Introduction to Separation Logic

Heavily borrows from slides by Cristiano Calcagno, Imperial College London

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Syntax of Separation Logic

- Given a decidable base-theory T, the syntax of separation logic $SL(T)_{Loc,Data}$ is presented
- Loc and $Data$ represent the type of the address and the values [Srivas: 1. Is T different from Loc, Data? I thought they would be part of T
  2. Data can be Loc too, right otherwise you couldn't express indirection]
- Loc and $Data$ can be any sorts, but Loc should be countably infinite, for the purpose of the decision procedure
- For example, Loc and $Data$ can be Int

$$P, Q ::= \ false \mid P \wedge Q \mid P \vee Q \mid P \rightarrow Q$$
$$\mid P * Q \mid P \mathbin{-\!*} Q$$
$$\mid E = E' \mid E \hookrightarrow E' \mid empty$$

We use E and E' to denote expressions in the base theory, where heap indirection is not used. This is needed to syntactically rule out formulas like $F : (x \hookrightarrow v_1) = (y \hookrightarrow v_2)$ [Srivas: What are the operators allowed in E? Obviously it scan't have dref(*); can it have arithmetic over Loc? I guess yes; Can it have &; I guess not?]

Semantics of Separation Logic

The model consists of an interpretation (I) and a heap (h)

$$I : \mathrm{Var} \rightarrow \mathrm{Loc}$$
$$h : \mathrm{Loc} \rightarrow \mathrm{Data}$$

$I, h \models \mathit{false}$	never satisfied
$I, h \models P \wedge Q$	$I, h \models P$ and $I, h \models Q$
$I, h \models P \vee Q$	$I, h \models P$ or $I, h \models Q$
$I, h \models P \rightarrow Q$	$I, h \models P$ implies $I, h \models Q$
$I, h \models E = E'$	$[\![E]\!]_I = [\![E']\!]_I$

We use $[\![E]\!]_I$, to denote the value of E under the interpretation I. Also, the domain of h should be a <u>finite</u> subset of Loc and it can be a partial function.

Semantics of Separation Logic:

Empty heap: $\quad\quad\quad I, h \models empty$ iff $h = \phi$

Points to: $\quad\quad\quad I, h \models E \hookrightarrow E'$ iff $h(\llbracket E \rrbracket_I) = \llbracket E' \rrbracket_I$

Separating conjunction: defines two separate areas of heap

$$I, h \models P * Q$$
$$\text{iff } \exists h_1, h_2.(h_1 \# h_2) \wedge (h = h1 \circ h2) \wedge I, h_1 \models P \wedge I, h_2 \models Q,$$

where $h_1 \# h_2$ denotes that the heap domains are disjoint and $h_1 \circ h_2$ means their union.

Examples

Points to,

$$F : x \hookrightarrow 10$$
$$I : \{(x, 0)\}$$
$$h : \{(0, 10)\}$$
$$I, h \models F$$

Separating conjunction,

$$F : x \hookrightarrow 10 * y \hookrightarrow 20$$
$$I : \{(x, 0), (y, 1)\}; \quad h : \{(0, 10), (1, 20)\}; \quad I, h \models F$$
$$I : \{(x, 0), (y, 1)\}; \quad h' : \{(0, 10), (1, 10)\}; \quad I, h' \models F \ ?$$

Examples

Another example,

$$F : x \hookrightarrow y * y \hookrightarrow x$$
$$I : \{(x, 0), (y, 1)\} \qquad\qquad h : \{(0, 1), (1, 0)\}$$
$$I, h \models F$$
$$I' : \{(x, 0), (y, 0)\}$$
$$h' : \{(0, 0)\}$$
$$I', h' \not\models F$$

Separating Implication: $\rightarrow\!\!\ast$

Separating Implication

$$I, h \models P \rightarrow\!\!\ast Q$$
$$\text{iff } \forall h'.(h\#h') \wedge (I, h' \models P \rightarrow I, h \circ h' \models Q)$$

Interpretation: Defines condition under which P is *separable* from Q

P is *separable* from Q if for every heap h' that satisfies P, we can make another heap h satisfy Q by *disjointedly extending* h by h'

Examples

Separating Implication

$$I, h \models P \twoheadrightarrow Q$$
$$\text{iff } \forall h'.(h \# h') \wedge (I, h' \models P) \rightarrow I, h \circ h' \models Q$$

Example,

$$F : (x \hookrightarrow 10) \twoheadrightarrow (x \hookrightarrow 10 * y \hookrightarrow 20)$$
$$I : \{(x, 0), (y, 1)\}$$
$$h' : \{(0, 10)\}$$
$$h : \{(1, 20)\}$$
$$h \circ h' : \{(0, 10), (1, 20)\}$$
$$I, h \models F$$

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Translating Separation Logic into Pointer Logic

Points to,

$$I, h \models x \hookrightarrow v$$
$$\Longleftrightarrow$$
$$L, M \models {}^{*}x = v$$

Separating conjunction,

$$I, h \models x \hookrightarrow v_1 * y \hookrightarrow v_2$$
$$\Longleftrightarrow$$
$$L, M \models {}^{*}x = v_1 \ \wedge \ {}^{*}y = v_2 \wedge x \neq y$$

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Need for inductive predicates

- Most interesting data structures in programs are defined as inductive systems
- For example : linked lists, trees, graphs
- Being able to reason about these in SL is useful
- But inductive predicates introduce quantifiers

This way of specifying a list is quite cumbersome,

$$p \hookrightarrow v_1, p_1 \wedge$$
$$p_1 \hookrightarrow v_2, p_2 \wedge$$
$$p_2 \hookrightarrow v_3, p_3 \wedge$$
$$\cdots$$

Lists in pointer logic

First try at defining lists without explicit quantifiers. We define the following predicates for the ith element of the list,

$$\text{list-elem}(p, 0) \equiv p$$
$$\text{list-elem}(p, i) \equiv *\text{list-elem}(p, i - 1)$$

In this formulation, a null-terminated list would be

$$\text{list-elem}(p, l) = NULL$$

and a cyclic list would be,

$$\text{list-elem}(p, l) = p$$

But the last predicate is also satisfied by an element with length 1.

To actually get a disjoint list with unique pointer elements, we need to add an extra constraint,

$$\text{overlap}(p, q) \equiv p = q \lor p + 1 = q \lor p = q + 1$$
$$\text{list-disjoint}(p, 0) \equiv \textit{TRUE}$$
$$\text{list-disjoint}(p, l) \equiv \text{list-disjoint}(p, l - 1) \land$$
$$\forall 0 \leq i < l - 1. \neg \, \text{overlap}(\text{list-elem}(p, i), \text{list-elem}(p, l - 1))$$

The set of clauses grows quadratically upon the quantifier instantiation.

Lists - Try 2

We now try to define a list with inductive predicates,

$$\text{list } 0 \ x \equiv x = nil$$
$$\text{list } n \ x \equiv \exists y.(x \hookrightarrow v, y) \wedge (list \ (n-1) \ y)$$

This is a satisfying assignment for $list \ 3 \ x$,

$$I = \{ \ (x, 0), (y, 0), (z, 1), (w, nil) \ \}$$
$$h = \{ \ (0, (v, 1)), (1, (v, w)) \ \}$$

The pointers x and y got aliased to point to z.

$$list \ 3 \ x \equiv$$
$$(x = 0) \hookrightarrow v, 1 \wedge$$
$$(y = 0) \hookrightarrow v, 1 \wedge$$
$$(z = 1) \hookrightarrow v, nil \wedge$$

Lists in Separation Logic

In separation logic, the separating conjunction takes care of ensuring the pointers don't alias.

$$\text{list } 0 \; x \equiv x = \textit{nil}$$
$$\text{list } n \; x \equiv \exists y.(x \hookrightarrow v, y) * (\textit{list } (n-1) \; y)$$

Applications: Hoare Logic

Separation logic has been useful in Hoare logic for program verification.
We will have a quick look at Hoare logic.

$$\{P\}S\{Q\}$$

P : Logical assertion on states - precondition

S : Code section that modifies state

Q : Logical assertion on states - postcondition

Example: $\{x = 1\}x := 2\{x = 2\}$

Hoare Triple : holds if "whenever you start in a state that satisfies the
precondition and execute the program statement S and terminate then you
will be in a state that satisfies the postcondition"

The programs S are defined against a language specification with
imperative commands such as skip, loops, variable including pointer
assignments.

Strongest Postcondition and Weakest Precondition

Now given a post-condition Q and program S, we want to calculate the set of states, that the program can be in before, to end-up in Q after execution.
Example,

$$\{x \geq 0\}x := x + 1\{x > 0\}$$

$(x > 0)$ is the *strongest* postcondition of $(x \geq 0)$: $SP(x := x + 1, x \geq 0)$

Similarly, we can reason backwards using *weakest* precondition defining all the states that the program *should* be in before, to guarantee Q to hold.
$WP(x := x + 1, x \geq 0) = x \geq -1$

The first application of separation logic in Hoare-style verification is to do local reasoning, which is defined as,

$$\frac{\{P\}S\{Q\}}{\{P * R\}S\{Q * R\}}$$

Example,

$$\{\text{root} \hookrightarrow (\text{left}, \text{right}) * \text{tree}(\text{left}) * \text{tree}(\text{right})\}$$
$$\text{deletetree}(\text{left})$$
$$\{\text{root} \hookrightarrow (\text{left}, \text{right}) * \text{emp} * \text{tree}(\text{right})\}$$
$$\text{deletetree}(\text{right})$$
$$\{\text{root} \hookrightarrow (\text{left}, \text{right}) * \text{emp} * \text{emp}\}$$
$$\text{free}(\text{root})$$
$$\{\text{emp} * \text{emp} * \text{emp}\}$$
$$\{\text{emp}\}$$

Applications

Another application is to have an operation similar to modus-ponens for heap predicates.

$$P * (P \twoheadrightarrow Q) \models Q$$

Example,

$$(x \hookrightarrow v_1) * (x \hookrightarrow v_1 \twoheadrightarrow y \hookrightarrow v_2) \models (y \hookrightarrow v_2)$$

Also, this is used in weakest pre-condition computation, where if we have a triple,

$$\{P\}x := 3\{Q\}$$

where the weakest precondition would be,

$$wp(x := 3, Q) \equiv (x \hookrightarrow -) * ((x \hookrightarrow 3) \twoheadrightarrow Q)$$

The trivial case being $P = x \hookrightarrow 3$.