MAT1847 Lecture Notes

Lecture notes on a course Holomorphic Dynamics

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Chapter 1

Riemann Surfaces

The next three sections are meant to be an overview of background material required for this course.

1.1 Simply Connected Surfaces

Definition 1.1.1 (Riemann Surface)**.** A **Riemann surface** is a connected, paracompact, Hausdorff topological space X equipped with an open covering $\{U_i\}$ and a collection of homeomorphisms $f_i : U_i \to \mathbb{C}$ such that there exist analytic (i.e. holomorphic) g_{ij} such that

$$f_i \circ f_j^{-1} = g_{ij}$$

where we have the assumption $U_i \cap U_j \neq 0$. We refer to the collection $\{(U_i, f_i)\}$ as the **chart** of the Riemann surface. An equivalent way defining a Riemann surface is as follows: A Riemann surface is a connected complex analytic manifold of complex dimension 1.

Example. *The complex plane $\mathbb{C}$ is the most basic Riemann surface. $\mathbb{C}$ is not only connected but also simply connected and is also clearly Hausdorff and paracompact. The chart is given by $\{(\mathbb{C}, i)\}$ where $i : \mathbb{C} \to \mathbb{C}$ is the identity function. In fact, any non-empty open set U of $\mathbb{C}$ (for example $\mathbb{H} = \{z \in \mathbb{C} : Im(z) > 0\}$) is also a Riemann surface.*

Example (Riemann sphere)**.** *The Riemann sphere is another example of a simply connected Riemann surface. This sphere is not a subset of $\mathbb{C}$ nor is it biholomorphic to $\mathbb{C}$. This is the extended complex plane (i.e. $\mathbb{C} \cup \{\pm\infty\}$). Geometrically, this can be idealized using stereographic projection:*
*Identify the xy−plane (in $\mathbb{R}^3$) with $\mathbb{C}$. Let $\mathbb{S}$ be the sphere sitting in $\mathbb{R}^3$ centered at $(0, 0, 1/2)$ and of radius $1/2$. Also, denote $\mathbb{N}$ to be the North pole of $\mathbb{S}$. Then, given any point $W = (X, Y, Z)$ on $\mathbb{S} - \mathbb{N}$, the line joining N and W intersects the xy-plane at a single point w. We say that w is the **stereographic projection** of W. The inverse map can also be derived similarly, therefore we have a bijective*

mapping between $\mathbb{S} - \mathbb{N}$ and $\mathbb{C}$. As the point w goes to infinity in $\mathbb{C}$ (i.e. $|w| \to \infty$) the corresponding point W on $\mathbb{S}$ comes arbitrarily close to $\mathbb{N}$. Thus, we can define $\mathbb{N}$ to be the point at infinity. Therefore, we can identify the extended complex plane with the sphere $\mathbb{S}$.

Remark. We can show that the projection mapping is not only a bijective mapping but a *diffeomorphic* mapping. This makes showing that transition maps are holomorphic not too hard.

Definition 1.1.2 (Conformal)**.** A function $f : \mathbb{C} \to \mathbb{C}$ is **conformal** if it preserves angles.

Remark. A function $f : \mathbb{U} \to \mathbb{C}$ is **holomorphic** if

1. the Cauchy-Riemann equations hold i.e.

$$\overline{\partial} f = 0 \qquad \text{and} \qquad {}^{\partial f}/_{\partial \overline{z}} = 0$$

2. $$\lim_{h \to 0} \frac{f(z+h) - f(z)}{h} = f'(z) \quad \text{exists}$$

3. For all $z_0 \in U$, there exists a convergent power series such that $f(z) = \sum_{n=0}^{\infty} a_n z^n$ for z in a neighbourhood of z_0.

Remark. A holomorphic function $f : V \to \mathbb{C}$ is **conformal** if the derivative $f'(z)$ never vanishes i.e. $f'(z_0) \neq 0$ for all $z \in V$. (note $f' : V \to \mathbb{C}$ also and $f'(z) \neq 0$ implies that f is locally injective). We can see this through:

$$f'(z_0) = \lim_{h \to 0} \frac{f(z+h) - f(z)}{h}$$
$$\Rightarrow \; f(z+h) \approx f(z) + f'(z)h$$

Recall that multiplying by a number (non-zero and complex) is a composition of rotation and dilation so f is conformal.

There are three main *simply* connected Riemann surfaces that are distinct (general case is uncountable).

Theorem 1 (Uniformization Theorem)**.** *Any simply connected Riemann surface is conformally isomorphic either*

1. *to the open disk $\mathbb{D} \subset \mathbb{C}$ consisting of all $|z|^2 < 1$ or,*

2. *to the plane $\mathbb{C}$*

3. *to the Riemann sphere $\hat{\mathbb{C}}$*

To see that these are indeed inequivalent surfaces, we have to talk about the following:

Theorem 2 (Liouville)**.** *Every entire, bounded function is constant.*

Corollary 2.1. $\mathbb{C}$ is not biholomorphic to $\mathbb{D}$.

Proof. Let f be a holomorphic function $f : \mathbb{C} \to \mathbb{D}$. Then, f is bounded since D is bounded. By Liouville's theorem, we have that f is constant. Since constant functions are clearly not bijective, we have that f is not biholomorphic. Therefore, there exist no biholomorphic maps between $\mathbb{C}$ and $\mathbb{D}$. □

It is clear that $\hat{\mathbb{C}}$ is not biholomorphic to any of the other Riemann surfaces on the list above since it is compact and the two other surfaces are not. Here is another way to arrive at the same conclusion:

Remark. There are no non-constant biholomorphic maps $f : \hat{\mathbb{C}} \to \mathbb{C}$.

Proof. It is clear that $f(\hat{\mathbb{C}})$ is compact since $\hat{\mathbb{C}}$ is and f is continuous. By the open mapping theorem, if f is non-constant, $f(\hat{\mathbb{C}})$ is open. However, there are no open compact subsets of $\mathbb{C}$. Therefore, f is constant as wanted. □

Remark. $\mathbb{D} \hookrightarrow \mathbb{C} \hookrightarrow \hat{\mathbb{C}}$.

Corollary 2.2. Every Riemann surface is the quotient of either $\mathbb{D}, \mathbb{C}$, or $\hat{\mathbb{C}}$ by a discrete group Γ of automorphisms of $\mathbb{D}, \mathbb{C}$, or $\hat{\mathbb{C}}$.

Definition 1.1.3 (Conformally Isomorphic)**.** Two Riemann surfaces S and S' are **conformally isomorphic** (or **biholomorphic**) if and only if there exists a homeomorphism from S to S' which is holomorphic in terms of the respecting local uniformizing parameters i.e. the transition maps. Equivalently, we can say that a **biholomorphism** is a bijective and holomorphic map (this forces the inverse to be holomorphic as well).
A **conformal automorphism** is a conformal isomorphism $f : U \to U$ (U a complex domain or Riemann surface).