COMPUTATIONAL HEAT AND FLUID FLOW (ME605)

Assignment 1: Finite Difference Methods

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Assignment 01 ME605

PAGE NO.:
DATE: / /

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Solving Poisson equation:

$$-\frac{\partial^2 u}{\partial x^2} + u = (\pi^2+1)\sin(\pi x)$$

Domain : $x \in [0,1]$

BC's : i) $u(x=0)=0$

ii) $u(x=1)=0$

The given equation is a second order, linear, non-homogenous differential equation of the form:

$$y''(t) + p(t)y' + q(t)y = g(t) \quad — ①$$

where $g(t)$ is a non-zero function

The solution for the above differential equation is given by:

$$y(t) = y_c(t) + y_p(t)$$

where $y_c(t)$ is the complementary solution + $y_p(t)$ is the particular solution.

i) $y_c(t)$ is the solution of the homogenous version of ①, namely:

$$y''(t) + p(t)y'(t) + q(t)y(t) = 0$$

given by $y(t) = c_1 y_1(t) + c_2 y_2(t)$

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for our case:

$$u''(x) - u(x) = -(\pi^2+1)\sin(\pi x) \quad \text{— (A)}$$

we have

$$p = 0$$
$$q = -1$$
$$g = -(\pi^2+1)\sin(\pi x)$$

$\therefore$ the homogenous version of (A) will be:

$$u''(x) - u(x) = 0 \quad \text{— (B)}$$

This is a second order, linear, homogenous differential equation with constant coefficients.

$\therefore$ Using the method of undertermined coefficients to solve the above equation

let $u(x) = e^{rx}$ & putting in (B) gives us the characteristic polynomial:

$$\boxed{r^2 - 1 = 0}$$

$$\Rightarrow r = 1 \text{ or } -1$$

$\therefore$ the general ~~sol~~ solution for (B) is of the form

$$u(x) = c_1 e^{x} + c_2 e^{-x}$$

$$\therefore \boxed{u_c(x) = c_1 e^{x} + c_2 e^{-x}}$$

ii) $y_p(t)$ is the solution of the non-homogenous eqn. (A)

$$u''(x) - u(x) = -(\pi^2+1)\sin(\pi x)$$

let the particular solution for (A) be of the form:

$$u_p(x) = A\cos(\pi x) + B\sin(\pi x)$$

$$\therefore u_p''(x) = -A\pi^2\cos(\pi x) - B\pi^2\sin(\pi x) \quad \text{— (B)}$$

putting (B) in (A)

$$-A\pi^2\cos\pi x - B\pi^2\sin(\pi x) - A\cos\pi x - B\sin\pi x = -(\pi^2+1)\sin\pi x$$

$$\Rightarrow -A\pi^2 - A = 0 \quad \text{and} \quad -B\pi^2 - B = -(\pi^2+1)$$

$$\Rightarrow A = 0 \quad \& \quad B = 1$$

$\Rightarrow u_p(x) = \sin(\pi x)$

$\therefore$ the total solution ~~of~~ $u(x)$ is:

$$u(x) = u_p(x) + u_c(x)$$

$$\boxed{u(x) = C_1 e^{x} + C_2 e^{-x} + \sin \pi x} \quad \text{— } \textcircled{C}$$

Now, solving Ⓒ for C_1 & C_2 by putting the BCs.

$$u(x=0) = 0 = C_1 + C_2 \quad \text{— } \textcircled{2}$$
$$u(x=1) = 0 = C_1 e + C_2 e^{-1} \quad \text{— } \textcircled{3}$$

From ② & ③, we conclude that

$$\boxed{C_1 = C_2 = 0}$$

$\therefore$ the solution for the given Poisson equation:

$$\frac{\partial^2 u}{\partial x^2} - u = -(\pi^2+1)\sin(\pi x)$$

is $\boxed{u(x) = \sin(\pi x)}$

The discretization employed for this equation is: second order finite difference scheme.

$\therefore \frac{\partial^2 u}{\partial x^2} \approx \frac{u_{i+1} - 2u_i + u_{i-1}}{(\Delta x)^2}$, which is a second order scheme.

we have an approximation for $\frac{\partial^2 u}{\partial x^2}$, because we are neglecting the higher order terms of $O((\Delta x)^2)$

∴ the discretized equation is :

$$\frac{u_{i+1} - 2u_i + u_{i-1}}{(\Delta x)^2} - u_i = -(\pi^2+1)\sin(\pi \cdot x_i)$$

where x_i = mesh point / grid point
u_{i+1} = value at $(i+1)^{th}$ node
u_i = value at i^{th} node
u_{i-1} = value at $(i-1)^{th}$ node
Δx = mesh spacing

BC: $u_0 = 0$
$u_{(num)} = 0$
where num = last grid point

∴ $$\boxed{u_{i-1} - u_i(2+(\Delta x)^2) + u_{i+1} = -(\Delta x)^2(\pi^2+1)\sin(\pi \cdot x_i)}$$

$$\begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 \\ 1 & -\alpha & 1 & \cdots & 1 & \cdots & \vdots & 0 \\ 0 & 1 & -\alpha & \cdots & & \cdots & & 0 \\ \vdots & 0 & 1 & \ddots & & & & 0 \\ \vdots & \vdots & 0 & & \ddots & & & 0 \\ 0 & \vdots & \vdots & \cdots & 1 & -\alpha & & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & & 1 \end{bmatrix} \cdot \begin{bmatrix} u_0 \\ u_1 \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ u_n \end{bmatrix} = \begin{bmatrix} \beta_0 \\ \beta_1 \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \beta_n \end{bmatrix}$$

where $\alpha = (2+(\Delta x)^2)$ & $\beta_i = -(\Delta x)^2(\pi^2+1)\sin(\pi x_i)$

exact solution of Ⓐ

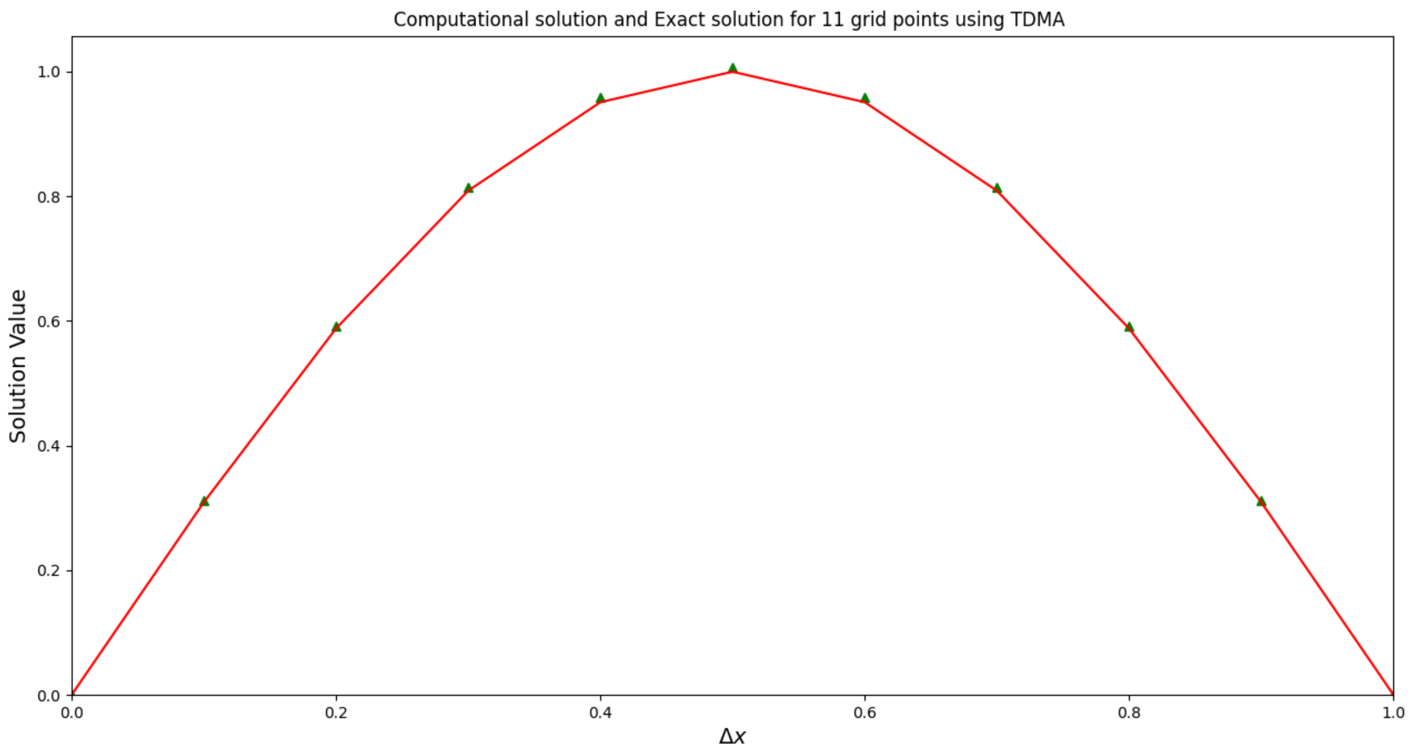
$$u(x) = \sin(\pi x)$$

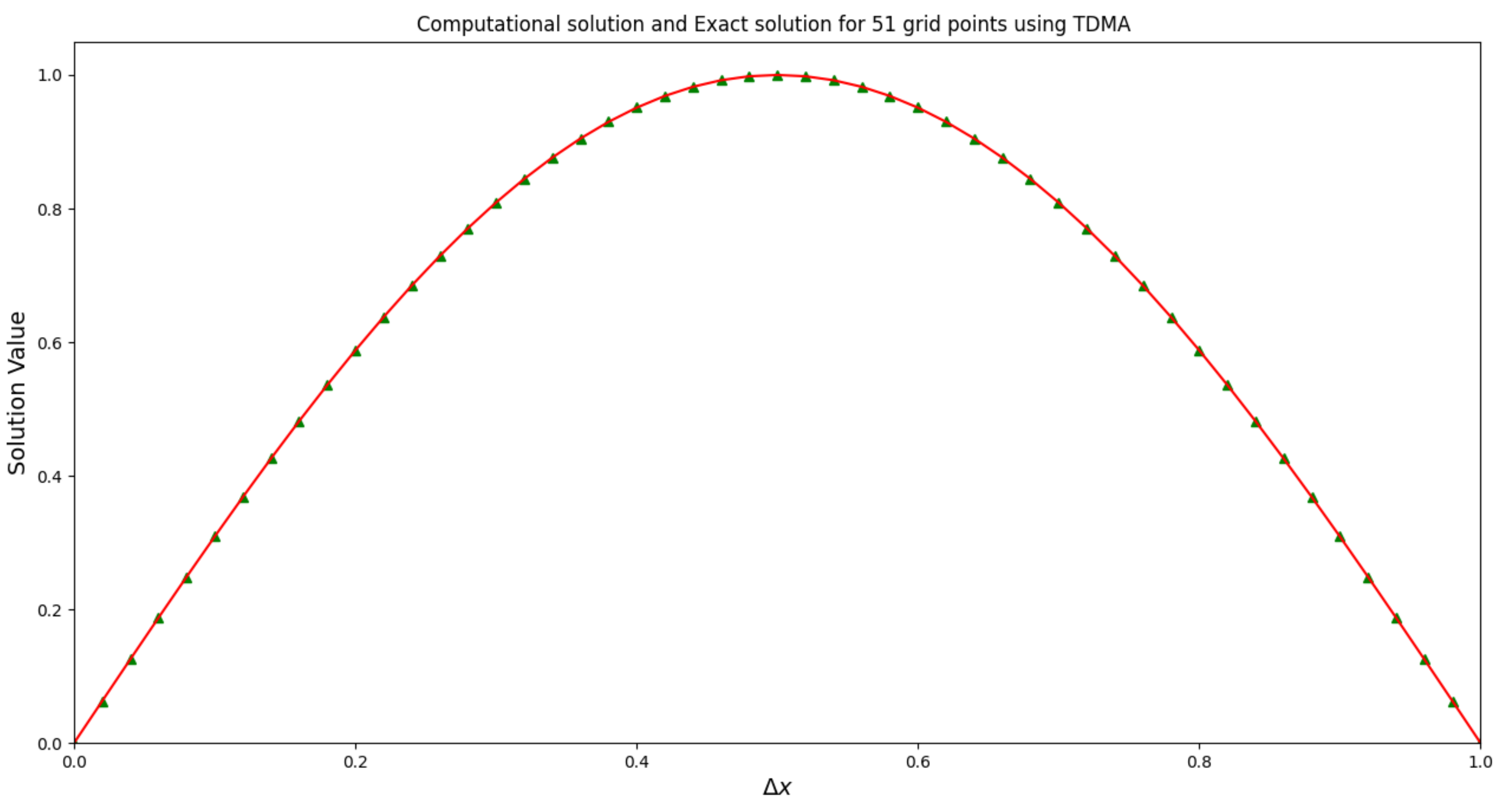
numerical solution expression for Ⓐ

$$u_{i-1} - u_i(2+(\Delta x)^2) + u_{i+1} = -(\Delta x)^2(\pi^2+1)\sin(\pi x_i)$$

Task1. Plot the distribution of the computational solution and the exact solution for 11 and 51 grid points using TDMA as the solver.

After observing the two graphs, we can conclude that greater the number of mesh points, the greater the computational solution approaches the exact solution.
This observation in conjunction with the fact that, as $\Delta x \rightarrow 0$ the computational solution approaches the exact solution.

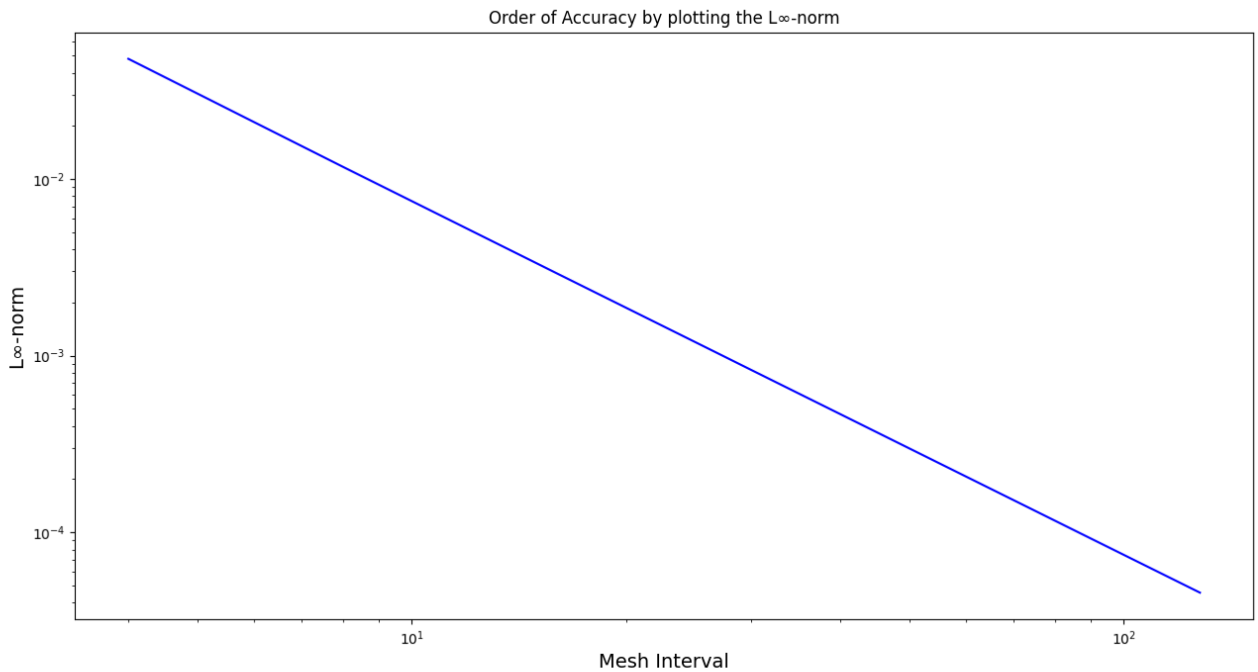




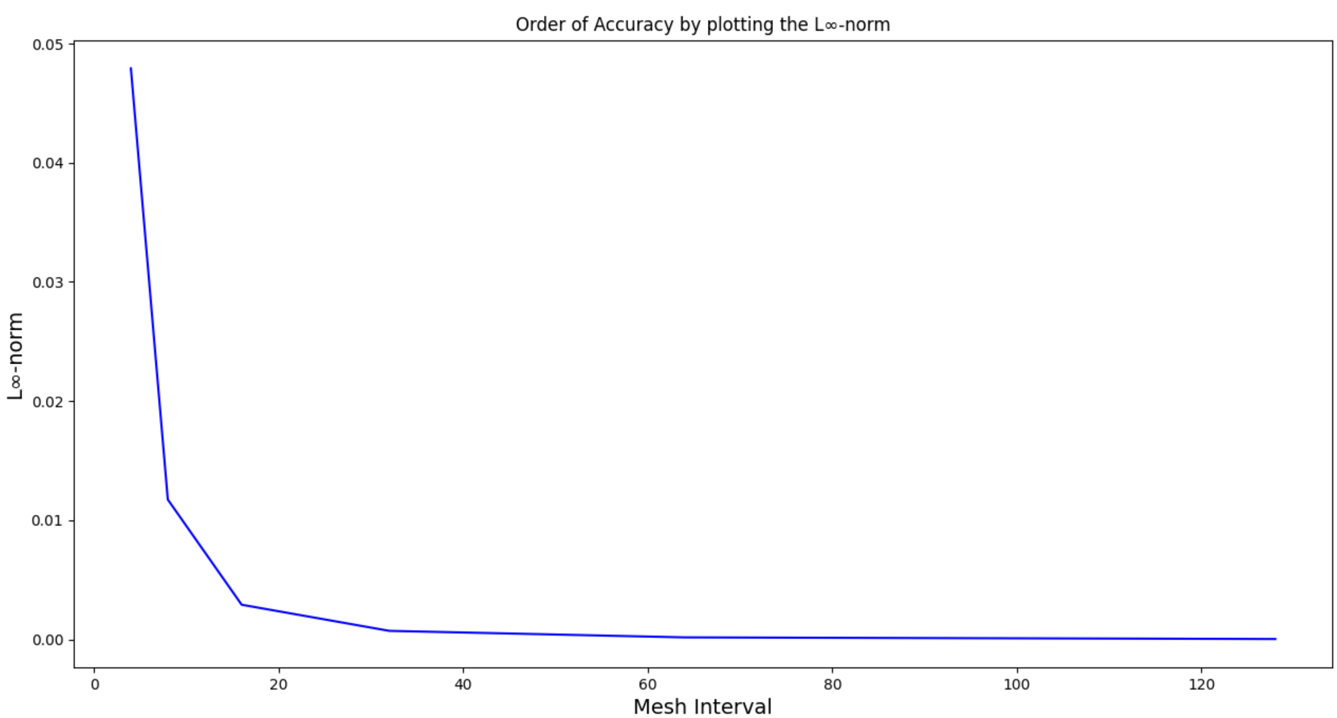
Task2. Solve the above using number of mesh intervals 4, 8, 16, 32, 64, and 128 with TDMA as the solver. Estimate the order of accuracy by plotting the L∞-norm of solution error against the number of mesh intervals.

Here we can see that as the mesh interval increases the error at each mesh point decreases. It is because of the fact that as the mesh gets finer and finer, the computational value approaches the theoretical at each grid point.

Logarithmic Graph



Linear Graph



Task3. Plot the convergence of all three iterative schemes: Jacobi, Gauss-Seidel, and Successive over-relaxation method.

The convergence of any iterative method depends on its spectral radius. Spectral radius of a square matrix is the largest absolute value of its eigenvalues, and for an iterative scheme to be convergent, the spectral radius must be less than 1. For an equation:

$$\mathrm{A}\boldsymbol{x} = \mathrm{b}$$

where A is the coefficient matrix, **x** is the matrix of unknowns and, b is the matrix of known values. We can break the coefficient matrix in the following manner:

$$\mathrm{A} = \mathrm{D} + \mathrm{R}$$

where D is the diagonal matrix of A and R is the matrix with all the diagonal entries equal to 0. Then, we can write the following iterative scheme

$$\boldsymbol{x}^{\mathrm{k+1}} = \mathrm{D}^{-1}(\mathrm{b} - \mathrm{R}\boldsymbol{x}^{\mathrm{k}})$$

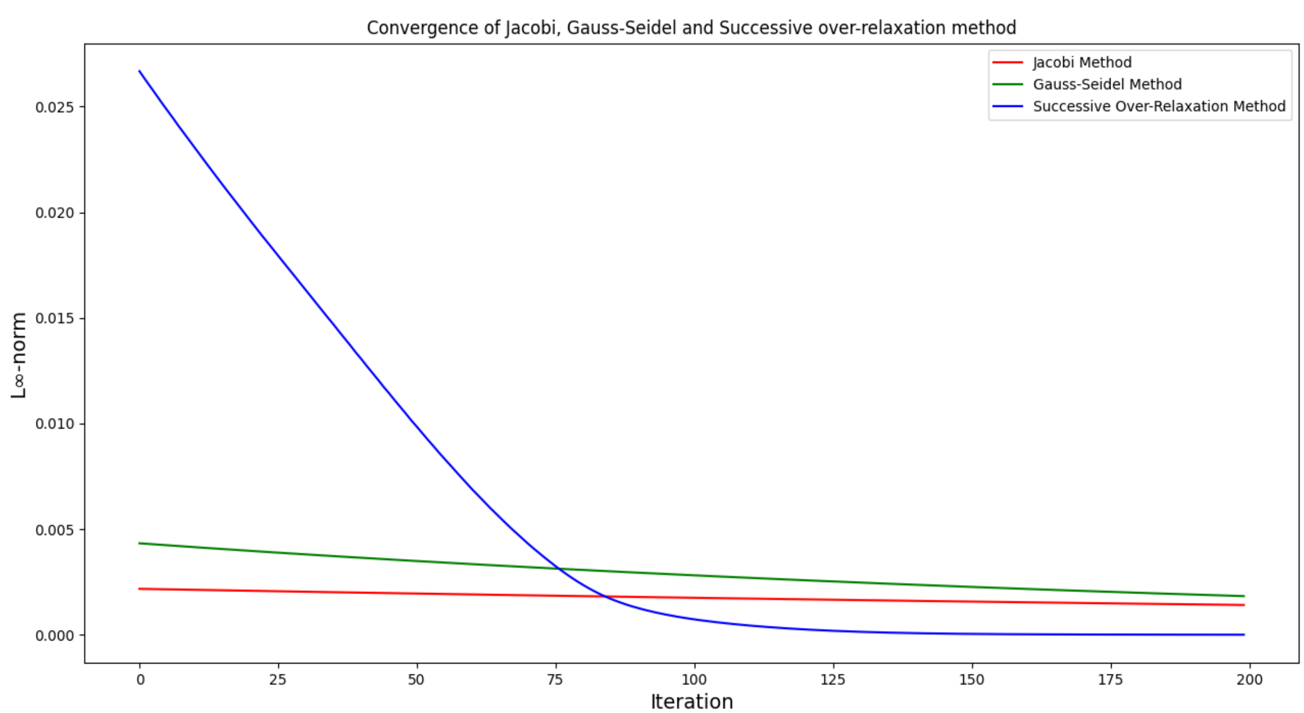
which can be further simplified to the following form:

$$\boldsymbol{x}^{\mathrm{k+1}} = \mathrm{T}\boldsymbol{x}^{\mathrm{k}} + \mathrm{C}$$

Here, T is known to govern the convergence of an iterative scheme. The largest eigenvalue of T must be smaller than 1 or the iterative method will not be convergent.

We can observe that the reduction in error is steepest for Successive over-relaxation method due to the fact that the relaxation parameter, ω causes the convergence of the scheme to speed up dramatically. We try to minimize the spectral radius of the T_{ω} matrix as much as we can. If we put ω=1, we end up with Gauss-Seidel method.

Linear Graph



Logarithmic Graph