COMPUTATIONAL HEAT AND FLUID FLOW (ME605)

Assignment 2: Finite Volume Method

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Assignment 02 ME605

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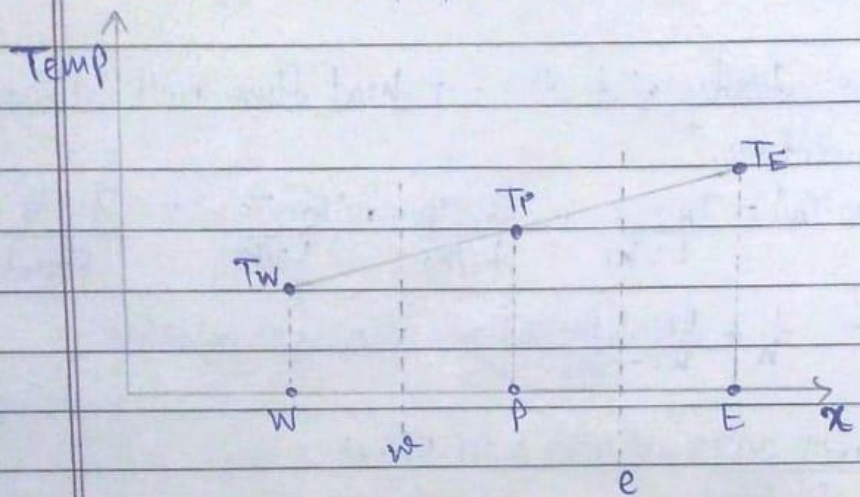
For all the questions that follows, we have used the following profile assumption for the temperature inside the control volume.

Piecewise linear profile



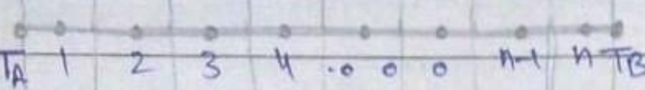
And for the cases, where there is a source term present, we use Negative-slope linearization ⇒

$$S = S_u + S_p T_p \quad ; \quad S_p \leq 0$$

By employing the above stated profile assumptions & the negative-slope linearization for ~~___~~ source term, we move forward & discretize the given problems & solve them using TDMA algorithm.

Ques 1

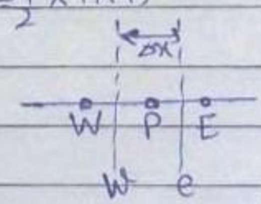
i)



The entire domain is divided into (n-1) equal intervals with the intervals between successive grid points, from 1 to n being of length Δx & intervals between A & 1 and n & B, being of length $\Delta x/2$. The boundaries of the control volumes are located at the mid-point of each grid-length.

ii) The discretization is done using finite volume method with each element being of volume $(\Delta x \times 1 \times 1)$ for interior & the boundary elements being of volume $(\frac{\Delta x}{2} \times 1 \times 1)$

for interior points 2, 3, ⋯, (n-2), (n-1)



$$\int_w^e \frac{d}{dx}\left(k\frac{dT}{dx}\right)dx = 0 \Rightarrow \left(\frac{kdT}{dx}\right)_e - \left(\frac{kdT}{dx}\right)_w$$

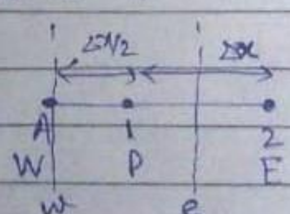
$$K_eA_e\left(\frac{T_E - T_P}{(\delta x)_e}\right) - K_wA_w\left(\frac{T_P - T_W}{(\delta x)_w}\right) = 0$$

$$\left(\frac{K_eA_e}{(\delta x)_e} + \frac{K_wA_w}{(\delta x)_w}\right)T_P = \frac{K_eA_e}{(\delta x)_e}T_E + \frac{K_wA_w}{(\delta x)_w}T_W$$

$$a_P = a_W + a_E \quad , a_E = \frac{K_eA_e}{(\delta x)_e} \quad , a_W = \frac{K_wA_w}{(\delta x)_w}$$

$$K_e = K_w = K \;\&\; (\delta x)_e = (\delta x)_w = \delta x \;\&\; A_w = A_e = A$$

iii) For boundary point 1.



$$\int_w^e \frac{d}{dx}\left(\frac{KdT}{dx}\right)dx = 0$$

$$\Rightarrow \left(\frac{KdT}{dx}\right)_e - \left(K\frac{dT}{dx}\right)_w = 0$$

$$\Rightarrow K_eA_e\left(\frac{dT}{dx}\right)_e - K_wA_w\left(\frac{dT}{dx}\right)_w = 0$$

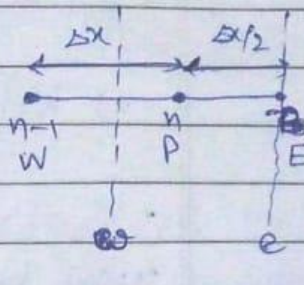
$$\Rightarrow K_eA_e\left(\frac{T_E - T_P}{(\delta x)_e}\right) - K_wA_w\left(\frac{T_P - T_A}{(\delta x/2)_w}\right) = 0$$

$$\Rightarrow \left(\frac{2K_wA_w}{(\delta x)_w} + \frac{K_eA_e}{(\delta x)_e} \right) T_P = \frac{K_eA_e}{(\delta x)_e} T_E + (0) T_W + \frac{2K_wA_w}{(\delta x)_w} T_A$$

$$a_e = \frac{K_eA_e}{(\delta x)_e} \quad a_w = 0 \quad S_u = \frac{2K_wA_w}{(\delta x)_w} T_A \quad S_p = -\frac{2K_wA_w}{(\delta x)_w}$$

$$a_p = a_e + a_w - S_p$$

For boundary point n:



$$\int_w^e \frac{d}{dx}\left(K\frac{dT}{dx}\right) dV = 0$$

$$\Rightarrow \left(\frac{KdTA}{dx}\right)_e - \left(KA\frac{dT}{dx}\right)_w$$

$$\Rightarrow K_eA_e\left(\frac{T_B - T_P}{(\delta x/2)_e}\right) - K_wA_w\left(\frac{T_P - T_W}{(\delta x)_w}\right) = 0$$

$$\left(\frac{2K_eA_e}{(\delta x)_e} + \frac{K_wA_w}{(\delta x)_w}\right) T_P = (0)T_E + \left(\frac{K_wA_w}{(\delta x)_w}\right) T_W + \frac{2K_eA_e}{(\delta x)_e} T_B$$

$$a_e = 0, \quad a_w = \frac{K_wA_w}{(\delta x)_w}, \quad S_u = \frac{2K_eA_e}{(\delta x)_e} T_B \quad S_p = -\frac{2K_eA_e}{(\delta x)_e}$$

$$a_p = a_e + a_w - S_p$$

$A_w = A_e = A$ + $(\delta x)_e = (\delta x)_w = \delta x$ & $K_e = K_w = K$

let $\frac{KA}{\delta x} = \beta$

for internal points: $2, 3, \dots, (n-1)$: $a_p = 2\beta$, $a_e = \beta = a_w$

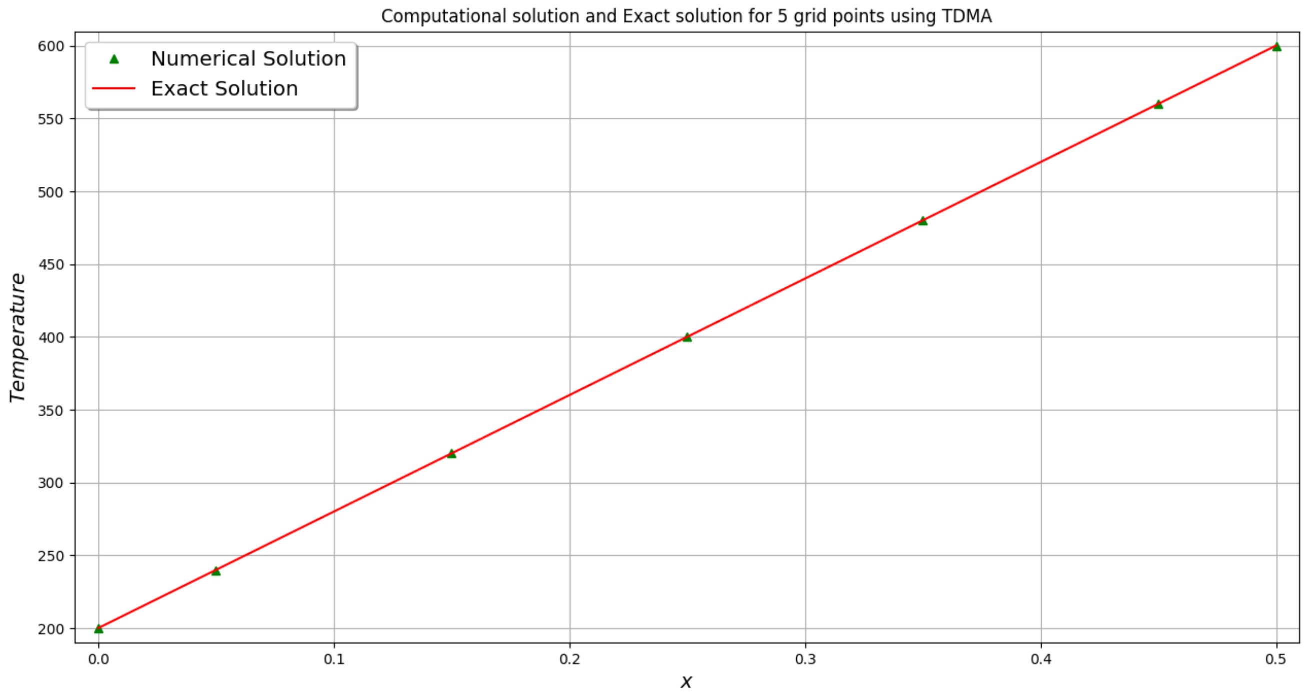
for boundary point 1: $a_p = 3\beta$, $a_e = \beta$, $a_w = 0$, $S_u = 2\beta T_A$, $S_p = -2\beta$

for boundary point n: $a_p = 3\beta$, $a_e = 0$, $a_w = \beta$, $S_u = 2\beta T_B$, $S_p = -2\beta$

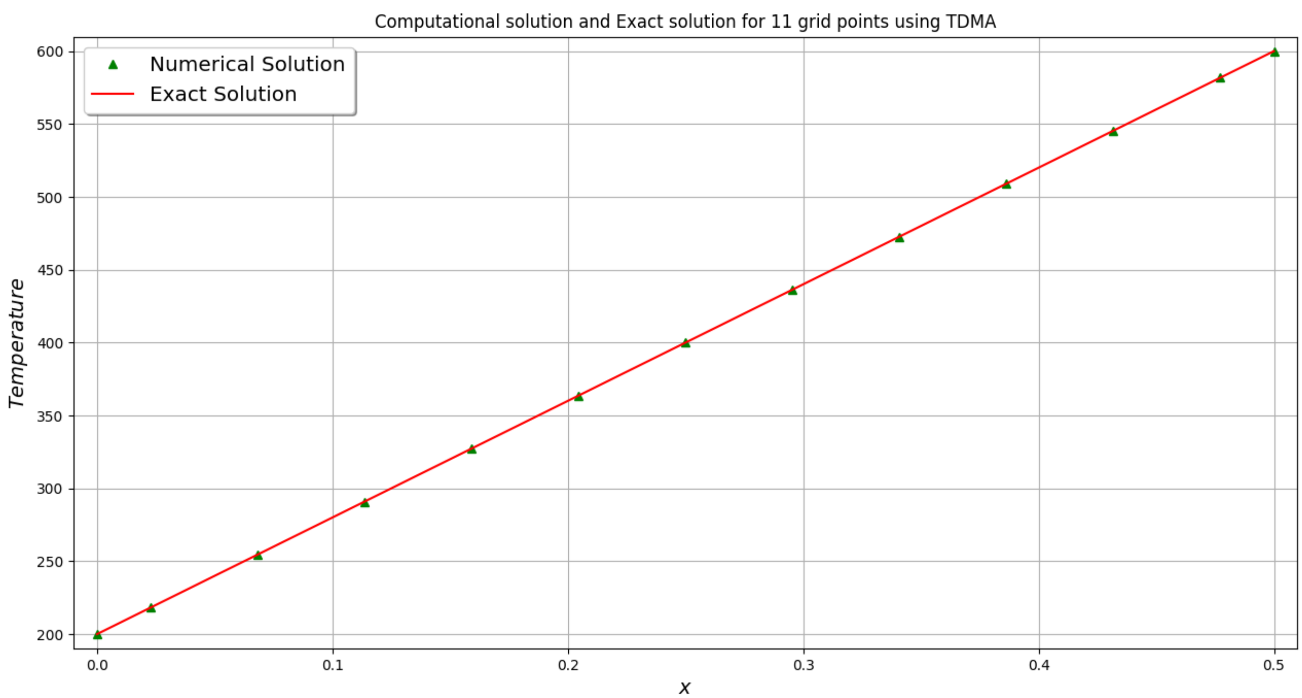
$$\begin{bmatrix} 3\beta & -\beta & 0 & \cdots & 0 \\ -\beta & 2\beta & -\beta & & \\ 0 & -\beta & 2\beta & & \\ 0 & & & \ddots & -\beta \\ \vdots & & & -\beta & 2\beta \\ 0 & \cdots & & & \end{bmatrix} \begin{bmatrix} T_1 \\ T_2 \\ \vdots \\ \\ T_n \end{bmatrix} = \begin{bmatrix} 2\beta T_A \\ 0 \\ 0 \\ \vdots \\ 2\beta T_B \end{bmatrix}$$

v) Required output (plots/any other means)

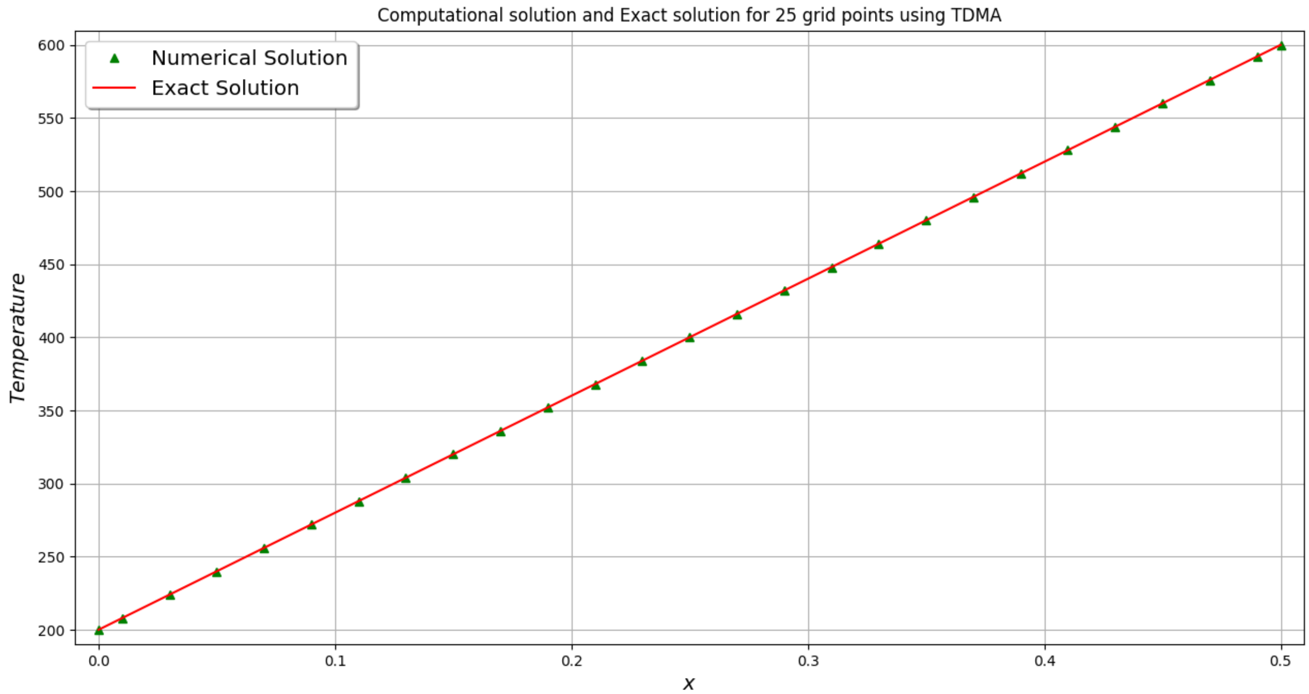
1. 5 grid points



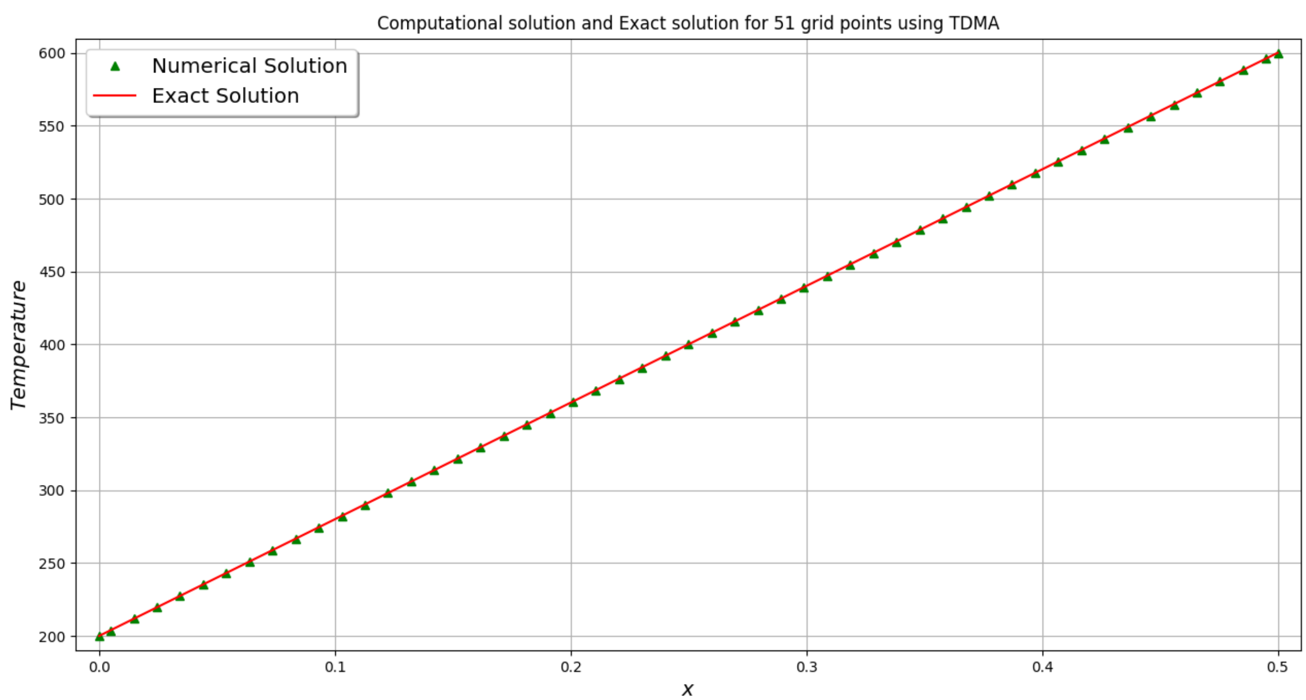
2. 11 grid points



3. 25 grid points



4. 51 grid points

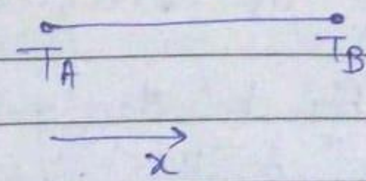


vi) Analysis of results

Deriving the exact Solution.

Using the one dimensional, steady state, no heat generation form of heat diffusion equation:

$$\frac{d}{dx}\left(K\frac{dT}{dx}\right) = 0$$



$$\Rightarrow K\frac{dT}{dx} = C$$

$$\Rightarrow T = C_1 x + C_2$$

using boundary conditions,

i) $x = 0,\ T = T_A$

$$\Rightarrow T_A = C_2$$

$$\Rightarrow T = C_1 x + T_A$$

ii) $x = L,\ T = T_B$

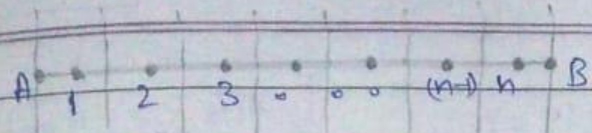
$$T_B = C_1 L + T_A$$

$$\Rightarrow C_1 = \frac{T_B - T_A}{L}$$

$$\Rightarrow \boxed{T = 800x + 200}$$

From the above plots and the analytical solution of the problem, we can conclude that as the number of the grid points are increased, the numerical solution approaches the exact solution and the error between the two decreases. We obtain a linear relationship between the temperature and length, as there is no heat generation and the system is in steady state and one dimensional.

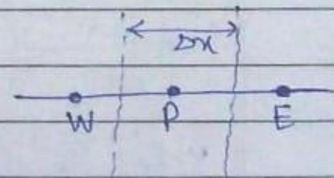
Ques 2 i)



The entire domain is divided into (n-1) equal intervals with the intervals between successive grid points, from 1 to n being of length Δx & intervals between A & 1 and n & B being of length $\Delta x/2$. The boundaries of the control volume are located at the mid-point of each grid-length.

ii) The discretization is done using finite volume method with each element being of volume $(\Delta x \times 1 \times 1)$ for interior & the boundary elements being of volume $(\Delta x/2 \times 1 \times 1)$. In this problem, we have an additional term of heat generation, therefore we will have to capture its effect in each control volume. The heat generation term can be treated as a source term and then integrated in the governing equation.

For interior points : 2, 3, ..., (n-1)



$$\int_w^e \frac{d}{dx}\left(K\frac{dT}{dx}\right)dV + \int_w^e \dot{q}\,dV = 0$$

$$\Rightarrow \frac{K_e A_e}{(\delta x)_e}\left(\frac{dT}{dx}\right)_e - \frac{K_w A_w}{(\delta x)_w}\left(\frac{dT}{dx}\right)_w + \dot{q}A\delta x = 0$$

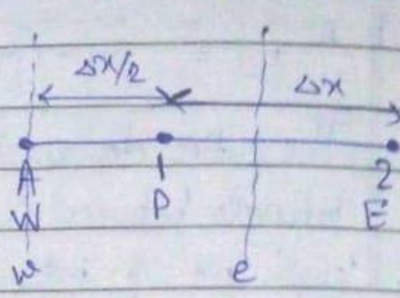
$$\frac{K_e A_e}{(\delta x)_e}(T_E - T_P) - \frac{K_w A_w}{(\delta x)_w}(T_P - T_W) + \dot{q}A\,\delta x = 0$$

$$\left(\frac{K_e A_e}{(\delta x)_e} + \frac{K_w A_w}{(\delta x)_w}\right)T_P = \left(\frac{K_e A_e}{(\delta x)_e}\right)T_E + \left(\frac{K_w A_w}{(\delta x)_w}\right)T_W + \dot{q}A\delta x$$

$$a_e = \frac{K_e A_e}{(\delta x)_e},\ a_w = \frac{K_w A_w}{(\delta x)_w},\ S_u = \dot{q}A\delta x\ ,\ S_p = 0\ ,\ a_p = a_e + a_w - S_p$$

$$K_e = K_w = K \ \& \ (\delta x)_e = (\delta x)_w = \delta x \ \& \ A_w = A_e = A$$

iii) For boundary point 1



$$\int_w^e \frac{d}{dx}\left(K\frac{dT}{dx}\right)dV + \int_w^e \dot{q}\,dV = 0$$

$$\left(AK\frac{dT}{dx}\right)_e - \left(AK\frac{dT}{dx}\right)_w + \dot{q}A\delta x = 0$$

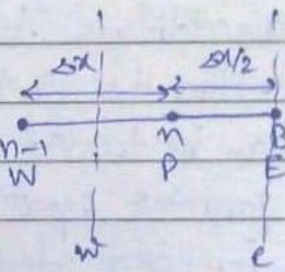
$$A_eK_e\left(\frac{T_E - T_P}{(\delta x)_e}\right) - A_wK_w\left(\frac{T_P - T_A}{(\delta x/2)_w}\right) + \dot{q}A\delta x = 0$$

$$\left(\frac{A_eK_e}{(\delta x)_e} + \frac{2K_wA_w}{(\delta x)_w}\right)T_P = \frac{A_eK_e}{(\delta x)_e}T_E + (0)T_W + \dot{q}A\delta x + \frac{2K_wA_w}{(\delta x)_w}T_A$$

$$a_e = \frac{A_eK_e}{(\delta x)_e};\ a_w = 0;\ a_p = a_e + a_w - S_p;\ S_u = \dot{q}A\delta x + \frac{2k_wA_w}{(\delta x)_w}T_A$$

$$S_p = -\frac{2k_wA_w}{(\delta x)_w}$$

For boundary point n



$$\int_w^e \frac{d}{dx}\left(K\frac{dT}{dx}\right)dV + \int_w^e \dot{q}\,dV = 0$$

$$\left(KA\frac{dT}{dx}\right)_e - \left(KA\frac{dT}{dx}\right)_w + \dot{q}A\delta x = 0$$

$$K_eA_e\left(\frac{T_B - T_P}{(\delta x/2)_e}\right) - K_wA_w\left(\frac{T_P - T_W}{(\delta x)_w}\right) + \dot{q}A\delta x = 0$$

$$\left(\frac{2K_eA_e}{(\delta x)_e} + \frac{K_wA_w}{(\delta x)_w}\right)T_P = T_E(0) + T_W\left(\frac{K_wA_w}{(\delta x)_w}\right) + \dot{q}A\delta x + \frac{2K_eA_e}{(\delta x)_e}T_B$$

$$a_e = 0;\ a_w = \frac{K_wA_w}{(\delta x)_w};\ a_p = a_e + a_w - S_p;\ S_u = \dot{q}A\delta x + \frac{2K_eA_e}{(\delta x)_e}T_B$$

$$S_p = -\frac{2k_eA_e}{(\delta x)_e}$$

$A_w = A_e = A$ & $(\delta x)_e = (\delta x)_w = (\delta x)$ & $K_e = K_w = K$

Let $KA/\delta x = \beta$ & $\dot{q}A\delta x = \alpha$

for interior points : $a_e = a_w = \beta$; $a_p = 2\beta$; $S_u = \alpha$; $S_p = 0$

for boundary point 1: $a_e = \beta$; $a_w = 0$; $S_u = \alpha + 2\beta T_A$; $S_p = -2\beta$; $a_p = \beta + 2\beta$

for boundary point n: $a_e = 0$; $a_w = \beta$; $S_u = \alpha + 2\beta T_B$; $S_p = -2\beta$; $a_p = 3\beta$

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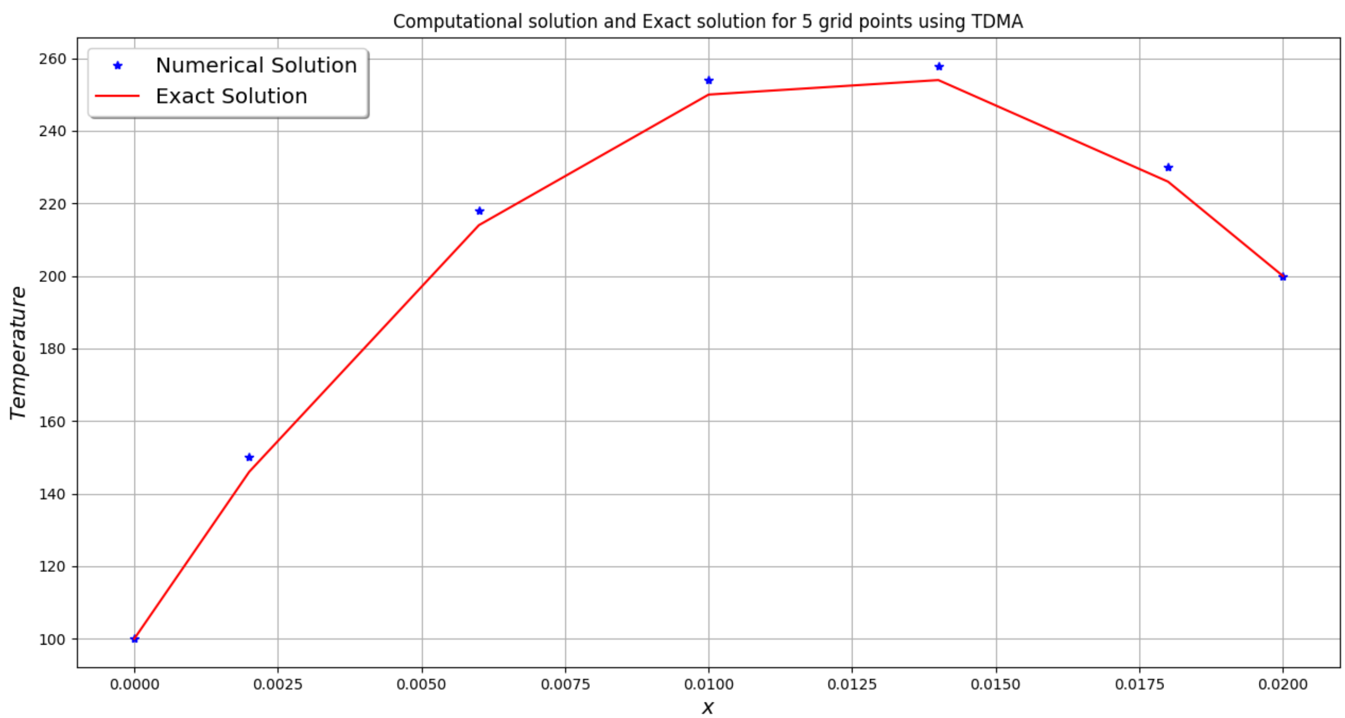
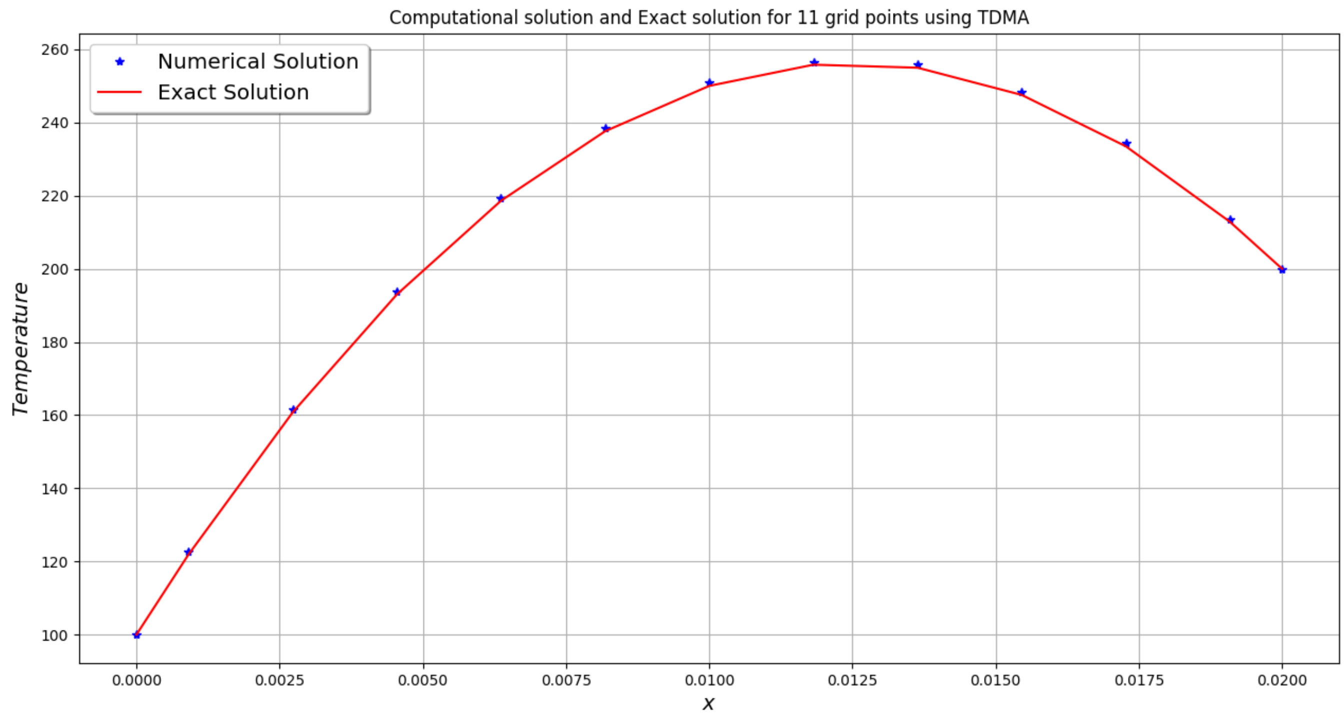
$$\begin{bmatrix} 3\beta & -\beta & 0 & 0 & \cdots & 0 & 0 & 0 \\ -\beta & 2\beta & -\beta & & & & & \\ 0 & -\beta & 2\beta & & & & & \\ 0 & & -\beta & \ddots & & & & \\ 0 & & & & \ddots & & & \\ \vdots & & & & & & & -\beta \\ 0 & 0 & \cdots & & & & -\beta & 2\beta \end{bmatrix} \begin{bmatrix} T_1 \\ T_2 \\ \vdots \\ \\ \\ \\ T_n \end{bmatrix} = \begin{bmatrix} \alpha + 2\beta T_A \\ \alpha \\ \alpha \\ \vdots \\ \\ \alpha \\ \alpha + 2\beta T_B \end{bmatrix}$$

v) Required output (plots/any other means)

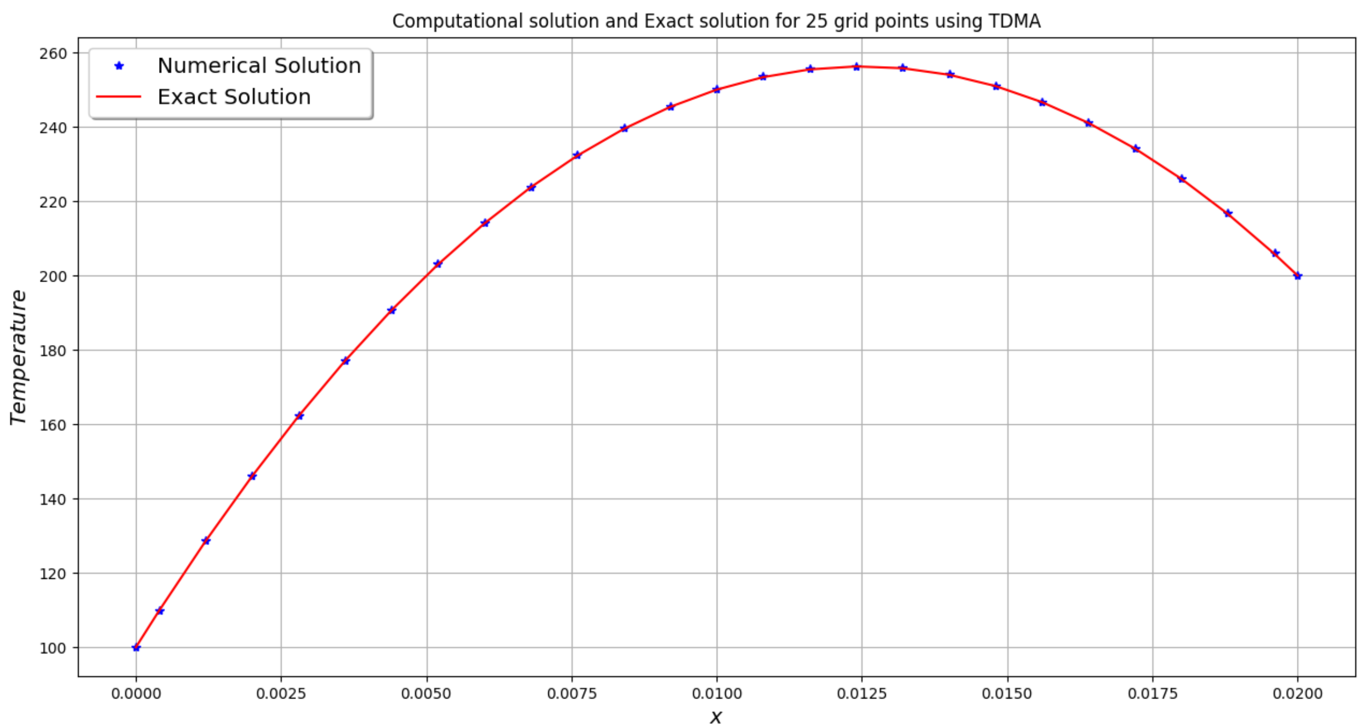
1. 5 grid points



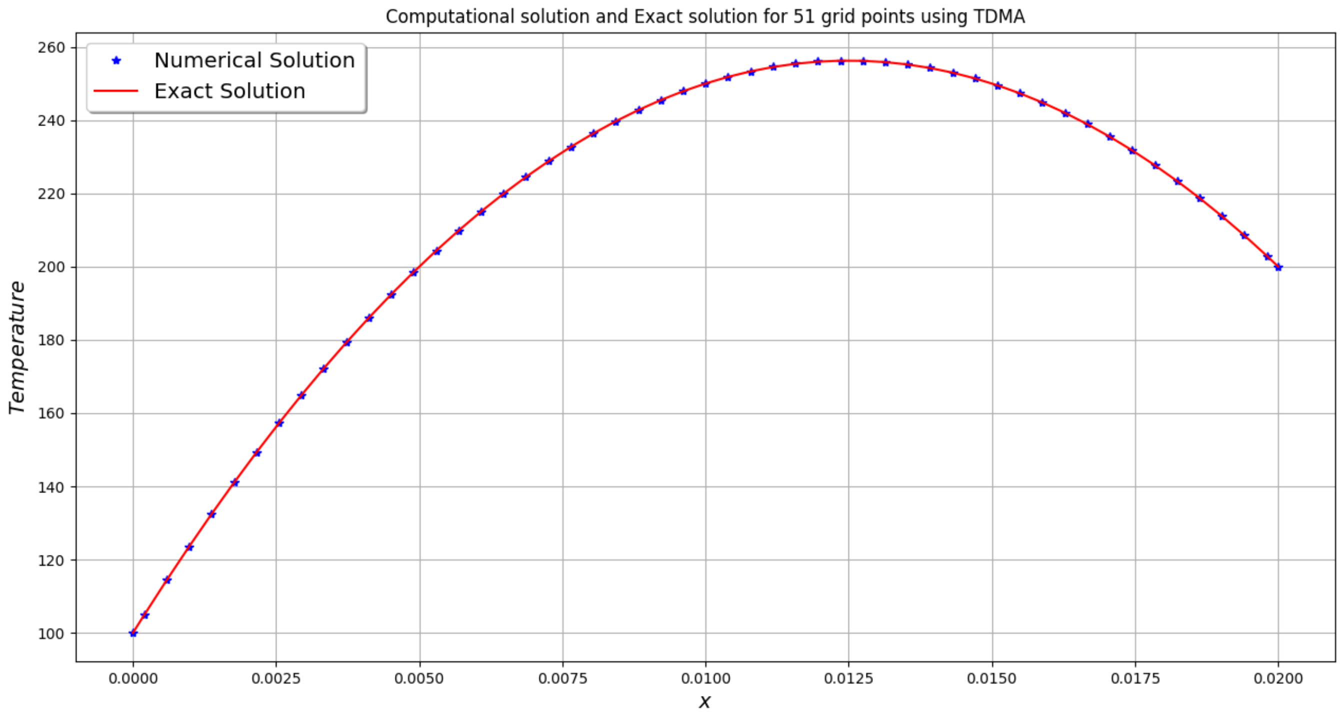
2. 11 grid points



3. 25 grid points

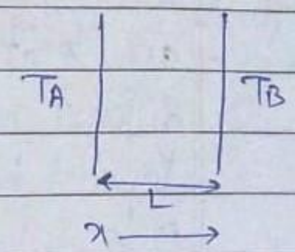


4. 51 grid points



vi) Analysis of results

Deriving the exact solution



Using the one-dimensional, steady state with heat generation form of head diffusion equation:

$$\frac{d}{dx}\left(k\frac{dT}{dx}\right) + \dot{q} = 0$$

$$k\frac{dT}{dx} = -\dot{q}x + C_1 \quad \Rightarrow \quad T = \frac{-\dot{q}x^2}{2K} + C_1x + C_2$$

using boundary conditions $T = T_A$ @ $x=0$ & $T = T_B$ @ $x = L$

$$T = T_A + (T_B - T_A)\frac{x}{L} + \frac{\dot{q}}{2K} Lx\left(1 - \frac{x}{L}\right)$$

$$T = 200 + 5000x + 40x(1 - 25x)$$

From the above plots and the analytical solution of the problem, we can conclude that as the number of the grid points are increased, the numerical solution approaches the exact solution and the error between the two decreases. We obtain a quadratic relationship between the temperature and length, as there is heat generation and the system is in steady state and one dimensional. Since we have considered the temperature profile to be linear while in reality it is quadratic and this assumption also incurs some error in the final answer.

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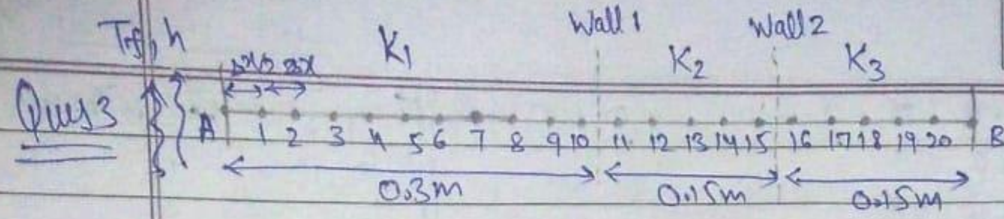
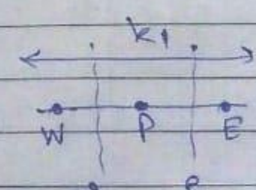
Ques 3



i) The entire domain divided into 19 equal intervals with the intervals between successive grid points from 1 to 20 being of length $\Delta x = 0.03\,m$ & intervals between A & 1 and B & 20 being of length 0.015m. The boundaries of each control volume are located at the center of two grid points. Here, we have different materials with varying values of K. The first wall is located between grid 10 and grid 11 while the second wall is located between grid 15 and grid 16.

ii) The discretization is done using finite volume method with each element being of volume $(\Delta x \times 1 \times 1)$ for all the interior point, which are points that are not located at the boundary, or aren't adjacent to a wall, and the boundary elements being of volume $(\Delta x/2 \times 1 \times 1)$. In this problem, we have a convective effect at one end namely at $x=0$, therefore we will have to capture its effect in the first control volume.



for interior points: 2, 3, ..., 8, 9

$$\int_w^e \frac{d}{dx}\left(K_1 \frac{dT}{dx}\right) dV = 0$$

$$\Rightarrow \left(K_1 A \frac{dT}{dx}\right)_e - \left(K_1 A \frac{dT}{dx}\right)_w = 0$$

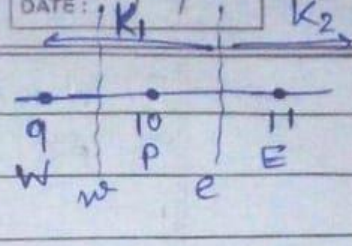
$$\Rightarrow K_{1e} A_{1e} \left(\frac{T_E - T_P}{(\delta x)_e}\right) - K_{1w} A_{1w} \left(\frac{T_P - T_W}{(\delta x)_w}\right) = 0$$

$$T_P\left(\frac{K_{1e} A_{1e}}{(\delta x)_e} + \frac{K_{1w} A_{1w}}{(\delta x)_w}\right) = \frac{A_{1e} K_{1e}}{(\delta x)_e} T_E + \frac{A_{1w} K_{1w}}{(\delta x)_w} T_W$$

$$a_e = \frac{A_{1e} K_{1e}}{(\delta x)_e} \; ; \; a_w = \frac{A_{1w} K_{1w}}{(\delta x)_w} \; ; \; a_P = a_e + a_w - S_P \; ; \; S_P = 0$$

$A_{1e} = A_{1w} = A$ & $K_{1e} = K_{1w} = K_1$ & $(\delta x)_e = (\delta x)_w = \delta x$

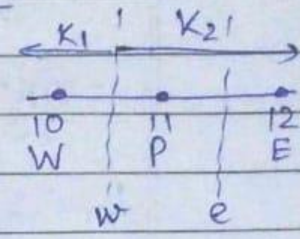
for point 10:



$$T_P\left(\frac{K_e A_e}{(\delta x)_e} + \frac{K_1 A_w}{(\delta x)_w}\right) = \frac{K_e A_e}{(\delta x)_e} T_E + \frac{K_w A_w}{(\delta x)_w} T_W$$

K_e can be : i) $\frac{2K_1K_2}{K_1+K_2}$ ii) $\frac{K_1+K_2}{2}$

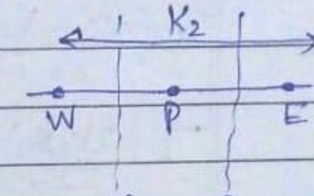
for point 11:



$$T_P\left(\frac{K_2 A}{\delta x} + \frac{K_w A}{\delta x}\right) = \frac{K_2 A}{\delta x} T_E + \frac{K_w A}{\delta x} T_W$$

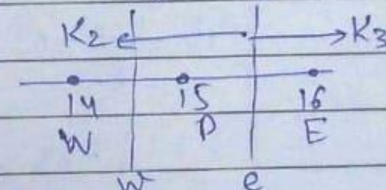
K_w can be: i) $\frac{2K_1K_2}{K_1+K_2}$ ii) $\frac{K_1+K_2}{2}$

for interior points 12, 13, 14 :



$$T_P\left(\frac{2K_2 A}{\delta x}\right) = \frac{AK_2}{\delta x} T_E + \frac{AK_2}{\delta x} T_W$$

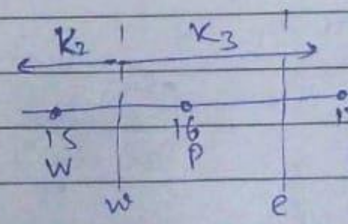
for point 15:



$$T_P\left(\frac{K_e A}{\delta x} + \frac{K_2 A}{\delta x}\right) = \frac{K_e A}{\delta x} T_E + \frac{K_2 A}{\delta x} T_W$$

K_e can be : i) $\frac{2k_2k_3}{k_2+k_3}$ ii) $\frac{k_2+k_3}{2}$

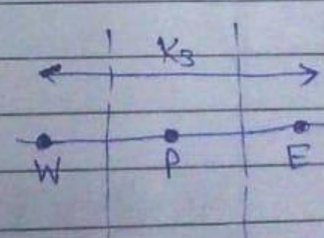
for point 16:



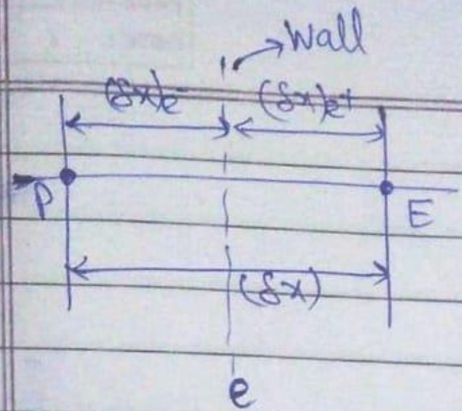
$$T_P\left(\frac{k_3 A}{\delta x} + \frac{K_w A}{\delta x}\right) = \frac{K_3 A}{\delta x} T_E + \frac{K_w A}{\delta x} T_W$$

K_w can be : i) $\frac{2k_2k_3}{k_2+k_3}$ ii) $\frac{K_2+k_3}{2}$

for interior points 17, 18, 19 :



$$T_P\left(\frac{2k_3 A}{\delta x}\right) = \frac{AK_3}{\delta x} T_E + \frac{AK_3}{\delta x} T_W$$



$$K_e = f_e K_P + (1-f_e) K_E$$

where $f_e = \dfrac{(\delta x)_{e^+}}{(\delta x)} = \dfrac{(\delta x)_{e^-}}{(\delta x)}$

From the above expressions, we can derive two different expressions for K_e, namely:

i) Arithmetic formulation,

$$K_e = \frac{K_P + K_E}{2}$$

ii) Harmonic formulation,

$$K_e = \frac{2 K_P K_E}{K_P + K_E}$$

Similarly, we can derive expressions for K_w as:

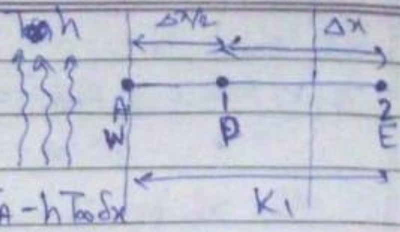
i) Arithmetic

$$K_w = \frac{K_P + K_W}{2}$$

ii) Harmonic

$$K_w = \frac{2 K_P K_W}{K_P + K_W}$$

iii) For boundary point 1

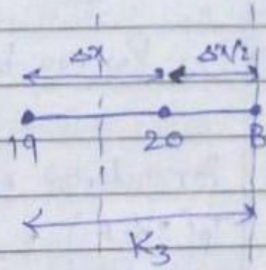


$$T_P\left(\frac{AK_1}{\delta x}+\frac{2AK_1}{\delta x}\right)$$

$$= T_E\frac{K_1A}{\delta x}+T_W(0)+\left(\frac{2K_1A}{\delta x}+h\delta x\right)T_A-hT_\infty\delta x$$

$$\Rightarrow \frac{3K_1A}{\delta x}T_P=\frac{K_1A}{\delta x}T_E+T_W(0)+\left(\frac{2K_1A}{\delta x}+hA\delta x\right)T_A-hT_\infty\delta xA$$

for boundary point 20



$$\cancel{T_P\left(\frac{2K_3A}{\delta x}\right)=\frac{AK_3}{\delta x}T_E+\frac{AK_3}{\delta x}T_B}$$

$$T_P\left(\frac{3K_3A}{\delta x}\right)=\frac{AK_3}{\delta x}T_W+T_E(0)+\frac{2K_3A}{\delta x}T_B$$

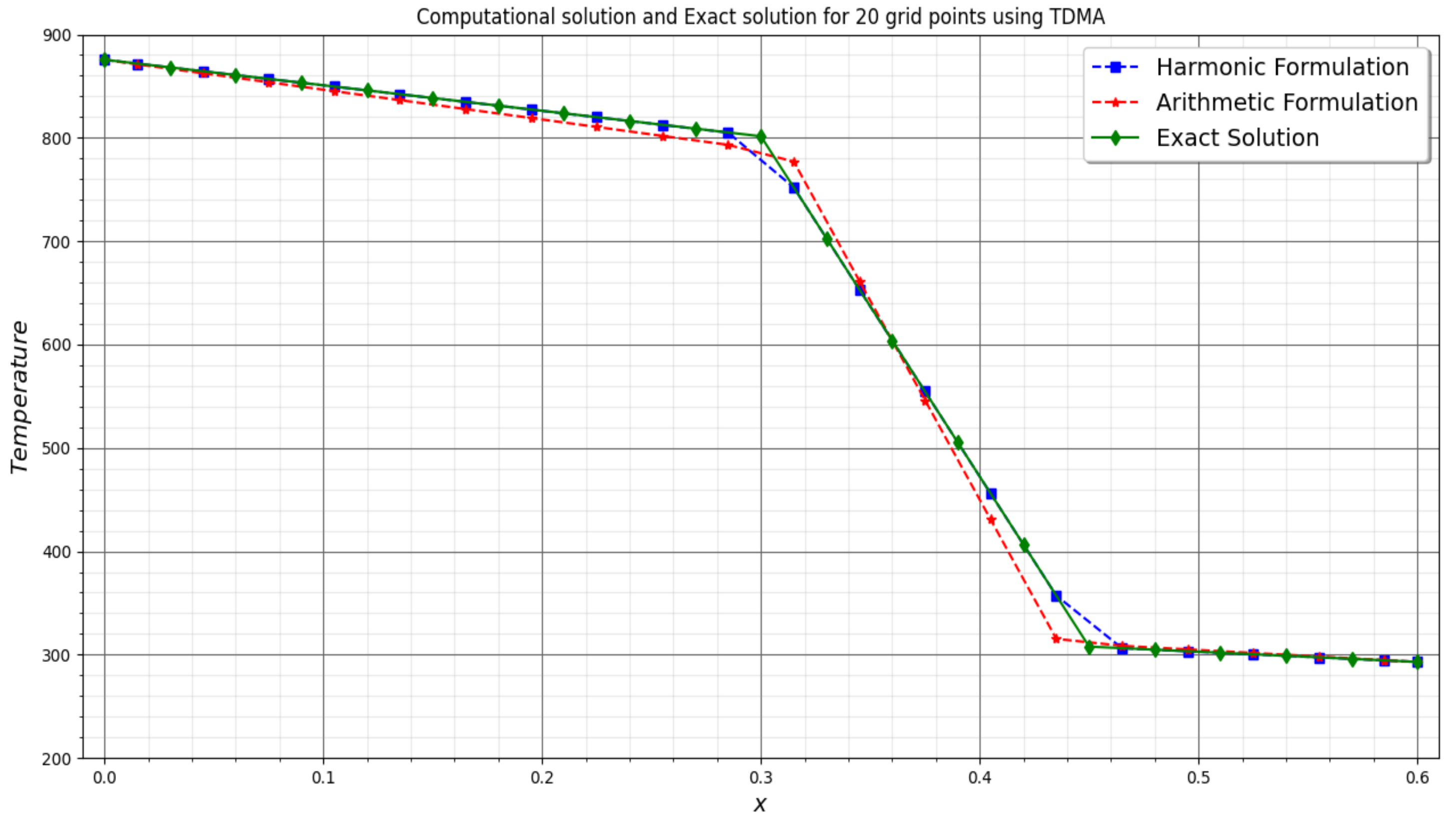
let $\frac{K_1A}{\delta x}=\beta_1$; $\frac{K_2A}{\delta x}=\beta_2$; $\frac{K_3A}{\delta x}=\beta_3$

$hA\delta x=\alpha$; $\frac{2K_1K_2}{K_1+K_2}\frac{A}{\delta x}=\gamma_{12}$; $\frac{2K_2K_3}{K_2+K_3}\frac{A}{\delta x}=\gamma_{23}$

$$\begin{bmatrix}
3\beta_1 & -\beta_1 & & & & & & & & \\
-\beta_1 & 2\beta_1 & -\beta_1 & & & & & & & \\
 & -\beta_1 & \ddots & -\beta_1 & & & & & 0 & \\
 & & & \gamma_{12}+\beta_1 & -\gamma_{12} & & & & & \\
 & & & -\gamma_{12} & \gamma_{12}+\beta_2 & -\beta_2 & & & & \\
 & & & & -\beta_2 & 2\beta_2 & -\beta_2 & & & \\
 & & & & & -\beta_2 & \gamma_{23}+\beta_2 & -\gamma_{23} & & \\
 & & & & & & -\gamma_{23} & \gamma_{23}+\beta_3 & -\beta_3 & \\
 & 0 & & & & & & -\beta_3 & 2\beta_3 & -\beta_3 \\
 & & & & & & & & -\beta_3 & 3\beta_3
\end{bmatrix}
\begin{bmatrix} T_1 \\ T_2 \\ \vdots \\ \vdots \\ T_{20} \end{bmatrix}
=
\begin{bmatrix} (2\beta_1+\alpha)T_A-\alpha T_\infty \\ 0 \\ 0 \\ \vdots \\ 0 \\ \vdots \\ 0 \\ 2\beta_3T_B \end{bmatrix}$$

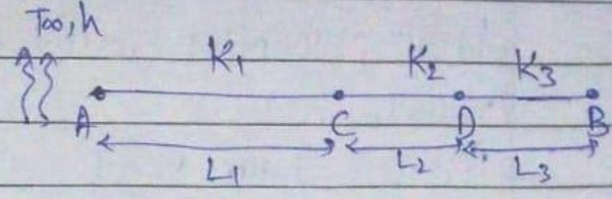
for interfaces, e.g. Wall I & Wall II, where the value of K changes from one control volume to another, we use the following formulations to arrive at an expression for K.

v) Required output (plots/any other means)



(vi) Analysis of result

Deriving the exact solution



Here, we don't have heat generation.

The system is in steady state, the net heat flux will be constant across all cross-sections.

$$\Rightarrow \quad q = h(T_\infty - T_A) = \frac{T_A - T_C}{L_1/K_1} = \frac{T_C - T_D}{L_2/K_2} = \frac{T_D - T_B}{L_3/K_3} = \frac{T_f - T_B}{R_{Total}}$$

$$\text{Now, } R_{Total} = \frac{1}{h} + \frac{L_1}{k_1} + \frac{L_2}{k_2} + \frac{L_3}{k_3} = 0.158 \ m^2 K/W$$

$$\Rightarrow q = \frac{1073 - 293}{0.158} = 4936.71 \ W/m^2$$

finding T_A, T_C & T_D

$$4936.71 = 25(1073 - T_A) \implies T_A = 875.5 \ K$$

$$4936.71 = \frac{20}{0.3}(875.5 - T_C) \Rightarrow T_C = 801.45 \ K$$

$$4936.71 = \frac{1.5}{0.15}(801.45 - T_D) \Rightarrow T_D = 307.78 \ K$$

$\therefore$ The temperature distribution in each segment AC, CD, DB respectively are:

$$T = 875.5 - 246.83x \quad ; \ x \in [0, 0.3]$$

$$T = 801.46 - 3291.13(x - 0.3) \quad ; \ x \in [0.3, 0.45]$$

$$T = 307.78 - 98.53(x - 0.45) \quad ; \ x \in [0.45, 0.6]$$

From the above plot and the analytical solution of the problem, we can conclude that as the number of the grid points are increased, the numerical solution approaches the exact solution and the error between the two decreases. We obtain a linear relationship between the temperature and length for each domain. As there exists a convective term which gives rise to a source of heat and the system is in steady state and one dimensional.

From the plot we can infer the fact that the harmonic formulation of thermal conductivity at the wall interface is better when compared to the arithmetic formulation as the plot using the harmonic formulation is much more close to the exact solution than the arithmetic formulation.