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Answer to the Question 1:

To show that $\sqrt{n} = \Omega(\log_2^3 n)$ using Definition 2; we proceed as follows:

Ω Notation:

$g(n) \in \Omega(f(n))$: exist constants $c > 0$ and $n_0 \geq 0$ such that:

$g(n) \geq c \cdot f(n)$ for all $n \geq n_0$

Here, needed to show:

$$\sqrt{n} \geq c \cdot \log_2^3 n$$

Let, $g(n) = \sqrt{n}$

$$f(n) = \log_2^3 n$$

$$= \left(\frac{\log_e n}{\log_e 2}\right)^3$$

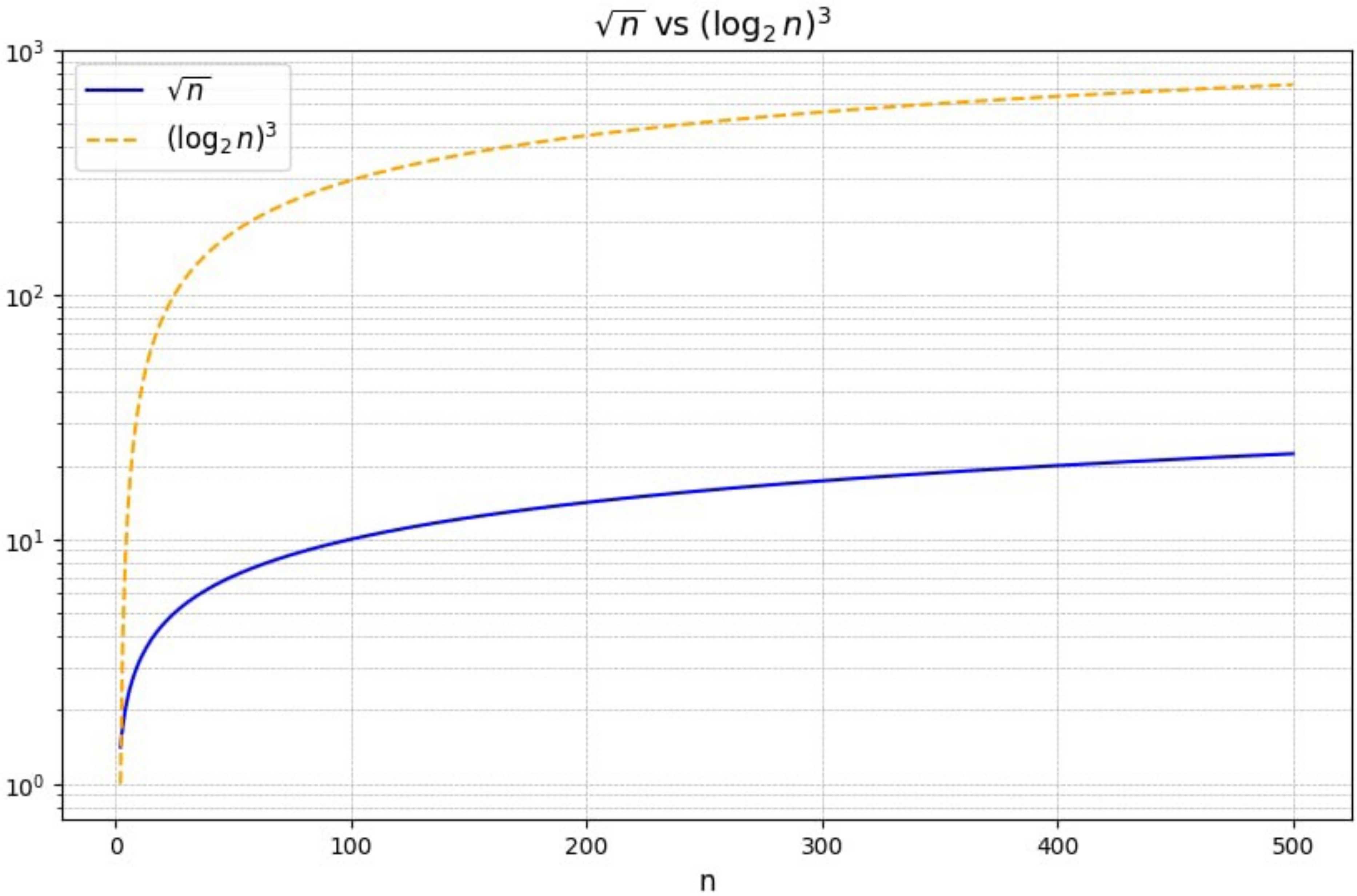
$$= \left(\frac{\ln n}{\ln 2}\right)^3$$

So, to show that:

$$\sqrt{n} \geq c \cdot \left(\frac{\ln n}{\ln 2}\right)^3$$

$$\Rightarrow \frac{\sqrt{n}}{\ln^3 n} \geq c \cdot \left(\frac{1}{\ln 2}\right)^3 \quad [\text{Dividing both sides by } \ln^3 n]$$

For choosing constants c and n_0



$\sqrt{n}$ vs $(\log_2 n)^3$
$\sqrt{n}$
$(\log_2 n)^3$
10^0
10^1
10^2
10^3
0
100
200
300
400
500
n

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Now, Let $c = (\ln 2)^3$ as a positive constant and we need to choose n_0 large enough so $n \geq n_0$.

~~As the~~ right side is constant. Now to analyze ~~the~~ left side.

Since $n \to \infty$, the term $\sqrt{n}$ grows ~~faster than~~ $\ln^3 n$. Because log functions grow slower ~~than~~ any polynomial value of n.

$$\cancel{\lim_{n\to\infty} \frac{\sqrt{n}}{}}$$

$$\lim_{n\to\infty} \frac{\sqrt{n}}{\ln^3 n}$$

$$= \lim_{n\to\infty} \frac{\frac{1}{2\sqrt{n}}}{\frac{3(\ln n)^2}{n}}$$

$$= \lim_{n\to\infty} \frac{\sqrt{n}}{6(\ln n)^2}$$

$$= \lim_{n\to\infty} \frac{\sqrt{n}}{24 \ln n} = \lim_{n\to\infty} \frac{\sqrt{n}}{48} = \infty$$

The proof shows that, $\lim_{n\to\infty} \frac{g(n)}{f(n)} = \infty$

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The statement for Ω-notation is satisfied because,

$$\sqrt{n} \geq c \cdot \log_2^3 n \quad \text{for all } n \geq n_0$$

So, by using definition 2, it is proven that,

$$\sqrt{n} = \Omega\left(\log_2^3 n\right)$$

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Questions: $\boxed{1.1}$

~~Answer to~~ the Question 2:

~~Answer~~: Now, to compare of running times in the following table -

where to convert times to microseconds

$1 sec = 10^6$ microseconds

$1 min = 60 \times 10^6$ micro

$1 hour = 3600 \times 10^6$ micro

$1 day = 86400 \times 10^6$ micro

$1 month = 259200 \times 10^6$ micro

$1 year = 31536000 \times 10^6$ micro

$1 century = 3153600000 \times 10^6$ micro

To solve each function

1. $f(n) = \log n \Rightarrow \frac{\log_{10} n}{\log_{10} 2} \times n = t$

$= 2^{10^6}$ [for 1 second, $t = 10^6$ micro]

2. $f(n) = \sqrt{n}$

So, $\sqrt{n} = t$

$\Rightarrow n = t^2 = (10^6)^2 = 10^{12}$ [1 second $= 10^6$]

$f(n) = 10^{12}$

3. $f(n) = n = t$

$\therefore f(n) = 10^6$

4. $f(n) = n \log_4 n = t$

$\Rightarrow n \times \frac{\log_{10} n}{\log_{10} 2} = t$

$\Rightarrow f(n) = 6.24 \times 10^4$

5. $f(n) = n^2 = t$

$\Rightarrow n^2 = 10^6$

$\Rightarrow n = \sqrt{10^6}$

$\therefore n = 1000$

6. $f(n) = n^3 = t$

$\therefore n = 100$

7. $f(n) = 2^n$

$= 19$

	1 second	1 minute	1 hour	1 day	1 month	1 year	1 century
$\lg n$	2^{10^6}	$2^{60\times10^6}$	$2^{3.6\times10^9}$	$2^{8.64\times10^{10}}$	$2^{2.59\times10^{12}}$	$2^{3.15\times10^{13}}$	$2^{3.15\times10^{15}}$
$\sqrt{n}$	10^{12}	3.6×10^{15}	1.3×10^{19}	7.46×10^{21}	6.72×10^{24}	9.95×10^{26}	9.95×10^{30}
n	10^6	6×10^7	3.6×10^9	8.64×10^{10}	2.59×10^{12}	3.15×10^{13}	3.15×10^{15}
$n\lg n$	6.24×10^4	2.8×10^6	1.33×10^8	2.76×10^9	7.19×10^{10}	7.98×10^{11}	6.86×10^{13}
n^2	1000	7745	60000	293938	1609968	5615692	56156922
n^3	100	391	1532	4420	13736	31593	146645
2^n	19	25	30	35	40	45	50
$n!$	9	11	12	13	15	16	17

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Answer to the Question 3

Answer: 3.2 →

Indicate for each pair of expressions (A, B) in the below with $O, o, \Omega, \omega, \Theta$ relationships, to analyze the asympatic behaviors -

We know that,

$O \rightarrow f(n) \in O(g(n))$; $c>0$ and n_0 such that $0 \leq f(n) \leq c \cdot g(n)$ for all $n \geq n_0$

$o \rightarrow f(n) \in o(g(n))$; $c>0$ there exists n_0 such that $0 \leq f(n) \cancel{\leq} < c \cdot g(n)$ for all $n \geq n_0$

$\Omega \rightarrow f(n) \in \Omega(g(n))$; $c>0$ and n_0 such that $0 \leq c \cdot g(n) \leq f(n)$ for all $n \geq n_0$

$\omega \rightarrow f(n) \in \omega(g(n))$; $c>0$ and n_0 such that $c \cdot g(n) < f(n)$ for all $n \geq n_0$

$\Theta \rightarrow f(n) \in \Theta(g(n))$; $f(n) = O(g(n))$ and
$f(n) = \Omega(g(n))$

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Here's the justification for each row to clarify how to prove the table -

First one, $\log^k n$ vs n^ϵ : The polynomial term n^ϵ grows significantly faster than any logarithmic term $\log^k n$ as $n \to \infty$

$$\lim_{n \to \infty} \frac{\log^k n}{n^\epsilon} = 0$$

This shows that A is $o(B)$,

← O; Yes; $A \leq c \cdot B$ for large n

← o: Yes: A grows slower than B

← Ω, ω, Θ, No; B dominates A

Then, n^k vs c^n : Exponential growth dominates polynomial growth for any k.

$$\lim_{n \to \infty} \frac{n^k}{c^n} = 0$$

This shows that →

O; Yes as $A \leq c \cdot B$

o: Yes as A grows strictly slower than B

Ω, ω, Θ; No as B dominates A

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$\sqrt{n}$ vs $n^{\sin n}$;

Now, $\sqrt{n}$: Polynomial growth with $n^{1/2}$, $n^{\sin n}$; Between n^{-1} and n.

$n^{\sin n}$ doesnot grow consistently faster or slower than $\sqrt{n}$ and there are intervals when $A > B$ and intervals where $A < B$.

So, $O, o, \Omega, \omega, \Theta$; All are No.

2^n vs $2^{n/2}$; $A/B = \frac{2^n}{2^{n/2}}$

$$\frac{2^n}{2^{n/2}} = 2^{n-\frac{n}{2}}$$

$$= 2^{n/2}$$

As $n \to \infty$; $\frac{A}{B} \to \infty$, So A dominates B

O, o ; No as A grows faster than B

Ω, ω ; Yes as $A \geq c \cdot B$ and grows strictly faster.

Θ ; No as $A \neq \Theta(B)$

$n^{\log c}$ vs $c^{\log n}$

Here, $n^{\log c}$ and $c^{\log n}$ are identical expressions, so they grow at the same rate.

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Lastly, $\log(n!)$ vs $\log(n^n)$

For $n!$

$$n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n$$

Taking logarithm:

$$\log(n!) \sim n\log n - n$$

So, A is $n\log n$, so $A \sim B$ asymptotically.

To complete the table ->

A	B	O	o	Ω	ω	Θ
$\log^k n$	n^ϵ	Yes	Yes	No	No	No
n^k	c^n	Yes	Yes	No	No	No
$\sqrt{n}$	$n^{\sin n}$	No	No	No	No	No
2^n	$2^{n/2}$	No	No	Yes	Yes	No
$n^{\log c}$	$c^{\log n}$	Yes	No	Yes	No	Yes
$\log(n!)$	$\log(n^n)$	Yes	No	Yes	No	Yes