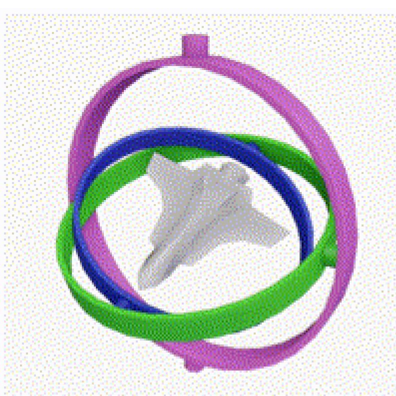
Q1) What is a Gimbal lock?
In case 2 axes (about which the rotation is capable of happening) get aligned exactly with each other, the problem of gimbal lock arises. In such a case, there is no way to rotate around one of the axes.
This problem becomes particularly evident while representing rotations using EULER angles as similar to the case of a gyroscope gimbal (where the inner gimbals move whenever the external gimbal moves), when we are using **INTRINSIC EULER ANGLES** (let's say Z-Y-X), then coordinate axes X and Y also rotate when Z rotates, thus displacing them from their original location and transporting them to a location where rotation has already taken place before.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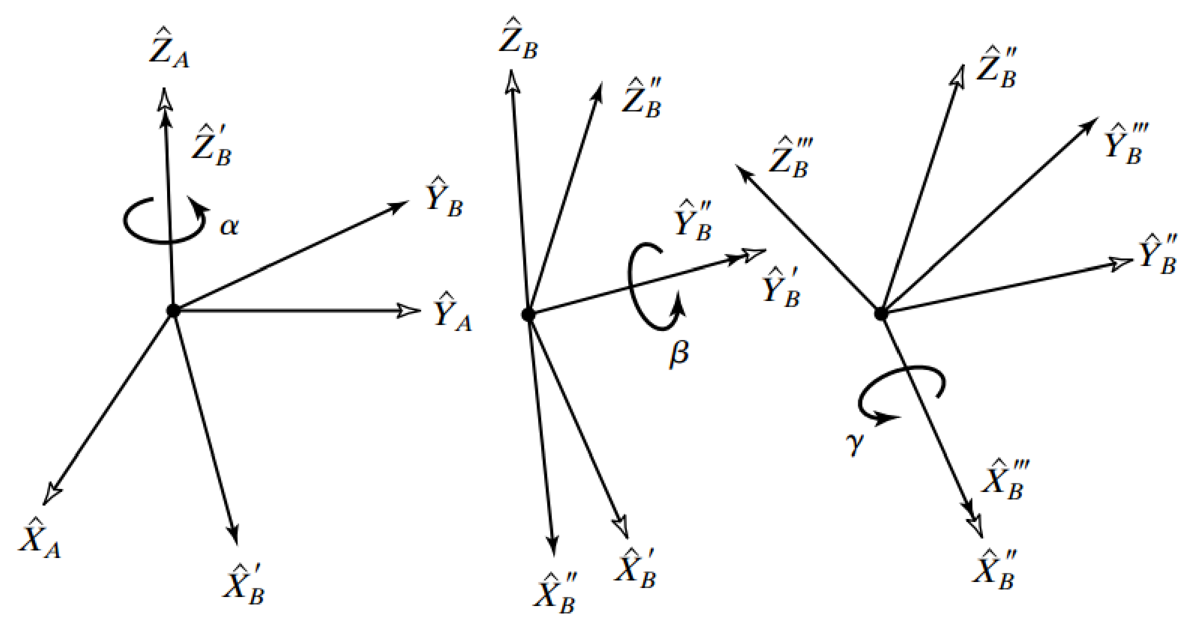


FIGURE 2.18: Z–Y–X Euler angles.

In case of a gimbal lock, it is NOT possible to uniquely determine the euler angles from the rotation matrix.
The gimbal lock leads to a loss in a degree of freedom.
I have elaborated more in the BETA = 90 degree case.

Q2) Why is β=90° a "certain problematic value"?
Consider an intrinsic rotation (X->Y->Z).
Then, the rotation matrix will be a multiplication of R_x, R_y and R_z.
Then,

$$R_x = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta_x & -\sin\theta_x \\ 0 & \sin\theta_x & \cos\theta_x \end{bmatrix} \quad R_y = \begin{bmatrix} \cos\theta_y & 0 & \sin\theta_y \\ 0 & 1 & 0 \\ -\sin\theta_y & 0 & \cos\theta_y \end{bmatrix} \quad R_z = \begin{bmatrix} \cos\theta_z & -\sin\theta_z & 0 \\ \sin\theta_z & \cos\theta_z & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

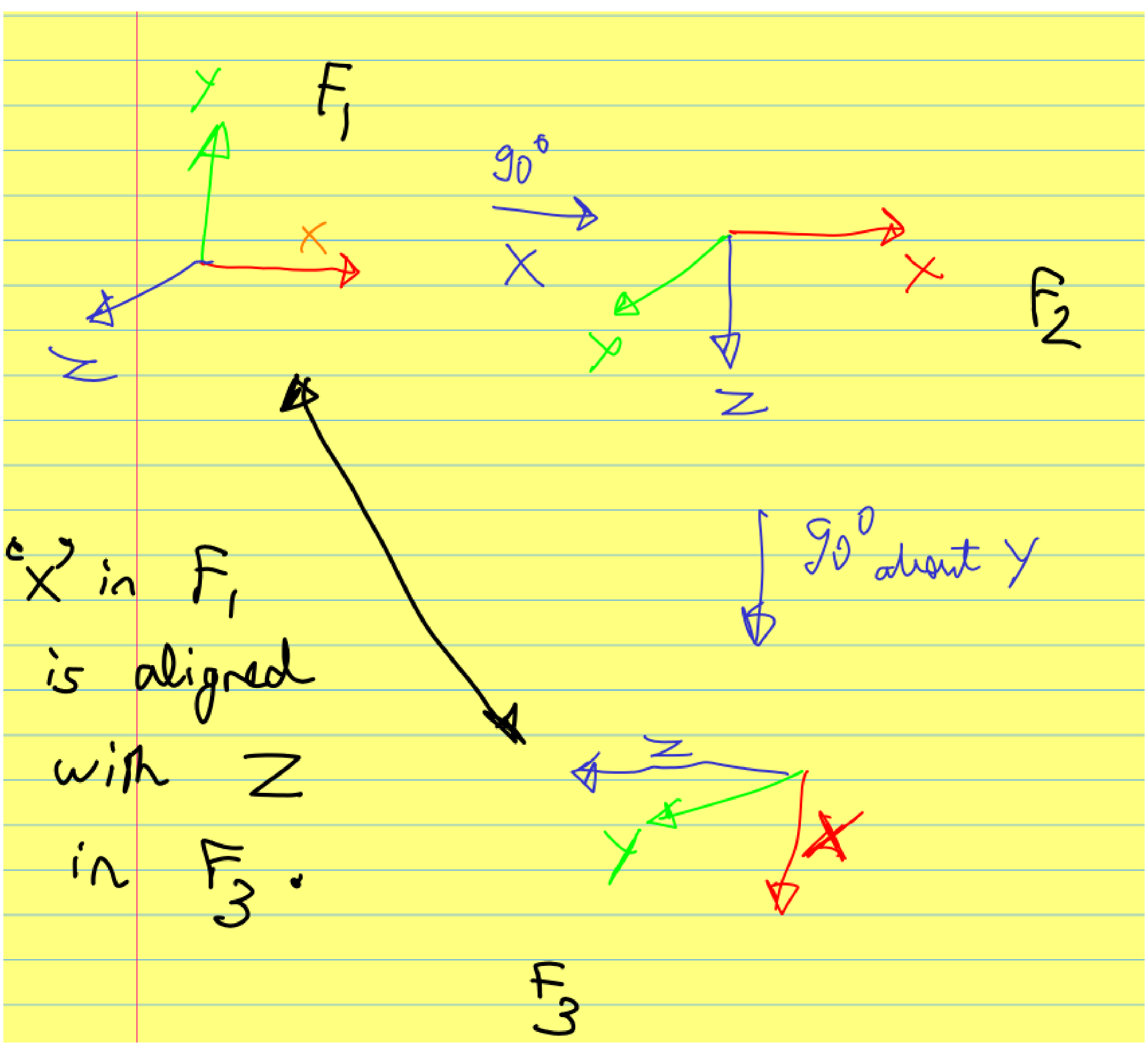
Final rotation matrix will be (in the case when θ_y=90 degrees):

$$\begin{bmatrix} 0 & 0 & 1 \\ \cos\theta_z \sin\theta_x + \cos\theta_x \sin\theta_z & \cos\theta_x \cos\theta_z - \sin\theta_x \sin\theta_z & 0 \\ -\cos\theta_x \cos\theta_z + \sin\theta_x \sin\theta_z & \cos\theta_z \sin\theta_x + \cos\theta_x \sin\theta_z & 0 \end{bmatrix}$$

This simplifies to:

$$\begin{bmatrix} 0 & 0 & 1 \\ sin(\theta_x + \theta_z) & cos(\theta_x + \theta_z) & 0 \\ -cos(\theta_x + \theta_z) & sin(\theta_x + \theta_z) & 0 \end{bmatrix}$$

Here, no matter how we adjust θ_x and θ_z, we will be able to rotate in one plane only.
So, in addition to the previous rotation about the "y" axis: we have just one more DOF, leading to a total of 2 DOFs.
It is for the same reason, that in this case, the EULER angles cannot be uniquely determined from the rotation matrix. In such a case, one can only determine the value of (θ_x + θ_z) but will not be able to derive the individual values uniquely.



Q2) Part 2

Let R_x, R_y, R_z be rotation matrices about R_x, R_y, R_z respectively:

$$R_x(\theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}$$

$$R_y(\theta) = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}$$

$$R_z(\theta) = \begin{bmatrix} \cos(\theta) & -\sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$P_0 \; T_0 \; 0_0$

Since rotation convention hasn't been mentioned in the question,

I will present my working with both the conventions :-

1st convention ?

Convention I : (ZYX intrinsic)

Let A be the initial frame and B be the frame after rotation.

Then,

$$ {}^{A}_{B}R_{ZYX} = R_Z(\alpha)\, R_Y(\beta)\, R_X(\gamma) $$

$$ \Rightarrow {}^{A}_{B}R_{ZYX} = \begin{bmatrix} c_\alpha c_\beta & c_\alpha s_\beta s_\gamma - s_\alpha c_\gamma & c_\alpha s_\beta c_\gamma + s_\alpha s_\gamma \\ s_\alpha c_\beta & s_\alpha s_\beta s_\gamma + c_\alpha c_\gamma & s_\alpha s_\beta c_\gamma - c_\alpha c_\gamma \\ -s_\beta & c_\beta s_\gamma & c_\beta c_\gamma \end{bmatrix} $$

When $\beta = 90^\circ$,

$$ \Rightarrow R^{A}_{B\,ZYX} = \begin{bmatrix} 0 & -\sin(\alpha-\gamma) & \cos(\alpha-\gamma) \\ 0 & \cos(\alpha-\gamma) & \sin(\alpha-\gamma) \\ -1 & 0 & 0 \end{bmatrix} $$

${}^{A}R_{gimbB}$

Let $\alpha - \gamma = \theta$

$$ \Rightarrow R^{A}_{B}\,ZYX = \begin{bmatrix} 0 & -\sin\theta & \cos\theta \\ 0 & \cos\theta & \sin\theta \\ -1 & 0 & 0 \end{bmatrix} $$

P.T.O.

Q2)

a) $$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 & -\sin(\theta) & \cos(\theta) \\ 0 & \cos(\theta) & \sin(\theta) \\ -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -\sin(\theta) \\ \cos(\theta) \\ 0 \end{bmatrix}$$

$\Rightarrow \theta = -90^0$ works

$\Rightarrow \gamma - \alpha = 90^0$ works

b) Let p_{new} lie on x-y unit circle

$\Rightarrow p_{new}$ can be parametrized as $(\cos t, \sin t)$

i.

$$\begin{pmatrix} \cos t \\ \sin(t) \\ 0 \end{pmatrix} = \begin{bmatrix} 0 & -\sin(\theta) & \cos\theta \\ 0 & \cos(\theta) & \sin(\theta) \\ -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix},$$

$$\Rightarrow \begin{bmatrix} \cos(t) \\ \sin(t) \\ 0 \end{bmatrix} = \begin{bmatrix} -\sin(\theta) \\ \cos(\theta) \\ 0 \end{bmatrix}$$

Now, $\sin(t) = \cos(\theta)$

and $\cos(t) = -\sin(\theta)$

$\Rightarrow \theta = -90^0 - t$

$\Rightarrow$ There does exist some (α, γ) for each point on XY unit circle such that it can be mapped to $\left[\begin{smallmatrix} 1 \\ 0 \\ 0 \end{smallmatrix}\right]$.

P.T.O.

P = (0, cos t, sin t

c) $$\begin{bmatrix} 0 \\ \cos(t) \\ \sin(t) \end{bmatrix} = \begin{bmatrix} -\sin(\theta) \\ \cos(\theta) \\ 0 \end{bmatrix}$$

$\Rightarrow$ again a soln exists: $\theta = 0^\circ$ works

$\Rightarrow$ when $t = 0, 180^\circ$

$$\Rightarrow \underbrace{P_{new} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \text{ and } \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}}$$

and $\boxed{\gamma = \alpha}$

This is because here, when $\beta = 90^\circ$, the Z-axis's rotation and the 'x' axis rotation take place along the same real world axis and are able to compensate for each other.

P.T.O.

2nd convention ?

b) Convention 2: XYZ intrinsic

$$\Rightarrow {}^{A}_{B}R_{XYZ} = R_X(\alpha)\, R_Y(\beta)\, R_Z(\gamma)$$

[Details in my doc pdf]

When $\beta = 90^\circ$

$$\Rightarrow {}^{A}_{B}R_{XYZ} = \begin{bmatrix} 0 & 0 & 1 \\ \sin(\alpha+\gamma) & \cos(\alpha+\gamma) & 0 \\ -\cos(\alpha+\gamma) & \sin(\alpha+\gamma) & 0 \end{bmatrix}$$

Let $\theta = (\alpha+\gamma)$

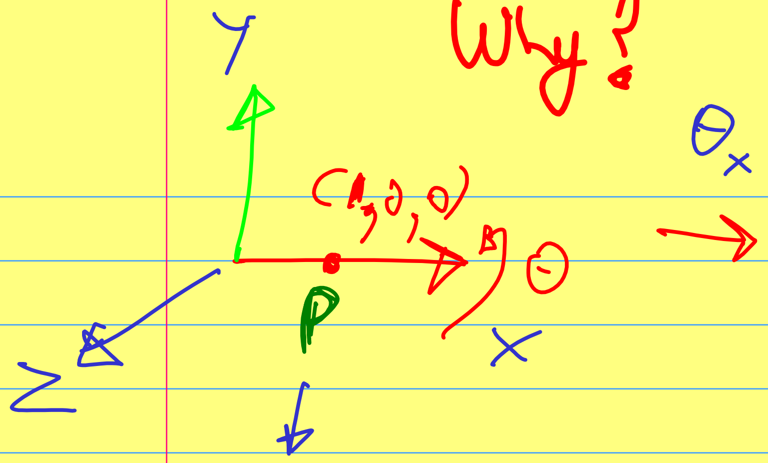
a) $$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ \sin(\theta) & \cos(\theta) & 0 \\ -\cos\theta & \sin(\theta) & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

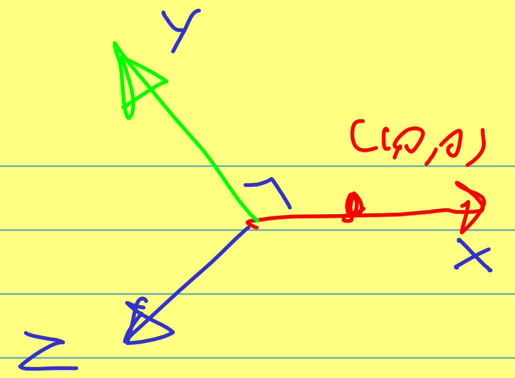
$$\Rightarrow \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ \cos\theta \\ \sin\theta \end{bmatrix}$$

$\Rightarrow$ No soln exists.

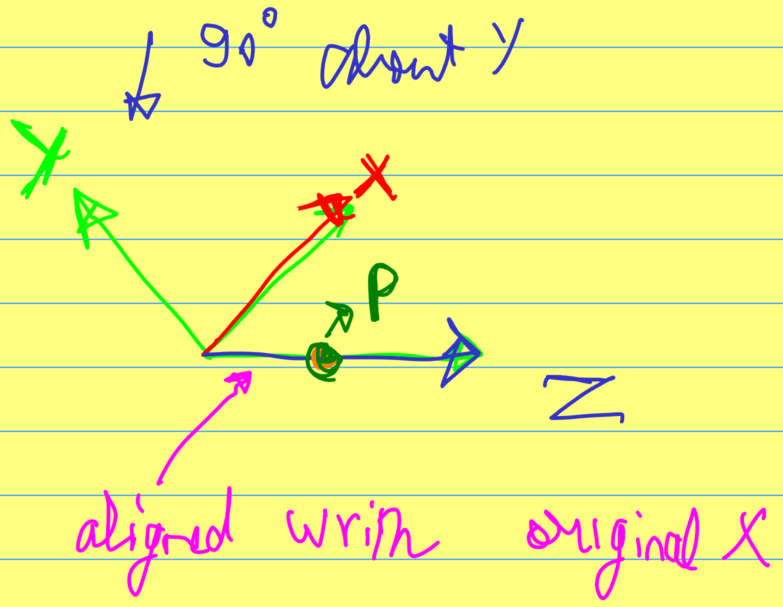
P. T. O.

Why?





Point is fixed in space and is not moving.



'p' lies on the axis around which the rotation is happening

⟹ there is no possible rotation about 'z' which can make Y ⟶ lie on P.

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b) $\begin{bmatrix} \cos t \\ \sin t \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ \cos(\theta) \\ \sin(\theta) \end{bmatrix} \Rightarrow$ Soln. exists

c) $\begin{bmatrix} 0 \\ \cos(t) \\ \sin(t) \end{bmatrix} = \begin{bmatrix} 0 \\ \cos(\theta) \\ \sin(\theta) \end{bmatrix} \Rightarrow$ Soln exists

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Q.T.D.