Computational Astrophysics

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Outline

2 Dimensional Advection

The two-dimensional linear advection equation is

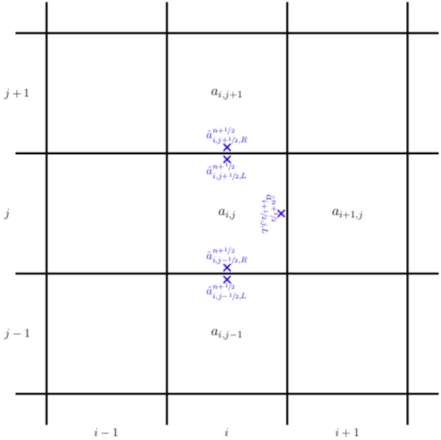
$$\partial_t a + u\partial_x a + v\partial_y a = 0 \tag{1}$$

where u is the velocity in the x-direction and v is the velocity in the y-direction.

We denote the average of $a(x, y, t)$ in a zone i, j as $a_{i,j}$. Here, i is the index in the x-direction and j is the index in the y-direction.

As in the one-dimensional case, we will extend the domain with a perimeter of ghost cells to set the boundary conditions.

2 Dimensional Grid



2 Dimensional Advection

Since u and v are constant, we can move them inside the derivatives,

$$\partial_t a + \partial_x(ua) + \partial_y(va) = 0. \tag{2}$$

We define the average of a in a zone by integrating it over the volume:

$$a_{i,j} = \frac{1}{\Delta x \Delta y} \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} \int_{y_{j-\frac{1}{2}}}^{y_{j+\frac{1}{2}}} a(x, y, t)\, dx\, dy \tag{3}$$

2 Dimensional Advection

Integrating the advection equation over x and y, we obtain

$$\begin{aligned}\frac{1}{\Delta x \Delta y}\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}}\int_{y_{j-\frac{1}{2}}}^{y_{j+\frac{1}{2}}} a_t\, dx\, dy = &-\frac{1}{\Delta x \Delta y}\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}}\int_{y_{j-\frac{1}{2}}}^{y_{j+\frac{1}{2}}} (ua)_x\, dx\, dy \\ &-\frac{1}{\Delta x \Delta y}\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}}\int_{y_{j-\frac{1}{2}}}^{y_{j+\frac{1}{2}}} (va)_y\, dx\, dy\end{aligned} \tag{4}$$

2 Dimensional Advection

Integration in the left hand side gives

$$\frac{\partial a_{i,j}}{\partial t} = -\frac{1}{\Delta x \Delta y}\int_{y_{j-\frac{1}{2}}}^{y_{j+\frac{1}{2}}} \left\{(ua)_{i+\frac{1}{2},j} - (ua)_{i-\frac{1}{2},j}\right\} dy - \frac{1}{\Delta x \Delta y}\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} \left\{(va)_{i,j+\frac{1}{2}} - (va)_{i,j-\frac{1}{2}}\right\} dx \tag{5}$$

2 Dimensional Advection

and integrating with respect to time

$$a_{i,j}^{n+1} - a_{i,j}^{n} = -\frac{1}{\Delta x \Delta y} \int_{t^n}^{t^{n+1}} \int_{y_{j-\frac{1}{2}}}^{y_{j+\frac{1}{2}}} \left\{ (ua)_{i+\frac{1}{2},j} - (ua)_{i-\frac{1}{2},j} \right\} dydt$$
$$- \frac{1}{\Delta x \Delta y} \int_{t^n}^{t^{n+1}} \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} \left\{ (va)_{i,j+\frac{1}{2}} - (va)_{i,j-\frac{1}{2}} \right\} dxdt \tag{6}$$

2 Dimensional Advection

We define the flux through the interface as the average over the face of that interface and time,

- x-face:

$$\langle (ua)_{i+\frac{1}{2},j} \rangle_{(t)} = \frac{1}{\Delta y \Delta t} \int_{t^n}^{t^{n+1}} \int_{y_{j-\frac{1}{2}}}^{y_{j+\frac{1}{2}}} (ua)_{i+\frac{1}{2},j} \, dydt \qquad (7)$$

- y-face

$$\langle (va)_{i,j+\frac{1}{2}} \rangle_{(t)} = \frac{1}{\Delta x \Delta t} \int_{t^n}^{t^{n+1}} \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} (va)_{i,j+\frac{1}{2}} \, dxdt \qquad (8)$$

where $\langle . \rangle_{(t)}$ denotes the time-average over the face.

2 Dimensional Advection

Now, we replace the average in time with the flux at the midpoint in time and the average over the face with the flux at the center of the face,

$$\langle (ua)_{i+\frac{1}{2},j} \rangle_{(t)} \approx (ua)^{n+\frac{1}{2}}_{i+\frac{1}{2},j} \tag{9}$$

2 Dimensional Advection

Then,

$$a_{i,j}^{n+1} = a_{i,j}^{n} - \Delta t \left[\frac{(ua)_{i+\frac{1}{2},j}^{n+\frac{1}{2}} - (ua)_{i-\frac{1}{2},j}^{n+\frac{1}{2}}}{\Delta x} + \frac{(va)_{i,j+\frac{1}{2}}^{n+\frac{1}{2}} - (va)_{i,j-\frac{1}{2}}^{n+\frac{1}{2}}}{\Delta y} \right] \tag{10}$$

Dimensionally split method

In a split method, we update the state in each coordinate direction independently. We will consider the *Strang splitting*, where we alternate the order of the dimensional updates each timestep. An update through Δt consists of x and y sweeps and appears as

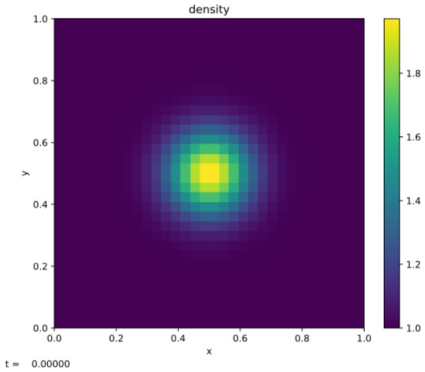
$$
\begin{aligned}
\frac{a^{\star}_{i,j} - a^{n}_{i,j}}{\Delta t} &= -\frac{ua^{n+\frac{1}{2}}_{i+\frac{1}{2},j} - ua^{n+\frac{1}{2}}_{i-\frac{1}{2},j}}{\Delta x} \qquad (11) \\
\frac{a^{n+1}_{i,j} - a^{\star}_{i,j}}{\Delta t} &= -\frac{va^{\star,n+\frac{1}{2}}_{i,j+\frac{1}{2}} - va^{\star,n+\frac{1}{2}}_{i,j-\frac{1}{2}}}{\Delta y} \qquad (12)
\end{aligned}
$$

Dimensionally split method

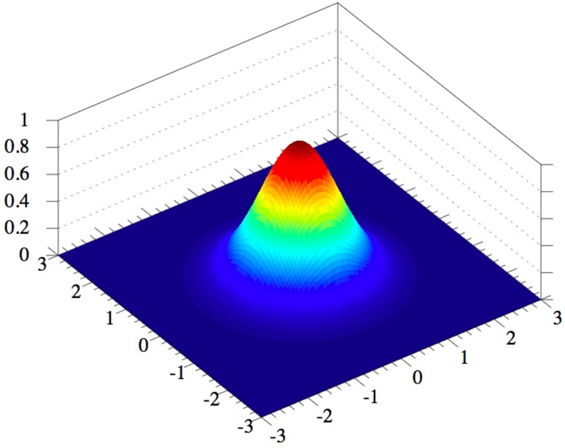
The states $a_{i+\frac{1}{2},j}^{n+\frac{1}{2}}$ are calculated as

$$\begin{aligned}
a_{i+\frac{1}{2},j}^{n+\frac{1}{2}} &= a_{i,j}^{n} + \frac{\Delta x}{2} \left.\frac{\partial a}{\partial x}\right|_{i,j} + \frac{\Delta t}{2} \left.\frac{\partial a}{\partial t}\right|_{i,j} + \mathcal{O}(\Delta x^2) + \mathcal{O}(\Delta t^2) \\
&= a_{i,j}^{n} + \frac{\Delta x}{2} \left.\frac{\partial a}{\partial x}\right|_{i,j} + \frac{\Delta t}{2} \left(-u \left.\frac{\partial a}{\partial x}\right|_{i,j} \right) + \dots \\
&= a_{i,j}^{n} + \frac{\Delta x}{2} \left(1 - \frac{\Delta t}{\Delta x} u \right) \left.\frac{\partial a}{\partial x}\right|_{i,j} + \dots \qquad (13)
\end{aligned}$$

2 Dimensional Grid



2 Dimensional Grid



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