AE 102: Data Analysis and Interpretation

Hypothesis testing

January - April 2016

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Hypothesis?

> A supposition or proposed explanation made on the basis of limited evidence as a starting point for further investigation.

Hypothesis testing?

- Use data to test a hypothesis
- Use probability theory to make this numerical

Examples

- Tara asks BSA to manufacture the cycle seats
- A shipment of 100 seats arrives
- Should the shipment be accepted or not?

Example from the text

- Company purchases cables which are supposed to take a load of 7000 psi.
- Large batch arrives, should they accept the shipment?

Examples

- An experiment to determine if vitamin C helps heal a cold.
- Is there statistical evidence in the data?

Examples

- A 4-sided die is tossed 1000 times and shows up 290 times
- Is it a fair die?

- Note: example is from stat414

Basic idea

- Make an initial assumption (hypothesis) about the parameter, called the **null hypothesis** or H_0
- Collect data
- Decide if we **accept** the hypothesis or **reject** it

Example

A 4-sided die is tossed 1000 times and shows up 290 times. Is it a fair die?

- Where do we start?

Solution

- First formulate the hypothesis, $H_0 : p = 0.25$
- Collect data
- Look at it, how?

Solution

- Let Y be RV which is the number of 4's.
- $\hat{p} = Y/n = 290/1000$
- This is normally distributed
- As per assumption, mean = $p_0 = 0.25$
- $\sigma = \sqrt{p_0(1 - p_0)/n}$
- Note that $\hat{p}$ is an MLE

Solution

- Convert to a standard normal $Z = \frac{\hat{p} - p_0}{\sigma}$
- How many std-devs account for 0.29? Z = 2.92!
- Is this reasonable or not?
- What if we obtained 0.255 instead of 0.29?

Critical regions

1. Identify **criticial region** where hypothesis is **rejected**
2. Use a **level of significance**, α

Critical regions

- Could define that if $Z > 1.645$, we reject the null
- That is if $\hat{p} > 0.273$, we reject the null

- What is the probability of finding $\hat{p} > 0.273$?
- Note that this is a one-tailed test (right-tailed test)

The process

1. Define the null hypothesis, H_0 and the alternative hypothesis H_A
2. Calculate the test statistic $Z = \frac{\hat{p}-p_0}{\sigma}$
3. Identify the critical region.
4. Decide to reject/accept H_0

Note that some point estimator is used to find the $\hat{p}$ value

Example

1. $H_0 : p_0 = 0.25$, $H_A : p_0 > 0.25$
2. $Z = 2.92$
3. Set critical region as: $Z_{critical} > 1.645$ (or $\hat{p} > 0.27$)
4. Reject null hypothesis

Errors

- Type I error: rejecting H_0 when it is true
- Type II error: accepting H_0 when it is false

Type I error

- If $Z_{critical} = 1.645$
  - Type I error is 0.05 (this is α)!
  - Why?

Type I error

- Fix α to set critical region!
- Typical α values are 0.01, 0.05 etc.
- Called **significance level of the test**

Type I error

- Given significance level α, Type I error probability is α

Type II error

- Say we got $p = 0.273$ and accepted H_0
- If actual $p = 0.27$, what would be the error?
- $\beta(p) = P_p\{\text{accept}H_0\}$
- $P\{\hat{p} < 0.273 \text{ given } p_0 = 0.27\} = 0.587$
- Clearly, this depends on p_0 the true value
- More on this later

The p-value

- Given the test-statistic value, its corresponding α value is the **p-value**
- If p-value < α accept H_0, else reject

Note: this is not the p of the example problem!

Example

- If $\alpha = 0.05$
- When $\hat{p} = 0.29$, $P(p \geq 0.29) = 0.0018$
- p-value = 0.0018 (which is < α)
- So reject the null

Problem

If we obtained 265 4 values in the example, what is the corresponding "p-value"?

Another Problem

- q is proportion of two-wheeler drivers wearing helmets
- It was found in a study that $q = 0.14$
- The traffic safety council: makes advertisements on TV to increase this
- After two months: 104/590 riders wear helmets
- Did the advertising help?

Recap

- Statistical hypothesis: statement about some parameters of a population
- Null Hypothesis, H_0
- Example: for a normal population with variance 1: - $H_0 : \theta = 1$
- $\theta > 1$ is the alternate hypothesis, H_1
- Critical region: values there reject H_0
- Check that the hypothesis is consistent with the data

Recap

- Type I error
- Type II error
- Choose a level of significance, α
- Test statistic should have probability of type I error < α to reject the null

Summary

1. Define the null hypothesis, H_0 and the alternative hypothesis H_A
2. Calculate the test statistic $Z = \frac{\hat{p}-p_0}{\sigma}$
3. Set a level of significance, α
4. Find the p value of the test statistic
5. Decide to reject H_0 if p-value $< \alpha$

One more example

A poll of 1000 people reveals that 47% of the population is very happy. The choices were, "very happy", "fairly happy" and "not happy". Can one say that the very happy people are a minority?

Kinds of alternate hypothesis

- Given some null hypothesis, $H_0 : \mu = \mu_0$
- The possible alternate hypothesis are:
  1. Two-sided or two-tailed test: $H_A : \mu \neq \mu_0$
  2. One-sided or one-sided test:
     - $H_A : \mu < \mu_0$
     - $H_A : \mu > \mu_0$

Examples

- Your turn!

A more formal discussion

Some Definitions

- Simple hypothesis: when true completely specifies the population
- Composite hyothesis

Tests for the mean

- n-sample from normal population, with known variance σ
- $H_0 : \mu = \mu_0$, some constant
- $H_1 : \mu \neq \mu_0$
- $Z = \frac{\bar{X}-\mu_0}{\sigma/\sqrt{n}}$
- $P_{\mu_0}\{|\bar{X} - \mu_0| > c\} = \alpha$
- $P\{|Z| > \frac{c\sqrt{n}}{\sigma}\} = \alpha$

Tests for the mean

- $P\{|Z| > \frac{c\sqrt{n}}{\sigma}\} = \alpha$
- $P\{Z > \frac{c\sqrt{n}}{\sigma}\} = \alpha/2$
- By definition $P\{Z > z_{\alpha/2}\} = \alpha/2$
- Thus, $\frac{c\sqrt{n}}{\sigma} = z_{\alpha/2}$
- Reject H_0 if $\frac{\sqrt{n}}{\sigma}|\bar{X} - \mu_0| > z_{\alpha/2}$

One-sided tests

- $H_0 : \mu = \mu_0$, some constant
- $H_1 : \mu > \mu_0$
- $P_{\mu_0}\{\bar{X} - \mu_0 > c\} = \alpha$
- In this case $\frac{c\sqrt{n}}{\sigma} = z_\alpha$
- Reject H_0 if $\frac{\sqrt{n}}{\sigma}(\bar{X} - \mu_0) > z_\alpha$

One-sided tests

- $H_0 : \mu = \mu_0$, some constant
- $H_1 : \mu < \mu_0$
- $P_{\mu_0}\{\bar{X} - \mu_0 < c\} = \alpha$
- In this case $\frac{c\sqrt{n}}{\sigma} = z_\alpha$
- Reject H_0 if $\frac{\sqrt{n}}{\sigma}(\bar{X} - \mu_0) < z_\alpha$

Tests for unknown variance

- Use the sample variance
- Use t-distribution of $n-1$ d.o.f.
- Called a t-test

Other cases

- Equality of means of normal populations
  1. Known variances
  2. Unknown variances which are same
  3. Unknown variances which are not same
- Paired t-test
  - Use when the samples are correlated.
- Test for variance of a normal population

Paired tests: an example

- A new device supposedly increases mileage
- Measure mileage before and after and compare means
- *But* the two samples are correlated!
  - If car 1 is a Nano its mileage is good already
- Solution?

What about Type II errors?

- Type II error: accepting H_0 when it is false
- $\beta(\mu) = P_\mu\{\text{accept} H_0\}$
- Assumed μ_0, but true mean is μ, i.e.

$$Z = \frac{\bar{X} - \mu}{\sigma/\sqrt{n}} \sim \mathcal{N}(0,1)$$

Type II error: two-sided test

$$\begin{aligned}
\beta(\mu) &= P_\mu \left\{ \frac{\sqrt{n}|\bar{X} - \mu_0|}{\sigma} \leq z_{\alpha/2} \right\} \\
&= P\left\{ -z_{\alpha/2} - \frac{\mu}{\sigma/\sqrt{n}} \leq \frac{\bar{X} - \mu_0 - \mu}{\sigma/\sqrt{n}} \leq z_{\alpha/2} - \frac{\mu}{\sigma/\sqrt{n}} \right\} \\
&= P\left\{ \frac{\mu_0 - \mu}{\sigma/\sqrt{n}} - z_{\alpha/2} \leq \frac{\bar{X} - \mu}{\sigma/\sqrt{n}} \leq z_{\alpha/2} + \frac{\mu_0 - \mu}{\sigma/\sqrt{n}} \right\} \\
&= P\left\{ \frac{\mu_0 - \mu}{\sigma/\sqrt{n}} - z_{\alpha/2} \leq Z \leq z_{\alpha/2} + \frac{\mu_0 - \mu}{\sigma/\sqrt{n}} \right\} \\
&= \phi(\frac{\mu_0 - \mu}{\sigma/\sqrt{n}} + z_{\alpha/2}) - \phi(\frac{\mu_0 - \mu}{\sigma/\sqrt{n}} - z_{\alpha/2})
\end{aligned}$$

Type II errors

- This is called the **operating characteristic curve (OC)**
- One tries to minimize β for a given case
- $1 - \beta$ is called the **power of the test**
- One typically tries to minimize both α, β
- The β is useful to detrmine the sample size

Type II errors: one-sided

$$
\begin{aligned}
\beta(\mu) &= P_\mu\left\{\bar{X} \le \mu_0 + z_\alpha \frac{\sigma}{\sqrt{n}}\right\} \\
&= P\{Z \le \frac{\mu_0 - \mu}{\sigma/\sqrt{n}} + z_\alpha\} \\
&= \phi(\frac{\mu_0 - \mu}{\sigma/\sqrt{n}} + z_\alpha)
\end{aligned}
$$