Computational Astrophysics

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Outline

Ordinary Differential Equations with Boundary Conditions

A boundary value problem consists of finding a solution of an ODE in an interval $[a, b]$ that satisfies constraints at both ends (boundary conditions).

Example

$$y'' = f(x, y, y') \ , \quad y(a) = A \ , \quad y(b) = B \ , \quad \text{and } x \in [a, b] \ . \tag{1}$$

Shooting Method

The shooting method solves a BVP by transforming it into an initial value problem by making an educated guess on unknown inner boundary conditions. Then, we iterate until a modified guessed inner boundary condition leads to the correct known outer boundary value.

Shooting Method

Example

Given the system in Eq. 1, the value of $y'(a)$ is unknown. We can make an initial guess $y'(a) = z_0$ to reduce the second-order problem to two first-order problems.

$$y' = u(x) \quad , \quad y'(a) = z_0 \tag{2}$$

$$u' = f(x, y, u) \quad , \quad y(a) = A. \tag{3}$$

Shooting Method

Then we just need to integrate the two ODEs out to b. Since we have chosen z_0, we have now solved to obtain the function $y = y(x, z_0)$, but our goal is to find y such that it satisfies the other boundary condition: $y(b, z_0) = B$.
In other words, we can define a new function

$$\Phi(z_0) = y(b, z_0) - B \tag{4}$$

and search for a z_0 so that $\Phi(z_0) = 0$. Hence, we are looking for the root of $\Phi(z_0)$!

Shooting Method

The full shooting algorithm for $y'' = f(x, y, y')$ goes as

1. Guess a starting value $z_0 = y'(a)$, set the iteration counter $i = 0$.
2. Compute $y = y(x, z_i)$ by integrating the IVP.
3. Compute $\Phi(z_i) = y(b, z_i) - B$. If z_i does not give a sufficiently accurate solution of the full problem, increment i to $i + 1$ and find a value for z_{i+1} using a root finder on $\Phi(z_i) = 0$. Then go back to (2).

Shooting Method

Note that one typically ends up with the secant method, since the derivative of $\Phi(z)$ is not known in the general case and one is stuck with having to numerically compute it. For this, at least two guesses for z are needed.

Finite-Difference Method

BVPs of the kind given by Eq. (1) can be solved by Taylor expanding the ODE itself to linear order (assuming there are no non-linearities in y and y'):

$$y'' = g(x) - p(x)y' - q(x)y \ , \tag{5}$$

where $g(x)$, $p(x)$, and $q(x)$ are functions of x only and the sign convention is arbitrary.

Finite-Difference Method

We can now discretize y' and y'' on an evenly spaced grid with step size h,

$$\begin{aligned} y'(x_i) &= \frac{y(x_{i+1}) - y(x_{i-1})}{2h} \ , \\ y''(x_i) &= \frac{y_{i+1} + y_{i-1} - 2y_i}{h^2} \ , \end{aligned} \tag{6}$$

where $x_i = a + ih$, $(i = 0, ..., n+1)$, and

$$h = \frac{b-a}{n+1}. \tag{7}$$

Finite-Difference Method

The discrete version of Eq. (5) is then a system of $n+2$ linear algebraic equations,

$$
\begin{aligned}
y_0 &= A \\
\left(1-\frac{h}{2}p_i\right) y_{i-1} - \left(2-h^2 q_i\right) y_i + \left(1+\frac{h}{2}p_i\right) y_{i+1} &= h^2 g_i \qquad (8) \\
y_{n+1} &= B \ ,
\end{aligned}
$$

where $p_i = p(x_i)$, $g_i = g(x_i)$, and $q_i = q(x_i)$.

Finite-Difference Method

The system is a tri-diagonal matrix of dimension $n \times n$:

$$
\begin{pmatrix}
-2+h^2q_1 & 1+\frac{h}{2}p_1 & 0 & & \cdots & & 0 \\
1-\frac{h}{2}p_2 & \ddots & \ddots & 0 & & \cdots & 0 \\
0 & \ddots & \ddots & \ddots & 0 & \cdots & \vdots \\
\vdots & 0 & \ddots & \ddots & \ddots & 0 & \vdots \\
\vdots & \vdots & 0 & \ddots & \ddots & \ddots & 0 \\
\vdots & \vdots & \vdots & 0 & \ddots & \ddots & 1+\frac{h}{2}p_{n-1} \\
\vdots & 0 & 0 & \cdots & 0 & 1-\frac{h}{2}p_n & -2+h^2q_n
\end{pmatrix}
\begin{pmatrix} y_1 \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ y_n \end{pmatrix}
=
\begin{pmatrix} h^2g_1 - A(1-\frac{h}{2}p_1) \\ h^2g_2 \\ \vdots \\ \vdots \\ \vdots \\ h^2g_{n-1} \\ h^2g_n - B(1+\frac{h}{2}p_n) \end{pmatrix}
\tag{9}
$$

Next Class

Partial Differential Equations.