2 SORTING BY RANKING

{**rank** of an item=number of items smaller}

rank each item by counting;

then place each item according to its rank.

If duplicate, then place it at the nearest slot to the right.

$W(n) = O(n(n-1)) = A(n) = B(n)$

$S(n) = O(n)$

To handle duplicates, redefine

rank(i) = number of items smaller than i or, if equal, occurring before i.

Oblivious Algorithm

1

3 SORTING BY SWAPPING

Bubble Sort–good when input is nearly sorted

$W(n) = O(n^2/2)$

$S(n) = O(1)$

Odd-Even Exchange Sort

a) odd-even compare & exchange

b) even-odd compare & exchange

c) repeat step (a) and (b) if exchange-count > 0

$W(n) = O(n^2)$

$S(n) = O(1)$

4 SORTING BY INSERTION

Online algorithms.

Linear Insertion Sort

Insert the next element in the ordered list prepared so far by sequential search & shifting.

$W(n) = O(n^2/2)$

$S(n) = O(1)$

$O(n)$ time on a sorted list

Binary Insertion sort

perform binary search to find location for insertion.

$W(n) = O(n \log n) + O(n^2)$.

Tree Sort

Insert into a binary search tree, then traverse tree in-order.

$W(n) = O(n^2)$

$S(n) = O(n)$

$A(n) = B(n) = O(n \log n) + O(n)$

5 SORTING BY SELECTION

Offline algorithms

Selection sort

find maximum & replace with the concurrent last

$W(n) = O(n^2)$

$S(n) = O(1)$

$O(n^2)$ even on a sorted list

Tournament Sort

$W(n) = A(n) = B(n) = O(n \log n)$

$S(n) = O(n)$ for the tournament tree

Heap Sort

- construct a max heap -$O(n)$
- delete root and update heap repeatedly -$O(n \log n)$

$W(n) = A(n) = O(n \log n)$

$S(n) = O(1)$

5.1 Heap Sort

- Restore-Heap(i): $O(h)$ where h is the height of the node i.

- Construct Heap:

  For $i = \left\lfloor \frac{n}{2} \right\rfloor$ down to 1   Restore-Heap(i)

  $O(n)$

- Heap Sort:

  1. Construct Heap – $O(n)$
  2. For i:=n down to 2 – $O(n \log n)$

     exchange $L[1]$ with $L[i]$

     decrement heap size

     Restore-Heap(1)

- Time Complexity of deletion phase in Heap Sort

$$\begin{aligned} W(n) &= 2\log n + W(n-1) \\ &= 2\sum_{i=1}^{n} \log i \\ &\leq 2\int_{1}^{n+1} \log x dx \end{aligned}$$

$$= \left[2(x\log x - x)\right]_1^{n+1}$$

$$= 2n\log n - 2n$$

5.1.1 Heapsort Construction

Construct Heap:

for $i = \lfloor n/2 \rfloor$ downto 1

Restore-heap(i)

Time Complexity of Iterative Algorithm for Heap Construction:

$$\sum_{h=1}^{\lfloor \log n \rfloor} \lceil n/2^{h+1} \rceil O(h)$$

$$= O\left(n \sum_{h=1}^{\log n} (h/2^h)\right)$$

$= O(n)$ (pp. 159)

Consider $\sum_{h=1}^{\log n} h/2^h$

Let

$$\begin{aligned}
x &= && 1/2 + 2/2^2 + 3/2^3 + 4/2^4 + \cdots + y/2^y && (1)\\
2x &= 1+ && 2/2^1 + 3/2^2 + 4/2^3 + \cdots + y/2^{y-1} && (2)\\
x &= 1+ && 1/2 + 1/2^2 + \cdots + 1/2^{y-1} && (3)\\
& && -y/2^y && (4)\\
&= && \frac{(1/2)^y - 1}{1/2 - 1} - y/2^y && (5)\\
&= && 2(1-(1/2)^y) - y/2^y && (6)\\
&\leq 2 && && (7)
\end{aligned}$$

5.1.2 Heapsort: Recursive Construction

Construct-Heap(n):

construct left subheap

construct right subheap

Restore-heap(1)

Time Complexity:

$$
\begin{aligned}
W(n) &= 2w(n/2) + \log n && (9)\\
&= \log n + 2w(n/2) && (10)\\
&= \log n + 2\left(\log(n/2) + 2w(\frac{n/2}{2})\right) && (11)\\
&= \log n + 2\log n - 2\log 2 + && (12)\\
& \quad 2^2 w(n/2^2) && (13)\\
&= \log n + 2\log n - 2\log 2 + && (14)\\
& \quad 2^2\left(\log(\frac{n}{2^2}) + 2w\left(\frac{n/2^2}{2}\right)\right) && (15)\\
&= \log n + 2\log n - 2\log 2 + && (16)\\
& \quad +2^2\log n - 2^2\log(2^2) + 2^3 w(n/2^3) && (17)\\
& \quad \cdots && (18)\\
&= \log n + 2\log n - 2\log 2 + && (19)\\
& \quad +2^2\log n - 2^2\log(2^2) + \cdots + && (20)\\
& \quad 2^k\log n - 2^k\log(2^k) + 2^{k+1}w(n/2^{k+1}) && (21)\\
&= \log n(1 + 2 + 2^2 + \cdots + 2^k) && (22)\\
& \quad -(2 + 2\cdot 2^2 + 3\cdot 2^3 + \cdots + k\cdot 2^k) && (23)
\end{aligned}
$$

Here, we assume that $n = 2^k$ so $k = \log n$.

Let $s = 2^0 + 2^1 + 2^2 + \cdots + 2^k$
$= \frac{2^{k+1}-1}{2-1} = 2^{k+1} - 1$

To derive this formula, one can follow these steps:

$$\begin{aligned}
s &= 2^0 + \quad 2^1 + 2^2 + \cdots + 2^k && (24)\\
2s &= \quad 2^1 + 2^2 + \cdots + 2^k + 2^{k+1} && (25)\\
s &= \quad -1 + 2^{k+1} && (26)
\end{aligned}$$

Thus, $\sum_{i=0}^{k} 2^i = 2^{k+1} - 1$

Likewise, let

$$\begin{aligned}
T &= 1 \cdot 2^1 + \quad 2 \cdot 2^2 + 3 \cdot 2^3 + \cdots + k \cdot 2^k && (27)\\
2T &= \quad 1 \cdot 2^2 + 2 \cdot 2^3 + \cdots + && (28)\\
& \quad (k-1) \cdot 2^k + k2^{k+1} && (29)\\
T &= \quad -2^1 - 2^2 - 2^3 - \cdots - 2^k + && (30)\\
& \quad k2^{k+1} && (31)\\
&= \quad k2^{k+1} - (2^1 + 2^2 + \cdots + 2^k) && (32)\\
&= \quad k2^{k+1} - (2^{k+1} - 2) && (33)\\
&= \quad (k-1)2^{k+1} + 2 && (34)\\
& && (35)
\end{aligned}$$

$\Rightarrow W(n) = \log n(2^{K+1} - 1) - ((k-1)2^{k+1} + 2)$
$= 2n \log n - \log n - 2n \log n + 2n - 2$
$= 2n - \log n - 2$
$= O(n)$

5.1.3 Priority Queue employing Heap

Read pp. 162 (Section 6.5)

Delete Max/Min

Insert

See Exercise 6-1, pp. 166

6 SORTING BY MERGING

- **Mergesort**

  — $W(n) = O(n) + 2w(n/2)$
  $= n + 2w(n/2)$
  $= n + 2(n/2 + 2w(n/2^2)$
  $= n + n + 2^2 w(n/2^2)$
  $= 2n + 2^2 w(n/2^2)$
  $= 2n + 2^2(n/2^2 + 2w(n/2^3))$
  $= 3n + 2^3 w(n/2^3)$
  $\cdots$
  $= kn + 2^k w(n/2^k)$
  $w(n/2^k) = w(1) = 0$
  $n/2^k = 1,\ k = \log_2 n$
  $\Rightarrow W(n) = \theta(n \log n)$

  — Recursion tree for $W(n)$

  — $S(n) = O(n)$
  Can be reduced to $O(1)$ but algorithm slows down.
  A temporary array of half the size is required, however.

7 SORTING BY SPLITING

$W(n) = O(n^2)$ but $A(n) = O(n \log n)$

Quicksort (A, p, r):

if $p < r$ then $q \leftarrow$ Partition(A, p, r)

Quicksort(A, p, q)

Quicksort$(A, q + 1, r)$

7.1 Quicksort: Partitioning

Partition I.

$x := A[p]$; $i := p - 1$; $j := r + 1$

while (TRUE) **do**

  Repeat Decrement j **until** $A[j] \leq x$

  Repeat Increment i **until** $A[i] \geq x$

  if $i < j$

    then exchange $A[i]$ and $A[j]$

    else return j

endwhile

$x = 15$

15 7 23 5 20 3

Why do i and j never get out of array bounds?

$W(n) = n + 2$ comparisons

$W(n) = O(n^2) = O(n) + W(n-1)$

$B(n) = O(n) + 2B(n/2) = O(n \log n)$

Balance Partitioning:

- Even if each split is 1% on one side and 99% on the other, recursion tree remains logarithmic.

- Alternate good and bad splits:

Quicksort Improvements

– random x

– median of first, middle and last items

– insertion sort for $n \leq 15$

$S(n) = O(n)$
(See 7-4; reduces $S(n)$ to $O(\log n)$)

7.2 Quicksort: Partitioning II

Input: A[p..r]

Invariants: {All items in $A[2..i]$ are $<$ the pivot.}

{All items in $A[(i+1)..(unknown-1)]$ are $\geq$ the pivot}

$x = A[p]$; $i := p$;

for $unknown := p + 1$ **to** r **do**

  if $A[unknown] < x$ **then**

    $i := i + 1$; swap$(A[i], A[unknown])$

swap$(A[1], A[i])$

$W(n) = n - 1$ comparisons

15 7 23 5 20 3

7.3 Av. Case Complexity of Quicksort

Assume

- all keys distinct
- all permutations equally likely

Probability that split point is i, for $1 \leq i \leq n$, is $1/n$

$$A(n) = (n-1) + \frac{1}{n}(A(0) + A(n-1) +$$

$$A(1) + A(n-2) + \cdots A(n-1) + A(0))$$

$$= n - 1 + \frac{2}{n}\left(A(0) + A(1) + \cdots + A(n-1)\right)$$

$$A(n) = n - 1 + \frac{2}{n}\Sigma_{i=2}^{n-1} A(i), \quad A(0) = A(1) = 0$$

$$(n)A(n) = (n)(n-1) + 2\Sigma_{i=2}^{n-1} A(i)$$

$$A(n-1) = n - 2 + \frac{2}{n-1}\Sigma_{i=2}^{n-2} A(i)$$

$$(n-1)A(n-1) = (n-1)(n-2) + 2\Sigma_{i=2}^{n-2} A(i)$$

$$nA(n) - (n-1)A(n-1)$$

$$= n(n-1) - (n-2)(n-1) + 2A(n-1)$$

$$nA(n) - (n+1)A(n-1) = 2(n-1)$$
$$\frac{A(n)}{n+1} - \frac{A(n-1)}{n} = \frac{2(n-1)}{n(n+1)}$$

Let $B(n) = \frac{A(n)}{n+1}$ $(Changing\ Variable)$

Since $A(1) = 0$, $B(1) = 0$

$$\Rightarrow B(n) - B(n-1) = \frac{2(n-1)}{n(n+1)}$$
$$B(n) = \frac{2(n-1)}{n(n+1)} + B(n-1)$$
$$= \frac{2(n-1)}{n(n+1)} + \left(\frac{2(n-2)}{(n-1)(n)} + B(n-2)\right)$$
$$= B(1) + \frac{2\cdot 1}{2\cdot 3} + \frac{2\cdot 2}{3\cdot 4} + \cdots + \frac{2(n-1)}{n(n+1)}, B(1) = 0$$
$$B(n) = \sum_{i=2}^{n} \frac{2(i-1)}{i(i+1)}$$

$f(n) = \frac{2(n-1)}{n(n+1)}$ continuous decreasing function Sum_a^b f(x) <= Int_{a-1}^b f(x)dx - pp. 51

$$B(n) \le \int_2^n f(x)dx, \quad f(1) = 0$$
$$\frac{2(x-1)}{x(x+1)} = \frac{A}{x} + \frac{B}{x+1}$$
$$= \frac{(A+B)x + A}{x(x+1)}$$
$$\Rightarrow A = -2$$
$$A + B = 2$$
$$\Rightarrow B = 4$$

$$\Rightarrow f(x) = \frac{4}{x+1} - \frac{2}{x}$$

$$\int_2^n f(x)dx = \int_2^n \left(\frac{4}{x+1} - \frac{2}{x}\right) dx$$

$$= (4\ln(x+1) - 2\ln x)\,|_2^n$$

$$= 4\ln(n+1) - 2\ln n - 4\ln 3 + 2\ln 2$$

$$\approx 2\ln n$$

$$\Rightarrow B(n) \leq 2\ln n$$

$$A(n) = (n+1)B(n) \leq 2(n+1)\ln n$$

$$\Rightarrow A(n) \leq 1.4(n+1)\log_2 n$$

8 LOWER BOUND

(a) Local exchange only

each accomplishes in undoing one inversion

5 1 4 7 2 has inversions (5,1),(5,4),(5,2),(4,2),(7,2)

(n n-1 … 2 1) has $n(n-1)/2$ inversions

$\Rightarrow O(n^2/2)$ lower bound

(b) lower bound on comparison based sorting

example: a b c - there are 3! outputs.

(for n numbers there are n! outputs)

Every decision tree to sort n numbers must have atleast n! leaf nodes

Every decision tree to sort n numbers must have atleast 2n! - 1 nodes

Every decision tree to sort n numbers must have depth atleast $\log_2 n!$ leaf nodes

Depth is the lower bound worst case time complexity for this class of algorithms

$\log_2 n! = log_2(n*(n-1)*(n-2)*(n-3)......1)$

$\log_2 n! = log_2(n) + log_2(n-1) + log_2(n-2) + log_2(n-3)...... + log_2(1)$

$\log_2 n! = \sum_{j=1}^{n} \log_2 j$

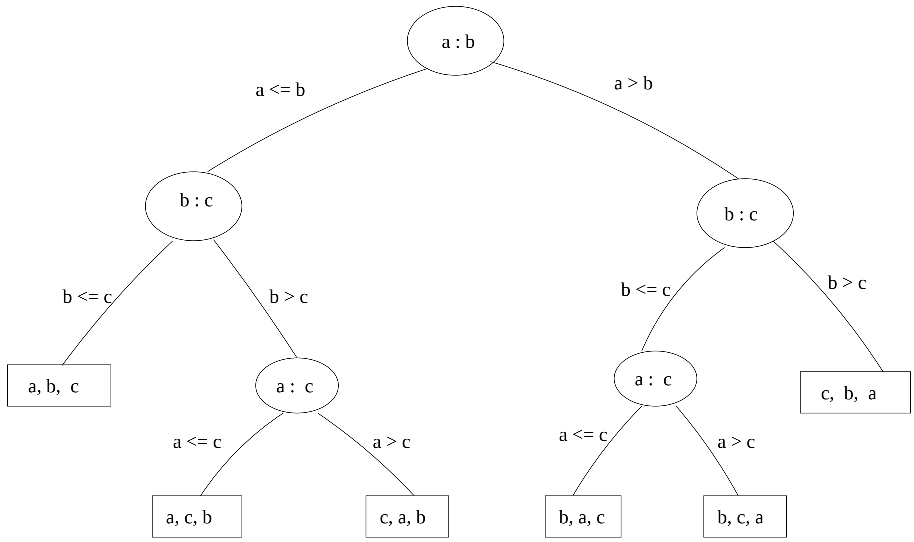
$\leq \int_1^n \log_2 x dx$

$= (x \log x - x)|_1^n$

$= n \log n - n$

Lower Bound on Comparison-Based Sorting Algorithm

Decision Tree for 3 numbers



Depth of binary tree with $n!$ leaves
$\geq \log_2 n!$
$\geq \int \log x dx$
$\geq n \log n - n$

9 SHELL SORT (Donald Shell)

Sort subarray comprising every h_i location, for a few selected hop sizes h_i, $k \geq i \geq 1$, and final $h_1 = 1$
$h_1 = 1$ always use insertion sort for sorting

with $h_2 = 1.72n^{1/3}$

$W(n) = (\frac{n}{1.72n^{1/3}})^2 + 1.72n^{1/3} + n^2$
$= \frac{n^2}{1.72n^{1/3}} + n^2$
$= \frac{n^{5/3}}{1.72} + n^2$
$= O(n^2)$

for $h_k = 2^k - 1$, $1 \leq k \leq \lfloor \log n \rfloor$
$W(n) = O(n$

$2^5 - 1 = 31 = (11111)_2$
$2^4 - 1 = 15 = (1111)_2$

for h_k is an integer of the form $2^i 3^j, h_k < n$
$W(n) = O(n(\log n)^2)$

$2^0 3^0, 2^1 3^0, 2^0 3^1, 2^1 3^1, 2^2 3^1$
1 2 3 6 12
too many h_k's, hence overhead is large.