1.3

1.2.1

Q1 Let $I(x, y, t)$ be the brightness of ~~point~~ pixel located at (x, y) on the image at the 't' time.

Assumption 1:

Brightness of a point remains constant over time.

i.e, $I(x+\delta x, y+\delta y, t+\delta t) = I(x, y, t)$

where $\delta x = \underbrace{V_x}_{\text{velocity in x-direction}} \delta t$ and $\delta y = \underbrace{V_y}_{\text{velocity in y-direction}} \delta t$

Assumption 2:

Displacement $(\delta x, \delta y)$ and timestep 'δt' are small. [Thus, allowing us to apply Taylor expansion]

$$I(x+\delta x, y+\delta y, t+\delta t) = I(x, y, t) + \left(\frac{\partial I}{\partial x}\right)(\delta x) + \left(\frac{\partial I}{\partial y}\right)(\delta y) + \left(\frac{\partial I}{\partial z}\right)(\delta z)$$

Applied

Assumption 3: Flow is locally smooth and hence, neighbouring pixels have the same displacement.

1.2.2

A3

W = window

Data term = assumes constancy of image property

Spatial term = models how flow is expected to vary across the image

As mentioned in A4, we are able to reduce the eqn for one pixel to be $I_x V_x + I_y V_y + I_t = 0$ — (1)

However, this is underdetermined and hence, we fetch similar eqns from neighbouring pixels in a window W in order to get :—

$$\cancel{\begin{bmatrix} \sum_W I_x I_x & \sum_W I_x I_y \\ \sum_W I_x I_y & \sum_W I_y I_y \end{bmatrix} \begin{bmatrix} V_x \\ V_y \end{bmatrix} + \begin{bmatrix} \sum_W I_x I_t \\ \sum_W I_y I_t \end{bmatrix} = 0}$$

~~This was obtained~~

Let the pixel under consideration be (i, j).

Let size of window = $n \times n$ where top leftmost pixel of window is 1×1 and bottom rightmost pixel is $(n \times n)$.

On stacking (1) for all points in the window, we get :-

$$\underbrace{\begin{bmatrix} I_x(1,1) & I_y(1,1) \\ I_x(1,2) & I_y(1,2) \\ \vdots & \vdots \\ I_x(i,j) & I_y(i,j) \\ \vdots & \vdots \\ I_x(n,n) & I_y(n,n) \end{bmatrix}}_{\text{Spatial term}} \begin{bmatrix} V_x \\ V_y \end{bmatrix} = - \underbrace{\begin{bmatrix} I_t(1,1) \\ I_t(2,2) \\ \vdots \\ I_t(i,j) \\ \vdots \\ I_t(n,n) \end{bmatrix}}_{\text{Data term}}$$

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Contd.

Now, we want to find the least squares soln which leads to the formulation of:-

$$\begin{bmatrix} \sum_w I_x I_x & \sum_w I_x I_y \\ \sum_w I_x I_y & \sum_w I_y I_y \end{bmatrix} \begin{bmatrix} V_x \\ V_y \end{bmatrix} = \begin{bmatrix} -\sum I_x I_t \\ -\sum I_y I_t \end{bmatrix}$$

Where the best estimate for (V_x, V_y) can be calculated by multiplying the inverse of leftmost matrix on both sides.

Also, sometimes, within the neighborhood (W) itself, the contribution of gradients of different pixels is modelled as a gaussian distribution.
i.e., if a pixel (p_1, p_2) is x, y far away from the central pixel,

then $W(p_1, p_2) \propto e^{-(x^2+y^2)}$.

- Including neighboring pixels in calculation of V_x, V_y for a central pixel allows the technique to be robust in case central pixel is an outlier (i.e., central pixel violates small motion assumption etc.).

1.2.3

A3 Assuming the movement to be small, optical flow involves expanding the intensity ~~in terms of~~ using Taylor series into:-

$$I(x+\Delta x, y+\Delta y, t+\Delta t) = I(x,y,t) + \left(\frac{\partial I}{\partial x}\right)(\Delta x) + \left(\frac{\partial I}{\partial y}\right)(\Delta y) + \left(\frac{\partial I}{\partial t}\right)(\Delta t) + \text{higher order terms}$$

Due to small motion, the higher order terms can be ignored.

As a result,

$$I(x+\Delta x, y+\Delta y, t+\Delta t) = I(x,y,t) + \frac{\partial I}{\partial x}(\Delta x) + \left(\frac{\partial I}{\partial y}\right)(\Delta y) + \left(\frac{\partial I}{\partial t}\right)(\Delta t) \quad -①$$

→ Taylor Expansion has allowed us to linearize the intensity function.

Combining ① with the brightness constancy assumption helps get

$$\left(\left(\frac{\partial I}{\partial x}\right)(V_x) + \left(\frac{\partial I}{\partial y}\right)V_y + \frac{\partial I}{\partial t} = 0\right)$$

Combined with assumption 3, this linear dependence leads to a convenient least squares based soln.

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1-2-4

A4

From the Taylor Expansion,

$$I(x+\delta x, y+\delta y, t+\delta t) = I(x,y,t) + \frac{\partial I}{\partial x}(\delta x) + \frac{\partial I}{\partial y}(\delta y) + \frac{\partial I}{\partial z}(\delta z) \quad (1)$$

Using brightness constancy assumption,

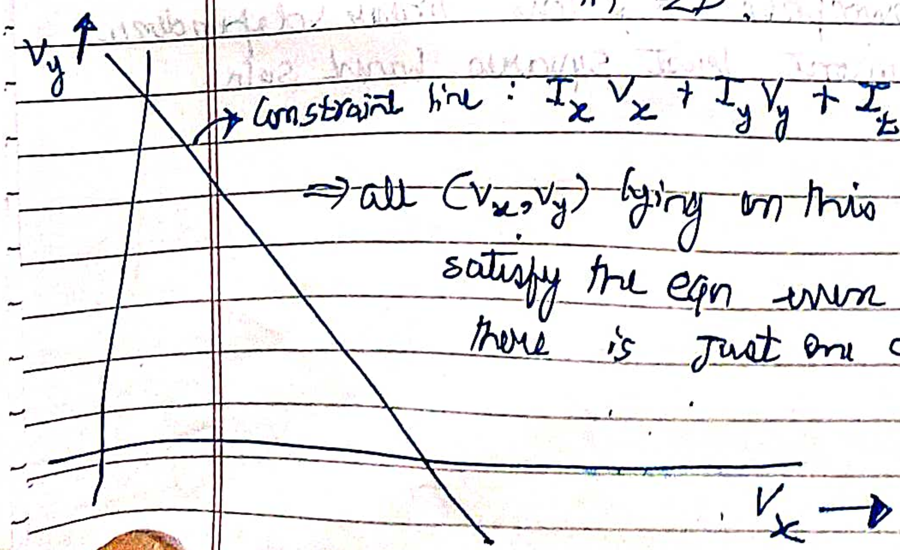
$$I(x+\delta x, y+\delta y, z+\delta z) = I(x,y,t) \quad (2)$$

Using (1), (2); we get

$$\underbrace{\left(\frac{\partial I}{\partial x}\right)}_{I_x}\underbrace{\left(\frac{dx}{dt}\right)}_{V_x} + \underbrace{\left(\frac{\partial I}{\partial y}\right)}_{I_y}\underbrace{\left(\frac{dy}{dt}\right)}_{V_y} + \underbrace{\frac{\partial I}{\partial t}}_{I_t} = 0$$

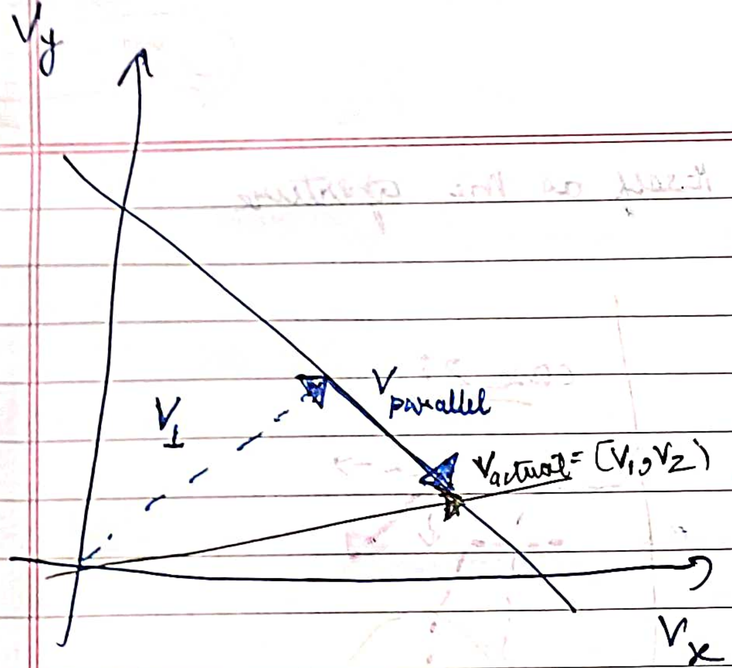
$$\Rightarrow \underbrace{\begin{bmatrix} I_x \\ I_y \end{bmatrix}^T}_{\text{known}} \underbrace{\begin{bmatrix} V_x \\ V_y \end{bmatrix}}_{\text{variables.}} + I_t = 0$$

This can be represented as a straight line eqn in 2D.



Constraint line : $I_x V_x + I_y V_y + I_t = 0$

$\Rightarrow$ all (V_x, V_y) lying on this line satisfy the eqn even though there is just one answer.



Let us assume
that actual velocity is (V_1, V_2) i.e, $V_x = V_1$
and $V_y = V_2$.

$\vec{V}_{actual}$ can be decomposed into :-

$\Rightarrow \vec{V}_{actual} = \vec{V}_{\perp} + \vec{V}_{parallel}$

perpendicular to the constraint line

parallel to the constraint line

On inspection,

$$\|V_\perp\| = \frac{\|I_t\|}{\sqrt{I_x^2 + I_y^2}}$$ = distance of origin from constraint line = known.

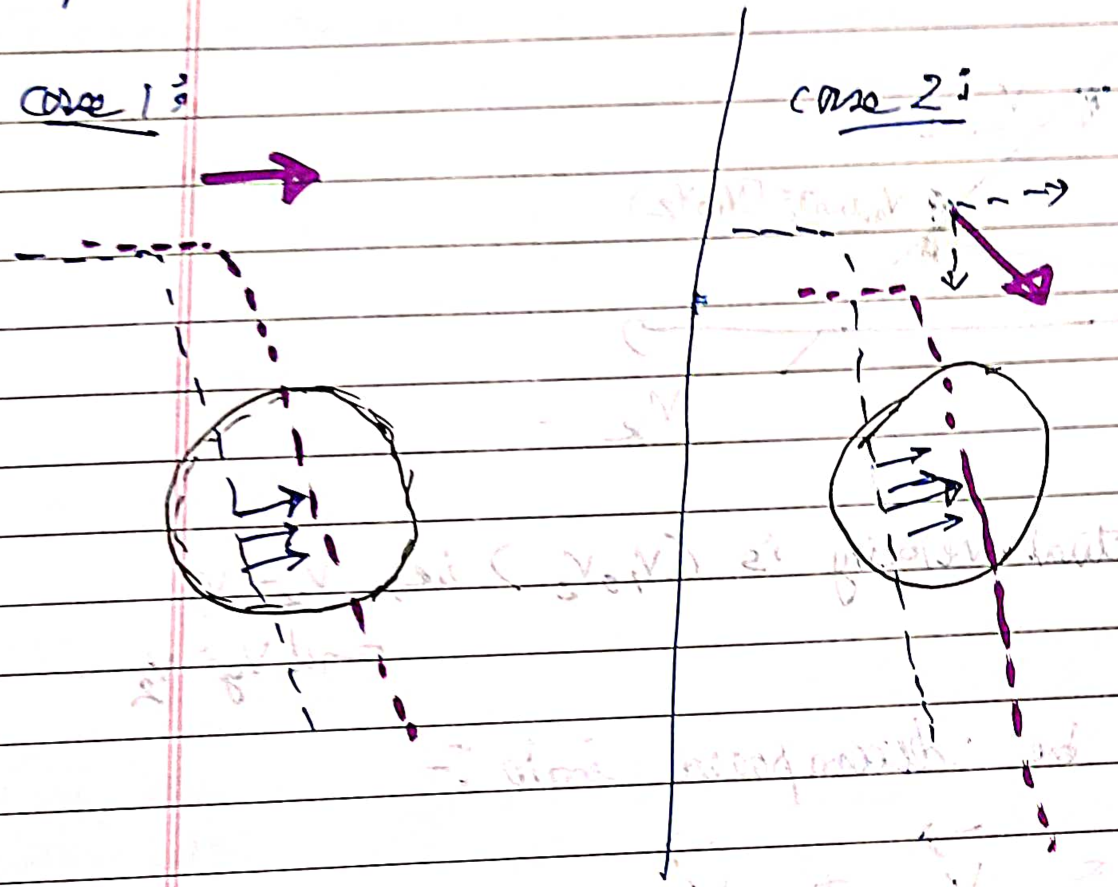
Hence, since we have just one eqn. but 2 variables we are only able ~~to~~ find the component of velocity $\perp$ to line i.e, $V_\perp$.

$\vec{V}_{parallel}$ remains unknown and is not able to be retrieved.

Visually, this manifests itself as the aperture problem where:-

Case 1:

Case 2:



In both cases, despite moving with a different velocity, the final positions of the 2 rods look the same.

In both cases, we are only able to determine the velocity of rod perpendicular to the lower end.

Due to the same final image in both cases, their different components of velocity parallel to the lower end of the rod is not able to be retrieved.