Answer to the Question No. - 1

To show that $\sqrt{n} = \Omega(\log_2^3 n)$ using Definition 2, we proceed as follows:

Ω-notation: $g(n) \in \Omega(f(n))$

If there exist constants $c > 0$ and $n_0 \geq 0$ such that:

$$g(n) \geq c \cdot f(n) \qquad \text{for all } n \geq n_0.$$

Here, we need to show:

$$\sqrt{n} \geq c \cdot \log_2^3 n \quad \text{for sufficiently large n (for some } n_0\text{)}$$

Let,

$$g(n) = \sqrt{n}$$

$$f(n) = \log_2^3 n = \left(\frac{\ln n}{\ln 2}\right)^3$$

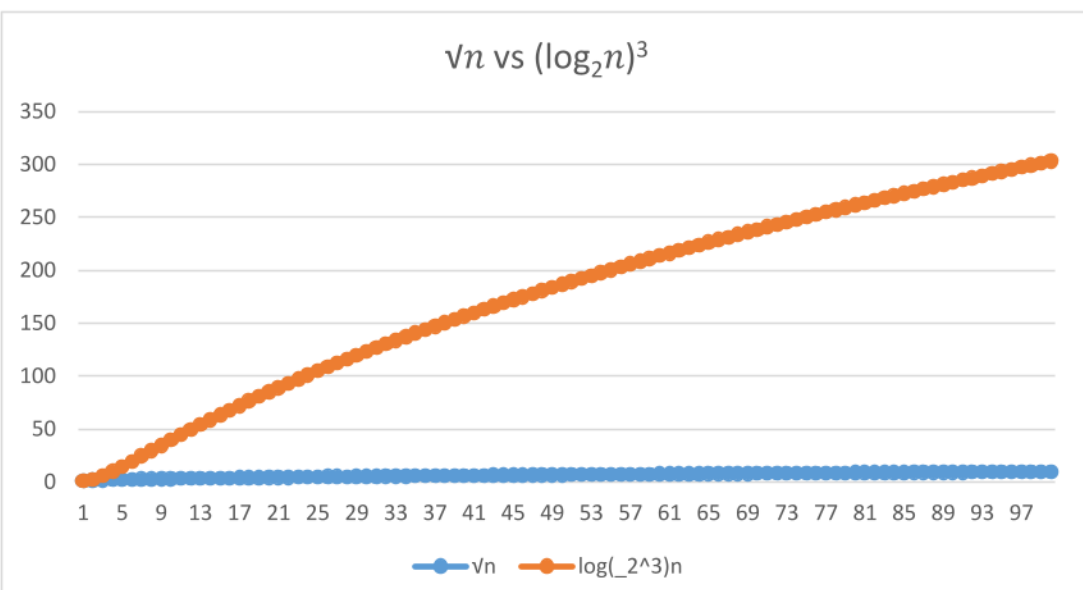
So, we need to show that:

$$\sqrt{n} \geq c \cdot \left(\frac{\ln n}{\ln 2}\right)^3$$

Dividing both sides by $\ln^3 n$, we get,

$$\frac{\sqrt{n}}{\ln^3 n} \geq c \cdot \left(\frac{1}{\ln 2}\right)^3$$

For choosing constants c and n_0,



We can let $c = (\ln 2)^3$ as a positive constant and we need to choose n_0 large enough such that the equation holds for all $n \geq n_0$.

As the right side is constant, let's analyze the left side.

Since, $n \to \infty$, the term $\sqrt{n}$ grows much faster than $\ln^3 n$. Because logarithmic functions grow much slower than any polynomial of n.

$$\Rightarrow \lim_{n\to\infty} \frac{\sqrt{n}}{\ln^3 n}$$

$$= \lim_{n\to\infty} \frac{\frac{1}{2\sqrt{n}}}{\frac{3(\ln n)^2}{n}} = \lim_{n\to\infty} \frac{\sqrt{n}}{6(\ln n)^2}$$

$$= \lim_{n\to\infty} \frac{\sqrt{n}}{24 \ln n}$$

$$= \lim_{n\to\infty} \frac{\sqrt{n}}{48} = \infty$$

The above proof shows that, $\lim_{n\to\infty} \frac{g(n)}{f(n)} = \infty$

The condition for Ω-notation is satisfied because,

$$\sqrt{n} \geq c \cdot \log_2^3 n \quad \text{for all } n \geq n_0.$$

So, by Definition 2, it is proven that,

$$\sqrt{n} = \Omega(\log_2^3 n)$$

Answer to the Question No. – 2

If we convert the given time into microseconds we get,

1 second = 10^6 microseconds

1 minute = 6×10^7 microseconds

1 hour = 3.6×10^9 microseconds

1 day = 8.64×10^{10} microseconds

1 month = 2.59×10^{12} microseconds

1 year = 3.15×10^{13} microseconds

1 century = 3.15×10^{15} microseconds

Let the time = t, then for each expression the result will be as following:

$$\begin{array}{cl} \log_2 n & \rightarrow\ 2^t \\ \sqrt{n} & \rightarrow\ t^2 \\ n & \rightarrow\ t \\ n\log n & \rightarrow\ \text{approximations by calculating } n\log n\ \sim t \\ n^2 & \rightarrow\ \sqrt{t} \\ n^3 & \rightarrow\ \sqrt[3]{t} \\ 2^n & \rightarrow\ \log_2 t \\ n! & \rightarrow\ \text{approximations by calculating } n! \sim t \end{array}$$

The value for the comparison of running times has been shown in the following table:

	1 second	1 minute	1 hour	1 day	1 month	1 year	1 century
$\log_2 n$	2^{10^6}	$2^{6\times10^7}$	$2^{3.6\times10^9}$	$2^{8.64\times10^{10}}$	$2^{2.59\times10^{12}}$	$2^{3.15\times10^{13}}$	$2^{3.15\times10^{15}}$
$\sqrt{n}$	10^{12}	3.6×10^{15}	1.3×10^{19}	7.46×10^{21}	6.72×10^{24}	9.95×10^{26}	9.95×10^{30}
n	10^6	6×10^7	3.6×10^9	8.64×10^{10}	2.59×10^{12}	3.15×10^{13}	3.15×10^{15}
$n\log n$	6.24×10^4	2.8×10^6	1.33×10^8	2.76×10^9	7.19×10^{10}	7.98×10^{11}	6.86×10^{13}
n^2	1000	7745	60000	293938	1.6×10^6	5.62×10^6	5.62×10^7
n^3	100	391	1532	4420	13736	31593	146645
2^n	19	25	31	36	41	44	51
$n!$	9	11	12	13	15	16	17

Answer to the Question No. – 3

Assuming that $k \geq 1, \epsilon > 0, and\ c > 1$ are constants, we need to find the relative asymptotic growths for each pair of expressions (A, B).

$\lg^k n$ vs n^ϵ:

$$\lim_{n\to\infty} \frac{\log^k n}{n^\epsilon}$$

Since n^ϵ grows much faster than any power of $\log n$, the limit goes to 0. We get,

$$\lim_{n\to\infty} \frac{\log^k n}{n^\epsilon} = 0$$

So, it will be **yes** for O, o but **no** for **Ω**, ω, θ.

n^k vs c^n:

$$\lim_{n \to \infty} \frac{n^k}{c^n}$$

Using logarithms,

$$= \lim_{n \to \infty} \frac{k \, log \, n}{n \, log \, c}$$

$$= \lim_{n \to \infty} e^{k \, log \, n - \, n \, log \, c}$$

Since, $n \, log \, c$ grows faster than $k \, log \, n$,

$$= \lim_{n \to \infty} e^{-\infty} = 0$$

So, it will be **yes** for O, o but **no** for **Ω**, ω, θ.

$\sqrt{n}$ vs $n^{\sin n}$:

The value for $\sin n$ is in the range between -1 to +1 and it doesn't have a consistent growth pattern. As there is no consistency, none of the asymptotic relations can be justified. So, it will be **no** for each one.

2^n vs $2^{n/2}$:

$$\lim_{n \to \infty} \frac{2^n}{2^{n/2}}$$

If we take the ratio, it would be $2^{n/2}$, which tends to ∞. It refers to the lower bound. So, it will be **yes** for **Ω**, ω but **no** for O, o and θ.

$n^{lg\,c}$vs $c^{lg\,n}$:

We can rewrite the expressions as,

$$n^{\lg c} = e^{\log c \, \cdot \, \log n}$$

$$c^{\lg n} = e^{\log n \, \cdot \, \log c}$$

Therefore,

$$n^{\lg c} = c^{\lg n}$$

As both the expressions are equal, they will grow at the same rate. So, it will be **yes** for O, **Ω** and θ but **no** for o and ω.

$\lg(n!)$ vs $\lg(n^n)$:

Both the functions represent the same output. As the difference is almost non-existent, they grow at the same rate. So, for this case, it will be **yes** for O, Ω and θ but **no** for o and ω.

Considering all the cases, the final results have been shown in the following table:

A	B	O	o	Ω	ω	θ
$\lg^k n$	n^ϵ	Yes	Yes	No	No	No
n^k	c^n	Yes	Yes	No	No	No
$\sqrt{n}$	$n^{\sin n}$	No	No	No	No	No
2^n	$2^{n/2}$	No	No	Yes	Yes	No
$n^{\lg c}$	$c^{\lg n}$	Yes	No	Yes	No	Yes
$\lg(n!)$	$\lg(n^n)$	Yes	No	Yes	No	Yes