## Any Questions?

zero elt of the vector space depends on the vector space.

Eg: $\mathbb{R}$ is V.S. over $\mathbb{R}$; $\mathbb{R}^2$ is V.S. over $\mathbb{R}$

$0 \in \mathbb{R} \leftarrow$ zero elt.

$\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ is the zero vector.

$\mathbb{R}^n$ $n \geq 1$ V.S. over $\mathbb{R}$

$\begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$ n-times is the zero elt.

$(V, +$ $\quad +: V \times V \to V$

field $\mathbb{R}$ or $\mathbb{C}$ $\quad *: \mathbb{R} \times V \to V$

V be the real V.S.

$(V := \{ f : \mathbb{R} \to \mathbb{R} \}, +)$ is V.S. over $\mathbb{R}$

$S = \{ f \in V : f(3) = 1 + f(-5) \}$

$r \in \mathbb{R}, f, g,$ $f + g \in S$?
$r f \in S$!

$(f+g)(1) = 1 + (f+g)(-5)$

$\Rightarrow f(1) + g(1) = 1 + f(-5) + g(-5)$

$f(1) = 1 + f(-5)$

$g(1) = 1 + g(-5)$

$$f(1) + g(1) = 2 + f(-5) + g(-5)$$
$$\neq 1 + f(-5) + g(-5)$$

Not a subspace.

On $\mathbb{R}^2$

$$(x, y) + (x_1, y_1) = (x + x_1, 0)$$
$$c(x, y) = (cx, 0)$$

I.(1) $\quad \underline{(a, b) + (x, y)} = (x, y)$

$\Rightarrow (a + x, 0) = (x, y)$

$\Rightarrow a + x = x, \quad y = 0$

$a = 0, \quad y = 0$

$\nexists (a, b) \in \mathbb{R}^2, \; (a, b) + (x, y) = (x, y)$

So the zero elt does not exist!

V is a real vector space and U, W are

two subspaces of V.

We are defining linear sum of U & W

$$\underbrace{U + W}_{\text{Notation}} := \{u + w : u \in U, w \in W\}$$

Claim: $U + W$ is also a subspace

$\alpha, \beta \in U+W, \; c \in \mathbb{R}$

(i) $\alpha+\beta \in U+W$?

(ii) $c\alpha \in U+\alpha$?

<u>For (i)</u>: $\alpha = u_1 + w_1$

$\beta = u_2 + w_2$

$$\alpha+\beta = u_1 + w_1 + u_2 + w_2$$

$$= \underbrace{u_1 + u_2}_{U} + \underbrace{w_1 + w_2}_{W}$$

$$= u' + w'$$

where $u' = u_1 + u_2$

$w' = w_1 + w_2$

$\Rightarrow \alpha+\beta \in U+W$

Similarly check that $c\alpha \in U+W$

$U+W$ is a subspace of $V$.

$U_1, U_2, \dots, U_n$ are subspace of $V$

<u>Define</u> $U_1 + U_2 + \dots + U_n$ in a similar way. Then

check that $U_1 + U_2 + \dots + U_n$ is a subspace of $V$.

$U+W \supseteq U$ and $W$

$u \in U$, $\underline{u = u + 0} \in U+W$

$U \leq U+W$

$W \leq U+W$

$U, W$ are two subspace of $V$.

$U + W \leq V$
$U + W = V$
$\alpha_1 + \alpha_2 = \alpha$ ($\alpha_1 \in U$, $\alpha_2 \in W$, $\alpha \in V$)

Why $\alpha = \alpha_1 + \alpha_2$ is a unique expression?

$U \cap W = (0)$.

$U, W \leq V$

① $U \cup W$ ③ $U^c$ Are ①, ②, ③ are subspaces of $V$?
② $U \cap W$

Answer! ① and ③ are not but ② is.

$0 \in U \leq V$

$U^c := V \setminus U$

$0 \notin U^c$. Hence $U^c$ can not be subspace of $V$.

Why $U \cup W$ is not a subspace $V$?

$V = (\mathbb{R}^2, +)$ over $\mathbb{R}$.

$U = \{(x, 0) : x \in \mathbb{R}\} \leftarrow$ x-axis

$W = \{(0, y) : y \in \mathbb{R}\} \leftarrow$ y-axis

$(1,0), (0,1) \in U \cup W \quad (1,0) + (0,1) = (1,1) \notin U \cup W$

Hence $U \cup W$ is not a subspace of $\mathbb{R}^2$.

$U, W \leq V$

$U \subseteq W$ or $W \subseteq U$

is a subspace of $V$.

Prop$^n$: $U \cup W$ is a subspace of $V$ if and only if $U \subset W$ or $W \subset U$.

Proof: Exercise.

Recall: L.I. and L.D. set $S \subseteq V$.

$$S = \{u_1, \ldots, u_n\} \subseteq V$$

$U \leq V$ a subspace of $V$.

Then a set $S \subseteq U$ is called $\boxed{\text{basis}}$ of $U$

if (i) $L(S) = U$

(ii) $S$ is linearly independent (L.I.) set.

dimension of $U$ which is denoted by

$$\dim(U) := |S|$$

$V = (\mathbb{R}^2, +)$ over $\mathbb{R}$.

$S = \{(1,0), (0,1)\} \subseteq \mathbb{R}^2$

Is $S$ a basis of $\mathbb{R}^2$?

Yes: (i) $L(S) = \mathbb{R}^2$? $(x,y) \in \mathbb{R}^2$

$L(S) \subseteq \mathbb{R}^2$ $\quad (x,y) = x(1,0) + y(0,1)$

$x, y \in \mathbb{R}$

$\Rightarrow (x,y) \in L(S)$. $\Rightarrow \mathbb{R}^2 \subseteq L(S)$

$\therefore L(S) = \mathbb{R}^2$

Need to check $S$ is L.I.

$c_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$\Rightarrow \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$\Rightarrow c_1 = 0$ & $c_2 = 0$ ✓

$\therefore S$ is L.I. ✓

So $S$ is a basis of $\mathbb{R}^2$.

$\dim(\mathbb{R}^2) = |S| = 2$.

$|S|$ = the number of elts inside $S$

$(\mathbb{R}^n, +, \cdot)$ over $\mathbb{R}$

$S = \{(1,0,\ldots,0), (0,1,\ldots,0), (0,0,\ldots,1)\}$

Similarly check that $L(S) = \mathbb{R}^n$
and $S$ is L.I.

$S$ is a basis of $\mathbb{R}^n$

$\dim(\mathbb{R}^n) = n$.

Is

$S$ be any set.

Cardinality of $S$ which is denoted by $|S|$ is equal to the number of elts in the set $S$.

$S = \{1, 2, 3\}$ $|S| = 3$

$S = \{1, 2, 3, \dots\}$ $|S| = \infty$

$S \subseteq \mathbb{R}^2$

$S = \left\{\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}1\\1\end{pmatrix}, \begin{pmatrix}2\\1\end{pmatrix}\right\}$

$|S| = 3$

$(\mathbb{R}^n, +, \cdot)$

$S = \{\underset{e_1}{(1, 0, \dots, 0)}, \underset{e_2}{(0, 1, \dots, 0)}, \dots, \underset{e_n}{(0, 0, \dots, 1)}\}$

So $S$ is a basis of $\mathbb{R}^n$

$S$ is called standard basis of $\mathbb{R}^n$.

Is basis of a V.Space V unique?

Answer is <u>NO</u>. Basis is not unique.

But the number of elements in the basis is unique.

$(\mathbb{R}^2, +, \cdot)$

$S = \{(1,0), (0,1)\}$ is a basis of $\mathbb{R}^2$.

Claim: $S_1 = \left\{ \begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}1\\1\end{pmatrix} \right\}$ is also a basis of $\mathbb{R}^2$.

Need to check (i) $L(S_1) = \mathbb{R}^2$

(ii) $S_1$ is L.I.

$L(S_1) \subseteq \mathbb{R}^2$

$\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}1\\1\end{pmatrix} \in$ Can show $\begin{pmatrix}0\\1\end{pmatrix} \in L(S_1)$

$$\begin{pmatrix}0\\1\end{pmatrix} = -1\begin{pmatrix}1\\0\end{pmatrix} + 1\begin{pmatrix}1\\1\end{pmatrix}$$

$$= \begin{pmatrix}-1+1\\0+1\end{pmatrix} = \begin{pmatrix}0\\1\end{pmatrix}$$

$$\Rightarrow \begin{pmatrix}0\\1\end{pmatrix} \in L(S_1) = L(S) = \mathbb{R}^2$$

$L(S_1) = \mathbb{R}^2$

$$\boxed{\begin{pmatrix}x\\y\end{pmatrix} \in \mathbb{R}^2 \quad \begin{pmatrix}x\\y\end{pmatrix} = c_1\begin{pmatrix}1\\0\end{pmatrix} + c_2\begin{pmatrix}1\\1\end{pmatrix}}$$

For L.I.

$$c_1\begin{pmatrix}1\\0\end{pmatrix}+c_2\begin{pmatrix}1\\1\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}$$

$$\Rightarrow \begin{pmatrix}c_1+c_2\\c_2\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}$$

$\Rightarrow c_2=0$ & $c_1+c_2=0$

$\Rightarrow c_1=0$

Hence $S_1$ is L.I.

$\therefore$ $S_1$ is a basis of $\mathbb{R}^2$.

$$T=\left\{\begin{pmatrix}1\\2\\3\end{pmatrix}\begin{pmatrix}4\\5\\6\end{pmatrix}\begin{pmatrix}9\\10\\12\end{pmatrix}\begin{pmatrix}3\\2\\0\end{pmatrix}\right\}$$

Is T L.I. ?

Tomorrow: echelon form ??
rowreduced " "

$\tilde{A}$

$Ax=b$ | $LUx=b$, $\underline{Ux=y}$
$A=LU$ | $\underline{Ly=b}$ solve for $x$