Uploaded Notes in the MS Team

$S = \left\{ \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}, \begin{pmatrix} 3 \\ 6 \\ -3 \end{pmatrix}, \begin{pmatrix} 3 \\ 9 \\ 3 \end{pmatrix}, \begin{pmatrix} 2 \\ 7 \\ 4 \end{pmatrix} \right\}$ L.I. or L.D. ?

Ans : L.D.

$$A = \begin{bmatrix} 1 & 3 & 3 & 2 \\ 2 & 6 & 9 & 7 \\ -1 & -3 & 3 & 4 \end{bmatrix}$$

Method-1: Note that $\begin{pmatrix} 3 \\ 6 \\ -3 \end{pmatrix} = 3 \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} \Rightarrow \left\{ \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}, \begin{pmatrix} 3 \\ 6 \\ -3 \end{pmatrix} \right\}$ L.D.

Prop^n 1: Superset of L.D. set is L.D.

Prop^n 2: Subset of L.I. set is L.I.

Conclusion! The S is L.D. by prop^n.

Method-2: make the matrix $A \rightsquigarrow$ echelon form

$$A \sim \begin{bmatrix} 1 & 3 & 3 & 2 \\ 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Fact: If there is atleast one zero rows, then the set S is L.D.
Otherwise L.I.

Since there is a zero row in the echelon form, so we conclude that S is L.D.

Prop^n: If two vectors $v_1, v_2 \in V$ are L.D., then then one is scalar multiple of the other one.

Proof: $c_1 v_1 + c_2 v_2 = 0$, c_1, c_2 are scalars.

L.D. $\Rightarrow$ either $c_1 \neq 0$, or $c_2 \neq 0$; We can assume $c_1 \neq 0$,

$$c_1^{-1} c_1 v_1 + c_1^{-1} c_2 v_2 = 0 \Rightarrow v_1 = \underbrace{\left(-\frac{c_2}{c_1}\right)}_{= r} v_2$$

$v_1 = r v_2$ $\blacksquare$

yesterday, we computed the null space $N(A)$

What is the dimension of $N(A)$?

$$N(A) = \left\{ c_1 \begin{pmatrix} -3 \\ 1 \\ 0 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ 0 \\ -1 \\ 1 \end{pmatrix} : c_1, c_2 \text{ are scalars} \right\}$$

$$B = \left\{ \begin{pmatrix} -3 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ -1 \\ 1 \end{pmatrix} \right\}; \quad L(B) = N(A)$$

B is L.I. because $\begin{pmatrix} 1 \\ 0 \\ -1 \\ 1 \end{pmatrix}$ can not be written as $c \begin{pmatrix} -3 \\ 1 \\ 0 \\ 0 \end{pmatrix}$.

$$\dim(N(A)) = |B| = 2$$

Observation: $\dim(N(A))$
= no. of columns − no. of pivots
$= 4 - 2 = 2$

Rank-Nullity theorem. Fundamental theorem of linear algebra

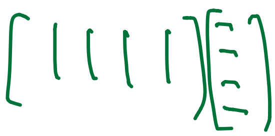
Fact: row-rank(A) = column-rank(A) = rank(A)
= the no of pivots

$AX = 0 \qquad AX = b$

What is the connection between these two system?

Pivot variables and free variables

Suppose a matrix $m \times n$ has more columns than rows $(n > m)$

 : Since m rows can hold at most m pivots, there must be atleast $n-m$ free variables. But no matter what, at least one variable must be free. This free variable can be assigne any value.

Conclusion:

Thm: If $AX = 0$ has more unknowns than equations (ie $n > m$), It has at least one special solution: there more solution than $x = 0$.

Caution ⚠ : this may not be true in the case of $AX = b$.

$$\begin{bmatrix} & & \\ & & \end{bmatrix}_{c_1 \cdots c_n}\begin{bmatrix}x_1\\ \vdots \\ x_n\end{bmatrix} = \underbrace{\begin{bmatrix}b_1\\ \vdots\\ b_m\end{bmatrix}}_{b} \qquad C(A) \ni b$$

$$x_1 C_1 + x_2 C_2 + \cdots + x_n C_n = b$$

$b \in C(A) \Leftrightarrow AX = b$ has a solution.

A as before, 3×4 matrix $\qquad b = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$ $\qquad R_2' = -2R_1 + R_2$, $\quad R_3' = R_1 + R_3$

$$b \sim \begin{bmatrix} b_1 \\ -2b_1 + b_2 \\ b_1 + b_3 \end{bmatrix} \underset{\sim}{R_3' = -2R_2 + R_3} \begin{bmatrix} b_1 \\ -2b_1 + b_2 \\ +4b_1 - 2b_2 + b_1 + b_3 \end{bmatrix}$$

$$\left[\begin{array}{cccc|c} 1 & 3 & 3 & 2 & b_1 \\ 0 & 0 & 3 & 3 & -2b_1 + b_2 \\ 0 & 0 & 0 & 0 & 5b_1 - 2b_2 + b_3 \end{array}\right]$$

First condition

$$0 = 5b_1 - 2b_2 + b_3$$

If $5b_1 - 2b_2 + b_3 \neq 0$, then $Ax = b$ can not have solution.

$$\begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 5 \\ 5 \end{pmatrix} \qquad 5 \times 1 - 2 \times 5 + 5 = 0$$

$$\begin{bmatrix} 1 & 3 & 3 & 2 \\ 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix}$$

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$Ax = 0$; $\quad Ax = b$, $b \neq 0$

$m \times n$, $n > m$: It has non-zero sol^n

$n > m$: It may not have sol^n.

$$\begin{bmatrix} \boxed{1} & 3 & 3 & 2 \\ 0 & 0 & \boxed{3} & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix} \Rightarrow \begin{aligned} u + 3v + 3w + 2y &= 1 \\ 3w + 3y &= 3 \\ 0 &= 0 \end{aligned}$$

– pivot – free

v, y are free variable. (1) $v = 1, \; y = 0$

$v = c_1, \; y = c_2$, c_1, c_2 are scalars $\begin{pmatrix} u \\ v \\ w \\ y \end{pmatrix} = \begin{pmatrix} -5 \\ 1 \\ 1 \\ 0 \end{pmatrix}$

$$w + y = 1 \Rightarrow w = 1 - c_2$$

$$u + 3v + 3w + 2y = 1$$

$$u = 1 - 3c_1 - 3(1 - c_2) - 2c_2 = -2 - 3c_1 + c_2$$

$$\begin{pmatrix} u \\ v \\ w \\ y \end{pmatrix} = \begin{pmatrix} -2 - 3c_1 + c_2 \\ c_1 \\ 1 - c_2 \\ c_2 \end{pmatrix} = \underbrace{\begin{pmatrix} -2 \\ 0 \\ 1 \\ 0 \end{pmatrix}}_{x_p} + \underbrace{c_1 \begin{pmatrix} -3 \\ 1 \\ 0 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ 0 \\ -1 \\ 1 \end{pmatrix}}_{x_n}$$

c_1, c_2 are scalars

$c_1 \begin{pmatrix} -3 \\ 1 \\ 0 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ 0 \\ -1 \\ 1 \end{pmatrix}$ is sol^n for $AX = 0$

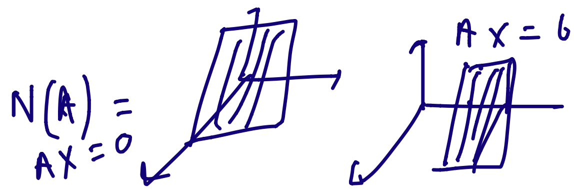
Sol^n of $AX = b$

$x_p + x_n$

x_p : particular solution

x_n : the sol^n of $Ax = 0$

Sol^n of $AX = b$. never become a space for $b \neq \underline{0}$. Because solution set in this case does not contain the zero vector.



Thm! Suppose $v_1, \dots, v_m$ and $w_1, \dots w_n$ are both bases of a vector space V. Then $m = n$.

Proof: Suppose $n > m$. (Then we will arrive at a contradiction, and say that it is not possible)

Since v's are a basis, they must span V.

$\therefore$ Every w's can be written as a linear combination of v's.

$$w_1 = a_{11} v_1 + \cdots + a_{m1} v_m$$
$$w_2 = a_{12} v_1 + \cdots + a_{m2} v_m$$
$$\vdots$$
$$w_n = a_{1n} v_1 + \cdots + a_{mn} v_m$$

$$\underbrace{\begin{bmatrix} w_1 & w_2 & \cdots & w_n \end{bmatrix}}_{W} = \underbrace{\begin{bmatrix} v_1 & \cdots & v_m \end{bmatrix}}_{V} \underbrace{\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & & a_{mn} \end{bmatrix}}_{A \quad m \times n \text{ matrix}}$$

$n > m$, $\exists X \neq 0$, $AX = 0$

$$WX = VAX = 0$$

$WX = 0 \Rightarrow$ there exists a non-zero linear combination of W's $\to$ this contradicts to the

Fact that w's are L.I.

$w_1, \dots, w_n \qquad x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$

$x_1 w_1 + \dots + x_n w_n = 0$ but not all x_i's are zero.

$\hookrightarrow$ non-zero linear combination.

$\Rightarrow \quad n \leq m.$

Similarly if we proceed with $m > n$, then we will arrive at a similar contradiction.

$m \leq n.$

Hence $m = n.$

We have proved that the number of elts in a basis is unique

Remark: In a subspace W $(\subseteq V)$ of dimension k. no set of more than k-vectors can be independent, and no set of fewer than k-vectors can span space.

Remark: A basis is maximal independent set and a minimal spanning set.

$\dim(\mathbb{R}^n) = n$ $\mathbb{R}^3$

$W = \{ (x, y, 0) \in \mathbb{R}^3 : \}$

$\dim(W) = 2$

Method of Calculating the dimension.

step-1: Find a basis B

step-2: $|B| = \dim$

How to find a basis? i.e. we have to find a set which is L.I. and spans.

$W = \{(x,y,0) \in \mathbb{R}^3\} \leq \mathbb{R}^3$

$= \{(x,y,0):$
$= \{x(1,0,0)+y(0,1,0) : x, y \in \mathbb{R}\}$

$\Rightarrow W = L\{(1,0,0),(0,1,0)\}$

$S=\{(1,0,0),(0,1,0),(0,0,1)\}$ L.I. set.

$S_1=\{(1,0,0),(0,1,0)\} \subseteq S$

Hence S_1 is L.I.

S_1 is a basis.

$\dim W = |S_1| = 2$.