

## 7. NETWORK FLOW II

- ▶ bipartite matching
- ▶ disjoint paths
- ▶ extensions to max flow
- ▶ survey design
- ▶ airline scheduling
- ▶ image segmentation
- ▶ project selection
- ▶ baseball elimination

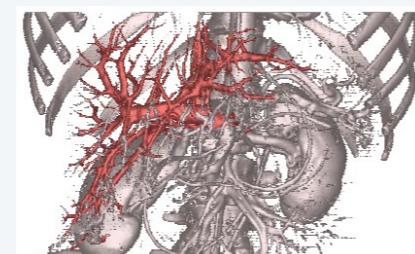
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<http://www.cs.princeton.edu/~wayne/kleinberg-tardos>

Last updated on Dec 23, 2013, 3:31 PM

### Max-flow and min-cut applications

Max-flow and min-cut are widely applicable problem-solving model.

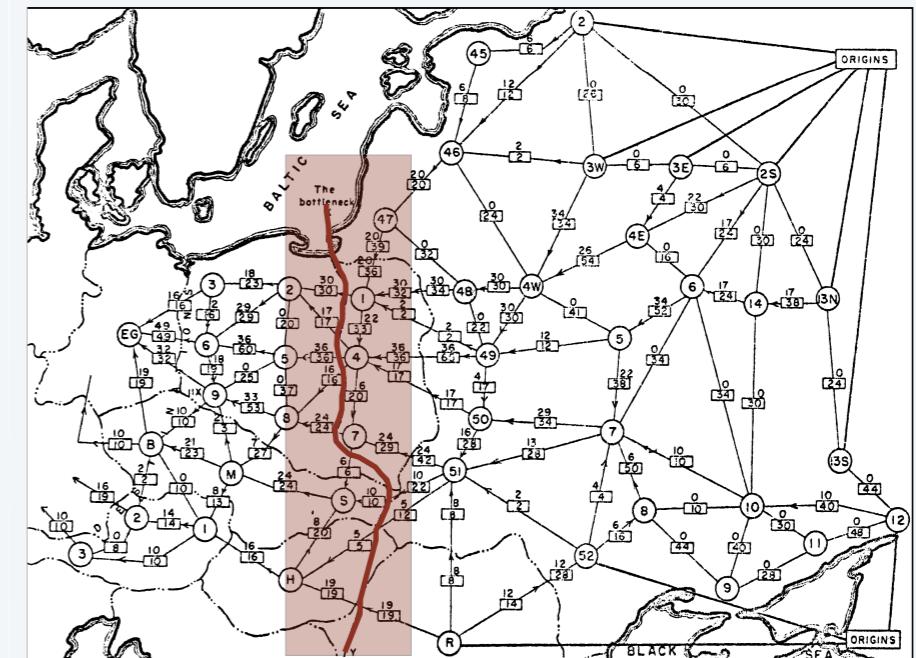
- Data mining.
- Open-pit mining.
- Bipartite matching.
- Network reliability.
- Baseball elimination.
- Image segmentation.
- Network connectivity.
- Distributed computing.
- Security of statistical data.
- Egalitarian stable matching.
- Network intrusion detection.
- Multi-camera scene reconstruction.
- Sensor placement for homeland security.
- Many, many, more.



liver and hepatic vascularization segmentation

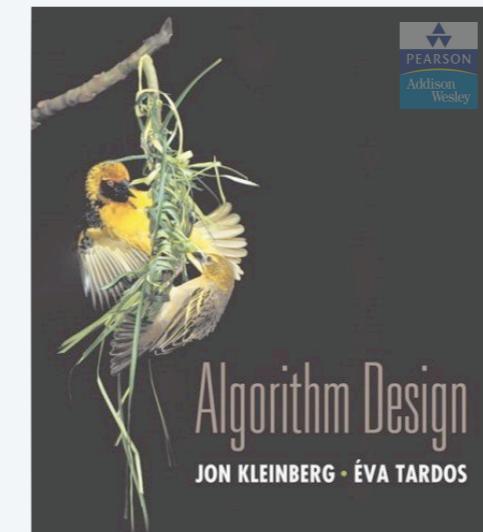
### Soviet rail network (1950s)

"Free world" goal. Cut supplies (if cold war turns into real war).



Reference: *On the history of the transportation and maximum flow problems*.  
 Alexander Schrijver in Math Programming, 91: 3, 2002.

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SECTION

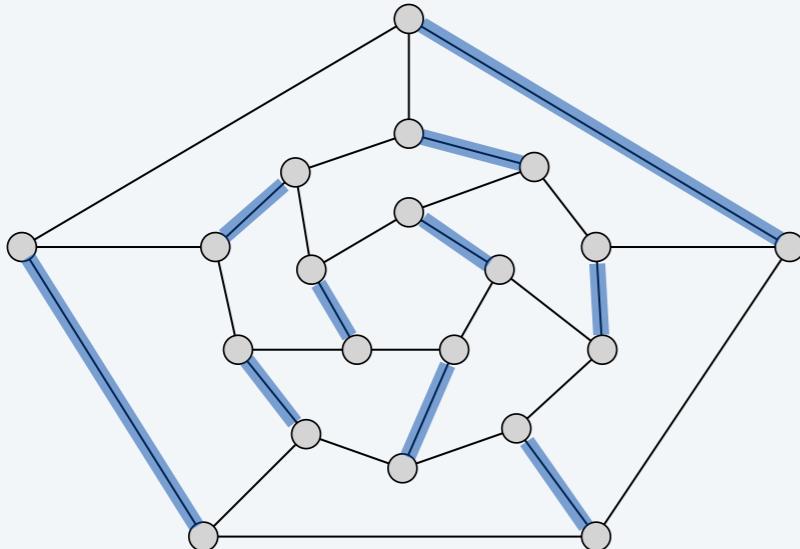
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## Matching

**Def.** Given an undirected graph  $G = (V, E)$  a subset of edges  $M \subseteq E$  is a **matching** if each node appears in at most one edge in  $M$ .

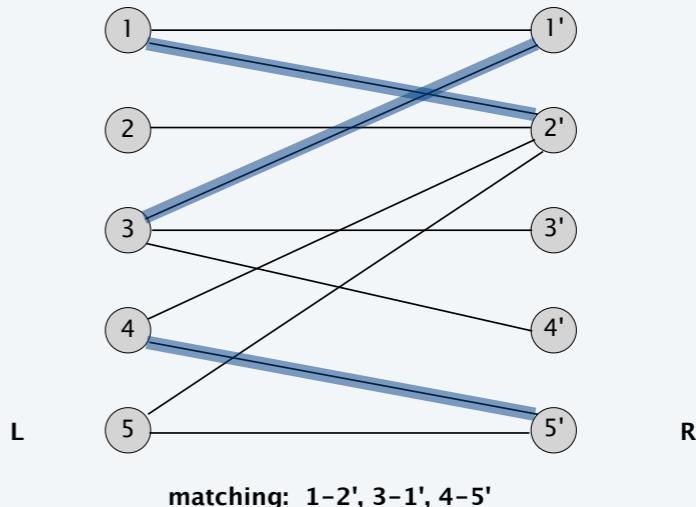
**Max matching.** Given a graph, find a max cardinality matching.



## Bipartite matching

**Def.** A graph  $G$  is **bipartite** if the nodes can be partitioned into two subsets  $L$  and  $R$  such that every edge connects a node in  $L$  to one in  $R$ .

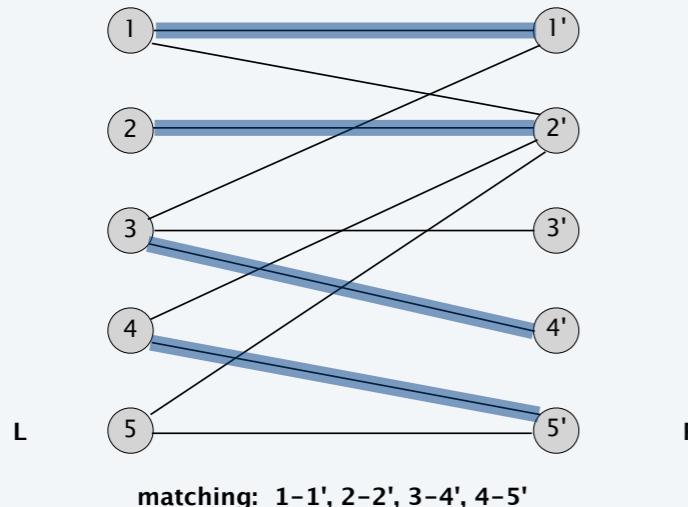
**Bipartite matching.** Given a bipartite graph  $G = (L \cup R, E)$ , find a max cardinality matching.



## Bipartite matching

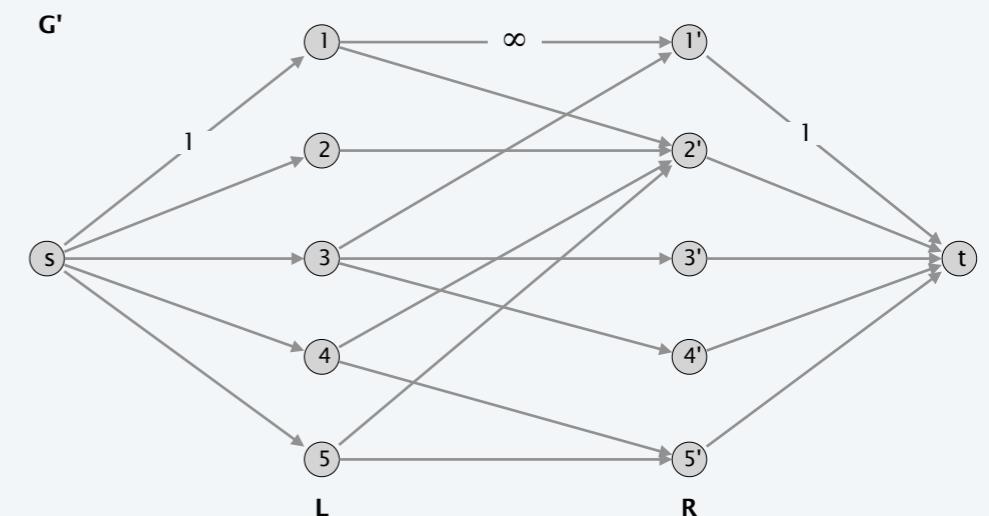
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**Bipartite matching.** Given a bipartite graph  $G = (L \cup R, E)$ , find a max cardinality matching.



## Bipartite matching: max flow formulation

- Create digraph  $G' = (L \cup R \cup \{s, t\}, E')$ .
- Direct all edges from  $L$  to  $R$ , and assign infinite (or unit) capacity.
- Add source  $s$ , and unit capacity edges from  $s$  to each node in  $L$ .
- Add sink  $t$ , and unit capacity edges from each node in  $R$  to  $t$ .

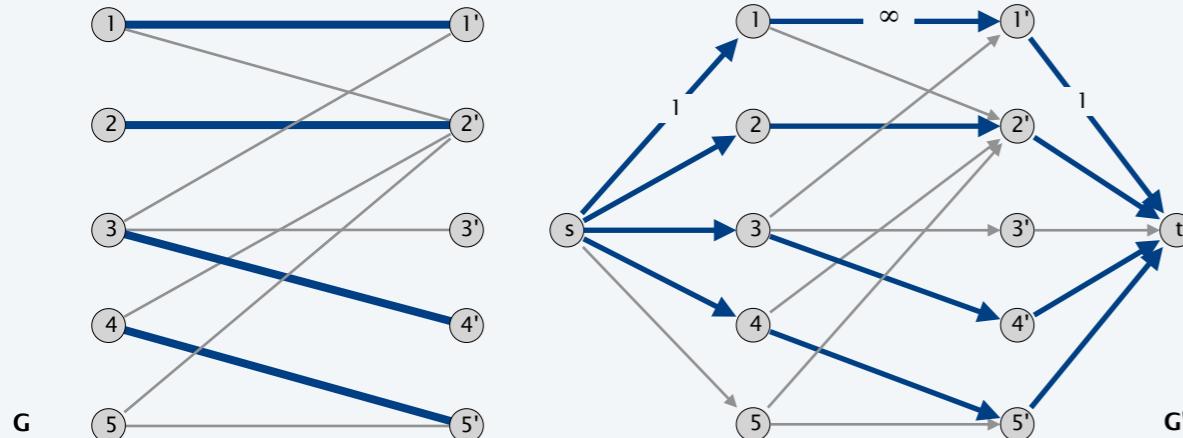


## Max flow formulation: proof of correctness

**Theorem.** Max cardinality of a matching in  $G$  = value of max flow in  $G'$ .

Pf.  $\leq$

- Given a max matching  $M$  of cardinality  $k$ .
- Consider flow  $f$  that sends 1 unit along each of  $k$  paths.
- $f$  is a flow, and has value  $k$ . ■



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## Perfect matching in a bipartite graph

**Def.** Given a graph  $G = (V, E)$  a subset of edges  $M \subseteq E$  is a **perfect matching** if each node appears in exactly one edge in  $M$ .

**Q.** When does a bipartite graph have a perfect matching?

### Structure of bipartite graphs with perfect matchings.

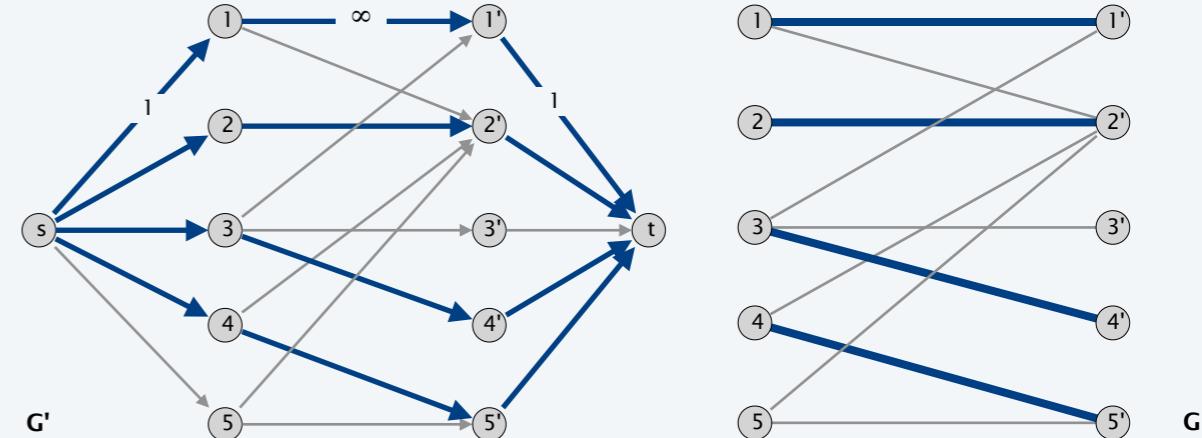
- Clearly we must have  $|L| = |R|$ .
- What other conditions are necessary?
- What conditions are sufficient?

## Max flow formulation: proof of correctness

**Theorem.** Max cardinality of a matching in  $G$  = value of max flow in  $G'$ .

Pf.  $\geq$

- Let  $f$  be a max flow in  $G'$  of value  $k$ .
- Integrality theorem  $\Rightarrow k$  is integral and can assume  $f$  is 0-1.
- Consider  $M = \text{set of edges from } L \text{ to } R \text{ with } f(e) = 1$ .
  - each node in  $L$  and  $R$  participates in at most one edge in  $M$
  - $|M| = k$ : consider cut  $(L \cup s, R \cup t)$  ■



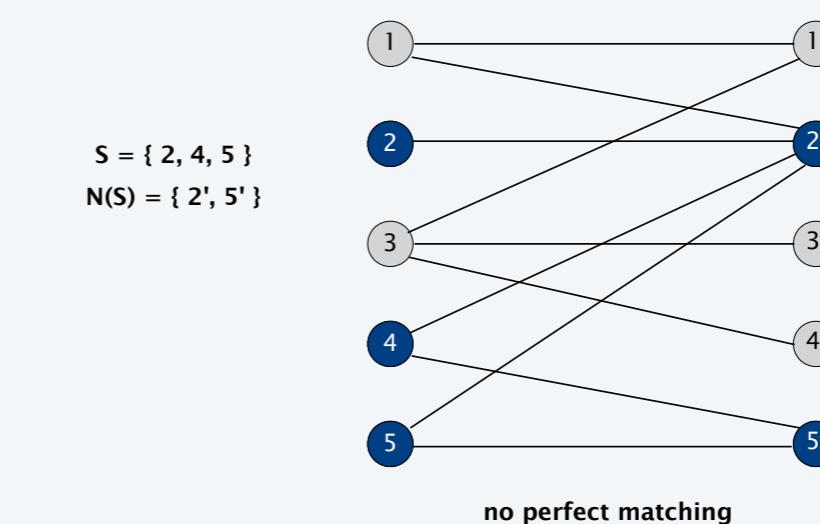
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## Perfect matching in a bipartite graph

**Notation.** Let  $S$  be a subset of nodes, and let  $N(S)$  be the set of nodes adjacent to nodes in  $S$ .

**Observation.** If a bipartite graph  $G = (L \cup R, E)$  has a perfect matching, then  $|N(S)| \geq |S|$  for all subsets  $S \subseteq L$ .

Pf. Each node in  $S$  has to be matched to a different node in  $N(S)$ . ■



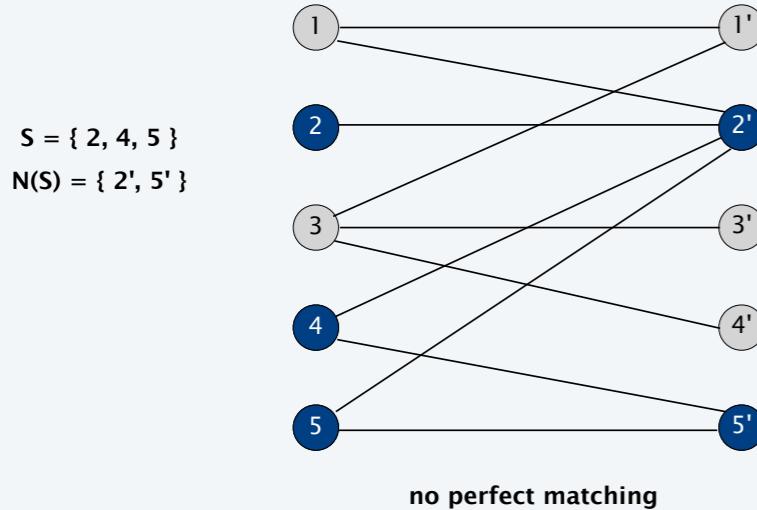
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## Hall's theorem

**Theorem.** Let  $G = (L \cup R, E)$  be a bipartite graph with  $|L| = |R|$ .  
 $G$  has a perfect matching iff  $|N(S)| \geq |S|$  for all subsets  $S \subseteq L$ .

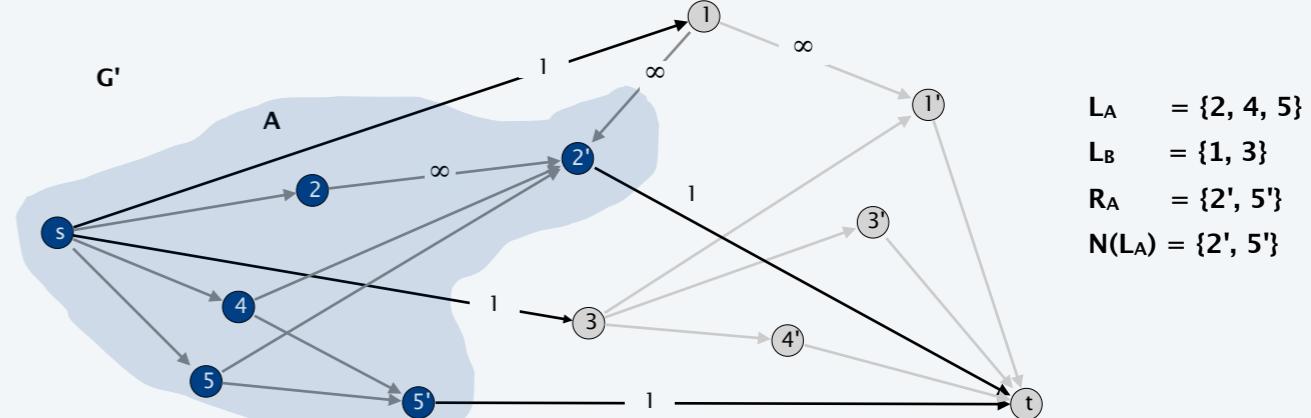
**Pf.**  $\Rightarrow$  This was the previous observation.



## Proof of Hall's theorem

**Pf.**  $\Leftarrow$  Suppose  $G$  does not have a perfect matching.

- Formulate as a max flow problem and let  $(A, B)$  be min cut in  $G'$ .
- By max-flow min-cut theorem,  $cap(A, B) < |L|$ .
- Define  $L_A = L \cap A$ ,  $L_B = L \cap B$ ,  $R_A = R \cap A$ .
- $cap(A, B) = |L_B| + |R_A|$ .
- Since min cut can't use  $\infty$  edges:  $N(L_A) \subseteq R_A$ .
- $|N(L_A)| \leq |R_A| = cap(A, B) - |L_B| < |L| - |L_B| = |L_A|$ .
- Choose  $S = L_A$ . ■



## Bipartite matching running time

**Theorem.** The Ford-Fulkerson algorithm solves the bipartite matching problem in  $O(mn)$  time.

**Theorem.** [Hopcroft-Karp 1973] The bipartite matching problem can be solved in  $O(mn^{1/2})$  time.

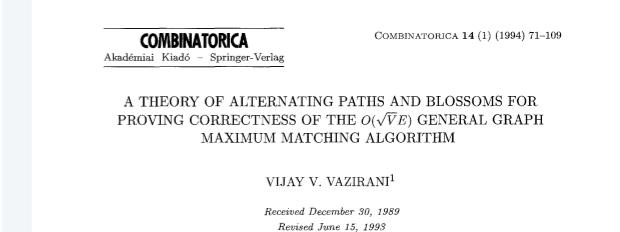
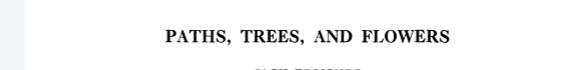
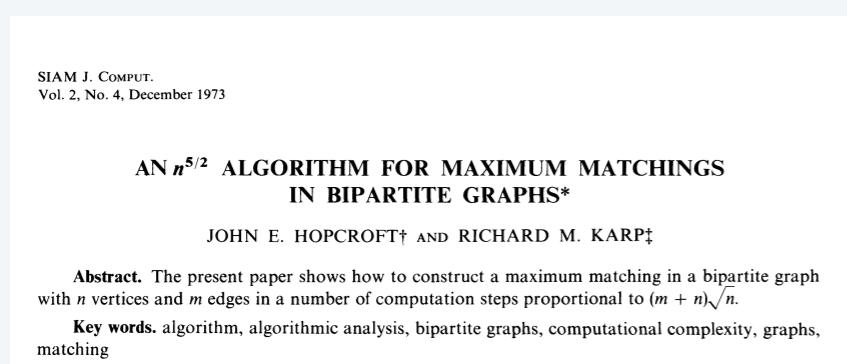
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## Nonbipartite matching

**Nonbipartite matching.** Given an undirected graph (not necessarily bipartite), find a matching of maximum cardinality.

- Structure of nonbipartite graphs is more complicated.
- But well-understood. [Tutte-Berge, Edmonds-Galai]
- Blossom algorithm:  $O(n^4)$ . [Edmonds 1965]
- Best known:  $O(mn^{1/2})$ . [Micali-Vazirani 1980, Vazirani 1994]



**2. Digression.** An explanation is due on the use of the words “efficient algorithm.” First, what I present is a conceptual description of an algorithm and not a particular formalized algorithm or “code.”

For practical purposes computational details are vital. However, my purpose is only to show as attractively as I can that there is an efficient algorithm. According to the dictionary, "efficient" means "adequate in operation or performance." This is roughly the meaning I want—in the sense that it is conceivable for maximum matching to have no efficient algorithm. Perhaps a better word is "good."

I am claiming, as a mathematical result, the existence of a *good* algorithm for finding a maximum cardinality matching in a graph.

There is an obvious finite algorithm, but that algorithm increases in difficulty exponentially with the size of the graph. It is by no means obvious whether or not there exists an algorithm whose difficulty increases only algebraically with the size of the graph.



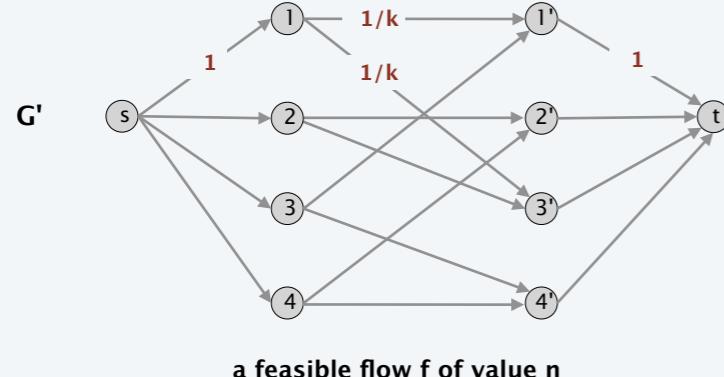
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**k-regular bipartite graphs have perfect matchings**

**Theorem.** Every  $k$ -regular bipartite graph  $G$  has a perfect matching.

Pf.

- Size of max matching = value of max flow in  $G'$ .
  - Consider flow
$$f(u, v) = \begin{cases} 1/k & \text{if } (u, v) \in E \\ 1 & \text{if } u = s \text{ or } v = t \\ 0 & \text{otherwise} \end{cases}$$
  - $f$  is a flow in  $G'$  and its value  $\equiv n \Rightarrow$  perfect matching.



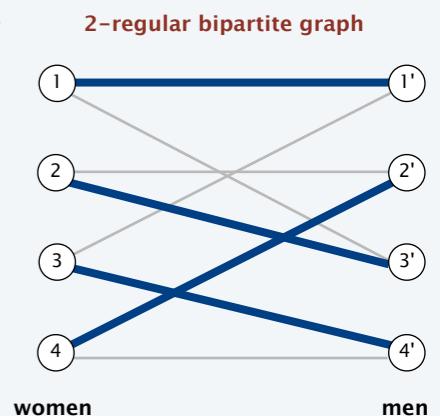
a feasible flow  $f$  of value  $n$

## Dancing problem

- Exclusive Ivy league party attended by  $n$  men and  $n$  women.
  - Each man knows exactly  $k$  women; each woman knows exactly  $k$  men.
  - Acquaintances are mutual.
  - Is it possible to arrange a dance so that each woman dances with a different man that she knows?

**Mathematical reformulation.** Does every  $k$ -regular bipartite graph have a perfect matching?

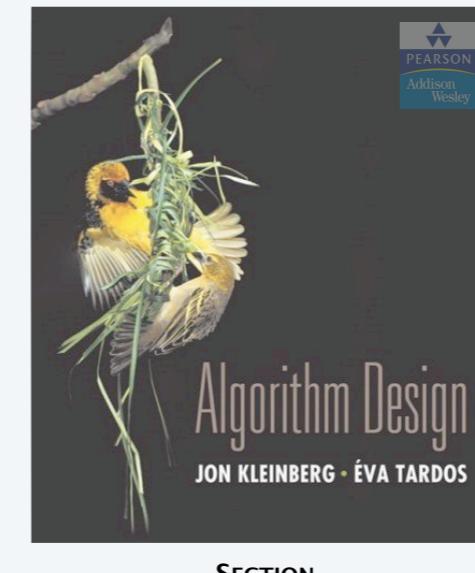
Ex. Boolean hypercube



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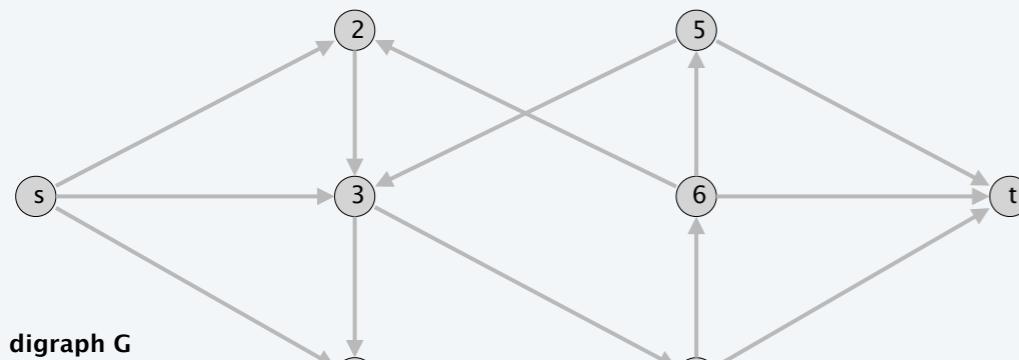


SECTION

## Edge-disjoint paths

**Def.** Two paths are **edge-disjoint** if they have no edge in common.

**Disjoint path problem.** Given a digraph  $G = (V, E)$  and two nodes  $s$  and  $t$ , find the max number of edge-disjoint  $s \rightarrow t$  paths.

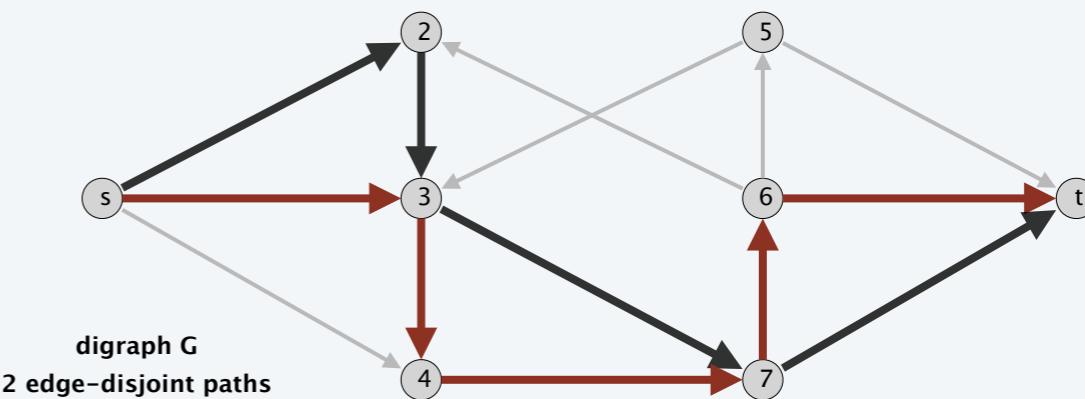


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**Ex.** Communication networks.



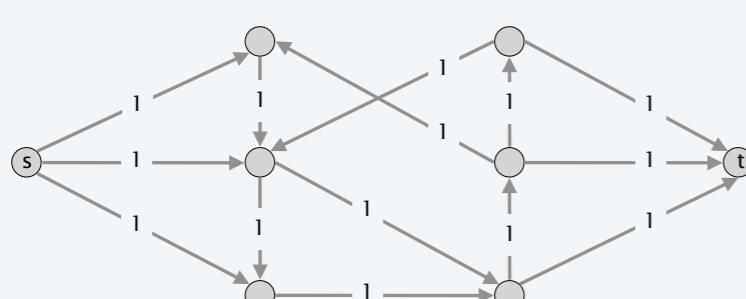
## Edge-disjoint paths

**Max flow formulation.** Assign unit capacity to every edge.

**Theorem.** Max number edge-disjoint  $s \rightarrow t$  paths equals value of max flow.

**Pf.**  $\leq$

- Suppose there are  $k$  edge-disjoint  $s \rightarrow t$  paths  $P_1, \dots, P_k$ .
- Set  $f(e) = 1$  if  $e$  participates in some path  $P_j$ ; else set  $f(e) = 0$ .
- Since paths are edge-disjoint,  $f$  is a flow of value  $k$ . ▀



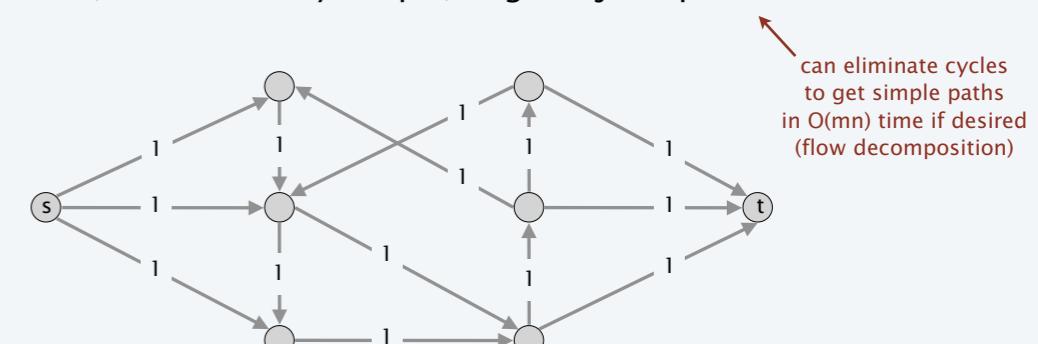
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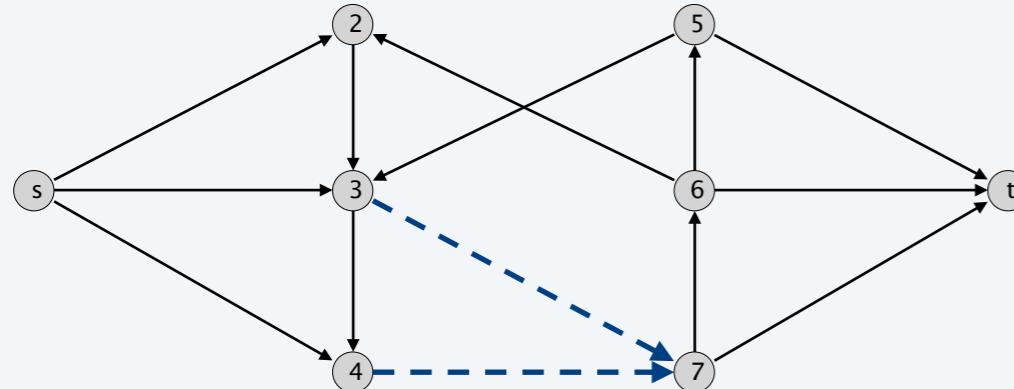
- Suppose max flow value is  $k$ .
- Integrality theorem  $\Rightarrow$  there exists 0-1 flow  $f$  of value  $k$ .
- Consider edge  $(s, u)$  with  $f(s, u) = 1$ .
  - by conservation, there exists an edge  $(u, v)$  with  $f(u, v) = 1$
  - continue until reach  $t$ , always choosing a new edge
- Produces  $k$  (not necessarily simple) edge-disjoint paths. ▀



## Network connectivity

**Def.** A set of edges  $F \subseteq E$  disconnects  $t$  from  $s$  if every  $s \rightarrow t$  path uses at least one edge in  $F$ .

**Network connectivity.** Given a digraph  $G = (V, E)$  and two nodes  $s$  and  $t$ , find min number of edges whose removal disconnects  $t$  from  $s$ .



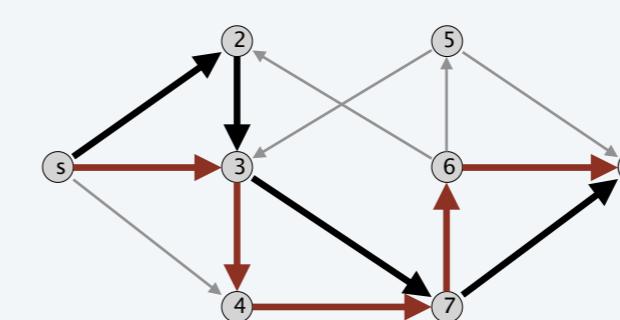
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## Menger's theorem

**Theorem.** [Menger 1927] The max number of edge-disjoint  $s \rightarrow t$  paths is equal to the min number of edges whose removal disconnects  $t$  from  $s$ .

**Pf.  $\leq$**

- Suppose the removal of  $F \subseteq E$  disconnects  $t$  from  $s$ , and  $|F| = k$ .
- Every  $s \rightarrow t$  path uses at least one edge in  $F$ .
- Hence, the number of edge-disjoint paths is  $\leq k$ . ■



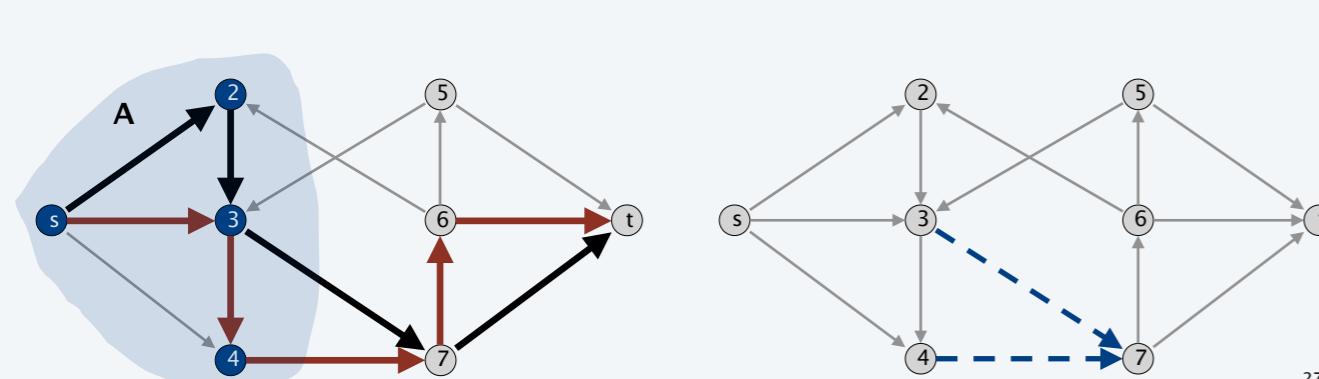
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**Pf.  $\geq$**

- Suppose max number of edge-disjoint paths is  $k$ .
- Then value of max flow =  $k$ .
- Max-flow min-cut theorem  $\Rightarrow$  there exists a cut  $(A, B)$  of capacity  $k$ .
- Let  $F$  be set of edges going from  $A$  to  $B$ .
- $|F| = k$  and disconnects  $t$  from  $s$ . ■

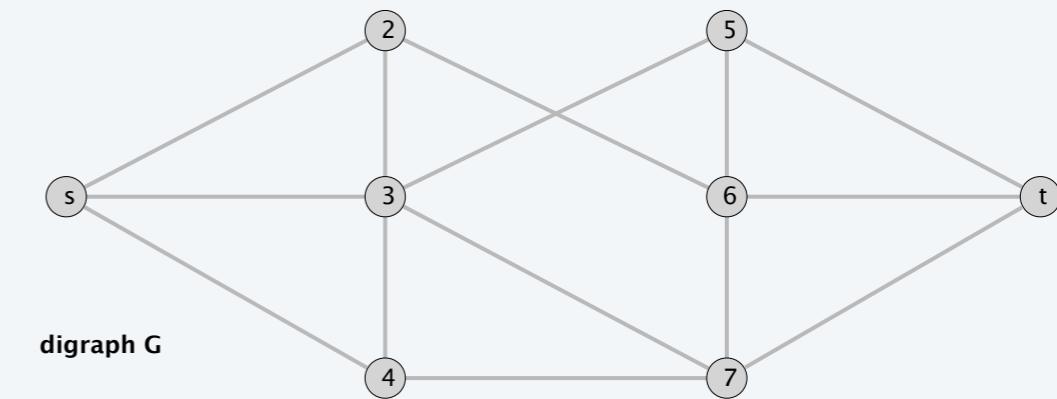


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## Edge-disjoint paths in undirected graphs

**Def.** Two paths are **edge-disjoint** if they have no edge in common.

**Disjoint path problem in undirected graphs.** Given a graph  $G = (V, E)$  and two nodes  $s$  and  $t$ , find the max number of edge-disjoint  $s-t$  paths.

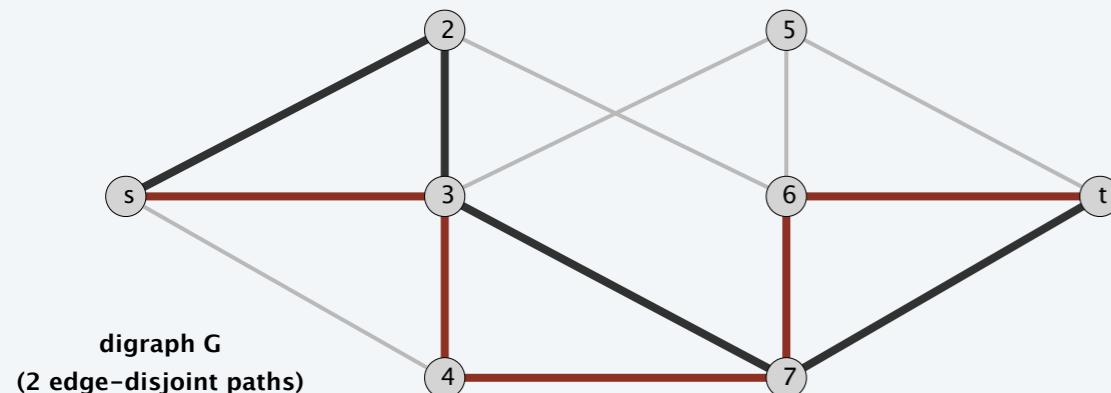


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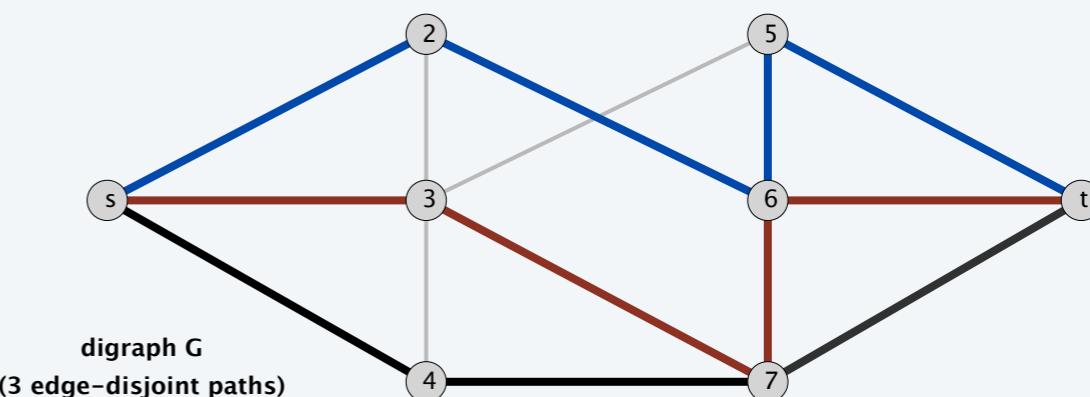
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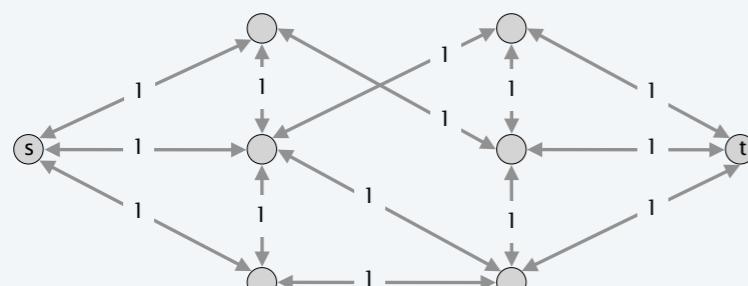


## Edge-disjoint paths in undirected graphs

**Max flow formulation.** Replace edge  $e$  with two antiparallel edges and assign unit capacity to every edge.

**Observation.** Two paths  $P_1$  and  $P_2$  may be edge-disjoint in the digraph but not edge-disjoint in the undirected graph.

if  $P_1$  uses edge  $(u, v)$   
and  $P_2$  uses its antiparallel edge  $(v, u)$



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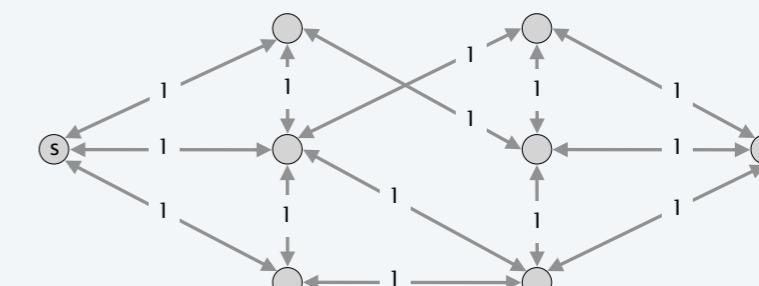
## Edge-disjoint paths in undirected graphs

**Max flow formulation.** Replace edge  $e$  with two antiparallel edges and assign unit capacity to every edge.

**Lemma.** In any flow network, there exists a maximum flow  $f$  in which for each pair of antiparallel edges  $e$  and  $e'$ , either  $f(e) = 0$  or  $f(e') = 0$  or both. Moreover, integrality theorem still holds.

**Pf.** [ by induction on number of such pairs of antiparallel edges ]

- Suppose  $f(e) > 0$  and  $f(e') > 0$  for a pair of antiparallel edges  $e$  and  $e'$ .
- Set  $f(e) = f(e) - \delta$  and  $f(e') = f(e') - \delta$ , where  $\delta = \min \{ f(e), f(e') \}$ .
- $f$  is still a flow of the same value but has one fewer such pair. ▀



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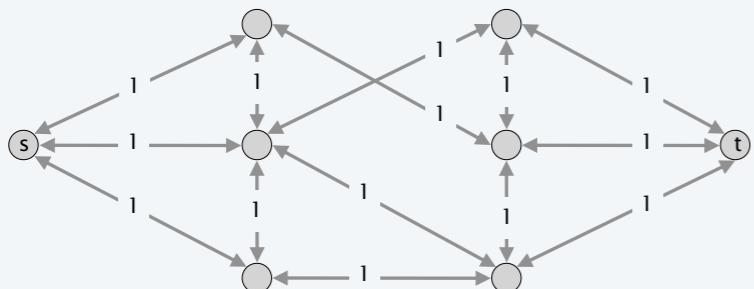
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**Theorem.** Max number edge-disjoint  $s \rightarrow t$  paths equals value of max flow.

**Pf.** Similar to proof in digraphs; use lemma.



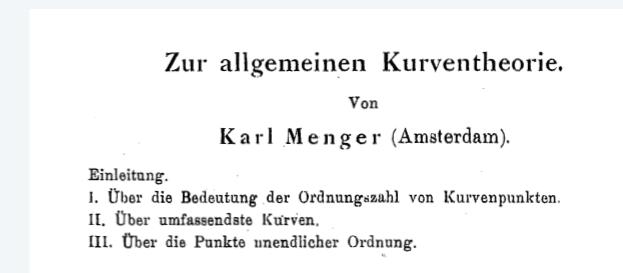
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## Menger's theorems

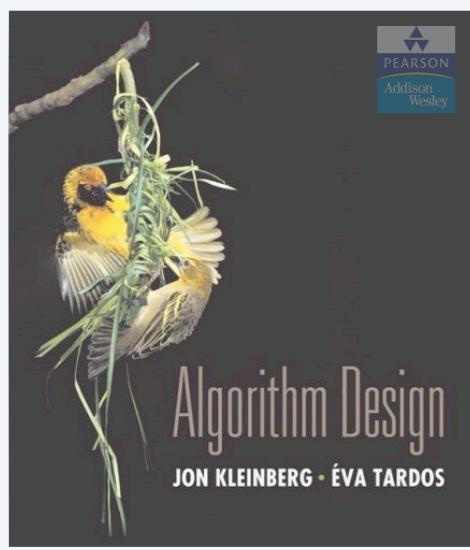
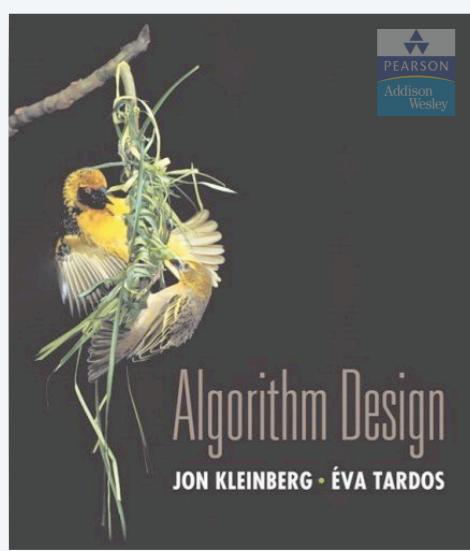
**Theorem.** Given an **undirected** graph with two nodes  $s$  and  $t$ , the max number of **edge-disjoint**  $s-t$  paths equals the min number of edges whose removal disconnects  $s$  and  $t$ .

**Theorem.** Given a **undirected** graph with two nonadjacent nodes  $s$  and  $t$ , the max number of internally **node-disjoint**  $s-t$  paths equals the min number of internal nodes whose removal disconnects  $s$  and  $t$ .

**Theorem.** Given an **directed** graph with two nonadjacent nodes  $s$  and  $t$ , the max number of internally **node-disjoint**  $s \rightarrow t$  paths equals the min number of internal nodes whose removal disconnects  $t$  from  $s$ .



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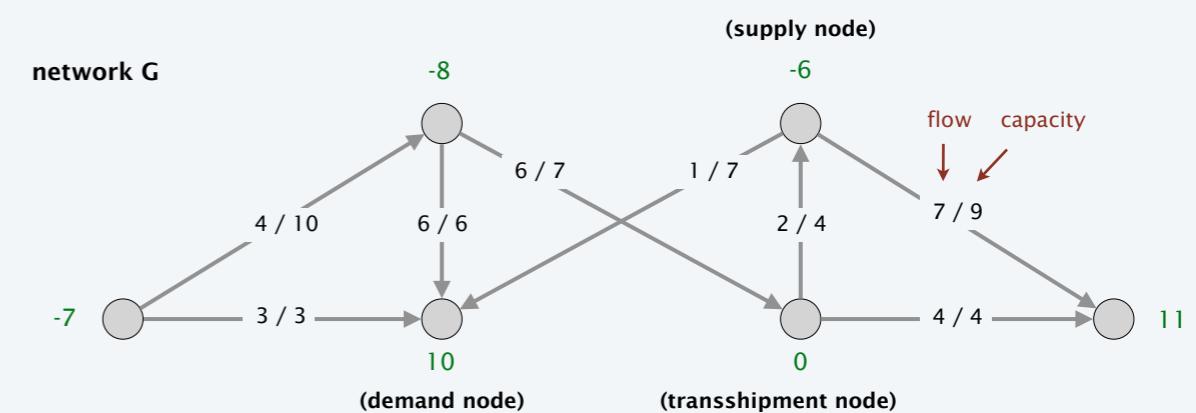
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## Circulation with demands

**Def.** Given a digraph  $G = (V, E)$  with nonnegative edge capacities  $c(e)$  and node supply and demands  $d(v)$ , a **circulation** is a function that satisfies:

- For each  $e \in E$ :  $0 \leq f(e) \leq c(e)$  (capacity)
- For each  $v \in V$ :  $\sum_{e \text{ in to } v} f(e) - \sum_{e \text{ out of } v} f(e) = d(v)$  (conservation)

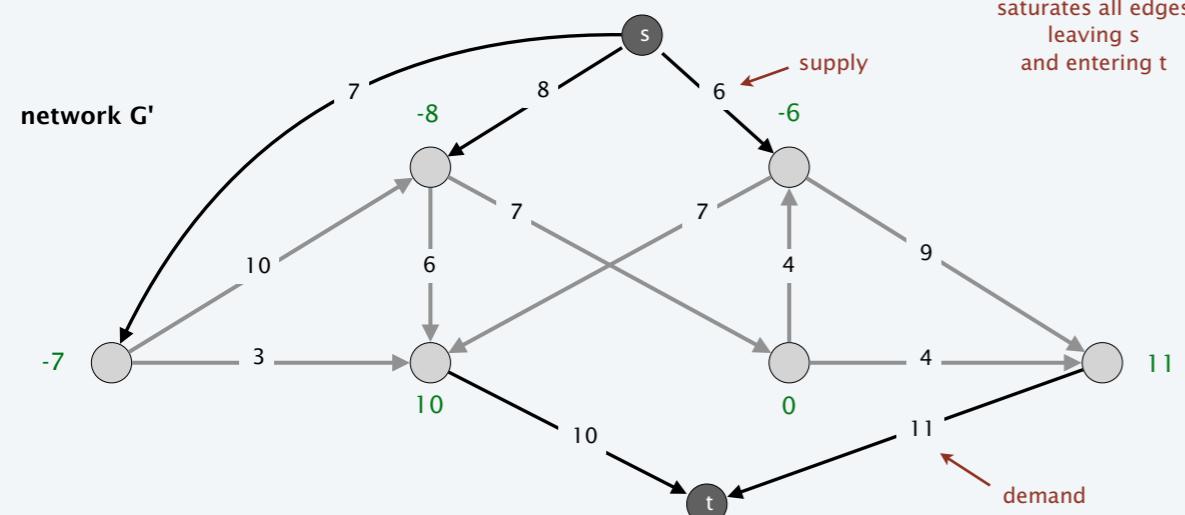


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## Circulation with demands: max-flow formulation

- Add new source  $s$  and sink  $t$ .
- For each  $v$  with  $d(v) < 0$ , add edge  $(s, v)$  with capacity  $-d(v)$ .
- For each  $v$  with  $d(v) > 0$ , add edge  $(v, t)$  with capacity  $d(v)$ .

**Claim.**  $G$  has circulation iff  $G'$  has max flow of value  $D = \sum_{v: d(v) > 0} d(v) = \sum_{v: d(v) < 0} -d(v)$



## Circulation with demands

**Integrality theorem.** If all capacities and demands are integers, and there exists a circulation, then there exists one that is integer-valued.

**Pf.** Follows from max-flow formulation + integrality theorem for max flow.

**Theorem.** Given  $(V, E, c, d)$ , there does **not** exist a circulation iff there exists a node partition  $(A, B)$  such that  $\sum_{v \in B} d(v) > \text{cap}(A, B)$ .

**Pf sketch.** Look at min cut in  $G'$ .

demand by nodes in  $B$  exceeds supply of nodes in  $B$  plus max capacity of edges going from  $A$  to  $B$

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## Circulation with demands and lower bounds

### Feasible circulation.

- Directed graph  $G = (V, E)$ .
- Edge capacities  $c(e)$  and lower bounds  $\ell(e)$  for each edge  $e \in E$ .
- Node supply and demands  $d(v)$  for each node  $v \in V$ .

**Def.** A **circulation** is a function that satisfies:

- For each  $e \in E$ :  $\ell(e) \leq f(e) \leq c(e)$  (capacity)
- For each  $v \in V$ :  $\sum_{e \text{ in to } v} f(e) - \sum_{e \text{ out of } v} f(e) = d(v)$  (conservation)

**Circulation problem with lower bounds.** Given  $(V, E, \ell, c, d)$ , does there exist a feasible circulation?

## Circulation with demands and lower bounds

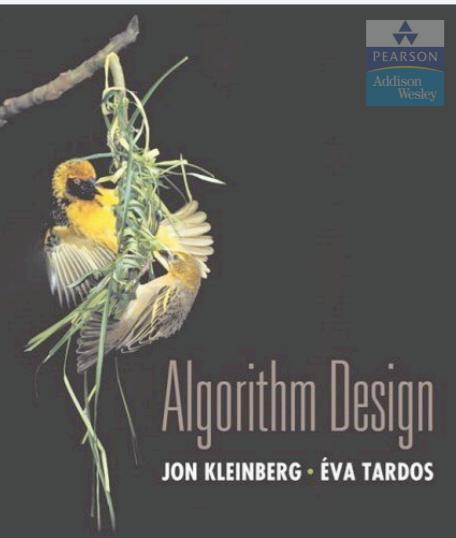
**Max flow formulation.** Model lower bounds as circulation with demands.

- Send  $\ell(e)$  units of flow along edge  $e$ .
- Update demands of both endpoints.



**Theorem.** There exists a circulation in  $G$  iff there exists a circulation in  $G'$ . Moreover, if all demands, capacities, and lower bounds in  $G$  are integers, then there is a circulation in  $G$  that is integer-valued.

**Pf sketch.**  $f(e)$  is a circulation in  $G$  iff  $f'(e) = f(e) - \ell(e)$  is a circulation in  $G'$ .



## 7. NETWORK FLOW II

SECTION

- ▶ bipartite matching
- ▶ disjoint paths
- ▶ extensions to max flow
- ▶ survey design
- ▶ airline scheduling
- ▶ image segmentation
- ▶ project selection
- ▶ baseball elimination

### Survey design

- Design survey asking  $n_1$  consumers about  $n_2$  products. ← one survey question per product
- Can only survey consumer  $i$  about product  $j$  if they own it.
- Ask consumer  $i$  between  $c_i$  and  $c_i'$  questions.
- Ask between  $p_j$  and  $p_j'$  consumers about product  $j$ .

**Goal.** Design a survey that meets these specs, if possible.

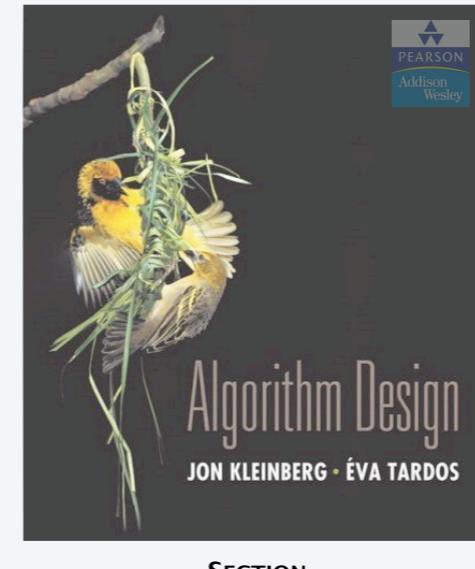
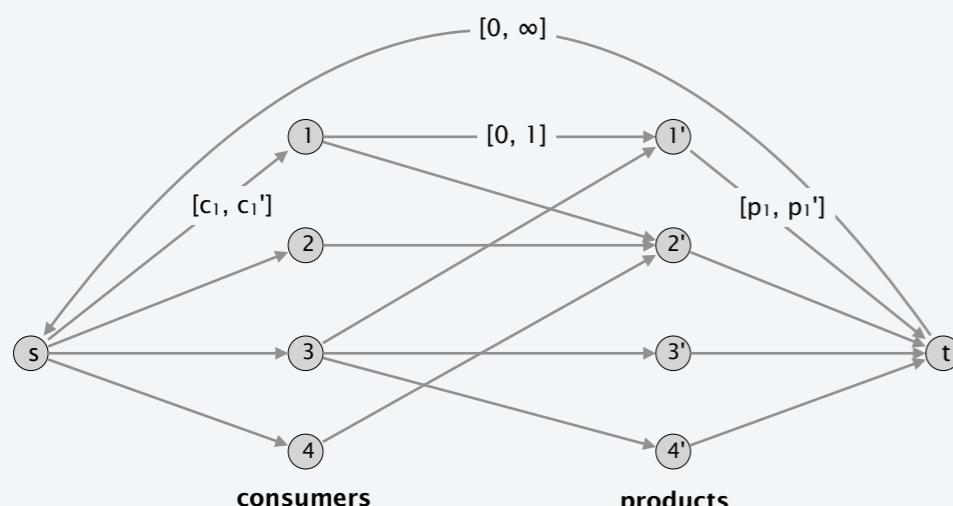
**Bipartite perfect matching.** Special case when  $c_i = c_i' = p_j = p_j' = 1$ .

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### Survey design

**Max-flow formulation.** Model as circulation problem with lower bounds.

- Add edge  $(i, j)$  if consumer  $j$  owns product  $i$ .
- Add edge from  $s$  to consumer  $j$ .
- Add edge from product  $i$  to  $t$ .
- Add edge from  $t$  to  $s$ .
- Integer circulation  $\Leftrightarrow$  feasible survey design.



SECTION

## 7. NETWORK FLOW II

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## Airline scheduling

### Airline scheduling.

- Complex computational problem faced by nation's airline carriers.
- Produces schedules that are efficient in terms of:
  - equipment usage, crew allocation, customer satisfaction
  - in presence of unpredictable issues like weather, breakdowns
- One of largest consumers of high-powered algorithmic techniques.

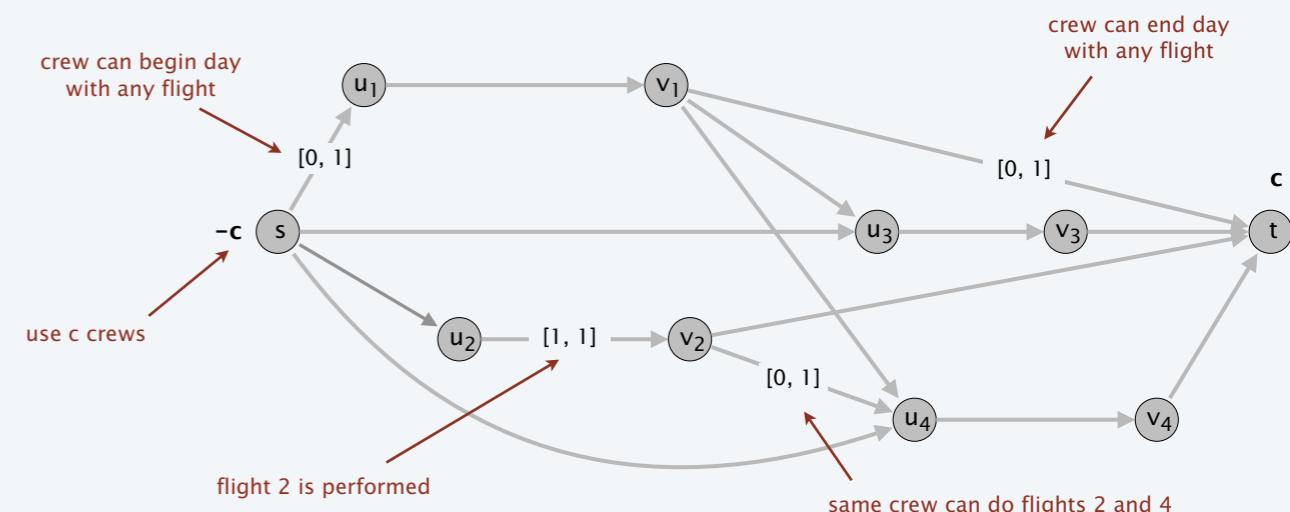
### "Toy problem."

- Manage flight crews by reusing them over multiple flights.
- Input: set of  $k$  flights for a given day.
- Flight  $i$  leaves origin  $o_i$  at time  $s_i$  and arrives at destination  $d_i$  destination at time  $f_i$ .
- Minimize number of flight crews.

## Airline scheduling

### Circulation formulation. [to see if $c$ crews suffice]

- For each flight  $i$ , include two nodes  $u_i$  and  $v_i$ .
- Add source  $s$  with demand  $-c$ , and edges  $(s, u_i)$  with capacity 1.
- Add sink  $t$  with demand  $c$ , and edges  $(v_i, t)$  with capacity 1.
- For each  $i$ , add edge  $(u_i, v_i)$  with lower bound and capacity 1.
- if flight  $j$  reachable from  $i$ , add edge  $(v_i, u_j)$  with capacity 1.



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## Airline scheduling: running time

**Theorem.** The airline scheduling problem can be solved in  $O(k^3 \log k)$  time.

**Pf.**

- $k =$  number of flights.
- $c =$  number of crews (unknown).
- $O(k)$  nodes,  $O(k^2)$  edges.
- At most  $k$  crews needed.  
⇒ solve  $\lg k$  circulation problems. ← binary search for optimal value  $c^*$
- Value of the flow is between 0 and  $k$ .  
⇒ at most  $k$  augmentations per circulation problem.
- Overall time =  $O(k^3 \log k)$ .

**Remark.** Can solve in  $O(k^3)$  time by formulating as **minimum flow problem**.

## Airline scheduling: postmortem

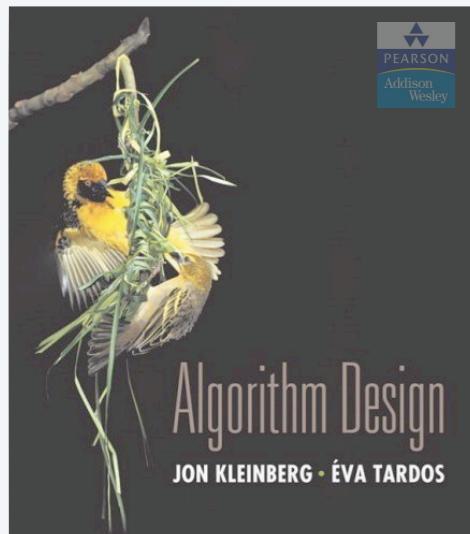
**Remark.** We solved a toy problem.

### Real-world problem models countless other factors:

- Union regulations: e.g., flight crews can only fly certain number of hours in given interval.
- Need optimal schedule over planning horizon, not just one day.
- Deadheading has a cost.
- Flights don't always leave or arrive on schedule.
- Simultaneously optimize both flight schedule and fare structure.

### Message.

- Our solution is a generally useful technique for efficient reuse of limited resources but trivializes real airline scheduling problem.
- Flow techniques useful for solving airline scheduling problems (and are widely used in practice).
- Running an airline efficiently is a very difficult problem.



SECTION

## 7. NETWORK FLOW II

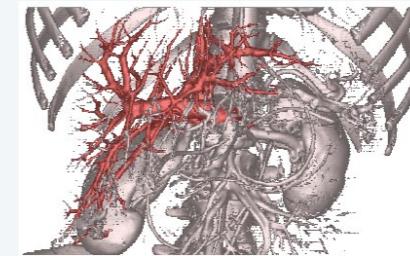
- ▶ bipartite matching
- ▶ disjoint paths
- ▶ extensions to max flow
- ▶ survey design
- ▶ airline scheduling
- ▶ image segmentation
- ▶ project selection
- ▶ baseball elimination

## Image segmentation

### Image segmentation.

- Central problem in image processing.
- Divide image into coherent regions.

**Ex.** Three people standing in front of complex background scene.  
Identify each person as a coherent object.

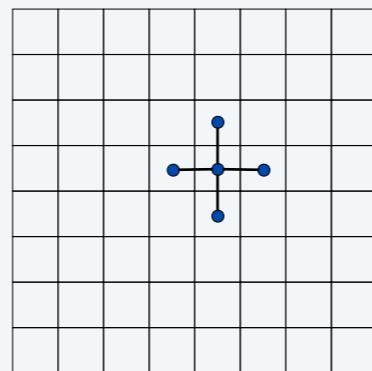


liver and hepatic vascularization segmentation

## Image segmentation

### Foreground / background segmentation.

- Label each pixel in picture as belonging to foreground or background.
- $V = \text{set of pixels}$ ,  $E = \text{pairs of neighboring pixels}$ .
- $a_i \geq 0$  is likelihood pixel  $i$  in foreground.
- $b_i \geq 0$  is likelihood pixel  $i$  in background.
- $p_{ij} \geq 0$  is separation penalty for labeling one of  $i$  and  $j$  as foreground, and the other as background.



### Goals.

- Accuracy: if  $a_i > b_i$  in isolation, prefer to label  $i$  in foreground.
- Smoothness: if many neighbors of  $i$  are labeled foreground, we should be inclined to label  $i$  as foreground.
- Find partition  $(A, B)$  that maximizes:  $\sum_{i \in A} a_i + \sum_{j \in B} b_j - \sum_{\substack{(i,j) \in E \\ |A \cap \{i,j\}|=1}} p_{ij}$

foreground      background

## Image segmentation

### Formulate as min cut problem.

- Maximization.
- No source or sink.
- Undirected graph.

### Turn into minimization problem.

$$\text{Maximizing } \sum_{i \in A} a_i + \sum_{j \in B} b_j - \sum_{\substack{(i,j) \in E \\ |A \cap \{i,j\}|=1}} p_{ij}$$

$$\bullet \text{ is equivalent to minimizing } \underbrace{\left( \sum_{i \in V} a_i + \sum_{j \in V} b_j \right)}_{\text{a constant}} - \sum_{i \in A} a_i - \sum_{j \in B} b_j + \sum_{\substack{(i,j) \in E \\ |A \cap \{i,j\}|=1}} p_{ij}$$

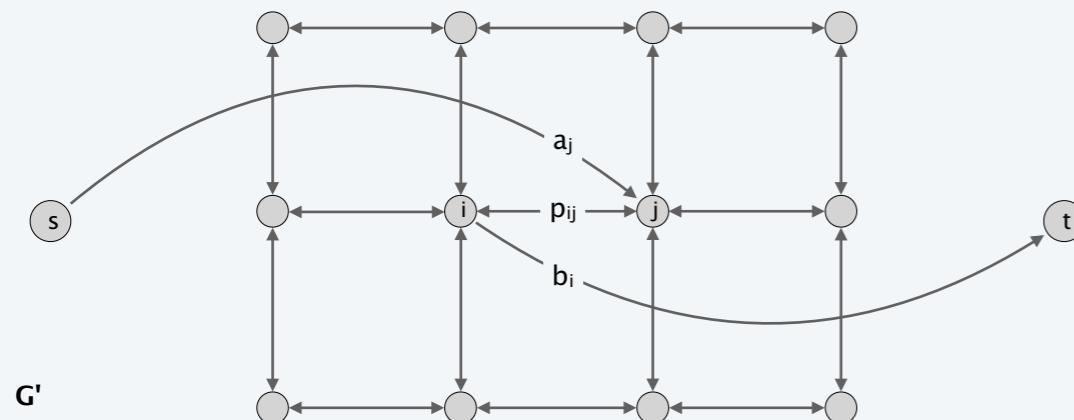
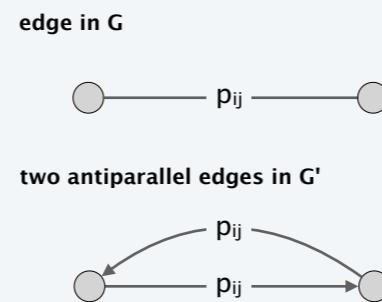
- or alternatively

$$\sum_{j \in B} a_j + \sum_{i \in A} b_i + \sum_{\substack{(i,j) \in E \\ |A \cap \{i,j\}|=1}} p_{ij}$$

## Image segmentation

Formulate as min cut problem  $G' = (V', E')$ .

- Include node for each pixel.
- Use two antiparallel edges instead of undirected edge.
- Add source  $s$  to correspond to foreground.
- Add sink  $t$  to correspond to background.



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## Image segmentation

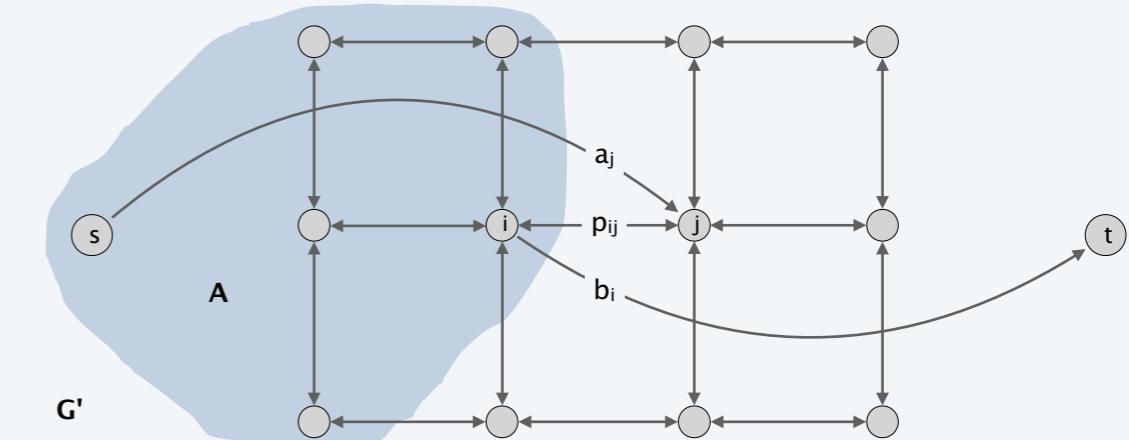
Consider min cut  $(A, B)$  in  $G'$ .

- $A$  = foreground.

$$cap(A, B) = \sum_{j \in B} a_j + \sum_{i \in A} b_i + \sum_{\substack{(i,j) \in E \\ i \in A, j \in B}} p_{ij}$$

if  $i$  and  $j$  on different sides,  
 $p_{ij}$  counted exactly once

- Precisely the quantity we want to minimize.



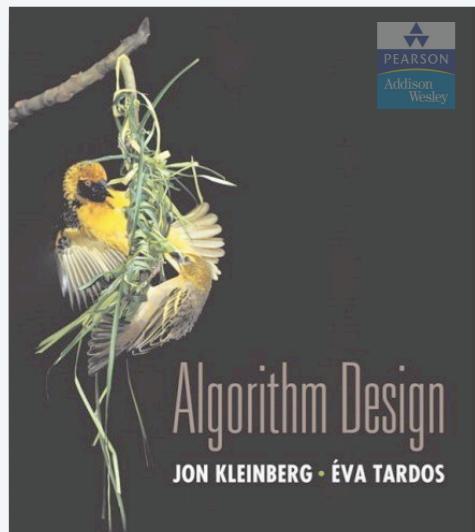
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## Project selection (maximum weight closure problem)

Projects with prerequisites.

- Set of possible projects  $P$ : project  $v$  has associated revenue  $p_v$ .
- Set of prerequisites  $E$ : if  $(v, w) \in E$ , cannot do project  $v$  unless also do project  $w$ .
- A subset of projects  $A \subseteq P$  is feasible if the prerequisite of every project in  $A$  also belongs to  $A$ .

can be positive or negative



SECTION

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Project selection problem. Given a set of projects  $P$  and prerequisites  $E$ , choose a feasible subset of projects to maximize revenue.

MANAGEMENT SCIENCE  
Vol. 22, No. 11, July, 1976  
Printed in U.S.A.

### MAXIMAL CLOSURE OF A GRAPH AND APPLICATIONS TO COMBINATORIAL PROBLEMS\*†

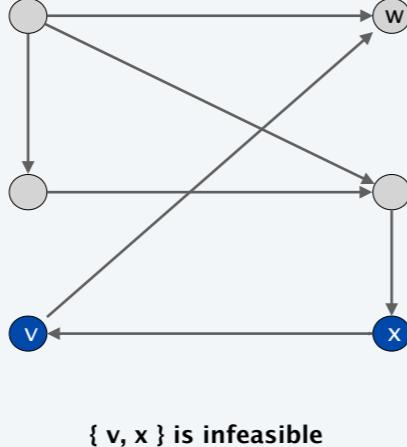
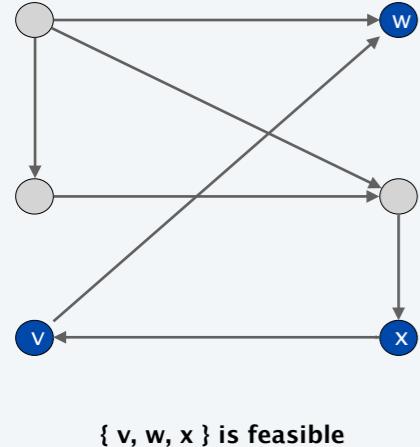
JEAN-CLAUDE PICARD  
Ecole Polytechnique, Montreal

This paper generalizes the selection problem discussed by J. M. Rhys [12], J. D. Murchland [9], M. L. Balinski [1] and P. Hansen [4]. Given a directed graph  $G$ , a closure of  $G$  is defined as a subset of nodes such that if a node belongs to the closure all its successors also belong to the set. If a real number is associated to each node of  $G$  a maximal closure is defined as a closure of maximal value.

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## Project selection: prerequisite graph

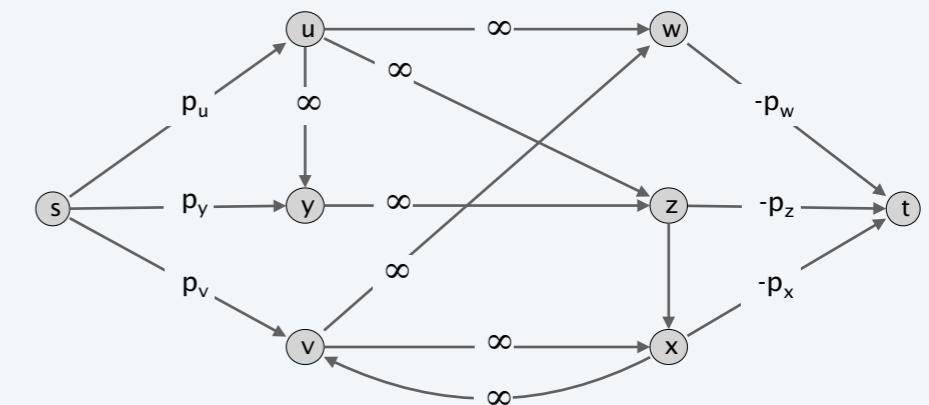
**Prerequisite graph.** Add edge  $(v, w)$  if can't do  $v$  without also doing  $w$ .



## Project selection: min-cut formulation

**Min-cut formulation.**

- Assign capacity  $\infty$  to all prerequisite edges.
- Add edge  $(s, v)$  with capacity  $p_v$  if  $p_v > 0$ .
- Add edge  $(v, t)$  with capacity  $-p_v$  if  $p_v < 0$ .
- For notational convenience, define  $p_s = p_t = 0$ .



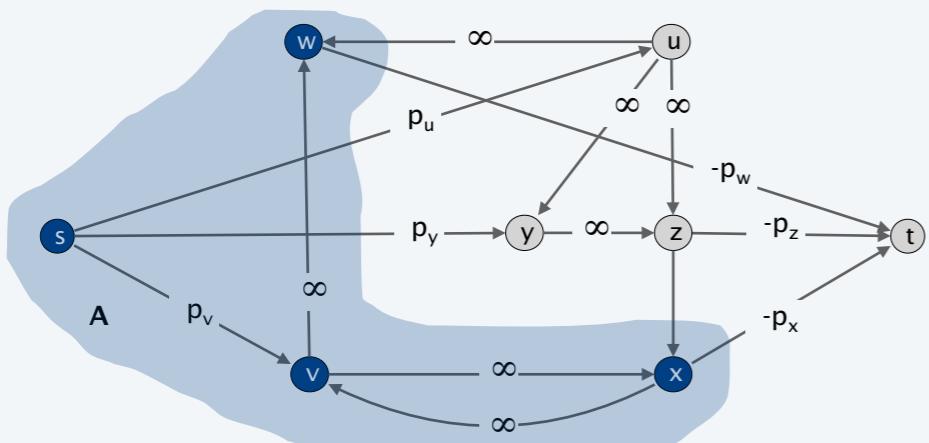
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## Project selection: min-cut formulation

**Claim.**  $(A, B)$  is min cut iff  $A - \{s\}$  is optimal set of projects.

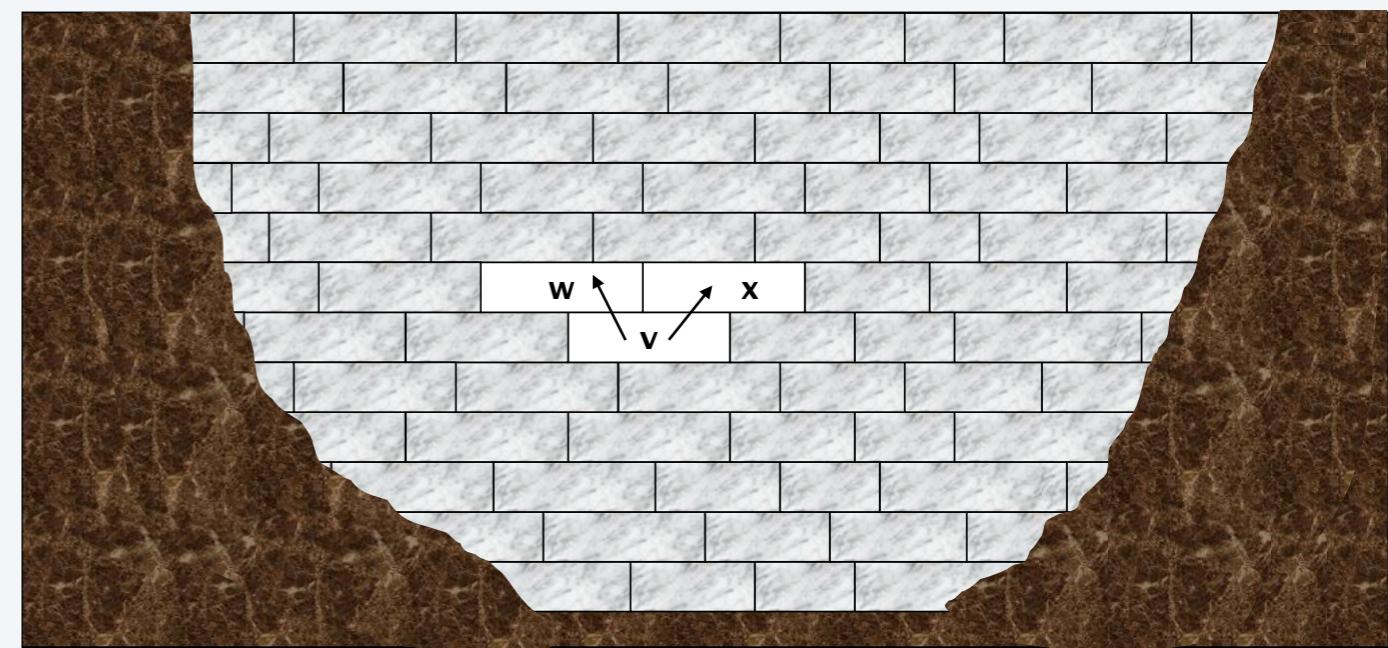
- Infinite capacity edges ensure  $A - \{s\}$  is feasible.
  - Max revenue because:  $cap(A, B) = \sum_{v \in B: p_v > 0} p_v + \sum_{v \in A: p_v < 0} (-p_v)$
- $$= \underbrace{\sum_{v: p_v > 0} p_v}_{\text{constant}} - \sum_{v \in A} p_v$$



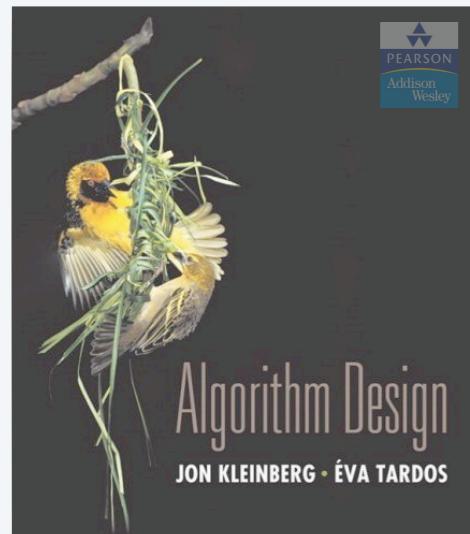
## Open-pit mining

**Open-pit mining.** (studied since early 1960s)

- Blocks of earth are extracted from surface to retrieve ore.
- Each block  $v$  has net value  $p_v$  = value of ore – processing cost.
- Can't remove block  $v$  before  $w$  or  $x$ .



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### Baseball elimination problem

Q. Which teams have a chance of finishing the season with the most wins?

i	team	wins	losses	to play	ATL	PHI	NYM	MON
0	Atlanta	83	71	8	-	1	6	1
1	Philly	80	79	3	1	-	0	2
2	New York	78	78	6	6	0	-	0
3	Montreal	77	82	3	1	2	0	-

Montreal is mathematically eliminated.

- Montreal finishes with  $\leq 80$  wins.
- Atlanta already has 83 wins.

Remark. This is the only reason sports writers appear to be aware of — conditions are sufficient but not necessary!

### Baseball elimination

TUESDAY, SEPTEMBER 10, 1996

**San Francisco Chronicle**

**SPORTING G**

**49ers, Young Get Big Bre**

Quarterback m

*By Gary Swan  
Chronicle Staff Writer*

The bye week has come at a perfect time for the 49ers and quarterback Steve Young. If they had a game next Sunday, there's a good chance Young would not play.

*...but the pulled groin muscle on his up-*

**Giants Officially Leave the NL West Race**

*By Nancy Gay  
Chronicle Staff Writer*

In San Diego, Greg Vaughn's three-run homer in the eighth pushed the Padres over the Pirates and officially shoved the rest of the Giants' season into the background. On the heels of their tedious 6-2 loss before an announced crowd of 10,307 at Candlestick Park, the Giants fell 19½ games off the lead.

With the smack of another National League West bat 500 miles away, the Giants' run at the division title ended last night, just as they were handing the visiting St. Louis Cardinals an even bigger lead in the NL Central.

**CARDINALS 6  
GIANTS 2**

In San Diego, Greg Vaughn's three-run homer in the eighth pushed the Padres over the Pirates and officially shoved the rest of the Giants' season into the background. On the heels of their tedious 6-2 loss before an announced crowd of 10,307 at Candlestick Park, the Giants fell 19½ games off the lead.

As it is, the worst the Padres (80-65) can finish is 80-82. The Giants have fallen to 59-83 with 20 games left; they cannot win 80 games. Coming off miserable 2-8 mark on a three-city road trip that saw their road record drop to 27-47, the Giants were hoping to get off on the right foot in their longest homestand of the year (15 games, 14 days).

"Where we are, you're going to be eliminated sooner or later," Baker said quietly. "But it doesn't alter the fact that we've still got to play ball. You've still got to play hard, the fans come out to watch you play. You've got to play for the fact of loving to play, no matter where you are in the standings."

"You've got to play the role of spoiler, to not make it easier on GIANTS; Page D5 Col. 3

### Baseball elimination problem

Q. Which teams have a chance of finishing the season with the most wins?

i	team	wins	losses	to play	ATL	PHI	NYM	MON
0	Atlanta	83	71	8	-	1	6	1
1	Philly	80	79	3	1	-	0	2
2	New York	78	78	6	6	0	-	0
3	Montreal	77	82	3	1	2	0	-

Philadelphia is mathematically eliminated.

- Philadelphia finishes with  $\leq 83$  wins.
- Either New York or Atlanta will finish with  $\geq 84$  wins.

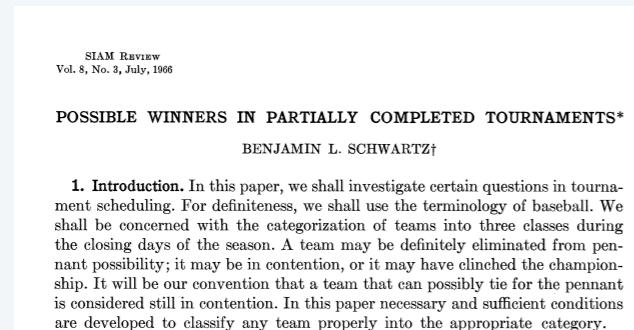
Observation. Answer depends not only on how many games already won and left to play, but on whom they're against.

## Baseball elimination problem

### Current standings.

- Set of teams  $S$ .
- Distinguished team  $z \in S$ .
- Team  $x$  has won  $w_x$  games already.
- Teams  $x$  and  $y$  play each other  $r_{xy}$  additional times.

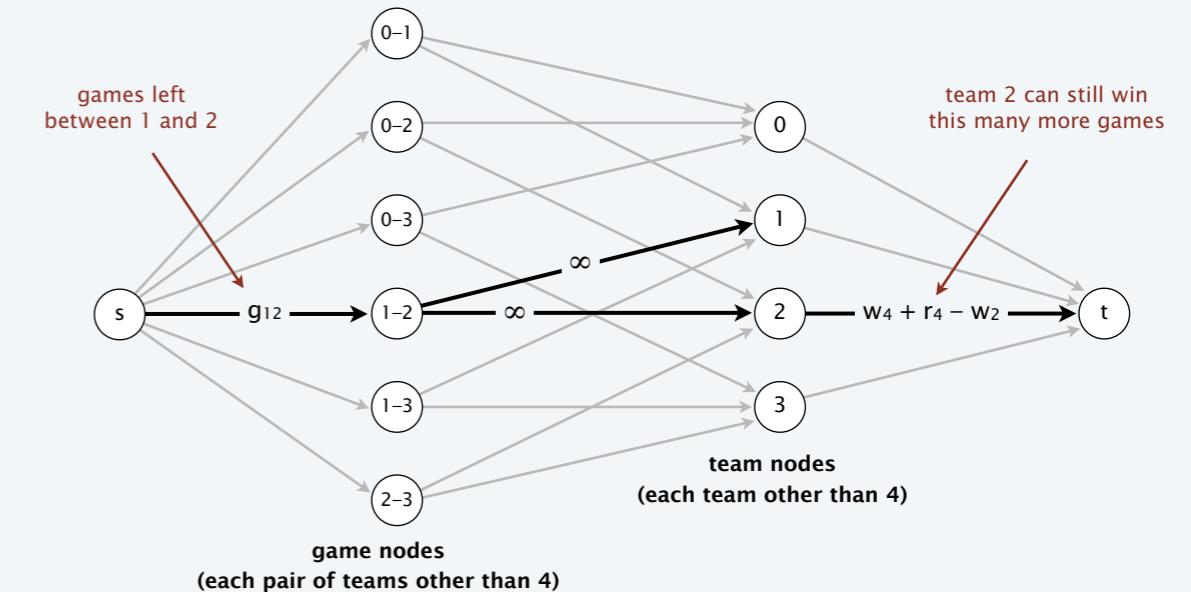
**Baseball elimination problem.** Given the current standings, is there any outcome of the remaining games in which team  $z$  finishes with the most (or tied for the most) wins?



## Baseball elimination problem: max-flow formulation

### Can team 4 finish with most wins?

- Assume team 4 wins all remaining games  $\Rightarrow w_4 + r_4$  wins.
- Divvy remaining games so that all teams have  $\leq w_4 + r_4$  wins.

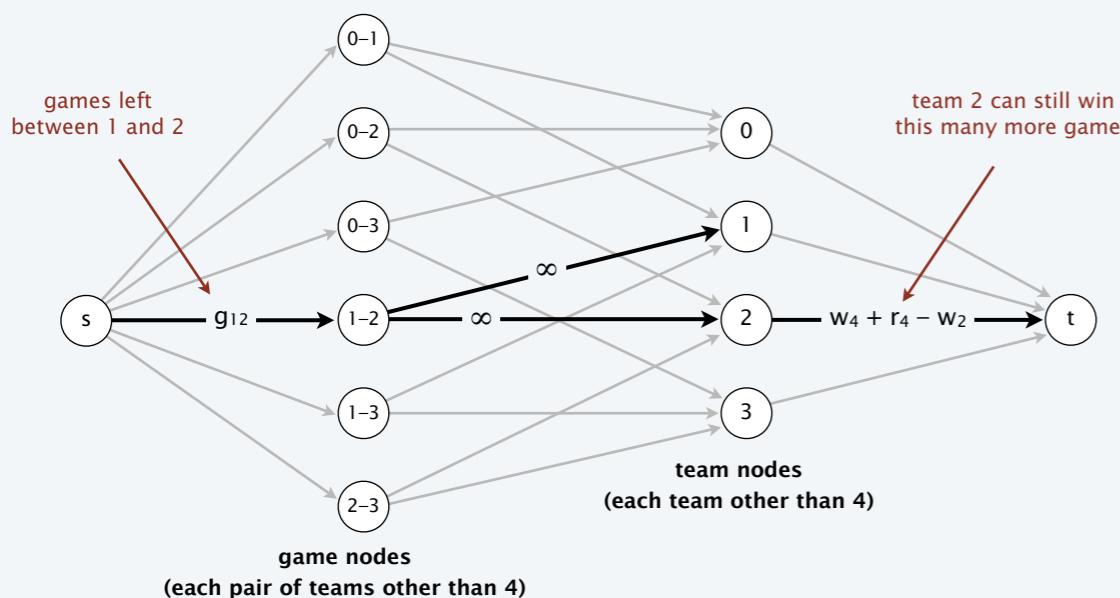


## Baseball elimination problem: max-flow formulation

**Theorem.** Team 4 not eliminated iff max flow saturates all edges leaving  $s$ .

Pf.

- Integrality theorem  $\Rightarrow$  each remaining game between  $x$  and  $y$  added to number of wins for team  $x$  or team  $y$ .
- Capacity on  $(x, t)$  edges ensure no team wins too many games. ■



## Baseball elimination: explanation for sports writers

**Q.** Which teams have a chance of finishing the season with the most wins?

i	team	wins	losses	to play	NYY	BAL	BOS	TOR	DET
0	New York	75	59	28	-	3	8	7	3
1	Baltimore	71	63	28	3	-	2	7	4
2	Boston	69	66	27	8	2	-	0	0
3	Toronto	63	72	27	7	7	0	-	0
4	Detroit	49	86	27	3	4	0	0	-

AL East (August 30, 1996)

**Detroit is mathematically eliminated.**

- Detroit finishes with  $\leq 76$  wins.
- Wins for  $R = \{ \text{NYY}, \text{BAL}, \text{BOS}, \text{TOR} \} = 278$ .
- Remaining games among  $\{ \text{NYY}, \text{BAL}, \text{BOS}, \text{TOR} \} = 3 + 8 + 7 + 2 + 7 = 27$ .
- Average team in  $R$  wins  $305/4 = 76.25$  games.

## Certificate of elimination.

$$T \subseteq S, w(T) := \overbrace{\sum_{i \in T} w_i}^{\# \text{ wins}}, \quad g(T) := \overbrace{\sum_{\{x,y\} \subseteq T} g_{xy}}^{\# \text{ remaining games}},$$

**Theorem.** [Hoffman-Rivlin 1967] Team  $z$  is eliminated iff there exists a subset  $T^*$  such that

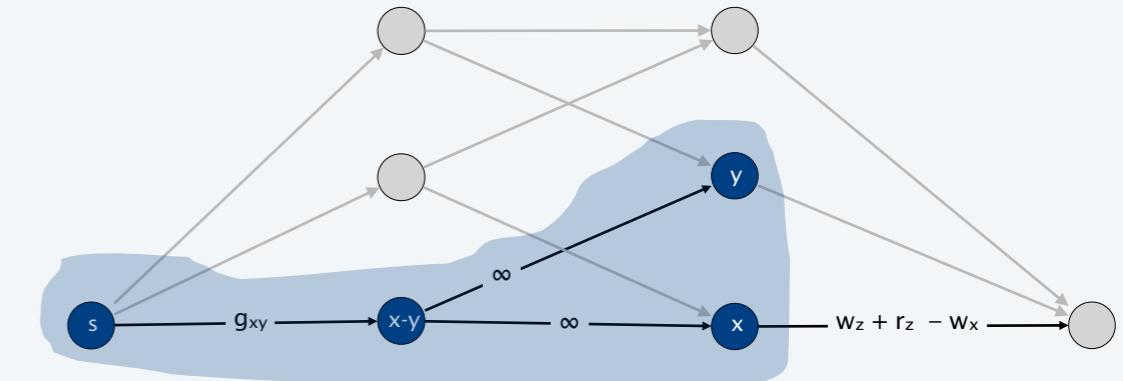
$$w_z + g_z < \frac{w(T^*) + g(T^*)}{|T^*|}$$

 Pf.  $\Leftarrow$ 

- Suppose there exists  $T^* \subseteq S$  such that  $w_z + g_z < \frac{w(T^*) + g(T^*)}{|T^*|}$ .
- Then, the teams in  $T^*$  win at least  $(w(T^*) + g(T^*)) / |T^*|$  games on average.
- This exceeds the maximum number that team  $z$  can win. ■

 Pf.  $\Rightarrow$ 

- Use max-flow formulation, and consider min cut  $(A, B)$ .
- Let  $T^* =$  team nodes on source side  $A$  of min cut.
- Observe that game node  $x-y \in A$  iff both  $x \in T^*$  and  $y \in T^*$ .
  - infinite capacity edges ensure if  $x-y \in A$ , then both  $x \in A$  and  $y \in A$
  - if  $x \in A$  and  $y \in A$  but  $x-y \notin A$ , then adding  $x-y$  to  $A$  decreases the capacity of the cut by  $g_{xy}$



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 Pf.  $\Rightarrow$ 

- Use max-flow formulation, and consider min cut  $(A, B)$ .
- Let  $T^* =$  team nodes on source side  $A$  of min cut.
- Observe that game node  $x-y \in A$  iff both  $x \in T^*$  and  $y \in T^*$ .
- Since team  $z$  is eliminated, by max-flow min-cut theorem,

$$\begin{aligned} g(S - \{z\}) &> \text{cap}(A, B) \\ &\quad \text{capacity of game edges leaving } s \quad \text{capacity of team edges entering } t \\ &= \overbrace{g(S - \{z\}) - g(T^*)}^{} + \overbrace{\sum_{x \in T^*} (w_z + g_z - w_x)}^{} \\ &= g(S - \{z\}) - g(T^*) - w(T^*) + |T^*|(w_z + g_z) \end{aligned}$$

- Rearranging terms:  $w_z + g_z < \frac{w(T^*) + g(T^*)}{|T^*|}$  ■