Littlewood) The maximal function satisfies:

1. f^* is measurable
2. $f^*(x) < \infty$ for a.e. x
3. $m(\{x \in \mathbb{R} : f^*(x) > a\} \le (3/a) \|f^*\|_{L^1(\mathbb{R}}$

Vitali Covering Lemma Suppose $B_1, \ldots, B_N$ is a finite collection of open balls in $\mathbb{R}^n$. Then there exists subcollection $B_{i_1}, \ldots, B_{i_k}$ such that

- $B_{i_1}, \ldots, B_{i_k}$ disjoint
- $\sum_{j=1}^k m(B_{i_j}) \ge (1/3^n) m(\sum_{l=1}^N b_l)$

Tchebychev's inequality $m(\{x : f(x) > \alpha\}) \le (1/\alpha) \|f\|_{L^1}$
Lebesgue's differentiation theorem If $f \in L^1(\mathbb{R})$, then $\lim_{|I|\to 0, x\in I} 1/|I| \int_I f(y)dy = f(x)$ for almost every x.
Absolutely Continuous $F : [a, b] \to \mathbb{R}$ is absolutely continuous if and only if $\forall\, \epsilon > 0$, there exists $\delta > 0$ such that if $(a_1, b_1), \ldots, (a_n, b_n) \subseteq [a, b]$ are disjoint intervals with $\sum_{k=1}^N (b_k - a_k) < \delta$, then $\sum_{k=1}^N |F(b_k) - F(a_k)| < \epsilon$.
We remark that Absolutely continuous implies uniformily continuous which implies normal continuity.
FTOC (General Version) Suppose $F : [a, b] \to \mathbb{R}$ is absolutely continuous, then $F'(x)$ exists for a.e. $x \in [a, b]$, F' is integrable and that $F(b) - F(a) = \int_a^b F'(x)dx$. Conversely, if $f : [a, b] \to \mathbb{R}$ is integrable, then $F(x) = \int_a^x f(t)dt$ is absolutely continuous, and $F'(x) = f(x)$ for a.e. x.

Variation of F

$$\mathrm{Var}_a^b(F) := \sup_{a=x_0<\ldots<x_n=b} \sum_{j=1}^n |F(x_j) - F(x_{j-1})|$$

If F is absolutely continuous, then it has bounded variation i.e. $\mathrm{Var}_a^b(F) < \infty$.

Fourier Analysis
Hilbert Space $\mathcal{H}$ Define $\mathcal{H} = L^2([-\pi, \pi]; \mathbb{C}) = \{f : [-\pi, \pi] \to \mathbb{C} : f$ measurable$\}/\sim$ with the inner product,

$$\langle f, g \rangle = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)\overline{g}(x)dx$$

Orthonormal Basis The functions $e_n(x) = e^{inx}$ $(n \in \mathbb{Z})$ are orthogonal i.e. $\langle e_n(x), e_m(x) \rangle$ equals 1 if $n = m$ and 0 if $n \ne m$. We can express any $f \in \mathcal{H}$ as

$$f(x) = \sum_{n=-\infty}^{n=\infty} a_n e^{inx}$$

This is called the fourier series where,

$$a_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)e^{-inx}dx$$

Each a_n is known as the fourier coefficients.
Theorem Let $f \in L^2([-\pi, \pi]; \mathbb{C})$. Define $a_n := \langle f, e_n \rangle$. Then,

1. The Fourier series of f converges to f in $\mathcal{H}$, i.e.

$$\lim_{N\to\infty} \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(x) - \sum_{n=-N}^{N} a_n e^{inx}|^2 dx = 0$$

2. We have Parseval's identity,

$$\sum_{n=-\infty}^{n=\infty} |a_n|^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(x)|^2 dx$$

Lemma If $f \in \mathcal{H}$, $\langle f, e_n \rangle = 0$ for all $n \in \mathbb{Z}$, then $f = 0$ in $\mathcal{H}$.
Theorem* Let $\{e_n\}_{n=-\infty}^\infty$ be an orthonormal set in a separable Hilbert space $\mathcal{H}$. Then, the following are equivalent:

1. Finite linear combinations of elements in $\{e_n\}_{n=1}^\infty$ are dense in $\mathcal{H}$
2. If $f \in \mathcal{H}$ and $\langle f, e_n \rangle = 0$ for all n, then $f = 0$
3. If $f \in \mathcal{H}$ and $e_n = \langle f, e_n \rangle$, then $\left\| \sum_{n=-N}^N a_n e_n - f \right\| \to 0$
4. If $f \in \mathcal{H}$ and $a_n = \langle f, e_n \rangle$, then $\|f\|^2 = \sum_{n=-\infty}^\infty |a_n|^2$

We remark that this theorem plus the previous lemma implies the previous theorem.

Cantor-Lebesgue function
Consider a function $F : [0, 1] \to \mathbb{R}$ defined so that: $F_1(x) =$ continuous increasing function on $[0, 1]$ that satisfies $F_1(0) = 0, F_1(x) = 1/2$ if $1/3 \le x \le 2/3$, $F_1(1) = 1$, and F_1 linear on C_1. Similarly, define F_2 using the second iteration of the ternary Cantor set C_2. This process yields a sequence of uniformily continuous increasing functions so that the sequence converges to a continuous limit F called the Cantor-Lebesgue function. This function is increasing but $F'(x) = 0$ almost everywhere, is constant on each interval of of the complement of the Cantor set. This function does not satisfy $\int_a^b F'(x)dx = F(b) - F(a)$.