Computational Astrophysics

2019

Exercises 13. Elliptic PDEs. Poisson Equation

Poisson Equation

In these exercises you will solve the Poisson equation,

$$\nabla^2 \Phi = 4\pi G \rho, \tag{1}$$

assuming spherical symmetry.

Homogeneous Sphere

Consider the Poisson equation as a second-order ODE and reduce it to first order (as described in class).

1. Integrate it with forward Euler using an appropriate initial condition. Remember, that the gravitational potential is determined only up to a constant, so you will need to adjust your result to match the boundary condition at $R = 10^9$ cm,

$$\Phi(R) = -\frac{GM(R)}{R} \ . \tag{2}$$

   Since you will have to compute the mass on your grid, assume that you are solving for the gravitational potential of a homogeneous sphere ($\rho = const$), which is given by

$$\Phi(r) = \frac{2}{3}\pi G \rho (r^2 - 3R^2) \ . \tag{3}$$

   Show that your code converges to the exact value of Φ at R_{outer} (the relative error must converge to zero at the right rate). Doing this is a bit tricky and you must be careful to adjust (via an additive constant) your solution at $r = 0$ (where your Φ will be zero).

2. Now you'll try the more sophisticated matrix method. Write a code to solve Poisson equation via the linear system

$$J\mathbf{\Phi} = \mathbf{b} \ , \tag{4}$$

   as described in class. Check that your implementation works for the particular case $\rho = const$.

Stellar Model

A supernova is a process in which a dying star liberates a huge quantity of energy and mass.
In this section we will solve Poisson equation to obtain the gravitational potential inside the mass distribution of a dying star.
In the data directory you will find a file with data called `presupernovaStar.dat`. The columns in this file are:

- First column: grid point index
- Second column: enclosed mass of the star (gr)
- Third column: radius of the distribution (cm)
- Fourth column: temperature (K)

- Fifth column: mass density (gr/cm^3)

1. Read the data and make a log-log plot of $\rho(r)$ for radii $r < 10^9$ cm.
2. Set up an equidistant grid with an outer radius of 10^9 cm and interpolate the density onto your new equidistant grid. You may choose what interpolation method to use.
3. Use the code that you write in the first exercise to obtain and plot the gravitational potential as a function of radius, solving the ODEs system.
4. Use the code to solve the Poisson equation using the matrix method. Remember that you must stagger your grid (shift it by $0.5dx$ to make sure you have no point at the origin). Obtain again Φ as a function of radius. The gravitational potential obtained with this method must match (within errors) the potential obtained with the ODE method. Plot both in the same figure.

Happy Coding :) !