Algebraic Topology Companion Notes

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Filling in some details or trying some proofs myself from James Munkres' *Topology*.

§51 Homotopy of Paths

Lemma 51.1. *The relations $\simeq$ and $\simeq_p$ are equivalence relations.*

Proof. First, we show $\simeq$ is an equivalence relation between homotopic continuous functions from $X \to Y$.
We will prove reflexivity first. Let $f : X \to Y$ be continuous. We show f is homotopic with itself. To do this, we have to show there exists a continuous mapping $F : X \times I \to Y$ such that $F(x, 0) = f(x)$ and $F(x, 1) = f(x)$ for all $x \in X$. Thus, define F to be given by $(x, t) \mapsto f(x)$. Since f is continuous, it follows that F is then also continuous.
Now, we show symmetry. Let f and g be any homotopic functions from X to Y. We show $g \simeq f$. To do this, we have to show there exists a continuous function $F : X \times I \to Y$ (not to be confused with the one above, we are no longer using it) such that $F(x, 0) = g(x)$ and $F(x, 1) = f(x)$. Thus, define F to be given by the mapping $(x, t) \mapsto G(x, 1 - t)$ where G is the homotopy between f and g. Thus, we immediately obtain that $F(x, 0) = G(x, 1) = g(x)$ and $F(x, 1) = G(x, 0) = f(x)$ as wanted. To see this is continuous, it suffices to see that the component functions are continuous. It is clear that $\pi_1(F(x, t))$ is continuous since G is continuous. The fact that $\pi_2(F(x, t))$ is continuous follows from the fact that it is a composition of continuous functions (namely $(x, t) \mapsto (x, 1 - t)$ and $(x, t) \mapsto G(x, t)$).
Now, we prove $\simeq$ is transitive to conclude that $\simeq$ is an equivalence relation. Let $f \simeq g$ and $g \simeq h$ for homotopic functions $f, g : X \to Y$ and homotopic functions $g, h : X \to Y$. Let F' and F'' be the homotopy of f, g and g, h respectively. Now, define $H : X \times [0, 2] \to Y$ via $(x, t) \mapsto F'(x, t)$ if $t \leq 1$ and $(x, t) \mapsto F''(x, t-1)$ if $t \geq 1$. We see that H is well defined since $H(x, 1) = F'(x, 1) = g(x) = F''(x, 0)$. H is also continuous since H is continuous on $t < 1$ via F' and continuous on $t > 1$ via F''. Thus, by the pasting lemma, we have that H is continuous on $X \times [0, 2]$. Using this as motivation, we define the homotopy $\delta : X \times I \to Y$ between f, h via $(x, t) \mapsto F'(x, 2t)$ if $t \leq 1/2$ and $(x, t) \mapsto F''(x, 2t-1)$ if $t \geq 1/2$.

Now, we show $\simeq_p$ is an equivalence relation. Let $f : I \to X$ be a continuous path from x_0 to x_1 and define F to be the homotopy from the reflexive proof of $\simeq$. Then, we only need to show the additional condition that $F(0,t) = x_0$ and $F(1,t) = x_1$. Note $F(x,t) = f(x)$ so $F(0,t) = f(0) = x_0$ and $F(1,t) = f(1) = x_1$ as wanted. Thus, $f \simeq_p f$ showing reflexivity. Now, we show symmetry. Let $f, f' : I \to X$ be two homotopic paths from x_0 to x_1. Define $F : I^2 \to X$ via $(s,t) \mapsto F'(s, 1-t)$ where F' is the path homotopy between f and f'. Then from the symmetry proof of $\simeq$ above, we have only left to prove that $F(0,t) = x_0$ and $F(1,t) = x_1$. Note $F(0,t) = F'(0, 1-t) = x_0$ and $F(1,t) = F'(1, 1-t) = x_1$ as wanted. We now, only need show transitivity to conclude that $\simeq_p$ is an equivalence relation. Let $f, g : I \to X$ and $g, h : I \to X$ be path homotopies between x_0 and x_1. We show f, h are path homotopic. Define δ to be a path homotopy between f and h the same way it was done in the transitive proof above. Thus, it is left to prove $\delta(0,t) = x_0$ and $\delta(1,t) = x_1$. If $t < 1/2$, we have $\delta(0,t) = F'(0, 2t) = x_0$ since F' is a path homotopy between f and g which are paths between x_0 and x_1. Similarly, $\delta(1,t) = x_1$. We can repeat this for $t \geq 1$ to again, get the same result. Thus, δ is a path homotopy as wanted. Therefore, $\simeq_p$ is an equivalence relation as wanted. □

Example 51.2. *Let f and g be any two maps of a space X into $\mathbb{R}^2$. It is easy to see that f and g are homotopic; the map*

$$F(x,t) = (1-t)f(x) + tg(x)$$

is a homotopy between them called the ***straight line homotopy***.

Proof. Note that $F(x,0) = (1-0)f(x) + 0g(x) = f(x)$ and $F(x,1) = (1-1)f(x) + 1g(x) = g(x)$ as wanted. F is continuous because f and g are continuous functions on X, thus, so are $(1-t)f(x)$ and $tg(x)$ for all $t \in \mathbb{R}$. $F(x,t)$ is the sum of these two functions so, we immediately obtain that F is a continuous function. Thus, F is a homotopy between f and g. □

Exercise 1

Show that if $h, h' : X \to Y$ are homotopic and $k, k' : Y \to Z$ are homotopic, then $k \circ h$ and $k' \circ h'$ are homotopic.

Proof. We want to deform $k \circ h : X \to Z$ into $k' \circ h' : X \to Z$ in a continuous manner. Let H be the homotopy between h and h' and K be the homotopy between k and k'. Let,

$$G : (x,t) \mapsto K(H(x,t), t)$$

If $t = 0$, we have $K(H(x,0), 0) = k(H(x,0)) = k \circ h(x)$. If $t = 1$, we have $K(H(x,1), 1) = k'(H(x,1)) = k' \circ h'(x)$. G is a composition of continuous functions, so it is continuous. Thus, $k \circ h$ and $k' \circ h'$ are homotopic. □

Exercise 2

Given spaces X and Y, let $[X, Y]$ denote the set of homotopy *classes* of maps of X into Y.

(a) Let $I = [0, 1]$. Show that for any X, the set $[X, I]$ has a single element.

(b) Show that if Y is path connected, the set $[I, Y]$ has a single element.

Proof. For (a), it suffices to show that any continuous maps $f, f' : X \to I$ is homotopic. Note I is convex so the straight line homotopy is enough to show that f and f' are homotopic. $\square$

Exercise 3

A space X is said to be **contractible** if the identity map $i_X : X \to X$ is nulhomotopic (able to be continuously deformed into a constant map).

(a) Show that I and $\mathbb{R}$ are contractible.

(b) Show that a contractible space is path-connected.

(c) Show that if Y is contractible, then for any X, the set $[X, Y]$ has a single element.

(d) Show that if X is contractible and Y is path connected, then $[X, Y]$ has a single element.

Proof. (a): Define $F : I \times I \to I$ to be given by $F(x, t) = (1 - t)x$. This is the straight line homotopy between the identity map i_I and the constant map $x \mapsto 0$. $\mathbb{R}$ is contractible for the same reason, simply apply the straight line homotopy between $i_{\mathbb{R}}$ and $x \mapsto 0$. Note, this is possible since both I and $\mathbb{R}$ are convex.

(b): Let X be a contractible space. Let F be the homotopy between i_X and some constant map f. Then, $\psi : t \mapsto F(x, t)$ is a continuous path with endpoints x and $f(x)$. Similarly, $\phi : t \mapsto F(y, t)$ is a continuous path with endpoints y and $f(y)$. Since f is a constant map, we have $f(x) = f(y)$ and so we can construct a continuous path $\Phi : I \to X$ using the pasting lemma with endpoints x and y.

(c): Let the identity map on Y be homotopic to the constant map, mapping everything to $y_0 \in Y$. Let $g : X \to Y$ be given by $x \mapsto y_0$ and $f : X \to Y$ be any continuous map. Let H be the homotopy between i_Y and the map $Y \mapsto y_0$. Define $F : X \times I \to Y$ be given by $F(x, t) = H(f(x), t)$. Then, F is a homotopy since H is and so $[X, Y]$ has a single element.

(d): Let H be the homotopy between the identity map on X and the map $X \mapsto x_0 \in X$. Then, $f \circ H : X \times I \to Y$ is a homotopy between f and $X \mapsto f(x_0)$ since $f \circ H(x, 0) = f(x)$ and $f \circ H(x, 1) = f \circ x_0 = f(x_0)$. Taking $g : X \to Y$

to be another continuous map, we can see $g \circ H$ is a homotopy between g and $X \mapsto g(x_0)$. Y is path connected so there exists a path between $g(x_0)$ and $f(x_0)$. Through this, we get a homotopy between f and g. Thus, $[X, Y]$ has a single element. $\square$

Side note: One could think that you may not need contractibility to be able to answer (d) if we let our homotopy $H(x, t)$ be equal to the path that connects some continuous functions $f(x)$ and $g(x)$ in Y. However, this mapping is not continuous.

52 The Fundamental Group

Exercise 1

A subset A of $\mathbb{R}^n$ is said to be **star convex** if for some point a_0 of A, all the line segments joining a_0 to other points of A lie in A.

(a) Find a star convex set that is not convex

(b) Show that if A is star convex, A is simply connected.

Proof.

(a) Let A be a star in $\mathbb{R}^2$ centered at $(0, 0)$. This is clearly not convex just by taking any two points on any (distinct) "tips" of the star.

(b) Let $a_0 \in A$ be the point which is connected to any other point of A via a straight line (lying entirely in A). From this property of a_0, we see that A is path connected. Now, we show that $|\pi_1(A, a_0)| = 1$ and so the fundamental group based at any other point also has order 1 (and so it must be the trivial fundamental group). To see this, let f and g be any two paths based at a_0. Define $F : I^2 \to A$ by $F(x, t) = [\phi_{f(x)} * \bar{\phi}_{g(x)}](t)$ where $\phi_a : I \to A$ is the straight line connecting a and a_0. Intuitively, F deforms f into g by first taking $f(x)$ to a_0, then takes a_0 to $g(x)$. Thus, $f \simeq_p g$ and so $|\pi_1(A, a_0)| = 1$ as wanted. $\square$

Exercise 2

Let α be a path in X from x_0 to x_1; let β be a path in X from x_1 to x_2. Show that if $\gamma = \alpha * \beta$, then $\hat{\gamma} = \hat{\beta} \circ \hat{\alpha}$.

Proof.

$$\begin{aligned}
\hat{\beta} \circ \alpha(\hat{[f]}) &= [\overline{\beta}] * ([\overline{\alpha}] * [f] * [\alpha]) * [\beta] \\
&= ([\overline{\beta}] * [\overline{\alpha}]) * [f] * ([\alpha] * [\beta]) \\
&= [\overline{\beta} * \overline{\alpha}] * [f] * [\alpha * \beta]
\end{aligned}$$

Recall from group theory that $(a \cdot b)^{-1} = b^{-1} \cdot a^{-1}$ so:

$$\hat{\beta} \circ \hat{\alpha}([f]) = [\overline{\alpha * \beta}] * [f] * [\alpha * \beta] = \hat{\gamma}([f])$$

as wanted. $\square$

Exercise 3

Let x_0 and x_1 be points of the path-connected space X. Show that $\pi_1(X, x_0)$ is abelian if and only if for every pair α and β of paths from x_0 to x_1, we have $\hat{\alpha} = \hat{\beta}$.

Proof. Assume the fundamental group at x_0 is abelian. Consider:

$$\begin{aligned}
\hat{\alpha}([f]) &= [\overline{\alpha}] * [f] * [\alpha] \\
&= ([\overline{\beta}] * [\beta]) * [\overline{\alpha}] * [f] * [\alpha] * ([\overline{\beta}] * [\beta]) \\
&= [\overline{\beta}] * ([f] * [\alpha] * [\overline{\beta}]) * ([\beta]) * [\overline{\alpha}]) * [\beta] \\
&= [\overline{\beta}] * [f] * [\beta] = \hat{\beta}([f])
\end{aligned}$$

where we have used the fact that $([f] * [\alpha] * [\overline{\beta}])$ is a path from x_0 to x_0 and so is $([\beta]) * [\overline{\alpha}])$ and so we can switch them using our abelian property. Now, assume instead that $\hat{\alpha} = \hat{\beta}$.

We know that $\gamma := f * \alpha$ is a path from x_0 to x_1 and so $\hat{\gamma} = \hat{\alpha}$. Note $[\overline{\gamma}] = [\overline{f * \alpha}] = [\overline{\alpha}] * [\overline{f}]$ and so,

$$\begin{aligned}
\hat{\gamma}([f * g]) &= [\overline{\gamma}] * [f * g] * [\gamma] \\
&= [\overline{\alpha}] * [\overline{f}] * [f] * [g] * [f] * [\alpha] \\
&= [\overline{\alpha}] * [g] * [f] * [\alpha] \\
&= [\overline{\alpha}] * [g * f] * [\alpha] = \hat{\alpha}([g * f]) = \hat{\gamma}([g * f])
\end{aligned}$$

and so $\hat{\gamma}([f * g]) = \hat{\gamma}([g * f])$. Using cancellation, we get $[f * g] = [g * f]$ (multiply by $[\gamma]$ on the left and by $[\overline{\gamma}]$ on the right) so that $\pi_1(X, x_0)$ is abelian. $\square$

Exercise 4

Let $A \subset X$; suppose that $r : A \to X$ is a continuous map such that $r(a) = a$ for each $a \in A$. (The map r is called a retraction of X onto A). If $a_0 \in A$, show that

$$r_* : \pi_1(X, a_0) \to \pi_1(A, a_0)$$

is surjective.

Proof. Choose any $[f] \in \pi_1(A, a_0)$. Since f (and any map, path homotopic to it) lies in A, we know that $r([f]) = [f]$ by definition of retraction. Thus, $[f] \in \pi_1(X, x_0)$ as wanted. $\square$

53 Covering Spaces

A fiber over a point in a covering space is the discrete space

Let $p : E \to B$ be a covering map. Let $b \in B$. Equip $p^{-1}(b)$ with the subspace topology. Then $p^{-1}(b)$ is precisely the discrete space.

Proof. A subspace $p^{-1}(b)$ of E is discrete if and only if for each $h \in p^{-1}(b)$, there exists an open set V in E such that $V \cap p^{-1}(b) = \{h\}$. Note p is a covering map, so in particular, there exists some neighbourhood U of B that p evenly covers. So, we partition $p^{-1}(U)$ into open disjoint sets and denote them by V_α. Thus, $h \in p^{-1}(b)$ falls in some unique V_α and so $\{h\} \subset V_\alpha \cap p^{-1}(b)$. Note that p is a covering map so $p|_{V_\alpha} : V_\alpha \to B$ is a homeomorphism. In particular, this means that $\#(p|_{V_\alpha}^{-1}) = 1$ or equivalently, $\{h\} = V_\alpha \cap p^{-1}(b)$. Thus, $p^{-1}(b)$ is a discrete space. $\square$