Are there any Questions ??

sets $\longleftrightarrow$ Space, Vector space (Real or Complex)

Solution sets $\rightarrow$ Solution Space

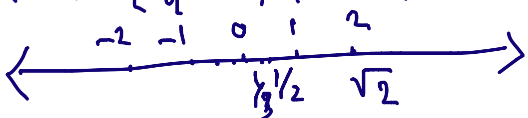
Real Vector Space: $\mathbb{R}$

$\mathbb{N} = \{0, 1, 2, 3, \ldots\}$

$\mathbb{Z} = \{0, \pm 1, \pm 2, \pm 3, \ldots\}$

$\mathbb{Q} = \{\frac{p}{q} : p, q \in \mathbb{Z}, q \neq 0\}$

scalar means either real numbers or complex number



$\mathbb{R}$, $\mathbb{C} = \{a + ib : a, b \in \mathbb{R},\ i = \sqrt{-1}\}$

Def^n: files $\rightarrow$ Class material

A non-empty set V of elts $a, b, c \ldots$ is called real vector space and these elts are called vectors

if in V, there two algebraic operations

(called vector addition and scalar multiplication) which satisfies following axioms

I. Vector addition associates with every pair of vectors a, b of to an another vector $a+b$ (called sum of a and b, such that the folling conditions are satisfied:

I.(1) $a + b = b + a$ (commutativity)

I.(2) $a + (b + c) = (a + b) + c = a + b + c$ (associativity)

I.3: $\exists!\, 0 \in V$, $a + 0 = a \ \forall a \in V$ "0" zero

I.4. zero vector for $a \in V$, $\exists\, (-a) \in V$ such that

$a + (-a) = 0$

II Scalar multiplication. The scalars means real numbers. Scalar multiplication associates to each $a \in V$ and to each scalar r a unique vector ra (or ar) $\in V$ s.t. the following are satisfied.

II.1: $r(a+b) = ra + rb$ distributive property
II.2: $(r+s)a = ra + sa$
II.3 $r(sa) = (rs)a$ holds
II.4 $\forall a \in V$ $1 \cdot a = a$, $1 \in \mathbb{R}$

I $\underset{a}{V} \times \underset{b}{V} \xrightarrow{+} \underset{a+b}{V}$

II $\underset{r}{\mathbb{R}} \times \underset{a}{V} \longrightarrow \underset{ra}{V}$

$V = \mathbb{R}^3 = \mathbb{R} \times \mathbb{R} \times \mathbb{R} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \in \mathbb{R}^3$, $\mathbb{R}$ - real numbers

$\underline{+}$: $a + b = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} + \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = \begin{pmatrix} a_1 + b_1 \\ a_2 + b_2 \\ a_3 + b_3 \end{pmatrix} \in \mathbb{R}^3$

$r \in \mathbb{R}$, $\underline{ra} = r \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} ra_1 \\ ra_2 \\ ra_3 \end{pmatrix} \in \mathbb{R}^3$

$(\mathbb{R}^3, +, \cdot)$ is a real vector space.

Yes: I.1 $a+b = b+a$

LHS $a+b = \begin{pmatrix} a_1+b_1 \\ a_2+b_2 \\ a_3+b_3 \end{pmatrix} = \begin{pmatrix} b_1+a_1 \\ b_2+a_2 \\ b_3+a_3 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} + \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = \text{RHS}$

I.2. $a + (b+c) = (a+b)+c$

$\begin{pmatrix} a_1+(b_1+c_1) \\ a_2+(b_2+c_2) \\ a_3+(b_3+c_3) \end{pmatrix} = \begin{pmatrix} (a_1+b_1)+c_1 \\ (a_2+b_2)+c_2 \\ (a_3+b_3)+c_3 \end{pmatrix} = (a+b)+c$
$= \text{RHS}$

I.3. $\underline{0} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \in \mathbb{R}^3 = V$

$a + \underline{0} = a$, L.H.S $\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} a_1+0 \\ a_2+0 \\ a_3+0 \end{pmatrix} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$

I.4: $a = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$, $-a = \begin{pmatrix} -a_1 \\ -a_2 \\ -a_3 \end{pmatrix}$ $a + (-a)$
$= \begin{pmatrix} a_1+(-a_1) \\ a_2+(-a_2) \\ a_3+(-a_3) \end{pmatrix}$

$= \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

II. scalar multiplications L.H.S $r(a+b)$

II.1 $r(a+b) = ra + rb$? $= r\begin{pmatrix} a_1+b_1 \\ a_2+b_2 \\ a_3+b_3 \end{pmatrix}$

$= \begin{pmatrix} r(a_1+b_1) \\ r(a_2+b_2) \\ r(a_3+b_3) \end{pmatrix} = \begin{pmatrix} ra_1+rb_1 \\ ra_2+rb_2 \\ ra_3+rb_3 \end{pmatrix} = \begin{pmatrix} ra_1 \\ ra_2 \\ ra_3 \end{pmatrix} + \begin{pmatrix} rb_1 \\ rb_2 \\ rb_3 \end{pmatrix}$

$= ra + rb = \text{R.H.S.}$

II.2 $(r+s)a = ra + sa$ (check)

II.3 $r(sa) = (rs)a$ (check)

II.4 $1a = a$, $1\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} 1a_1 \\ 1a_2 \\ 1a_3 \end{pmatrix} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = a$

We have checked that $(\mathbb{R}^3, +, \cdot)$ is a real vector space.

Observe that: $(\mathbb{R}^n, +, \cdot)$ is a real vector space for all $n \geq 1$.

$\mathbb{R}, \mathbb{R}^2, \mathbb{R}^3, \ldots$

$M_{m \times n} := \{$ set of all $m \times n$ matrices $\}$

$A + B \rightarrow$ matrix addition

$A = (a_{ij})$

$r \in \mathbb{R}$, $rA = [ra_{ij}]_{\substack{1 \leq i \leq m \\ 1 \leq j \leq n}}$

$(M_{m \times n}, +, \cdot)$ is a real vector space.

Exercise!

Replace real numbers by complex numbers in the definition, we get complex vector space

Fundamental Idea of Linear algebra is

" to take <u>linear combinations</u>"

$a = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \in \mathbb{R}^3, \quad b = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}. \quad a \pm b = \begin{pmatrix} a_1 + b_1^2 \\ a_2 + b_2^2 \\ a_3 + b_3^2 \end{pmatrix}$

$b + a = \begin{pmatrix} b_1 + a_1^2 \\ b_2 + a_2^2 \\ b_3 + a_3^2 \end{pmatrix} \neq a + b$

$a = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \quad b = \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix} \Rightarrow a + b \neq b + a$

$V, \quad v_1, .., v_n \in V$

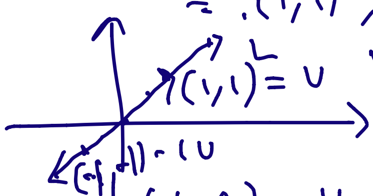
$\boxed{c_1 v_1 + c_2 v_2 + \cdots + c_n v_n}$, c_i's are scalar

This is one linear combination of $v_1, .., v_n$

$\mathbb{R}^2 \quad v = (1,1) \in \mathbb{R}^2$

$c \in \mathbb{R}, \quad c(1,1) \quad c = 1 \quad (1,1) \quad c = 5 \quad (5,5)$

$= (1,1), \quad c = 2 \quad (2,2)$

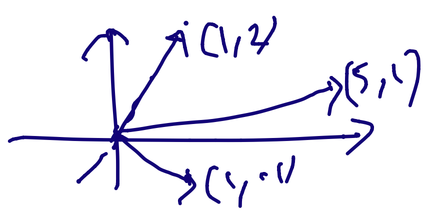


vary c

You will get whole line L

$v_1 = (1,2), \quad v_2 = (1,-1)$

$2(1,2) + 3(1,-1) = (5,1)$



set $S \subseteq V$

linear span of $S = L(S)$

$= \{ c_1 v_1 -$

$$L(S) := \left\{ c_1 v_1 + \cdots + c_m v_m : \begin{array}{l} c_i \in \mathbb{R} \\ v_i \in S \end{array} \right\}$$