(a) $|z| < 2$

$$\frac{1}{z-1}-\frac{1}{z-2}=-\frac{1}{(1-z)}-\frac{1}{-2\left(1-\frac{z}{2}\right)}=-(1-z)^{-1}+\frac{1}{2}\left(1-\frac{z}{2}\right)^{-1}$$

$$=-\left(1+z+z^2+z^3+\ldots\ldots\right)+\frac{1}{2}\left(1+\frac{z}{2}+\left(\frac{z}{2}\right)^2+\ldots\ldots\right)$$

(b) $|z| > 2$

$$\frac{1}{z-1}-\frac{1}{z-2}=\frac{1}{z\left(1-\frac{1}{z}\right)}-\frac{1}{z\left(1-\frac{2}{z}\right)}=\frac{1}{z}\left(1-\frac{1}{z}\right)^{-1}-\frac{1}{z}\left(1-\frac{2}{z}\right)^{-1}$$

$$=\frac{1}{z}\left(1+\frac{1}{z}+\frac{1}{z^2}+\ldots\ldots\right)-\frac{1}{z}\left(1+\frac{2}{z}+\left(\frac{2}{z}\right)^2+\ldots\ldots\right)=-\frac{1}{z^2}-\frac{3}{z^3}-\ldots\ldots$$

(c) $1 < |z| < 2$

$$\frac{1}{z-1}-\frac{1}{z-2}=\frac{1}{z\left(1-\frac{1}{z}\right)}+\frac{1}{2\left(1-\frac{z}{2}\right)}=\frac{1}{z}\left(1-\frac{1}{z}\right)^{-1}+\frac{1}{2}\left(1-\frac{z}{2}\right)^{-1}$$

$$=\frac{1}{z}\left(1+\frac{1}{z}+\frac{1}{z^2}+\ldots\ldots\right)+\frac{1}{2}\left(1+\frac{z}{2}+\left(\frac{z}{2}\right)^2+\ldots\ldots\right)=\left(\frac{1}{z}+\frac{1}{z^2}+\frac{1}{z^3}+\ldots\ldots\right)+\left(\frac{1}{2}+\frac{z}{4}+\frac{z^2}{8}+\ldots\ldots\right)$$

SOLVED EXAMPLES - 6.1

Find the Laurent series that converges for $0 < |z| < R$ and determine the precise region of convergence.

Q.1. $f(z)=\dfrac{1}{z^4-z^5}$

Soln. $$\frac{1}{z^4-z^5}=\frac{1}{z^4(1-z)}=\frac{1}{z^4}(1-z)^{-1}=\frac{1}{z^4}\left(1+z+z^2+z^3+z^4+z^5+\ldots\ldots\right)$$

$$=\frac{1}{z^4}+\frac{1}{z^3}+\frac{1}{z^2}+\frac{1}{z}+1+z+\ldots\ldots\ldots=\sum_{n=0}^{\infty}z^{-n+4}$$

$$a_n=1,\ a_{n+1}=1,\ \frac{1}{R}=\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|=\lim_{n\to\infty}\left|\frac{1}{1}\right|=1,\ R=1$$

Q.2. $f(z)=z\cos\frac{1}{z}$

Soln. $$z\cos\frac{1}{z}=z\left(1-\frac{\left(\frac{1}{z}\right)^2}{\lfloor 2}+\frac{\left(\frac{1}{z}\right)^4}{\lfloor 4}-\ldots\ldots\ldots\right)=z-\frac{1}{z\lfloor 2}+\frac{1}{z^3\lfloor 4}=\sum_{n=0}^{\infty}(-1)^n\frac{z^{-2n+1}}{\lfloor 2n}$$

$$a_n=\frac{(-1)^n}{\lfloor 2n},\quad a_{n+1}=\frac{(-1)^{n+1}}{\lfloor 2n+2}$$

$$\frac{1}{R}=\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|=\lim_{n\to\infty}\left|\frac{(-1)^{n+1}}{\lfloor 2n+2}\times\frac{\lfloor 2n}{(-1)^n}\right|=\lim_{n\to\infty}\left|\frac{(-1)}{(2n+2)(2n+1)}\right|=0,\qquad R=\infty$$

Q.3. $f(z)=\frac{e^{-z}}{z^3}$

Soln. $$\frac{e^{-z}}{z^3}=\frac{1}{z^3}\left(1-z+\frac{z^2}{\lfloor 2}-\frac{z^3}{\lfloor 3}+\ldots\ldots\right)=\frac{1}{z^3}-\frac{1}{z^2}+\frac{1}{z\lfloor 2}-\frac{1}{\lfloor 3}+\frac{z}{\lfloor 4}+\ldots\ldots=\sum_{n=0}^{\infty}\frac{(-1)^n}{\lfloor n}z^{n-3}$$

$$a_n=\frac{(-1)^n}{\lfloor n},\quad a_{n+1}=\frac{(-1)^{n+1}}{\lfloor n+1}$$

$$\frac{1}{R}=\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|=\lim_{n\to\infty}\left|\frac{(-1)^{n+1}}{\lfloor n+1}\cdot\frac{\lfloor n}{(-1)^n}\right|=\lim_{n\to\infty}\left|\frac{(-1)}{n+1}\right|=0,\quad R=\infty$$

Q.4. $f(z)=\frac{\cosh 2z}{z^2}$

Soln. $$\frac{\cosh 2z}{z^2}=\frac{1}{z^2}\left(1+\frac{(2z)^2}{\lfloor 2}+\frac{(2z)^4}{\lfloor 4}+\ldots\ldots\right)=\frac{1}{z^2}+\frac{2}{\lfloor 2}+\frac{2^4z^2}{\lfloor 4}+\ldots\ldots\ldots=\sum_{n=0}^{\infty}\frac{2^{2n}}{\lfloor 2n-2}z^{2n-2}$$

$$a_n=\frac{2^{2n}}{\lfloor 2n-2},\quad a_{n+1}=\frac{2^{2n+2}}{\lfloor 2n}$$

$$\frac{1}{R}=\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|=\lim_{n\to\infty}\left|\frac{2^{2n+2}}{\lfloor 2n}\cdot\frac{\lfloor 2n-2}{2^{2n}}\right|=\lim_{n\to\infty}\left|\frac{4}{2n(2n-1)}\right|=0,\qquad R=\infty$$

Q.5. $f(z)=z^{-3}e^{1/z^2}$

Soln. $z^{-3}e^{1/z^2}=z^{-3}\left(1+\left(\frac{1}{z^2}\right)+\frac{\left(\frac{1}{z^2}\right)^2}{\lfloor 2}+\frac{\left(\frac{1}{z^2}\right)^3}{\lfloor 3}+\ldots\ldots\right)=z^{-3}+z^{-5}+\frac{z^{-7}}{\lfloor 2}+\frac{z^{-9}}{\lfloor 3}+\ldots\ldots=\sum_{n=0}^{\infty}\frac{z^{-2n-3}}{\lfloor n}$

$$a_n=\frac{1}{\lfloor n},\ a_{n+1}=\frac{1}{\lfloor n+1}$$

$$\frac{1}{R}=\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|=\lim_{n\to\infty}\left|\frac{1}{\lfloor n+1}\frac{\lfloor n}{1}\right|=\lim_{n\to\infty}\left|\frac{1}{n+1}\right|=0,\ R=\infty$$

Q.6. $f(z)=\frac{e^z}{z^2-z^3}$

Soln. $\frac{e^z}{z^2-z^3}=\frac{e^z}{z^2(1-z)}=\frac{1}{z^2}\left(1+z+\frac{z^2}{\lfloor 2}+\ldots\right)(1+z+z^2+\ldots)=\frac{1}{z^2}+\frac{2}{z}+\frac{5}{2}+\ldots\ldots$

Find the Laurent series that converges for $0<|z-z_0|<R$ ***and determine the precise region of convergence.***

Q.7. $f(z)=\frac{e^z}{z-1},\ z_0=1$

Soln. $\frac{e^z}{z-1}=\frac{e^{z-1+1}}{z-1}=\frac{e^{z-1}e^1}{z-1}=\frac{e}{z-1}\left(1+(z-1)+\frac{(z-1)^2}{\lfloor 2}+\ldots\ldots\right)$

$$=e\left[\frac{1}{z-1}+1+\frac{(z-1)}{\lfloor 2}+\ldots\ldots\right]=\sum_{n=0}^{\infty}\frac{e(z-1)^{n-1}}{\lfloor n}$$

$$a_n=\frac{e}{\lfloor n},\ a_{n+1}=\frac{e}{\lfloor n+1}$$

$$\frac{1}{R}=\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|=\lim_{n\to\infty}\left|\frac{e}{\lfloor n+1}\cdot\frac{\lfloor n}{e}\right|=\lim_{n\to\infty}\left|\frac{1}{n+1}\right|=0,\ R=\infty$$

Region of convergence $0<|z-1|<\infty$

Q.8. $f(z)=\dfrac{\sin z}{\left(z-\frac{\pi}{4}\right)^2}$, $z_0=\dfrac{\pi}{4}$

Soln. $$\frac{\sin z}{\left(z-\frac{\pi}{4}\right)^2}=\frac{\sin\left(z-\frac{\pi}{4}+\frac{\pi}{4}\right)}{\left(z-\frac{\pi}{4}\right)^2}=\frac{\sin\left(z-\frac{\pi}{4}\right)\cdot\cos\frac{\pi}{4}+\cos\left(z-\frac{\pi}{4}\right)\sin\frac{\pi}{4}}{\left(z-\frac{\pi}{4}\right)^2}$$

$$=\frac{1}{\sqrt{2}}\frac{\left(\sin\left(z-\frac{\pi}{4}\right)+\cos\left(z-\frac{\pi}{4}\right)\right)}{(z-\pi/4)^2}$$

$$=\frac{1}{\sqrt{2}(z-\pi/4)^2}\left(1+(z-\pi/4)-\frac{(z-\pi/4)^2}{\lfloor 2}-\frac{(z-\pi/4)^3}{\lfloor 3}+\ldots\ldots\right)$$

$$=\frac{1}{\sqrt{2}}\left(\frac{1}{\left(z-\frac{\pi}{4}\right)^2}+\frac{1}{(z-\pi/4)}-\frac{1}{\sqrt{2}\lfloor 2}-\frac{(z-\pi/4)}{\lfloor 3}+\ldots\ldots\right)$$

Region of convergence $\left|z-\dfrac{\pi}{4}\right|>0$

Q.9. $f(z)=\dfrac{1}{z^2+1}$, $z_0=i$

Soln. $$\frac{1}{z^2+1}=\frac{1}{(z-i)(z+i)}=\frac{1}{(z-i)(z-i+2i)}=\frac{1}{(z-i)2i\left(1+\frac{z-i}{2i}\right)}=\frac{1}{2i(z-i)}\left(1+\frac{z-i}{2i}\right)^{-1}$$

$$=\frac{1}{2i(z-i)}\left[1-\left(\frac{z-i}{2i}\right)+\left(\frac{z-i}{2i}\right)^2-\ldots\ldots\right]=\frac{(z-i)^{-1}}{2i}-\frac{1}{(2i)^2}+\frac{(z-i)}{(2i)^3}-\ldots\ldots$$

$$=\sum_{n=0}^{\infty}\frac{(-1)^n(z-i)^{n-1}}{(2i)^{n+1}}$$

$$a_n=\frac{(-1)^n}{(2i)^{n+1}},\quad a_{n+1}=\frac{(-1)^{n+1}}{(2i)^{n+2}}$$

$$\frac{1}{R}=\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|=\lim_{n\to\infty}\left|\frac{(-1)^{n+1}}{(2i)^{n+2}}\frac{(2i)^{n+1}}{(-1)^n}\right|=\lim_{n\to\infty}\left|\frac{(-1)}{(2i)}\right|=\frac{1}{2}$$

$R = 2$, Region of convergence $0<|z-i|<2$

Q.10. $f(z)=\dfrac{\cos z}{(z-\pi)^4}=\dfrac{\cos((z-\pi)+\pi)}{(z-\pi)^4}=\dfrac{\cos(z-\pi)\cdot\cos\pi-\sin(z-\pi)\sin\pi}{(z-\pi)^4}$

$$=\frac{1}{(z-\pi)^4}(-\cos(z-\pi))=-\frac{1}{(z-\pi)^4}\left(1-\frac{(z-\pi)^2}{\lfloor 2}+\frac{(z-\pi)^4}{\lfloor 4}-\ldots\ldots\right)$$

$$=-\frac{1}{(z-\pi)^4}+\frac{1}{(z-\pi)^2\lfloor 2}-\frac{1}{\lfloor 4}-\ldots\ldots=\sum_{n=0}^{\infty}\frac{(-1)^{n+1}}{\lfloor 2n}(z-\pi)^{2n-4}$$

$$a_n=\frac{(-1)^{n+1}}{\lfloor 2n},\quad a_{n+1}=\frac{(-1)^{n+2}}{\lfloor 2n+2}$$

$$\frac{1}{R}=\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|=\lim_{n\to\infty}\left|\frac{(-1)^{n+2}}{\lfloor 2n+2}\frac{\lfloor 2n}{(-1)^{n+1}}\right|=\lim_{n\to\infty}\left|\frac{(-1)}{(2n+2)(2n+1)}\right|=0$$

$R=\infty$, Region of convergence $0<|z-\pi|<\infty$

Q.11. $f(z)=\dfrac{1}{(z+i)^2-(z+i)}$, $z_0=-i$

$$=\frac{1}{-(z+i)(1-(z+i))}=-\frac{1}{(z+i)}(1-(z+i))^{-1}=-\frac{1}{(z+i)}\left(1+(z+i)+(z+i)^2+\ldots\right)$$

$$=-\frac{1}{(z+i)}-1-(z+i)-\ldots\ldots=\sum_{n=0}^{\infty}-(z+i)^{n-1}$$

$a_n=(-1)$, $a_{n+1}=(-1)$

$$\frac{1}{R}=\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|=\lim_{n\to\infty}\left|\frac{(-1)}{(-1)}\right|=1$$

$R = 1$, Region of convergence $= 1$

Q.12. $f(z)=\dfrac{z^2}{(z+i)^2}$, $z_0=-i$

Soln. $\dfrac{z^2}{(z+i)^2}=\dfrac{(z+i-i)^2}{(z+i)^2}=\dfrac{(z+i)^2-1+2i(z+i)}{(z+i)^2}=1-\dfrac{1}{(z+i)^2}+\dfrac{2i}{(z+i)}$

Region of convergence $|z+i|>0$.

Q.13. $f(z)=\frac{z^2-4}{z-1}=\frac{((z-1)+1)^2-4}{z-1}=\frac{(z-1)^2+1+2(z-1)-4}{z-1}=(z-1)+2-\frac{3}{z-1}$

Region of convergence $|z-1|>0$

Q.14. $f(z)=z^2\sinh\frac{1}{z}$, $z_0=0$

Soln. $z^2\sinh\frac{1}{z}=z^2\left(\frac{1}{z}+\frac{\left(\frac{1}{z}\right)^3}{\lfloor 3}+\frac{\left(\frac{1}{z}\right)^5}{\lfloor 5}+\dots\dots\right)=z+\frac{1}{z\lfloor 3}+\frac{1}{z^3\lfloor 5}+\dots\dots=\sum_{n=1}^{\infty}\frac{z^{-2n+3}}{\lfloor 2n+1}$

$a_n=\frac{1}{\lfloor 2n+1}$, $a_{n+1}=\frac{1}{\lfloor 2n+3}$

$$\frac{1}{R}=\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|=\lim_{n\to\infty}\left|\frac{1}{\lfloor 2n+3}\cdot\frac{\lfloor 2n+1}{1}\right|=\lim_{n\to\infty}\left|\frac{1}{(2n+3)(2n+2)}\right|=0,\ R=\infty$$

Find all Taylor and Laurent series with center $z=z_0$ and determine the precise regions of convergence.

Q.15. $f(z)=\frac{1}{1-z^3}$, $z_0=0$

Soln. $\frac{1}{1-z^3}=(1-z^3)^{-1}=1+z^3+z^6+z^9+\dots\dots=\sum_{n=0}^{\infty}z^{3n}$, $|z|<1$

$$\frac{1}{1-z^3}=\frac{1}{-z^3\left(1-\frac{1}{z^3}\right)}=-\frac{1}{z^3}\left(1-\frac{1}{z^3}\right)^{-1}=-\frac{1}{z^3}\left(1+\frac{1}{z^3}+\frac{1}{z^6}+\dots\dots\right)=-\left(\frac{1}{z^3}+\frac{1}{z^6}+\frac{1}{z^9}+\dots\dots\right)$$

$$=-\sum_{n=0}^{\infty}\frac{1}{z^{3n+3}},\qquad |z|>1$$

Q.16. $f(z)=\frac{1}{1-z^2}$, $z_0=1$

Soln. $\frac{1}{(1-z^2)}=(1-z^2)^{-1}=1+z^2+z^4+\dots\dots=\sum_{n=0}^{\infty}z^{2n}$ $\quad |z|<1$

$$\frac{1}{1-z^2}=\frac{1}{-z^2\left(1-\frac{1}{z^2}\right)}=-\frac{1}{z^2}\left(1-\frac{1}{z^2}\right)^{-1}=-\frac{1}{z^2}\left(1+\frac{1}{z^2}+\frac{1}{z^4}+\dots\dots\right)$$

$$=-\left(\frac{1}{z^2}+\frac{1}{z^4}+\frac{1}{z^6}+\dots\dots\right)=-\sum_{n=0}^{\infty}\frac{1}{z^{2n+2}},\ |z|>1$$

Q.17. $f(z)=\dfrac{z^2}{1-z^4}, \qquad z_0=0$

Soln. $\dfrac{z^2}{1-z^4}=z^2\left(1-z^4\right)^{-1}=z^2\left(1+z^4+z^8+\ldots\ldots\right)=z^2+z^6+z^{10}+\ldots\ldots$

$$=\sum_{n=0}^{\infty} z^{2n+2}, \qquad |z|<1$$

$$\frac{z^2}{1-z^4}=\frac{z^2}{-z^4\left(1-\frac{1}{z^4}\right)}=-\frac{1}{z^2}\left(1-\frac{1}{z^4}\right)^{-1}=-\frac{1}{z^2}\left(1+\frac{1}{z^4}+\frac{1}{z^8}+\ldots\ldots\right)$$

$$=-\left(\frac{1}{z^2}+\frac{1}{z^6}+\frac{1}{z^{10}}+\ldots\ldots\right)=-\sum_{n=0}^{\infty} z^{-4n-2}, \qquad |z|>1$$

Q.18. $f(z)=\dfrac{1}{z}$, $z_0=1$

Soln. $\dfrac{1}{z}=\sum_{n=1}^{\infty} z^{-n} \qquad |z|<1$

Q.19. $f(z)=\dfrac{z^3-2iz^2}{(z-i)^2}$, $z_0=i$

Soln. $\dfrac{z^2(z-2i)}{(z-i)^2}=\dfrac{(z-i+i)^2(z-i-i)}{(z-i)^2}=\dfrac{\left((z-i)^2-1+2i(z-i)\right)\left((z-i)-i\right)}{(z-i)^2}$

$$=\frac{(z-i)^3-i(z-i)^2-(z-i)+i+2i(z-i)^2+2(z-i)}{(z-i)^2}$$

$$=\frac{(z-i)^3+i(z-i)^2+(z-i)+i}{(z-i)^2}=(z-i)+i+\frac{1}{z-i}+\frac{i}{(z-i)^2}$$

Q.20. $f(z)=\dfrac{\sinh z}{(z-1)^4}$, $z_0=1$

Soln. $\dfrac{\sinh z}{(z-1)^4}=\dfrac{\sinh(z-1+1)}{(z-1)^4}=\dfrac{\sinh(z-1)\cosh 1+\cosh(z-1)\sinh 1}{(z-1)^4}$

$$=\frac{1}{(z-1)^4}\left(\left((z-1)+\frac{(z-1)^3}{\lfloor 3}+\frac{(z-1)^5}{\lfloor 5}+\ldots\right)\cosh 1+\left(1+\frac{(z-1)^2}{\lfloor 2}+\frac{(z-1)^4}{\lfloor 4}+\ldots\ldots\right)\sinh$$

Q.21. $f(z)=\dfrac{4z-1}{z^4-1}$, $z_0=0$

Soln. $\dfrac{4z-1}{z^4-1}=\dfrac{(4z-1)}{-(1-z^4)}=(1-4z)(1-z^4)^{-1}$

$$=(1-4z)(1+z^4+z^8+........)=(1-4z)\sum_{n=0}^{\infty}z^{4n}\ ,\ |z|<1$$

$$\frac{4z-1}{z^4-1}=\frac{4z-1}{z^4\left(1-\frac{1}{z^4}\right)}=\frac{4z-1}{z^4}\left(1-\frac{1}{z^4}\right)^{-1}$$

$$=\left(\frac{4}{z^3}-\frac{1}{z^4}\right)\left(1+\frac{1}{z^4}+\frac{1}{z^8}+.......\right)$$

$$=\left(\frac{4}{z^3}-\frac{1}{z^4}\right)\sum_{n=0}^{\infty}\frac{1}{z^{4n}}\ ,\ |z|>1$$

Q.22. $f(z)=\dfrac{1}{z^2}$, $z_0=i$

Soln. $\dfrac{1}{z^2}=\dfrac{1}{(z-i+i)^2}=\dfrac{1}{i^2\left(1+\frac{z-i}{i}\right)^2}=-1\left(1+\dfrac{z-i}{i}\right)^{-2}=-\left(1-\dfrac{2}{i}(z-i)+.......\right)$

$$=-1-2i(z-i)-........$$

Q.23. $f(z)=\dfrac{\sin z}{z+\frac{\pi}{2}}$, $z_0=-\dfrac{1}{2}\pi$

Soln. $\dfrac{\sin z}{z+\frac{\pi}{2}}=\dfrac{\sin\left(z+\frac{\pi}{2}-\frac{\pi}{2}\right)}{z+\frac{\pi}{2}}=\dfrac{\sin\left(z+\frac{\pi}{2}\right)\cos\frac{\pi}{2}+\cos\left(z+\frac{\pi}{2}\right)\sin\frac{\pi}{2}}{z+\frac{\pi}{2}}=\dfrac{\cos\left(z+\frac{\pi}{2}\right)}{z+\frac{\pi}{2}}$

$$=\frac{1-\dfrac{\left(z+\frac{\pi}{2}\right)^2}{\lfloor 2}+\dfrac{\left(z+\frac{\pi}{2}\right)^4}{\lfloor 4}-..........}{z+\frac{\pi}{2}}=\frac{1}{\left(z+\frac{\pi}{2}\right)}-\frac{\left(z+\frac{\pi}{2}\right)}{\lfloor 2}+\frac{\left(z+\frac{\pi}{2}\right)^3}{\lfloor 4}......$$

Abhi —

SOLVED EXAMPLES - 6.2

Determine the location and kind of the singularities of the following functions in the finite plane and at infinity.

Q.1. $f(z)=\tan^2 \pi z$

Soln. $f(z)=\tan^2 \pi z=\dfrac{\sin^2 \pi z}{\cos^2 \pi z}$, $\cos^2 \pi z=0$, $\cos \pi z=0$, $\pi z=\pm(2n+1)\dfrac{\pi}{2}$

$z=\pm\frac{1}{2}(2n+1)$, $n=0, 1, 2,\ldots\ldots$

Pole of order = 2

At $z=\infty$, it is essential singularity.

Q.2. $f(z)=z+\dfrac{2}{z}-\dfrac{3}{z^2}$

Soln. At $z=0$, Pole of order 2

At $z=\infty$, Pole of order 1

Q.3. $f(z)=\cot z^2$

Soln. $f(z)=\cot z^2=\dfrac{\cos z^2}{\sin z^2}$ $\sin z^2=0$, $z^2=0$, $z^2=\pm n\pi$, $n=0, 1, 2,\ldots..$

$z=0, \pm\sqrt{\pi}, \pm\sqrt{2\pi},\ldots\ldots$ Pole of order = 1

At $z=\infty$, it is essential singularity.

Q.4. $f(z)=z^3 e^{1/(z-1)}$

Soln. $f(z)=(z-1+1)^3 e^{1/(z-1)}=\left((z-1)^3+3(z-1)^2+3(z-1)+1\right)\left(1+\dfrac{1}{z-1}+\dfrac{1}{\lfloor 2(z-1)^2}+\ldots\ldots\right.$

At $z=1$, it is essential singularity.

Q.5. $f(z)=\cos z-\sin z$

Soln. At $z=\infty$, it is essential singularity.

Q.6. $f(z)=\dfrac{1}{(\cos z-\sin z)}$

Soln. $f(z)=\dfrac{1}{\cos z-\sin z}$, $\cos z-\sin z=0$, $\tan z=1$, $z=n\pi+\dfrac{\pi}{4}$ $n=0, 1, 2\ldots.$

It is a simple pole.

(because $z\to\dfrac{\pi}{4}, f(z)=\left|\dfrac{1}{\cos z-\sin z}\right|\to\infty$)

Q.7. $f(z)=\dfrac{\sin 3z}{\left(z^4-1\right)^4}$

Soln. $f(z)=\dfrac{\sin 3z}{\left(z^4-1\right)^4}$, $z^4-1=0$, $\left(z^2-1\right),\left(z^2+1\right)=0$, $z=1,-1,i,-i$

At $z=1,-1,i,-i$ is a pole of order $=4$

At $z=\infty$, it is essential singularity.

Q.8. $f(z)=\dfrac{4}{z-1}+\dfrac{2}{(z-1)^2}-\dfrac{8}{(z-1)^3}$

Soln. At $z=1$, it is pole of order $=3$

Q.9. $f(z)=\cosh\left[1/\left(z^2+1\right)\right]$

Soln. $f(z)=\cosh\left[1/\left(z^2+1\right)\right]$

At $z=\pm i$ it is essential singularity.

Q.10. $f(z)=\dfrac{e^{1/(z-1)}}{e^z-1}$

Soln. $f(z)=\dfrac{e^{1/(z-1)}}{e^z-1}$, $e^z-1=0$, $e^z=1$, $z=0$

$z=0$ is a pole of order $=1$

Q.11. Discuss e^{1/z^2} in a similar way as $e^{1/z}$

Soln. $e^{1/z}=1+\dfrac{1}{z}+\dfrac{\left(\frac{1}{z}\right)^2}{\lfloor 2}+\dfrac{\left(\frac{1}{z}\right)^3}{\lfloor 3}+\ldots\ldots=1+\dfrac{1}{z}+\dfrac{1}{z^2\lfloor 2}+\dfrac{1}{z^3\lfloor 3}+\ldots\ldots$

$z=0$ is an essential singularity.

$e^{1/z}=1+\dfrac{1}{z^2}+\dfrac{\left(\frac{1}{z^2}\right)^2}{\lfloor 2}+\dfrac{\left(\frac{1}{z^2}\right)^3}{\lfloor 3}+\ldots\ldots=1+\dfrac{1}{z^2}+\dfrac{1}{z^4\lfloor 2}+\dfrac{1}{z^6\lfloor 3}+\ldots\ldots$

$z=0$ is an essential singularity.

Q.12. $f(z)=z^{-3}-z^{-1}$

Soln. $f(z)=\dfrac{1}{z^3}-\dfrac{1}{z}$

$z=0$, pole of order $=3$

Determine the location and order of the zeros.

Q.13. $f(z)=(z+16i)^4$

Soln. $f(z)=0$, $(z+16i)^4=0$, $z=-16i$, It has order four.

Q.14. $f(z)=(z^4-16)^4$

Soln. $f(z)=0$, $(z^4-16)^4=0$, $((z^2-4)(z^2+4))^2=0$, $z=\pm 2, \pm 2i$, It has order = 4

Q.15. $f(z)=\dfrac{\sin^3 \pi z}{z^3}$

Soln. $f(z)=0$, $\dfrac{\sin^3 \pi z}{z^3}=0$, $\sin^3 \pi z=0$

$\pi z=\pm n\pi$, n = 0, 1, 2,........., $z=\pm 1, \pm 2$,....., It has order = 3

Q.16. $f(z)=\cosh^2 z$

Soln. $f(z)=0$, $\cosh^2 z=0$, $\cosh z=0$, $\dfrac{e^z+e^{-z}}{2}=0$, $e^{2z}+1=0$

$2z=\ell n(-1)=\ell n 1+\pi i+2n\pi i$, $z=\dfrac{(2n+1)\pi i}{2}$, n = 0, 1, 2,...., It has order = 2

Q.17. $f(z)=(3z^2+1)e^{-z}$

Soln. $f(z)=0$, $(3z^2+1)e^{-z}=0$, $z=\pm i/\sqrt{3}$, It has order = 1

Q.18. $f(z)=(z^2-1)^2(e^{z^2}-1)$

Soln. $f(z)=0$, $(z^2-1)^2(e^{z^2}-1)=0$

$z=\pm 1$, it has order = 2

$z=0$, it has order = 1

e^z

$\ln z = \ln|z| + i \arg z + 2n\pi i$

Q.19. $f(z)=(z^2+4)(e^z-1)^2$

Soln. $f(z)=0$, $(z^2+4)(e^z-1)^2=0$, $z^2+4=0$, $z=\pm 2i$ It has order = 1

$e^z-1=0$, $z=\ell n1=\pm 2n\pi i$ (n = 0, 1, 2....) It has order = 2

Q.20. $f(z)=(\sin z-1)^3$.

Soln. $f(z)=0$, $(\sin z-1)^3=0$, $\sin z=1$

$z=(4n+1)\frac{\pi}{2}$, It has order = 6

Q.21. $f(z)=(1-\cos z)^2$

Soln. $f(z)=0$, $(1-\cos z)^2=0$, $1-\cos z=0$, $\cos z=1$, $z=\pm 2n\pi$, n = 0, 1, 2.........

$z=0$, $\pm 2\pi$, $\pm 4\pi$,......., it has order = 4

Q.22. $f(z)=e^z-e^{2z}$

Soln. $f(z)=0$, $e^z-e^{2z}=0$, $e^z(1-e^z)=0$

$z=\pm 2n\pi i$ (n = 0, 1, 2,......), It has order = 1

Q.23. If f(z) is analytic and has a zero of order n at $z=z_0$. Show that $f^2(z)$ has a zero of order 2n.

Soln. $f(z)=(z-z_0)^n g(z)$, $g(z_0)\neq 0$

Hence $f^2(z)=(z-z_0)^{2n} g^2(z)$