• Proof of why a Procrustes based SVD alignment gives the best aligning transform between two point clouds with known correspondences.

Ans

Let $P = \{\vec{p}_1, \vec{p}_2, \ldots, \vec{p}_n\}$ → Point cloud 1 ~~in R^d~~

$Q = \{\vec{q}_1, \vec{q}_2, \ldots, \vec{q}_n\}$ → Point cloud 2

Each point belongs to R^d.

Aim: To find a rigid transformation that optimally aligns the 2 sets in least squares sense i,e, we seek a rotation R and a translation 't' such that:-

$$(R, t) = \underset{R \in SO(d),\ t \in R^d}{\arg\min} \sum_{i=1}^{n} w_i \, \| (Rp_i + t) - q_i \|^2$$

where $w_i > 0$ for each point pair.

STEP 1: Computing the translation

Assume R is fixed.

Let $F(t) = \sum_{i=1}^{n} w_i \| Rp_i + t - q_i \|^2$.

We can find the optimal translation by taking the derivative of F with respect to 't' and searching for roots as:-

$$\cancel{0 = \frac{\partial F}{\partial t} = \sum_{i=1}^{n} 2 w_i (Rp_i}$$

Now, is d and $d \times 1$

$$\|Rp_i + t - q_i\|^2 = (p_i^T R^T + t^T - q_i^T)(Rp_i + t - q_i)$$

$$= p_i^T R^T R p_i + p_i^T R^T t - p_i^T R^T q$$
$$+ t^T R p_i + t^T t - t^T q_i$$
$$- q_i^T R p_i - q_i^T t + q_i^T q_i$$

Only the circled terms are dependent on 't'.

$$\therefore \frac{\partial F}{\partial t} = 2R\left(\sum_{i=1}^{n} w_i p_i\right) + 2t\left(\sum_{i=1}^{n} w_i\right) - 2\sum_{i=1}^{n} w_i q_i = 0$$

$$\Rightarrow t = \frac{\sum_{i=1}^{n} w_i q_i}{\sum_{i=1}^{n} w_i} - R\left(\frac{\sum_{i=1}^{n} w_i p_i}{\sum_{i=1}^{n} w_i}\right)$$

$\bar{q} \rightarrow$ centroid of pcl 2

$\bar{p} \rightarrow$ centroid of point cloud 1

$$\Rightarrow \boxed{t = \bar{q} - R\bar{p}} \quad \text{— (1)}$$

$\Rightarrow$ for a fixed rotation matrix 'R', the most optimal "T" is $\bar{q} - R\bar{p}$

Without loss of generality,
we can center both point clouds around their centroids i.e,

$$x_i := p_i - \bar{p} \quad \Big| \quad y_i := q_i - \bar{q} \qquad \text{—②}$$

∴ The problem can now be stated as:–

$$\left.\begin{aligned} R &= \underset{R\in(SO(d))}{\operatorname{argmin}} \sum_{i=1}^{n} w_i \left\| R(\bar{p}+x_i) + \bar{q} - R\bar{p} - (\bar{q}+y_i) \right\|^2 \\ &= \underset{R\in SO(d)}{\operatorname{argmin}} \sum_{i=1}^{n} w_i \left\| Rx_i - y_i \right\|^2 \end{aligned}\right\} \text{—③}$$

Now,

$$\|Rx_i - y_i\|^2 = \cancel{(Rx_i - y_i)^T} (x_i^T R^T - y_i^T)(Rx_i - y_i) \qquad [RR^T = I]$$

$$= x_i^T x_i + y_i^T y_i - \underbrace{x_i^T R^T y_i}_{\text{scalar}} - \underbrace{y_i^T R x_i}_{\text{scalar}}$$

$$\Rightarrow \boxed{\|Rx_i - y_i\|^2 = \underbrace{x_i^T x_i + y_i^T y_i}_{\text{Independent of } R} - 2y_i^T R x_i} \qquad \text{—④}$$

Using ③ and ④,

$$\left.\begin{aligned} R &= \underset{R\in SO(d)}{\operatorname{argmin}} \left(- \sum_{i=1}^{n} w_i y_i^T R x_i \right) \\ \Rightarrow R &= \underset{R\in SO(d)}{\operatorname{argmax}} \sum_{i=1}^{n} \left(w_i y_i^T R x_i \right) \end{aligned}\right\} \text{—⑤}$$

Let $W = \text{diag}(w_1, \dots, w_n) \rightarrow n \times n$ diagonal matrix with w_i as its diagonal entry i

$Y = d \times n$ matrix with y_i as its columns

$X = d \times n$ matrix with x_i as its columns

Then,

$$\underbrace{\begin{bmatrix} w_1 & & \\ & \ddots & \\ & & w_n \end{bmatrix}}_{n \times n} \overset{n \times 3}{\begin{bmatrix} \leftarrow y_1^T \rightarrow \\ \leftarrow y_2^T \rightarrow \\ \vdots \\ \leftarrow y_n^T \rightarrow \end{bmatrix}} \overset{3 \times 3}{\begin{bmatrix} R \end{bmatrix}} \overset{3 \times n}{\begin{bmatrix} \uparrow & \uparrow & & \uparrow \\ x_1 & x_2 & \cdots & x_n \\ \downarrow & \downarrow & & \downarrow \end{bmatrix}}$$

$$= \begin{bmatrix} w_1 y_1^T R x_1 & & \text{we do not care} \\ & \ddots & \\ & & w_n y_n^T R x_n \end{bmatrix}$$

$$= W Y^T R X \qquad \text{— (6)}$$

Now, $tr(WY^TRX) = \sum_{i=1}^{n} w_i y_i^T R x_i$ — (7)

Using (5), (6), (7)

$$\Rightarrow \boxed{R = \underset{R \in SO(d)}{\text{argmax}}\ tr(WY^TRX)}$$

$$\Rightarrow R = \underset{R \in SO(d)}{\text{argmax}}\ tr(RXWY^T) \qquad \text{— (8)}$$

Since $tr(AB) = tr(BA)$

Let us denote $d \times d$ covariance matrix $S = XWY^T$

Then, SVD of $S = U \Sigma V^T$ — (9)

U → orthogonal, Σ → diagonal, V^T → orthogonal

Using (8) and (9),

$R = \text{argmax}\left(RU\Sigma V^T\right)$

$\Rightarrow R = \text{argmax}\left(\Sigma V^T R U\right)$ — (10)

$V^T R U =$ orthogonal as $V^T R U U^T R^T V = I$

Let $V^T R U = M$ be orthogonal matrix

~~$\Rightarrow$ All entries m_{ij} of $M \leq 1$ is~~

Since $M =$ orthogonal

$\Rightarrow$ columns of M are orthonormal vector

$\Rightarrow$ Each $|m_{ij}| \leq 1$ — (11)

Using (10), (11);

$R = \underset{R \in SO(d)}{\text{argmax}}\left(\text{tr}(\Sigma M)\right)$

$$\Rightarrow R = \text{argmax}\begin{pmatrix}\sigma_1 & & & \\ & \sigma_2 & & \\ & & \ddots & \\ & & & \sigma_d\end{pmatrix}\begin{pmatrix}m_{11} & m_{12} & \cdots & \\ & & & \\ & & & m_{dd}\end{pmatrix}$$

$\Rightarrow R = \text{argmax}\left(\sum_{i=1}^{d} \sigma_i m_{ii}\right) \leq \sum_{i=1}^{d} \sigma_i$ [with maxima when all $m_{ii} = 1$]

$\Rightarrow M = I$

$\Rightarrow V^T R U = I$

$\Rightarrow \boxed{R = VU^T}$ ✓

orthogonal matrix

Now, orthogonal matrices include both rotation + reflections.

If $\det(VU^T) = +1$, then there is no reflection.

If $\det(VU^T) = -1$

$\Rightarrow$ we want $\det(M) = -1$

$\rightarrow$ we want 'M' such that $\det(M) = -1$

and $(\sigma_{11} m_{11} + \cdots + \sigma_d m_{dd})$ is maximized.

The set of all diagonal of rotation matrices of order 'n' is equal to:- convex hull of the points $(\pm 1, \pm 1, \ldots, \pm 1)$ with an even number of coordinates which are '-1'.

Now, for $\det(M) = -1$, we want odd number of '-1's.

Since we want to maximize and σ_d is smallest singular value, we want σ_{dd} to be -1.

$$\therefore M = \begin{pmatrix} 1 & & \\ & 1 & \\ & & -1 \end{pmatrix}$$

$$\Rightarrow \left. R = V \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & \det(VU^T) \end{pmatrix} U^T \right\} \rightarrow \text{Finally.}$$

Hence, this SVD based soln via the procrustes argument leads to least squared error.