Computational Astrophysics

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Outline

Linear Advection Equation

The linear advection equation is

$$\partial_t u + v\partial_x u = 0 \tag{1}$$

where $u(t,x)$ is some scalar quantity and v is the constant velocity at which it is advected ($v > 0$ advects to the right).
The solution to this equation is to simply take the initial data, $u(t=0,x)$, and displace it to the right at a speed v. The shape of the initial data is preserved in the advection.

Linear Advection Equation

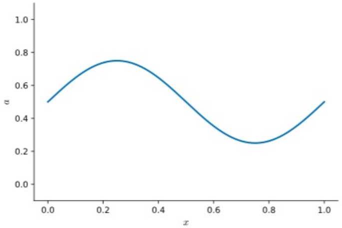
Many hyperbolic systems of PDEs, e.g. the equations of hydrodynamics, can be written in a form that looks like a system of (nonlinear) advection equations, so the advection equation provides important insight into the methods used for these systems.

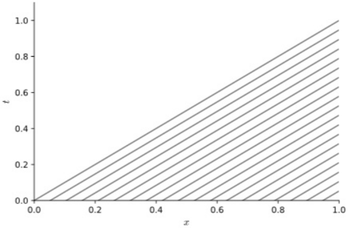
Linear Advection Equation

Direct substitution shows that $u(x - vt)$ is a solution to advection equation for any choice of u.

This means that the solution is constant along the lines $x = vt$ (the curves along which the solution is constant are called the characteristics).

Linear Advection Equation





Non-linear Hyperbolic Differential Equations. Burguer's Equation

Burgers' equation is the simplest *nonlinear* hyperbolic equation,

$$\partial_t u + u\partial_x u = 0. \tag{2}$$

It is almost identical to the advection equation treated before, but this time the wave speed is NOT a constant v but is given by the field u itself.

Hence, u is both the quantity being advected and the speed at which it is moving.

Non-linear Hyperbolic Differential Equations. Burguer's Equation

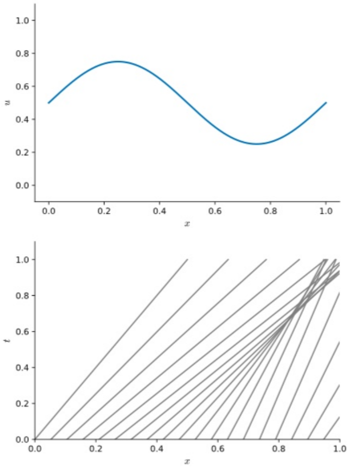
For the linear advection equation, the solution was constant along lines $x = vt + x_0$, which are parallel, since v is spatially constant.

For Burgers' equation, this is no longer the case, and the characteristic lines are now given by $\frac{dx}{dt} = u$, with $x(0) = x_0$. Since $u = u(t)$, we cannot integrate this directly.

Burguer's Equation. Shocks

If we take $u_0 = u(t = 0)$, then we can look at how the characteristic behave over a small time interval (before $u(x, t)$ changes significantly).
Next figure shows the behavior for an initial velocity with sinusoidal profile.

Burguer's Equation. Shocks

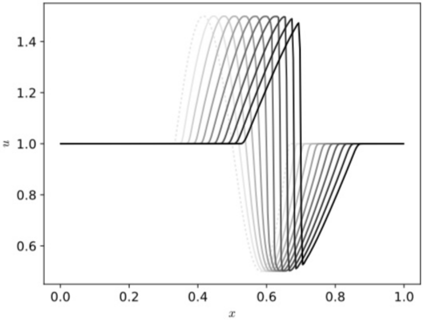


Burguer's Equation. Shocks

We see that after a short period of time, the characteristics intersect. At the point, (x_s, t_s) where they intersect, there is no way to trace backwards along the characteristics to find a unique initial state.

This merging of the characteristics in the x-t plane is a *shock*, and represents just one way that nonlinear problems can differ from linear ones.

Burguer's Equation. Shocks

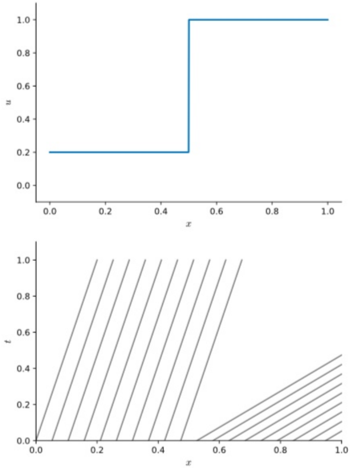


Burguer's Equation. Rarefaction

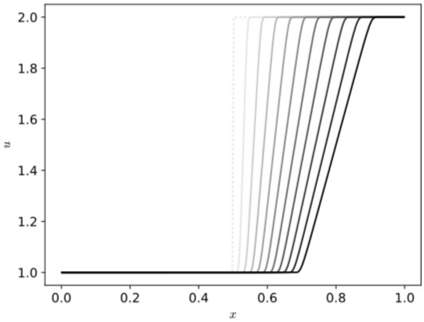
Another type of wave not present in a linear system is a *rarefaction*. Next figure shows initial conditions of slower velocity to the left of faster velocity.

We see that the characteristics diverge in this case, and we will be left with having to fill in the solution inbetween as some intermediate state.

Burguer's Equation. Rarefaction



Burguer's Equation. Rarefaction



Matrix Method

In the matrix method, we discretize Poisson Equation with centered differences. Then, for an interior (non-boundary) grid point i we have

$$\begin{aligned} \left.\frac{\partial \Phi}{\partial r}\right|_i &\approx \frac{1}{2\Delta r}\left(\Phi_{i+1} - \Phi_{i-1}\right) \ , \\ \left.\frac{\partial^2 \Phi}{\partial r^2}\right|_i &\approx \frac{1}{(\Delta r)^2}\left(\Phi_{i+1} - 2\Phi_i + \Phi_{i-1}\right) \ . \end{aligned} \tag{3}$$

Then, Poisson equation is

$$\frac{1}{r_i}\frac{\left(\Phi_{i+1} - \Phi_{i-1}\right)}{\Delta r} + \frac{\left(\Phi_{i+1} - 2\Phi_i + \Phi_{i-1}\right)}{(\Delta r)^2} = 4\pi G \rho_i \ . \tag{4}$$

Matrix Method

Inner boundary condition:
Because of the $1/r$ term in the Laplacian, it is clear that the behavior at $r = 0$ must be regularized.

It is possible to use an expansion or we can stagger the grid so that there is no point exactly at $r = 0$, for example by moving the entire grid over by $0.5\Delta r$.

Matrix Method

Inner boundary condition:
At the inner boundary, we have the condition $\frac{\partial \Phi}{\partial r} = 0$, so we assume

$$\Phi_{-1} = \Phi_0 \ . \tag{5}$$

Matrix Method

Hence, Poisson equation can be written as linear system

$$J\mathbf{\Phi} = \mathbf{b} \ , \tag{6}$$

where $\Phi = (\Phi_0, \cdots, \Phi_{n-1})^T$ (for a grid with n points labeled 0 to $n-1$) and $\mathbf{b} = 4\pi G(\rho_0, \cdots, \rho_{n-1})^T$.

Matrix Method

The matrix J has tri-diagonal form and can be explicitely given as:

(a) $i = j = 0$:

$$J_{00} = -\frac{1}{(\Delta r)^2} - \frac{1}{r_0 \Delta r} \,, \tag{7}$$

(b) $i = j$:

$$J_{ij} = \frac{-2}{(\Delta r)^2} \,, \tag{8}$$

(c) $i + 1 = j$:

$$J_{ij} = \frac{1}{(\Delta r)^2} + \frac{1}{r_i \Delta r} \,, \tag{9}$$

(d) $i - 1 = j$:

$$J_{ij} = \frac{1}{(\Delta r)^2} - \frac{1}{r_i \Delta r} \,. \tag{10}$$

Matrix Method

After finding the solution to the linear system, it is neccesary to correct for the analytic outer boundary value

$$\Phi(R_{\text{surface}}) = -\frac{GM}{R_{\text{surface}}}. \tag{11}$$

Next Class

Poisson Equation