

## 1. REPRESENTATIVE PROBLEMS

- ▶ *stable matching*
- ▶ *five representative problems*

Lecture slides by Kevin Wayne  
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<http://www.cs.princeton.edu/~wayne/kleinberg-tardos>

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### Matching med-school students to hospitals

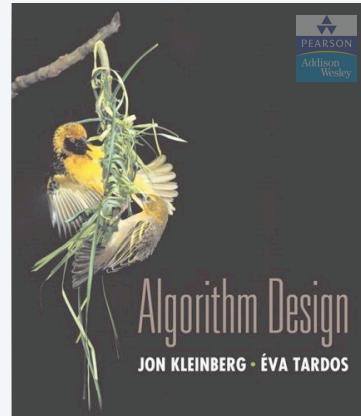
**Goal.** Given a set of preferences among hospitals and med-school students, design a **self-reinforcing** admissions process.

**Unstable pair:** student  $x$  and hospital  $y$  are **unstable** if:

- $x$  prefers  $y$  to its assigned hospital.
- $y$  prefers  $x$  to one of its admitted students.

**Stable assignment.** Assignment with no unstable pairs.

- Natural and desirable condition.
- Individual self-interest prevents any hospital–student side deal.



## 1. REPRESENTATIVE PROBLEMS

- ▶ *stable matching*
- ▶ *five representative problems*

### SECTION 1.1

### Stable matching problem

**Goal.** Given a set of  $n$  men and a set of  $n$  women, find a "suitable" matching.

- Participants rank members of opposite sex.
- Each man lists women in order of preference from best to worst.
- Each woman lists men in order of preference from best to worst.

	1 <sup>st</sup>	2 <sup>nd</sup>	3 <sup>rd</sup>
Xavier	Amy	Bertha	Clare
Yancey	Bertha	Amy	Clare
Zeus	Amy	Bertha	Clare

men's preference list

	1 <sup>st</sup>	2 <sup>nd</sup>	3 <sup>rd</sup>
Amy	Yancey	Xavier	Zeus
Bertha	Xavier	Yancey	Zeus
Clare	Xavier	Yancey	Zeus

women's preference list

## Perfect matching

**Def.** A **matching**  $S$  is a set of ordered pairs  $m-w$  with  $m \in M$  and  $w \in W$  s.t.

- Each man  $m \in M$  appears in at most one pair of  $S$ .
- Each woman  $w \in W$  appears in at most one pair of  $S$ .

**Def.** A matching  $S$  is **perfect** if  $|S|=|M|=|W|=n$ .

	1 <sup>st</sup>	2 <sup>nd</sup>	3 <sup>rd</sup>
Xavier	Amy	Bertha	Clare
Yancey	Bertha	Amy	Clare
Zeus	Amy	Bertha	Clare

	1 <sup>st</sup>	2 <sup>nd</sup>	3 <sup>rd</sup>
Amy	Yancey	Xavier	Zeus
Bertha	Xavier	Yancey	Zeus
Clare	Xavier	Yancey	Zeus

a perfect matching  $S = \{ X-C, Y-B, Z-A \}$

## Unstable pair

**Def.** Given a perfect matching  $S$ , man  $m$  and woman  $w$  are **unstable** if:

- $m$  prefers  $w$  to his current partner.
- $w$  prefers  $m$  to her current partner.

**Key point.** An unstable pair  $m-w$  could each improve partner by joint action.

	1 <sup>st</sup>	2 <sup>nd</sup>	3 <sup>rd</sup>
Xavier	Amy	Bertha	Clare
Yancey	Bertha	Amy	Clare
Zeus	Amy	Bertha	Clare

	1 <sup>st</sup>	2 <sup>nd</sup>	3 <sup>rd</sup>
Amy	Yancey	Xavier	Zeus
Bertha	Xavier	Yancey	Zeus
Clare	Xavier	Yancey	Zeus

Bertha and Xavier are an unstable pair

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## Stable matching problem

**Def.** A **stable matching** is a perfect matching with no unstable pairs.

**Stable matching problem.** Given the preference lists of  $n$  men and  $n$  women, find a stable matching (if one exists).

- Natural, desirable, and self-reinforcing condition.
- Individual self-interest prevents any man–woman pair from eloping.

	1 <sup>st</sup>	2 <sup>nd</sup>	3 <sup>rd</sup>
Xavier	Amy	Bertha	Clare
Yancey	Bertha	Amy	Clare
Zeus	Amy	Bertha	Clare

	1 <sup>st</sup>	2 <sup>nd</sup>	3 <sup>rd</sup>
Amy	Yancey	Xavier	Zeus
Bertha	Xavier	Yancey	Zeus
Clare	Xavier	Yancey	Zeus

a perfect matching  $S = \{ X-A, Y-B, Z-C \}$

## Stable roommate problem

**Q.** Do stable matchings always exist?

**A.** Not obvious a priori.

### Stable roommate problem.

- $2n$  people; each person ranks others from 1 to  $2n-1$ .
- Assign roommate pairs so that no unstable pairs.

	1 <sup>st</sup>	2 <sup>nd</sup>	3 <sup>rd</sup>
Adam	B	C	D
Bob	C	A	D
Chris	A	B	D
Doofus	A	B	C

no perfect matching is stable

$A-B, C-D \Rightarrow B-C$  unstable

$A-C, B-D \Rightarrow A-B$  unstable

$A-D, B-C \Rightarrow A-C$  unstable

**Observation.** Stable matchings need not exist for stable roommate problem.

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## Gale-Shapley deferred acceptance algorithm

An intuitive method that **guarantees** to find a stable matching.



GALE-SHAPLEY (preference lists for men and women)

INITIALIZE  $S$  to empty matching.

WHILE (some man  $m$  is unmatched and hasn't proposed to every woman)

$w \leftarrow$  first woman on  $m$ 's list to whom  $m$  has not yet proposed.

IF ( $w$  is unmatched)

Add pair  $m-w$  to matching  $S$ .

ELSE IF ( $w$  prefers  $m$  to her current partner  $m'$ )

Remove pair  $m'-w$  from matching  $S$ .

Add pair  $m-w$  to matching  $S$ .

ELSE

$w$  rejects  $m$ .

RETURN stable matching  $S$ .

## Proof of correctness: termination

**Observation 1.** Men propose to women in decreasing order of preference.

**Observation 2.** Once a woman is matched, she never becomes unmatched; she only "trades up."

**Claim.** Algorithm terminates after at most  $n^2$  iterations of while loop.

**Pf.** Each time through the while loop a man proposes to a new woman.

There are only  $n^2$  possible proposals. ▀

	1 <sup>st</sup>	2 <sup>nd</sup>	3 <sup>rd</sup>	4 <sup>th</sup>	5 <sup>th</sup>
Victor	A	B	C	D	E
Wyatt	B	C	D	A	E
Xavier	C	D	A	B	E
Yancey	D	A	B	C	E
Zeus	A	B	C	D	E

	1 <sup>st</sup>	2 <sup>nd</sup>	3 <sup>rd</sup>	4 <sup>th</sup>	5 <sup>th</sup>
Amy	W	X	Y	Z	V
Bertha	X	Y	Z	V	W
Clare	Y	Z	V	W	X
Diane	Z	V	W	X	Y
Erika	V	W	X	Y	Z

$n(n-1) + 1$  proposals required

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## Proof of correctness: perfection

**Claim.** In Gale-Shapley matching, all men and women get matched.

**Pf.** [by contradiction]

- Suppose, for sake of contradiction, that Zeus is not matched upon termination of GS algorithm.
- Then some woman, say Amy, is not matched upon termination.
- By Observation 2, Amy was never proposed to.
- But, Zeus proposes to everyone, since he ends up unmatched. ▀

## Proof of correctness: stability

**Claim.** In Gale-Shapley matching, there are no unstable pairs.

**Pf.** Suppose the GS matching  $S^*$  does not contain the pair  $A-Z$ .

- Case 1:  $Z$  never proposed to  $A$ .  
⇒  $Z$  prefers his GS partner  $B$  to  $A$ . ← men propose in decreasing order of preference  
⇒  $A-Z$  is stable.
- Case 2:  $Z$  proposed to  $A$ .  
⇒  $A$  rejected  $Z$  (right away or later)  
⇒  $A$  prefers her GS partner  $Y$  to  $Z$ . ← women only trade up  
⇒  $A-Z$  is stable.
- In either case, the pair  $A-Z$  is stable. ▀

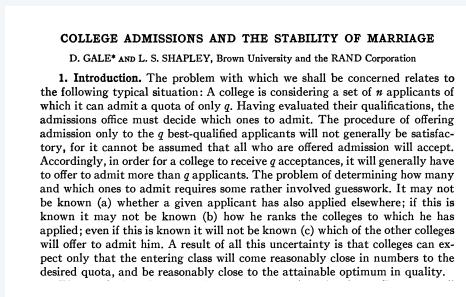
Gale-Shapley matching  $S^*$

## Summary

**Stable matching problem.** Given  $n$  men and  $n$  women, and their preferences, find a stable matching if one exists.

**Theorem.** [Gale-Shapley 1962] The Gale-Shapley algorithm guarantees to find a stable matching for **any** problem instance.

- Q. How to implement GS algorithm efficiently?
- Q. If there are multiple stable matchings, which one does GS find?



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## Efficient implementation (continued)

### Women rejecting/accepting.

- Does woman  $w$  prefer man  $m$  to man  $m'$ ?
- For each woman, create **inverse** of preference list of men.
- Constant time access for each query after  $O(n)$  preprocessing.

pref[]	1 <sup>st</sup>	2 <sup>nd</sup>	3 <sup>rd</sup>	4 <sup>th</sup>	5 <sup>th</sup>	6 <sup>th</sup>	7 <sup>th</sup>	8 <sup>th</sup>
	8	3	7	1	4	5	6	2

inverse[]	1	2	3	4	5	6	7	8
	4 <sup>th</sup>	8 <sup>th</sup>	2 <sup>nd</sup>	5 <sup>th</sup>	6 <sup>th</sup>	7 <sup>th</sup>	3 <sup>rd</sup>	1 <sup>st</sup>

```
for i = 1 to n
    inverse[pref[i]] = i
```

woman prefers man 3 to 6  
since  $\text{inverse}[3] < \text{inverse}[6]$

## Efficient implementation

**Efficient implementation.** We describe an  $O(n^2)$  time implementation.

### Representing men and women.

- Assume men are named  $1, \dots, n$ .
- Assume women are named  $1', \dots, n'$ .

### Representing the matching.

- Maintain a list of free men (in a stack or queue).
- Maintain two arrays  $wife[m]$  and  $husband[w]$ .
  - if  $m$  matched to  $w$ , then  $wife[m] = w$  and  $husband[w] = m$   
set entry to 0 if unmatched

### Men proposing.

- For each man, maintain a list of women, ordered by preference.
- For each man, maintain a pointer to woman in list for next proposal.

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## Understanding the solution

For a given problem instance, there may be several stable matchings.

- Do all executions of GS algorithm yield the same stable matching?
- If so, which one?

	1 <sup>st</sup>	2 <sup>nd</sup>	3 <sup>rd</sup>		1 <sup>st</sup>	2 <sup>nd</sup>	3 <sup>rd</sup>	
Xavier	Amy	Bertha	Clare		Amy	Yancey	Xavier	Zeus
Yancey	Bertha	Amy	Clare		Bertha	Xavier	Yancey	Zeus
					Clare	Xavier	Yancey	Zeus

an instance with two stable matching:  $M = \{ A-X, B-Y, C-Z \}$  and  $M' = \{ A-Y, B-X, C-Z \}$

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## Understanding the solution

**Def.** Woman  $w$  is a **valid partner** of man  $m$  if there exists some stable matching in which  $m$  and  $w$  are matched.

**Ex.**

- Both Amy and Bertha are valid partners for Xavier.
- Both Amy and Bertha are valid partners for Yancey.
- Clare is the only valid partner for Zeus.

	1 <sup>st</sup>	2 <sup>nd</sup>	3 <sup>rd</sup>
Xavier	Amy	Bertha	Clare
Yancey	Bertha	Amy	Clare
Zeus	Amy	Bertha	Clare

	1 <sup>st</sup>	2 <sup>nd</sup>	3 <sup>rd</sup>
Amy	Yancey	Xavier	Zeus
Bertha	Xavier	Yancey	Zeus
Clare	Xavier	Yancey	Zeus

an instance with two stable matching:  $M = \{ A-X, B-Y, C-Z \}$  and  $M' = \{ A-Y, B-X, C-Z \}$

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## Man optimality

**Claim.** GS matching  $S^*$  is man-optimal.

**Pf.** [by contradiction]

- Suppose a man is matched with someone other than best valid partner.
- Men propose in decreasing order of preference  
⇒ some man is rejected by valid partner during GS.
- Let  $Y$  be first such man, and let  $A$  be the first valid woman that rejects him.
- Let  $S$  be a stable matching where  $A$  and  $Y$  are matched.
- When  $Y$  is rejected by  $A$  in GS,  $A$  forms (or reaffirms) engagement with a man, say  $Z$ .  
⇒ **A prefers  $Z$  to  $Y$ .**
- Let  $B$  be partner of  $Z$  in  $S$ .
- $Z$  has not been rejected by any valid partner (including  $B$ ) at the point when  $Y$  is rejected by  $A$ . ← because this is the first rejection by a valid partner
- Thus,  $Z$  has not yet proposed to  $B$  when he proposes to  $A$ .  
⇒ **Z prefers  $A$  to  $B$ .**
- Thus  $A-Z$  is unstable in  $S$ , a contradiction. ▀

$A - Y$   
 $B - Z$   
 ...  
**stable matching  $S$**

## Understanding the solution

**Def.** Woman  $w$  is a **valid partner** of man  $m$  if there exists some stable matching in which  $m$  and  $w$  are matched.

**Man-optimal assignment.** Each man receives best valid partner.

- Is it perfect?
- Is it stable?

**Claim.** All executions of GS yield **man-optimal** assignment.

**Corollary.** Man-optimal assignment is a stable matching!

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## Woman pessimality

**Q.** Does man-optimality come at the expense of the women?

**A.** Yes.

**Woman-pessimal assignment.** Each woman receives worst valid partner.

**Claim.** GS finds **woman-pessimal** stable matching  $S^*$ .

**Pf.** [by contradiction]

- Suppose  $A-Z$  matched in  $S^*$  but  $Z$  is not worst valid partner for  $A$ .
- There exists stable matching  $S$  in which  $A$  is paired with a man, say  $Y$ , whom she likes less than  $Z$ .  
⇒ **A prefers  $Z$  to  $Y$ .**
- Let  $B$  be the partner of  $Z$  in  $S$ . By man-optimality,  $A$  is the best valid partner for  $Z$ .  
⇒ **Z prefers  $A$  to  $B$ .**
- Thus,  $A-Z$  is an unstable pair in  $S$ , a contradiction. ▀

$A - Y$   
 $B - Z$   
 ...  
**stable matching  $S$**

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## Deceit: Machiavelli meets Gale-Shapley

Q. Can there be an incentive to misrepresent your preference list?

- Assume you know men's propose-and-reject algorithm will be run.
- Assume preference lists of all other participants are known.

Fact. No, for any man; yes, for some women.

men's preference list

	1 <sup>st</sup>	2 <sup>nd</sup>	3 <sup>rd</sup>
X	A	B	C
Y	B	A	C
Z	A	B	C

women's preference list

	1 <sup>st</sup>	2 <sup>nd</sup>	3 <sup>rd</sup>
A	Y	X	Z
B	X	Y	Z
C	X	Y	Z

Amy lies

	1 <sup>st</sup>	2 <sup>nd</sup>	3 <sup>rd</sup>
A	Y	(Z)	(X)
B	X	Y	Z
C	X	Y	Z

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## Extensions: matching residents to hospitals

Ex: Men ≈ hospitals, Women ≈ med school residents.

Variant 1. Some participants declare others as unacceptable.

resident A unwilling to work in Cleveland

Variant 2. Unequal number of men and women.

Variant 3. Limited polygamy. ← hospital X wants to hire 3 residents

Def. Matching is  $S$  unstable if there is a hospital  $h$  and resident  $r$  such that:

- $h$  and  $r$  are acceptable to each other; and
- Either  $r$  is unmatched, or  $r$  prefers  $h$  to her assigned hospital; and
- Either  $h$  does not have all its places filled, or  $h$  prefers  $r$  to at least one of its assigned residents.

## Historical context

### National resident matching program (NRMP).

- Centralized clearinghouse to match med-school students to hospitals.
- Began in 1952 to fix unraveling of offer dates.
- Originally used the "Boston Pool" algorithm.
- Algorithm overhauled in 1998.
  - med-school student optimal
  - deals with various side constraints (e.g., allow couples to match together) ← stable matching is no longer guaranteed to exist
- 38,000+ residents for 26,000+ positions.

hospitals began making offers earlier and earlier, up to 2 years in advance

stable matching is no longer guaranteed to exist

The Redesign of the Matching Market for American Physicians:  
Some Engineering Aspects of Economic Design

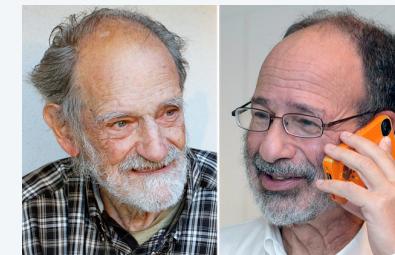
By ALVIN E. ROTH AND ELLIOTT PERANSON\*

We report on the design of the new clearinghouse adopted by the National Resident Matching Program, which annually fills approximately 20,000 jobs for new physicians. Because the market has complementarities between applicants and between positions, the theory of simple matching markets does not apply directly. However, computational experiments show the theory provides good approximations. Furthermore, the set of stable matchings, and the opportunities for strategic manipulation, are surprisingly small. A new kind of "core convergence" result explains this: that each applicant interviews only a small fraction of available positions is important. We also describe engineering aspects of the design process. *JEL C78, B41, J44*

## 2012 Nobel Prize in Economics

Lloyd Shapley. Stable matching theory and Gale-Shapley algorithm.

Alvin Roth. Applied Gale-Shapley to matching new doctors with hospitals, students with schools, and organ donors with patients.



Lloyd Shapley

Alvin Roth



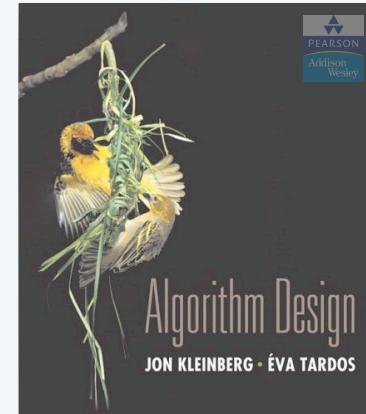
## Lessons learned

### Powerful ideas learned in course.

- Isolate underlying structure of problem.
- Create useful and efficient algorithms.

### Potentially deep social ramifications. [legal disclaimer]

- Historically, men propose to women. Why not vice versa?
- Men: propose early and often; be honest.
- Women: ask out the men.
- Theory can be socially enriching and fun!
- COS majors get the best partners (and jobs)!



SECTION 1.2

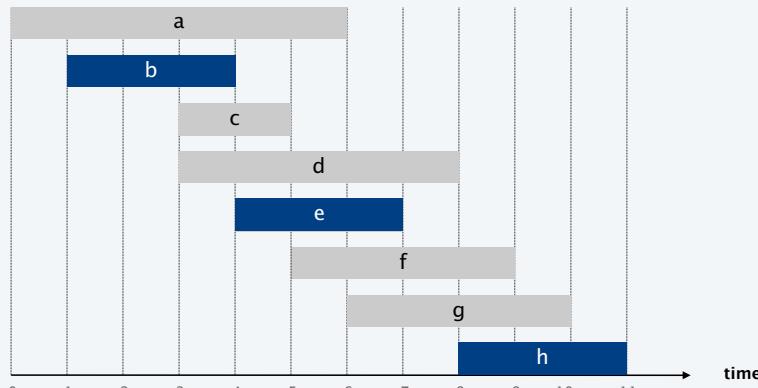
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## Interval scheduling

**Input.** Set of jobs with start times and finish times.

**Goal.** Find maximum cardinality subset of mutually compatible jobs.

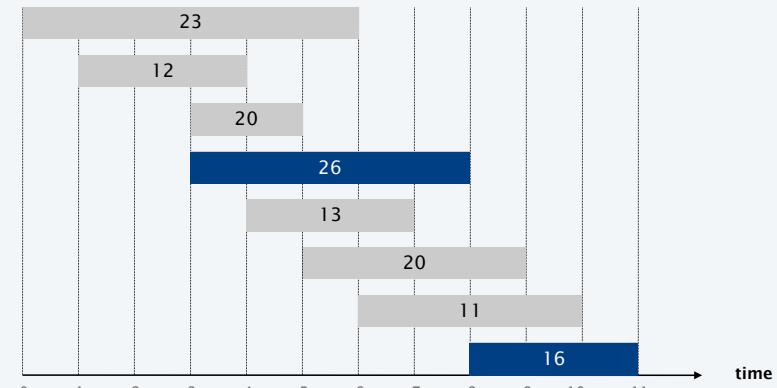
jobs don't overlap



## Weighted interval scheduling

**Input.** Set of jobs with start times, finish times, and weights.

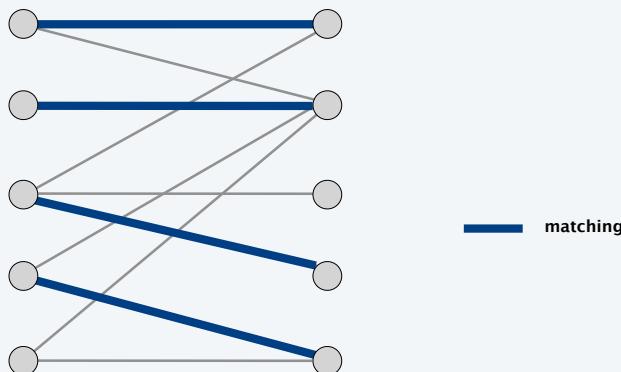
**Goal.** Find maximum weight subset of mutually compatible jobs.



## Bipartite matching

**Problem.** Given a bipartite graph  $G = (L \cup R, E)$ , find a max cardinality matching.

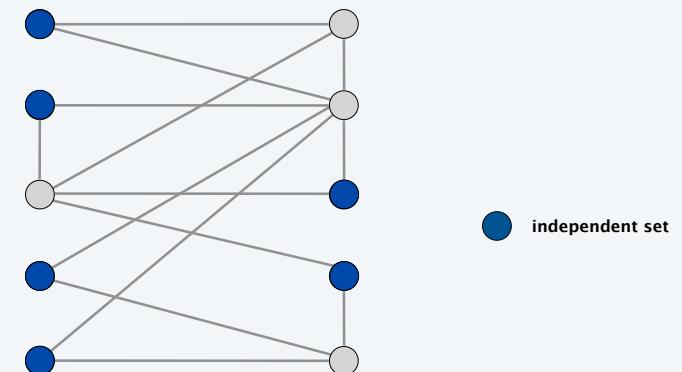
**Def.** A subset of edges  $M \subseteq E$  is a **matching** if each node appears in exactly one edge in  $M$ .



## Independent set

**Problem.** Given a graph  $G = (V, E)$ , find a max cardinality independent set.

**Def.** A subset  $S \subseteq V$  is **independent** if for every  $(u, v) \in E$ , either  $u \notin S$  or  $v \notin S$  (or both).



## Competitive facility location

**Input.** Graph with weight on each node.

**Game.** Two competing players alternate in selecting nodes.

Not allowed to select a node if any of its neighbors have been selected.

**Goal.** Select a **maximum weight** subset of nodes.



Second player can guarantee 20, but not 25.

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## Five representative problems

**Variations on a theme:** independent set.

**Interval scheduling:**  $O(n \log n)$  greedy algorithm.

**Weighted interval scheduling:**  $O(n \log n)$  dynamic programming algorithm.

**Bipartite matching:**  $O(n^k)$  max-flow based algorithm.

**Independent set:** NP-complete.

**Competitive facility location:** PSPACE-complete.

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