

Computational Astrophysics

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Outline

- 1 Linear Advection Equation
- 2 Non-linear Hyperbolic Differential Equations. Burguer's Equation
- 3 Matrix Method

Linear Advection Equation

The linear advection equation is

$$\partial_t u + v \partial_x u = 0 \quad (1)$$

where $u(t, x)$ is some scalar quantity and v is the constant velocity at which it is advected ($v > 0$ advects to the right).

The solution to this equation is to simply take the initial data, $u(t = 0, x)$, and displace it to the right at a speed v . The shape of the initial data is preserved in the advection.

Linear Advection Equation

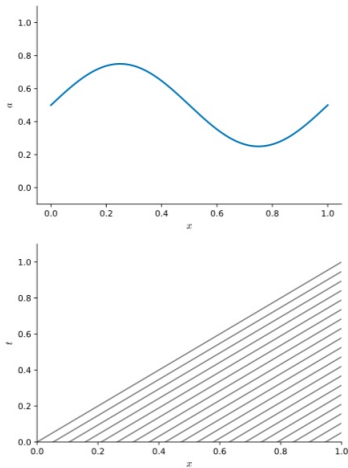
Many hyperbolic systems of PDEs, e.g. the equations of hydrodynamics, can be written in a form that looks like a system of (nonlinear) advection equations, so the advection equation provides important insight into the methods used for these systems.

Linear Advection Equation

Direct substitution shows that $u(x - vt)$ is a solution to advection equation for any choice of u .

This means that the solution is constant along the lines $x = vt$ (the curves along which the solution is constant are called the characteristics).

Linear Advection Equation



Non-linear Hyperbolic Differential Equations. Burguer's Equation

Burgers' equation is the simplest *nonlinear* hyperbolic equation,

$$\partial_t u + u \partial_x u = 0. \quad (2)$$

It is almost identical to the advection equation treated before, but this time the wave speed is NOT a constant v but is given by the field u itself.

Hence, u is both the quantity being advected and the speed at which it is moving.

Non-linear Hyperbolic Differential Equations. Burguer's Equation

For the linear advection equation, the solution was constant along lines $x = vt + x_0$, which are parallel, since v is spatially constant.

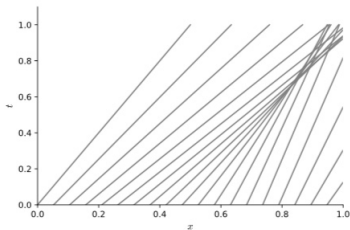
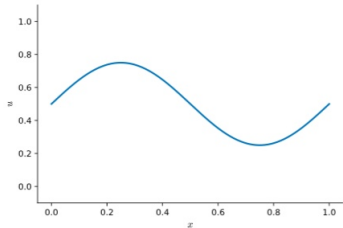
For Burgers' equation, this is no longer the case, and the characteristic lines are now given by $\frac{dx}{dt} = u$, with $x(0) = x_0$. Since $u = u(t)$, we cannot integrate this directly.

Burguer's Equation. Shocks

If we take $u_0 = u(t = 0)$, then we can look at how the characteristic behave over a small time interval (before $u(x, t)$ changes significantly).

Next figure shows the behavior for an initial velocity with sinusoidal profile.

Burguer's Equation. Shocks

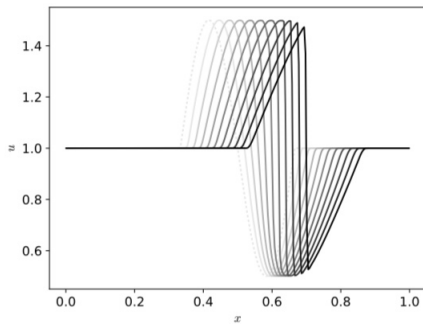


Burguer's Equation. Shocks

We see that after a short period of time, the characteristics intersect. At the point, (x_s, t_s) where they intersect, there is no way to trace backwards along the characteristics to find a unique initial state.

This merging of the characteristics in the x - t plane is a *shock*, and represents just one way that nonlinear problems can differ from linear ones.

Burguer's Equation. Shocks

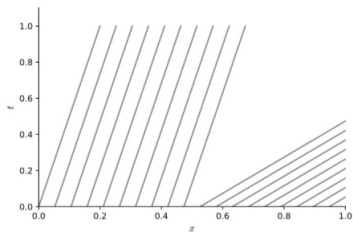
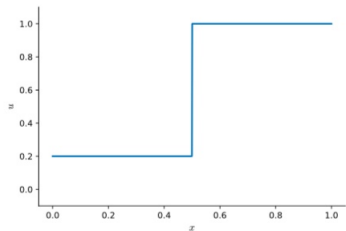


Burguer's Equation. Rarefaction

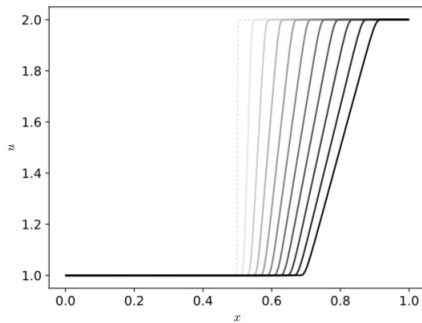
Another type of wave not present in a linear system is a *rarefaction*. Next figure shows initial conditions of slower velocity to the left of faster velocity.

We see that the characteristics diverge in this case, and we will be left with having to fill in the solution inbetween as some intermediate state.

Burguer's Equation. Rarefaction



Burguer's Equation. Rarefaction



Next Class