Computational Astrophysics

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June 9, 2019

Outline

Elliptic Partial Differential Equations

The simplest elliptic partial differential equation is the linear Poisson equation.
For the Newtonian gravitational potential Φ, this is

$$\nabla^2\Phi = 4\pi G\rho \ . \tag{1}$$

Poisson Equation with Spherical Symmetry

Consider a spherically symmetric mass distribution. The Laplacian ∇^2 reduces to

$$\begin{aligned}\nabla^2[\cdot] &= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}[\cdot]\right) \ , \\ &= \frac{2}{r}\frac{\partial}{\partial r}[\cdot] + \frac{\partial^2}{\partial r^2}[\cdot] \ .\end{aligned} \tag{2}$$

ODE Method

Since the PDE reduces to just one variable, r, we write it as a second-order ODE

$$\frac{d^2\Phi}{dr^2} + \frac{2}{r}\frac{d\Phi}{dr} = 4\pi G\rho \ . \tag{3}$$

This equation is reduced to the system

$$\begin{aligned} \frac{d\Phi}{dr} &= z \ , \\ \frac{dz}{dr} + \frac{2}{r}z &= 4\pi G\rho \ . \end{aligned} \tag{4}$$

ODE Method

In order to solve inside the matter distribution, we will use inner and outer boundary conditions,

$$\left.\frac{d\Phi}{dr}\right|_{r=0} = 0 \ ,$$
$$\Phi(R_{\rm surface}) = -\frac{GM(R_{\rm surface})}{R_{\rm surface}} \ . \tag{5}$$

ODE Method

In practice, one sets $\Phi(r = 0) = 0$ and integrates out. Then, the result is corrected by adding a constant term to obtain the outer boundary condition, which is the analytic value for the gravitational potential outside a spherical mass distribution.

Remember that this added constant doesn't affect the field value because of gauge invariance!

Matrix Method

In the matrix method, we discretize Poisson Equation with centered differences. Then, for an interior (non-boundary) grid point i we have

$$\begin{aligned} \left.\frac{\partial\Phi}{\partial r}\right|_i &\approx \frac{1}{2\Delta r}\left(\Phi_{i+1}-\Phi_{i-1}\right) \ , \\ \left.\frac{\partial^2\Phi}{\partial r^2}\right|_i &\approx \frac{1}{(\Delta r)^2}\left(\Phi_{i+1}-2\Phi_i+\Phi_{i-1}\right) \ . \end{aligned} \tag{6}$$

Then, Poisson equation is

$$\frac{1}{r_i}\frac{(\Phi_{i+1}-\Phi_{i-1})}{\Delta r}+\frac{(\Phi_{i+1}-2\Phi_i+\Phi_{i-1})}{(\Delta r)^2}=4\pi G\rho_i \ . \tag{7}$$

Matrix Method

Inner boundary condition:
Because of the $1/r$ term in the Laplacian, it is clear that the behavior at $r = 0$ must be regularized.

It is possible to use an expansion or we can stagger the grid so that there is no point exactly at $r = 0$, for example by moving the entire grid over by $0.5\Delta r$.

Matrix Method

Inner boundary condition:
At the inner boundary, we have the condition $\frac{\partial \Phi}{\partial r} = 0$, so we assume

$$\Phi_{-1} = \Phi_0 \ . \tag{8}$$

Matrix Method

Hence, Poisson equation can be written as linear system

$$J\mathbf{\Phi} = \mathbf{b} \ , \tag{9}$$

where $\Phi = (\Phi_0, \cdots, \Phi_{n-1})^T$ (for a grid with n points labeled 0 to $n-1$) and $\mathbf{b} = 4\pi G(\rho_0, \cdots, \rho_{n-1})^T$.

Matrix Method

The matrix J has tri-diagonal form and can be explicitely given as:

(a) $i = j = 0$:

$$J_{00} = -\frac{1}{(\Delta r)^2} - \frac{1}{r_0 \Delta r} \,, \tag{10}$$

(b) $i = j$:

$$J_{ij} = \frac{-2}{(\Delta r)^2} \,, \tag{11}$$

(c) $i + 1 = j$:

$$J_{ij} = \frac{1}{(\Delta r)^2} + \frac{1}{r_i \Delta r} \,, \tag{12}$$

(d) $i - 1 = j$:

$$J_{ij} = \frac{1}{(\Delta r)^2} - \frac{1}{r_i \Delta r} \,. \tag{13}$$

Matrix Method

After finding the solution to the linear system, it is neccesary to correct for the analytic outer boundary value

$$\Phi(R_{\text{surface}}) = -\frac{GM}{R_{\text{surface}}}. \tag{14}$$

Parabolic PDEs

Parabolic PDEs describe processes that are slowly changing, such as the slow diffusion of heat in a medium, sediments in ground water, and radiation in an opaque medium. Parabolic PDEs are second order and have generally the shape

$$\partial_t u - k\partial_{xx}^2 u = f \ . \tag{15}$$

Parabolic PDEs Conditions

Initial Conditions for Parabolic Problems
One must specify $u(x, t = 0)$ at all x.

Boundary Conditions for Parabolic Problems
Dirichlet, von Neumann or Robin boundary conditions. If the boundary conditions are independent of time, the system will evolve towards a steady state ($\partial_t u = 0$). In this case, one may set $\partial_t u = 0$ for all times and treat Eq. (15) as an elliptic equation.

Elliptic PDEs

Elliptic equations describe systems that are static, in steady state and/or in equilibrium. There is no time dependence. A typical elliptic equation is the Poisson equation:

$$\nabla^2 u = f \ , \tag{16}$$

which one encounters in Newtonian gravity and in electrodynamics. ∇^2 is the Laplace operator, and f is a given scalar function of position. Elliptic problems may be linear (f does not depend on u or its derivatives) or non-linear (f depends on u or its derivatives).

Elliptic PDEs Conditions

Initial Conditions for Elliptic Problems
Do not apply, since there is no time dependence.

Boundary Conditions for Elliptic Problems
Dirichlet, von Neumann or Robin boundary conditions.

Numerical Methods for PDEs

There is no such thing as a general robust method for the solution of generic PDEs. Each type (and each sub-type) of PDE requires a different approach. Real-life PDEs may be of mixed type or may have special properties that require knowledge about the underlying physics for their successful solution.

There are three general classes of approaches to solving PDEs

1. Finite Difference Methods.
The differential operators are approximated using their finite-difference representation on a given grid. A sub-class of finite-difference methods, so-called finite-volume methods, can be used for PDEs arising from conservation laws (e.g., the hydrodynamics equations).
Finite difference/volume methods have polynomial convergence for smooth functions.

2. Finite Element Methods.
The domain is divided into cells ("*elements*"). The solution is represented as a single function (e.g., a polynomial) on each cell and the PDE is transformed to an algebraic problem for the matching conditions of the simple functions at cell interfaces. Finite element methods can have polynomial or exponential convergence for smooth functions.

3. Spectral Methods.

The solution is represented by a linear combination of known functions (e.g. trigonometric functions or special polynomials). The PDE is transformed to a set of algebraic equations (or ODEs) for the amplitudes of the component functions. A sub-class of these methods are the collocation methods. In them, the solution is represented on a grid and the spectral decomposition of the solution in known functions is used to estimate to a high degree of accuracy the partial derivatives of the solution on the grid points.

Linear Advection Equation

Linear Advection Equation

Consider the linear advection equation,

$$\frac{\partial u}{\partial t} + v\frac{\partial u}{\partial x} = 0 \ . \tag{17}$$

This is the simplest example of a hyperbolic equation and its exact solution is simply given by

$$u(t, x) = u(t = 0, x - vt) \ , \tag{18}$$

hence, the advection equation does (as one might expect) just translate the given data along the x-axis with constant advection velocity v.

FTCS Discretization

First-order in Time, Centered in Space (FTCS) Discretization gives

$$u_j^{(n+1)} = u_j^{(n)} - \frac{v\Delta t}{2\Delta x}\left(u_{j+1}^{(n)} - u_{j-1}^{(n)}\right) \ . \tag{19}$$

FTCS Discretization

Introducing $u(x, t^n) = e^{ikx}$ into the discretized equation,

$$\begin{aligned} u_j^{(n+1)} &= e^{ik\Delta x j} - \frac{v\Delta t}{2\Delta x}\left(e^{ik\Delta x(j+1)} - e^{ik\Delta x(j-1)}\right) \, , \\ &= \left(1 - \frac{v\Delta t}{2\Delta x}\left(e^{ik\Delta x} - e^{-ik\Delta x}\right)\right) e^{ik\Delta x j} \, , \\ &= \underbrace{\left(1 - \frac{v\Delta t}{\Delta x} i \sin(k\Delta x)\right)}_{=\xi} e^{ik\Delta x j} \, , \end{aligned} \tag{20}$$

and

$$|\xi| = \sqrt{\xi\xi^*} = \sqrt{1 + \left(\frac{v\Delta t}{\Delta x}\sin(k\Delta x)\right)^2} > 1 \, . \tag{21}$$

Hence, the FTCS method is *unconditionally unstable* for the advection equation!

Upwind Method

We consider now a first-order in space, first-order in time method to test if it could be stable.
The spatial derivative, to first order, has two possiblities:

$$\begin{aligned} \frac{\partial u}{\partial x} &\approx \frac{1}{\Delta x}(u_j - u_{j-1}) && \text{upwind finite difference,} \\ \frac{\partial u}{\partial x} &\approx \frac{1}{\Delta x}(u_{j+1} - u_j) && \text{downwind finite difference.} \end{aligned} \tag{22}$$

These are both so-called one-sided approximations, because they use data only from one side or the other of point x_j.

Upwind Method

Coupling one of these equations with forward Euler time integration yields

$$
\begin{aligned}
u_j^{(n+1)} &= u_j^{(n)} - \frac{v\Delta t}{\Delta x}\left(u_j^{(n)} - u_{j-1}^{(n)}\right) \qquad \text{upwind}, \\
u_j^{(n+1)} &= u_j^{(n)} - \frac{v\Delta t}{\Delta x}\left(u_{j+1}^{(n)} - u_j^{(n)}\right) \qquad \text{downwind} .
\end{aligned}
\tag{23}
$$

Upwind Method

Stability analysis shows that the upwind method is stable for

$$0 \leq \frac{v\Delta t}{\Delta x} \leq 1 \; , \tag{24}$$

while the downwind method is stable for

$$-1 \leq \frac{v\Delta t}{\Delta x} \leq 0 \; , \tag{25}$$

which just confirms that for $v > 0$ one should use the upwind method and for $v < 0$ the downwind method.

Upwind Method

The condition

$$\alpha = \left| \frac{v \Delta t}{\Delta x} \right| \leq 1 \tag{26}$$

is a mathematical realization of the physical causality principle: the propagation of information (via advection) in one time step Δt must not jump ahead more than one grid interval of size Δx.
The demand that $\alpha \leq 1$ is usually referred to as the *Courant-Friedrics-Lewy (CFL) condition*. The CFL condition is used to determine the allowable time step for a spatial grid size Δx for a certain accuracy and stability,

$$\Delta t = c_{\mathrm{CFL}} \frac{\Delta x}{|v|} \ , \tag{27}$$

where $c_{\mathrm{CFL}} \leq 1$ is the CFL factor.

Lax-Friedrich Method

The Lax-Friedrichs method, like FTCS, is first order in time, but second order in space. It is given by

$$u_j^{(n+1)} = \frac{1}{2}\left(u_{j+1}^{(n)} + u_{j-1}^{(n)}\right) - \frac{v\Delta t}{2\Delta x}\left(u_{j+1}^{(n)} - u_{j-1}^{(n)}\right) \ , \tag{28}$$

and, compared to FTCS, has been made stable for $\alpha \leq 1$ (Eq. 26) by using the average of the old result at points $j+1$ and $j-1$ to compute the update at point j. This is equivalent to adding a dissipative term to the equations (which damps the instability of the FTCS method) and leads to rather poor accuracy.

Leapfrog Method

A fully second-order method is the *Leapfrog* Method (also called the midpoint method) given by

$$u_j^{(n+1)} = u_j^{(n-1)} - \frac{v\Delta t}{\Delta x}\left(u_{j+1}^{(n)} - u_{j-1}^{(n)}\right) \ . \tag{29}$$

One can show that it is stable for $\alpha < 1$ (Eq. 26). A special feature of this method is that it is *non-dissipative*. This means that any initial condition simply translates unchanged. All modes (in a decomposition picture) travel unchanged, but they do not generally travel at the correct speed, which can lead to high-frequency oscillations that cannot damp (since there is no numerical viscosity).

Lax-Wendroff Method

The *Lax-Wendroff* method is an extension of the Lax-Friedrich method to second order in space and time. It is given by

$$u_j^{(n+1)} = u_j^n - \frac{v\Delta t}{2\Delta x}\left(u_{j+1}^{(n)} - u_{j-1}^{(n)}\right) + \frac{v^2(\Delta t)^2}{2(\Delta x)^2}\left(u_{j-1}^{(n)} - 2u_j^{(n)} + u_{j+1}^{(n)}\right) \ . \tag{30}$$

It has considerably better accuracy than the Lax-Friedrich method and is stable for $\alpha \leq 1$ (Eq. 26).

Next Class

Poisson Equation