Exercises

Smooth Manifolds - Lee

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1 Topological Manifolds

Exercise 1.1. Show that equivalent definitions of locally Euclidean spaces are obtained if, instead of requiring U to be homeomorphic to an open subset of $\mathbb{R}^n$, we require it to be homeomorphic to an open ball in $\mathbb{R}^n$, or to $\mathbb{R}^n$ itself.

Solution 1.1.

Let $\phi : U \to \tilde{U}$ where $p \in U \subset M$ and $\tilde{U} \subset \mathbb{R}^n$. Since $\tilde{U}$ is open, we can find an open ball around $\phi(p)$ (denoted $B_r(\phi(p))$ for some $r > 0$) such that $B_r(\phi(p)) \subset \tilde{U}$. We know $p \in \phi^{-1}(B_r(\phi(p)))$ and is open since ϕ and ϕ^{-1} is continuous. Therefore, the map $\phi' : \phi^{-1}(B_r(\phi(p))) \to B_r(\phi(p))$ is a homeomorphism. Using this along with the fact that open balls are of course open sets in $\mathbb{R}^n$, we obtain that requiring U to be homeomorphic to open subsets of $\mathbb{R}^n$ or open balls of $\mathbb{R}^n$ makes no difference.

Next, consider a ball $B_r(x)$ centered at $x \neq 0$ of radius $r > 0$. The function $f : B_r(x) \to B_r(0)$ translates this ball to the origin via $z \mapsto z - x$. This map is clearly bijective and linear (so it is differentiable and so, continuous) with a continuous inverse, thus it is a homeomorphism. Therefore, it suffices to consider to balls centered at the origin. Now, consider the map $g : B_r(0) \to B_1(0)$ defined by $z \mapsto {}^{z}\!/_{r}$ which is clearly still a homeomorphism. Now, consider a map $\pi : B_1(0) \to \mathbb{R}^n$ defined by $x \mapsto (\tan({}^{\pi|x|}\!/_{2})x)$.

Exercise 1.2. Show that any topological subspace of a Hausdorff space is Hausdorff, and any finite product of Hausdorff spaces is Hausdorff.

Solution 1.2.

Let A be a subspace of a Hausdorff space T. Let $x \neq y \in A$, since $x, y \in T$, then there exist open sets U, V in T such that $x \in U$, $y \in V$ and $V \cap U = \emptyset$. Since $x \in U$ and $x \in A$, then $x \in U \cap A$ and similarly, for $y \in V \cap A$ and since $U \cap A \subset U \cap A$ and similarly for V, we have that $U \cap A \cap V \cap A = \emptyset$. Therefore, since $U \cap A$ and $V \cap A$ are open sets in A, we have the existence of two disjoint sets which contain x and y so that A is also Hausdorff. A similar proof follows for finite products except we take the product of the disjoint sets i.e. if $U_1 \times \ldots \times U_n$ is our space, then take disjoint sets from each U_i and product them together.

Exercise 1.3. Show that any topological subspace of a second countable space is second countable, and any finite product of second countable spaces is second countable.

Solution 1.3.

Let A be a subspace of the second countable space T. Since a basis of A can be given by $B_i = \{A \cap T_i : T_i \in \{T_i\}\}$, we can simply let $\{T_i\}$ be our countable basis of T. Thus, A is second countable. Note if $U_1 \times \ldots \times U_n$ is a finite product of second countable space, its basis is given by $B = \{U_{1_i} \times \ldots \times U_{n_i} : i \in \mathbb{N}$, U_{k_i} is the ith basis element of $U_k\}$. From this, choosing the appropriate basis yields the fact that $U_1 \times \ldots \times U_n$ is second countable.

Work it out. Show that any subset of a topological n-manifolds equipped with the subspace topology is itself a topological n-manifold.

Solution.

Let $A \subset M$ be a subset of M (non-empty), then we can equip A with the subspace topology. Since every subspace of a second countable space is second countable and similarly for Hausdroff, it follows that a subspace of a Hausdorff and second countable space is Hausdorff and second countable. It is left to prove that A is locally a Euclidean space. If $p \in A$, then there exists $\phi_p : U \subset M \to \tilde{U} \subset \mathbb{R}^n$. It then follows $\phi_p' : U \cap A \to \phi_p'(U \cap A)$ is bijective and continuous with a continuous inverse since it is the restriction of a homeomorphism. Thus, A is a manifold equipped with the subspace topology.