Computational Astrophysics

E. Larrañaga

Observatorio Astronómico Nacional
Universidad Nacional de Colombia

June 10, 2019

Outline

Elliptic Partial Differential Equations

The simplest elliptic partial differential equation is the linear Poisson equation.
For the Newtonian gravitational potential Φ, this is

$$\nabla^2\Phi = 4\pi G\rho \ . \tag{1}$$

Poisson Equation with Spherical Symmetry

Consider a spherically symmetric mass distribution. The Laplacian ∇^2 reduces to

$$\begin{aligned} \nabla^2[\cdot] &= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}[\cdot]\right) \ , \\ &= \frac{2}{r}\frac{\partial}{\partial r}[\cdot] + \frac{\partial^2}{\partial r^2}[\cdot] \ . \end{aligned} \tag{2}$$

ODE Method

Since the PDE reduces to just one variable, r, we write it as a second-order ODE

$$\frac{d^2\Phi}{dr^2} + \frac{2}{r}\frac{d\Phi}{dr} = 4\pi G\rho \ . \tag{3}$$

This equation is reduced to the system

$$\begin{aligned} \frac{d\Phi}{dr} &= z \ , \\ \frac{dz}{dr} + \frac{2}{r}z &= 4\pi G\rho \ . \end{aligned} \tag{4}$$

ODE Method

In order to solve inside the matter distribution, we will use inner and outer boundary conditions,

$$
\begin{aligned}
\left.\frac{d\Phi}{dr}\right|_{r=0} &= 0 \ , \\
\Phi(R_{\mathrm{surface}}) &= -\frac{GM(R_{\mathrm{surface}})}{R_{\mathrm{surface}}} \ .
\end{aligned}
\tag{5}
$$

ODE Method

In practice, one sets $\Phi(r = 0) = 0$ and integrates out. Then, the result is corrected by adding a constant term to obtain the outer boundary condition, which is the analytic value for the gravitational potential outside a spherical mass distribution.

Remember that this added constant doesn't affect the field value because of gauge invariance!

Matrix Method

In the matrix method, we discretize Poisson Equation with centered differences. Then, for an interior (non-boundary) grid point i we have

$$\begin{aligned} \left.\frac{\partial \Phi}{\partial r}\right|_i &\approx \frac{1}{2\Delta r}\left(\Phi_{i+1} - \Phi_{i-1}\right) \ , \\ \left.\frac{\partial^2 \Phi}{\partial r^2}\right|_i &\approx \frac{1}{(\Delta r)^2}\left(\Phi_{i+1} - 2\Phi_i + \Phi_{i-1}\right) \ . \end{aligned} \tag{6}$$

Then, Poisson equation is

$$\frac{1}{r_i}\frac{\left(\Phi_{i+1} - \Phi_{i-1}\right)}{\Delta r} + \frac{\left(\Phi_{i+1} - 2\Phi_i + \Phi_{i-1}\right)}{(\Delta r)^2} = 4\pi G \rho_i \ . \tag{7}$$

Matrix Method

Inner boundary condition:
Because of the $1/r$ term in the Laplacian, it is clear that the behavior at $r = 0$ must be regularized.

It is possible to use an expansion or we can stagger the grid so that there is no point exactly at $r = 0$, for example by moving the entire grid over by $0.5\Delta r$.

Matrix Method

Inner boundary condition:
At the inner boundary, we have the condition $\frac{\partial \Phi}{\partial r} = 0$, so we assume

$$\Phi_{-1} = \Phi_0 \ . \tag{8}$$

Matrix Method

Hence, Poisson equation can be written as linear system

$$J\mathbf{\Phi} = \mathbf{b} \ , \tag{9}$$

where $\Phi = (\Phi_0, \cdots, \Phi_{n-1})^T$ (for a grid with n points labeled 0 to $n-1$) and $\mathbf{b} = 4\pi G(\rho_0, \cdots, \rho_{n-1})^T$.

Matrix Method

The matrix J has tri-diagonal form and can be explicitely given as:

(a) $i = j = 0$:

$$J_{00} = -\frac{1}{(\Delta r)^2} - \frac{1}{r_0 \Delta r} \,, \tag{10}$$

(b) $i = j$:

$$J_{ij} = \frac{-2}{(\Delta r)^2} \,, \tag{11}$$

(c) $i + 1 = j$:

$$J_{ij} = \frac{1}{(\Delta r)^2} + \frac{1}{r_i \Delta r} \,, \tag{12}$$

(d) $i - 1 = j$:

$$J_{ij} = \frac{1}{(\Delta r)^2} - \frac{1}{r_i \Delta r} \,. \tag{13}$$

Matrix Method

After finding the solution to the linear system, it is neccesary to correct for the analytic outer boundary value

$$\Phi(R_{\text{surface}}) = -\frac{GM}{R_{\text{surface}}}. \tag{14}$$

Next Class

Poisson Equation