Linear Algebra MA 20105

Recall : $AX = b$ $\quad X=\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$ $\quad b=\begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix}$

A $m \times n$ matrix

Gaussian Elimination

Step-1: Form $\tilde{A} = [A \mid b]$

Step-2: Apply elementary row operations to arrive in the following

What are the elementary row operations?

(1) $R_j' = cR_i + R_j$ (2) $R_j = cR_j$, $c \neq 0$

(3) $R_i \longleftrightarrow R_j$: Interchanging ith row & jth row

A = LU decomposition

$$x + y + z = 1$$
$$2x + 2y + 5z = 3$$
$$4x + 6y + 8z = 8$$

$A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 5 \\ 4 & 6 & 8 \end{bmatrix}$ $\quad b = \begin{bmatrix} 1 \\ 3 \\ 8 \end{bmatrix}$

pivot

$$\tilde{A} = \left[\begin{array}{ccc|c} \boxed{1} & 1 & 1 & 1 \\ 2 & 2 & 5 & 3 \\ 4 & 6 & 8 & 8 \end{array}\right] \xrightarrow[R_3' = -4R_1 + R_3]{R_2' = -2R_1 + R_2} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 0 & 3 & 1 \\ 0 & 2 & 4 & 4 \end{array}\right] \xrightarrow{R_2 \leftrightarrow R_3}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = E_1$$

$$\left[\begin{array}{ccc|c} \boxed{1} & 1 & 1 & 1 \\ 0 & \boxed{2} & 4 & 4 \\ 0 & 0 & \boxed{3} & 1 \end{array}\right]$$

by back substitution we can solve for x, y, z

A matrix $n \times n$ is called singular if $\det(A) = 0$

otherwise, we call it non-singular

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc, \quad \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} =$$

$$\begin{vmatrix} & & \\ c_1 & c_2 & c_3 \end{vmatrix}$$

$$= a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$$

Properties: $\det(AB) = \det(A)\det(B)$

$\det(cA) = c^n \det(A)$

In the case of a singular matrix, we will have some pivots to be zero. $A \sim U$

$\text{rank}(A) :=$ the highest order of the square sub-matrix of the given matrix A whose minor is non-zero.

For example: $\begin{bmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{bmatrix} = I_{n \times n}$

$O = [0]_n$ $\quad \text{rank}(I) = n \quad \text{rank}(O) = 0$

$$\tilde{A} \quad \underbrace{\overbrace{\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 3 & 4 \\ 0 & 1 & 3 & 4 \end{bmatrix}}^{A}}_{A_1} \; 3 \times 4 \qquad rk(\tilde{A}) = 2$$

$rk(A) \neq rk(\tilde{A})$

$$\tilde{A} = \left[\begin{array}{cc|c} 1 & 1 & 1 \\ 2 & 2 & 3 \end{array}\right] \qquad rk(\tilde{A}) = 2$$

$-2R_1 + R_2$ $\quad \underline{A}\; 0 \;\; 0 \;\; 1 \quad 0 = 1 \qquad rk(A) = 1$

$Ax = b$ $\quad m$ equ^{n}s & n unknowns

Case 1: Unique sol^n if $rk(A) = rk(\tilde{A}) = n$

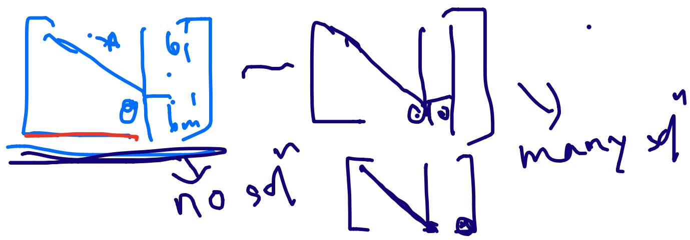
Case 2: $rk(A) < rk(\tilde{A}) \Rightarrow$ No sol^n

Case 3: $rk(A) = rk(\tilde{A}) = r < n \Rightarrow$ infinitely many solutions.

$\times \longleftrightarrow rk(A) = rk(\tilde{A}) = 2$

$/\!/ \longleftrightarrow rk(A) < rk(\tilde{A})$

$\not\!/ \longleftrightarrow rk(A) = rk(\tilde{A}) < 2$



A matrix B is said to be elementary if B can be obtained from I by finite elementary row operations

Eg: $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{R_2 = 2R_1 + R_2} \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

What is the connection b/w elry row op^{n}s & elry matrices

$$\begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 5 \\ 4 & 6 & 8 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 3 \\ 4 & 6 & 8 \end{bmatrix}$$

Note: doing row oper^{n}s $\equiv$ multiply elry matrix from the left.

$$E_3 E_2 E_1 A \sim\sim\sim U$$

$$E_3 E_2 E_1 A = U$$ ← upper triangular matrix.

E_1, E_2, E_3. How do they look?

In the Gaussian Elimination
The elmry matrix we get,
they are lower triangular.

Qn! How the inverses of
lower & upper triangular
matrices look like?

Lecture - 3

Recall: $AX = b$

Gaussian elt^n, row oper^n's, elmry matrices
rank(A), 2D, 3D Geometrical Interpretation

when system has a
unique sol^n, No solution &
infinite sol^n.

A $n \times n$ matrix

$A = LU$ ← LU decomposition

Lower   Uper triangular

How does it help? $AX = b$

$LUx = b$ $Ux = y$ upper triangular

$Ly = b$, $y = c$ $Ux = c$ Backward substitution

Forward substitution   We get "x".

$$A \sim\sim\sim U$$

$$E_1^{-1}E_2^{-1}E_3^{-1}E_3E_2E_1A = E_1^{-1}E_2^{-1}E_3^{-1}U$$

$A = LU$ L lower triangular

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 5 \\ 4 & 6 & 8 \end{bmatrix} \xrightarrow{-2R_1+R_2=R_2'} \boxed{\begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 3 \\ 4 & 6 & 8 \end{bmatrix}} = E_1 A$$

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{-2R_1+R_2=R_2'} \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = E_1$$

A $n \times n$ matrix

An inverse of A is an another matrix

B such that $AB = BA = I$

we say that A is invertible if A^{-1}.

exists.

$$R_2' = +2R_1 + R_2$$

$$\text{matrix} \xrightarrow{\text{Elry}} F_1 = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

$$F_1 E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow F_1 = E_1^{-1}$$

Uniqueness of inverse: $AB = BA = I$, $AC = CA = I$

$$(BA)C = B(AC) \Rightarrow \text{Inverse is unique}$$
$$IC = BI$$
$$C = B$$

$$E_1 A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 3 \\ 4 & 6 & 8 \end{bmatrix} \xrightarrow{R_3' = -4R_1 + R_3} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 3 \\ 0 & 2 & 4 \end{bmatrix}$$

$$E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -4 & 0 & 1 \end{bmatrix}, \quad E_2^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 4 & 0 & 1 \end{bmatrix}$$

$$E_2 E_1 A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 3 \\ 0 & 2 & 4 \end{bmatrix} \xrightarrow{R_2 \leftrightarrow R_3} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 4 \\ 0 & 0 & 3 \end{bmatrix} = U \qquad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} = E_3$$

$E_3^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$ $\quad E_3 E_2 E_1 A = U$

Problem: E_3 is neither upper triag. nor Lower triag

Claim: If in the G.E. type 3 operation comes, then LU factorization is not possible

(type 3 = interchanging rows)

Ex: $A = \begin{bmatrix} 1 & 2 \\ 3 & 8 \end{bmatrix}$ $\quad A = LU$

$R_2' = -3R_1 + R_2$ $\quad \begin{bmatrix} 1 & 2 \\ 0 & 2 \end{bmatrix} = U$

$E_1 A = U$

$E_1 = \begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix}$, $E_1^{-1} = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}$

$R_2' = 3R_1 + R_2$

$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}$

$E_1 A = U$

$A = E_1^{-1} U$; $\begin{bmatrix} 1 & 2 \\ 3 & 8 \end{bmatrix} = \underset{L}{\begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}} \underset{U}{\begin{bmatrix} 1 & 2 \\ 0 & 2 \end{bmatrix}}$

$\begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$BA = I \qquad AX = b$

$\begin{bmatrix} a_{11} & & 0 \\ a_{21} & c_2 & \\ a_{n1} & \cdots & a_{nn} \end{bmatrix}$ $\quad x_1 = b$, $\quad x_1 = \frac{b_1}{a_{11}}$

$LUX = b$

$LY = b$

$Y = \ldots$

$Ux = c$

$$\begin{bmatrix} \diagdown & \\ 0 & \end{bmatrix} x = c$$

Ex: $A = \begin{bmatrix} 0 & 2 \\ 3 & 4 \end{bmatrix} \neq LU$