Computational Astrophysics

2019

Exercises 10. Ordinary Differential Equations (ODEs)

ODE Integration: Stellar Structure

Consider the system of two ODEs describing a simple model of stellar structure

$$\frac{dP}{dr} = -\frac{GM(r)}{r^2}\rho \ , \qquad \text{Hydrostatic Equilbrium} \qquad (1)$$

$$\frac{dM}{dr} = 4\pi\rho r^2 \ . \qquad \text{Mass Conservation} \qquad (2)$$

To complete the set of equations also requires an Equation Of State (EOS), i.e. a relation between density and pressure. Since we will not solve an internal energy equation, we will use a simplified EOS that is barotropic: $P = P(\rho)$. A possible choice is the polytropic EOS,

$$P = K\rho^\Gamma \ , \qquad (3)$$

where K is the polytropic constant and Γ is the adiabatic index, i.e. the ratio of the specific heats.
A good kind of star to model with a barotropic EOS is a white dwarf, which is supported by the pressure of degenerate electrons. In the case of a fully relativistically degenerate white dwarf, we have the values

$$\begin{aligned} K &= 1.244 \times 10^{15} (0.5)^\Gamma \text{dyne cm}^{-2} \ (g^{-1}\text{cm}^3)^\Gamma \\ \Gamma &= 4/3 \ . \end{aligned} \qquad (4)$$

In this exercise, you will solve the simplified stellar structure equations by integrating them from the origin to the stellar surface using Forward Euler, RK2 and, optionally also RK3, and RK4. You will demonstrate convergence.

1. In the code directory you will finde a code skeleton for you to work from. Download it and study its structure.

2. Fill in the missing code segments (marked with `[ FILL IN CODE ]`). You will have to implement the grid setup, the right-hand-side (RHS) for both equations, the Euler integrator, and the EOS inversion to obtain a new density from the updated pressure.

   Use a central density of $\rho_c = 10^{10}\,\text{g}\,\text{cm}^{-3}$ and a cut-off pressure corresponding to $10^{-10} P_c$. Test the code using $\sim$1000 points and an outer radius of 2000 km. You should get something close to 1.45 $M_\odot$ and a radius of $\sim$ 1500 km.

3. Add an RK2 integrator and show convergence on the mass of the star using multiple resolutions.

   Implement also an RK3 or RK4 integrator.

4. Make a plot of $\rho(r)$ and $P(r)$ and $M(r)$ of a well-resolved RK2 (or RK3 or RK4) run. By intelligently rescaling them and using two y-axes, you can fit all of them on one plot. You may want to use a logarithmic scale for density and/or radius.

Happy Coding :) !