Computational Astrophysics

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Outline

Ordinary Differential Equations with Boundary Conditions

A boundary value problem consists of finding a solution of an ODE in an interval $[a, b]$ that satisfies constraints at both ends (boundary conditions).

Example

$$y'' = f(x, y, y')\ , \quad y(a) = A\ , \quad y(b) = B\ , \quad \text{and } x \in [a, b]\ . \tag{1}$$

Shooting Method

The shooting method solves a BVP by transforming it into an initial value problem by making an educated guess on unknown inner boundary conditions. Then, we iterate until a modified guessed inner boundary condition leads to the correct known outer boundary value.

Shooting Method

Example

Given the system in Eq. 1, the value of $y'(a)$ is unknown.
We can make a guess $y'(a) = z$ to write $z' = f(a, y(a), y'(a))$.
This reduces the second-order problem to two first-order problems.

Shooting Method

Then we just need to integrate the two ODEs out to b. Since we have chosen z, we have now solved $y = y(x, z)$, but our goal is to find y such that $y(b, z) = B$.
In other words, we can define a new function

$$\Phi(z) = y(b, z) - B \tag{2}$$

and search for a z so that $\Phi(z) = 0$. Hence, we are looking for the root of $\Phi(z)$!

Shooting Method

The full shooting algorithm for $y'' = f(x, y, y')$ goes as

1. Guess a starting value $z_0 = y'(a)$, set the iteration counter $i = 0$.
2. Compute $y = y(x, z_i)$ by integrating the IVP.
3. Compute $\Phi(z_i) = y(b, z_i) - B$. If z_i does not give a sufficiently accurate solution of the full problem, increment i to $i + 1$ and find a value for z_{i+1} using a root finder on $\Phi(z_i) = 0$. Then go back to (2).

Shooting Method

Note that one typically ends up with the secant method, since the derivative of $\Phi(z)$ is not known in the general case and one is stuck with having to numerically compute it. For this, at least two guesses for z are needed.

Finite-Difference Method

BVPs of the kind given by Eq. (1) can be solved by Taylor expanding the ODE itself to linear order (assuming there are no non-linearities in y and y'):

$$y'' = g(x) - p(x)y' - q(x)y \ , \tag{3}$$

where $g(x)$, $p(x)$, and $q(x)$ are functions of x only and the sign convention is arbitrary.

Finite-Difference Method

We can now discretize y' and y'' on an evenly spaced grid with step size h,

$$
\begin{aligned}
y'(x_i) &= \frac{y(x_{i+1}) - y(x_{i-1})}{2h} \ , \\
y''(x_i) &= \frac{y_{i+1} + y_{i-1} - 2y_i}{h^2} \ ,
\end{aligned}
\tag{4}
$$

where $x_i = a + ih$, $(i = 0, ..., n+1)$, and

$$
h = \frac{b-a}{n+1}. \tag{5}
$$

Finite-Difference Method

The system is a tri-diagonal matrix of dimension $n \times n$:

$$
\begin{pmatrix}
-2+h^2q_1 & 1+\frac{h}{2}p_1 & 0 & & \cdots & & 0 \\
1-\frac{h}{2}p_2 & \ddots & \ddots & 0 & & \cdots & 0 \\
0 & \ddots & \ddots & \ddots & 0 & \cdots & \vdots \\
\vdots & 0 & \ddots & \ddots & \ddots & 0 & \vdots \\
\vdots & \vdots & 0 & \ddots & \ddots & \ddots & 0 \\
\vdots & \vdots & \vdots & 0 & \ddots & \ddots & 1+\frac{h}{2}p_{n-1} \\
\vdots & 0 & 0 & \cdots & 0 & 1-\frac{h}{2}p_n & -2+h^2q_n
\end{pmatrix}
\begin{pmatrix} y_1 \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ y_n \end{pmatrix}
=
\begin{pmatrix} h^2g_1 - A(1-\frac{h}{2}p_1) \\ h^2g_2 \\ \vdots \\ \vdots \\ \vdots \\ h^2g_{n-1} \\ h^2g_n - B(1+\frac{h}{2}p_n) \end{pmatrix}
\tag{6}
$$

Next Class

Partial Differential Equations.