Integración de geodésicas en el espacio-tiempo de Kerr :

Dinámica Molecular
&
Relatividad Numérica y Método de Lattice Boltzmann

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DINAMICA MOLECULAR

Aproximación a primer orden

$$x\big|_{t+\Delta t} = x\big|_t + (\Delta t)\, v\big|_t$$
$$v\big|_{t+\Delta t} = v\big|_t + (\Delta t)\, (F/m)\big|_t$$

Para este caso

$$x^\alpha\big|_{\lambda+\Delta\lambda} = x^\alpha\big|_\lambda + (\Delta\lambda)\, v^\alpha\big|_\lambda$$

$$v^\alpha\big|_{\lambda+\Delta\lambda} = v^\alpha\big|_\lambda + (\Delta\lambda)\, F^\alpha\big|_\lambda$$

¿CÓMO DETERMINAR LA "FUERZA"?

A traves de la ecuación

$$\frac{d^2x^\alpha}{d\lambda^2} + \Gamma^\alpha_{\beta\gamma}\frac{dx^\beta}{d\lambda}\frac{dx^\gamma}{d\lambda} = 0$$

Con lo que

$$\frac{d^2x^\alpha}{d\lambda^2} = -\Gamma^\alpha_{\beta\gamma}\frac{dx^\beta}{d\lambda}\frac{dx^\gamma}{d\lambda}$$

Por lo tanto

$$x^\alpha|_{\lambda+\Delta\lambda} = x^\alpha|_\lambda + (\Delta\lambda)\, v^\alpha|_\lambda$$

$$v^\alpha|_{\lambda+\Delta\lambda} = v^\alpha|_\lambda - \Delta\lambda \left(\Gamma^\alpha_{\beta\gamma}v^\beta v^\gamma\right)\Big|_\lambda$$

MÉTODO A UTILIZAR

Omelyan-PEFRL

$$x = x + \xi \Delta t v$$
$$v = v + (1 - 2\lambda)\frac{\Delta t}{2}F$$
$$x = x + \chi \Delta t v$$
$$v = v + \lambda \Delta t F$$
$$x = x + (1 - 2(\chi + \xi))\Delta t v$$
$$v = v + \lambda \Delta t F$$
$$x = x + \chi \Delta t v$$
$$v = v + (1 - 2\lambda)\frac{\Delta t}{2}F$$

$$\xi = 0.1786178958448091$$
$$\lambda = -0.2123418310626054$$
$$\chi = -0.06626458266981849$$

Solución de Kerr

$$\boxed{\begin{aligned} ds^2 &= -\left(1-\frac{r_s r}{\Sigma}\right)c^2dt^2 - \frac{2r_s a r\sin^2\vartheta}{\Sigma}c\,dt\,d\varphi + \frac{\Sigma}{\Delta}dr^2 + \Sigma d\vartheta^2 \\ &\quad + \left(r^2+a^2+\frac{r_s a^2 r\sin^2\vartheta}{\Sigma}\right)\sin^2\vartheta\,d\varphi^2, \end{aligned}}$$

$$\Sigma = r^2 + a^2\cos^2\theta$$

$$\Delta = r^2 - r_s r + a^2$$

$$r_s = 2GM/c^2$$

$$\frac{d^2x^\alpha}{d\lambda^2} + \Gamma^\alpha_{\beta\gamma}\frac{dx^\beta}{d\lambda}\frac{dx^\gamma}{d\lambda} = 0$$

Christoffel symbols:

$$\Gamma^r_{tt} = \frac{c^2 r_s\Delta(r^2-a^2\cos^2\vartheta)}{2\Sigma^3}, \qquad \Gamma^\vartheta_{tt} = -\frac{c^2 r_s a^2 r\sin\vartheta\cos\vartheta}{\Sigma^3},$$

$$\Gamma^t_{tr} = \frac{r_s(r^2+a^2)(r^2-a^2\cos^2\vartheta)}{2\Sigma^2\Delta}, \qquad \Gamma^\varphi_{tr} = \frac{c r_s a(r^2-a^2\cos^2\vartheta)}{2\Sigma^2\Delta},$$

$$\Gamma^t_{t\vartheta} = -\frac{r_s a^2 r\sin\vartheta\cos\vartheta}{\Sigma^2}, \qquad \Gamma^\varphi_{t\vartheta} = -\frac{c r_s a r\cot\vartheta}{\Sigma^2},$$

$$\Gamma^r_{t\varphi} = -\frac{c\Delta r_s a\sin^2\vartheta(r^2-a^2\cos^2\vartheta)}{2\Sigma^3}, \qquad \Gamma^\vartheta_{t\varphi} = \frac{c r_s a r(r^2+a^2)\sin\vartheta\cos\vartheta}{\Sigma^3},$$

$$\Gamma^r_{rr} = \frac{2ra^2\sin^2\vartheta - r_s(r^2-a^2\cos^2\vartheta)}{2\Sigma\Delta}, \qquad \Gamma^\vartheta_{rr} = \frac{a^2\sin\vartheta\cos\vartheta}{\Sigma\Delta},$$

$$\Gamma^r_{r\vartheta} = -\frac{a^2\sin\vartheta\cos\vartheta}{\Sigma}, \qquad \Gamma^\vartheta_{r\vartheta} = \frac{r}{\Sigma},$$

$$\Gamma^r_{\vartheta\vartheta} = -\frac{r\Delta}{\Sigma}, \qquad \Gamma^\vartheta_{\vartheta\vartheta} = -\frac{a^2\sin\vartheta\cos\vartheta}{\Sigma},$$

$$\Gamma^\varphi_{\vartheta\varphi} = \frac{\cot\vartheta}{\Sigma^2}\left[\Sigma^2 + r_s a^2 r\sin^2\vartheta\right], \qquad \Gamma^t_{\vartheta\varphi} = \frac{r_s a^3 r\sin^3\vartheta\cos\vartheta}{c\Sigma^2},$$

$$\Gamma^t_{r\varphi} = \frac{r_s a\sin^2\vartheta\left[a^2\cos^2\vartheta(a^2-r^2) - r^2(a^2+3r^2)\right]}{2c\Sigma^2\Delta},$$

$$\Gamma^\varphi_{r\varphi} = \frac{2r\Sigma^2 + r_s\left[a^4\sin^2\vartheta\cos^2\vartheta - r^2(\Sigma+r^2+a^2)\right]}{2\Sigma^2\Delta},$$

$$\Gamma^r_{\varphi\varphi} = \frac{\Delta\sin^2\vartheta}{2\Sigma^3}\left[-2r\Sigma^2 + r_s a^2\sin^2\vartheta(r^2-a^2\cos^2\vartheta)\right],$$

$$\Gamma^\vartheta_{\varphi\varphi} = -\frac{\sin\vartheta\cos\vartheta}{\Sigma^3}\left[A\Sigma + (r^2+a^2)\,r_s a^2 r\sin^2\vartheta\right],$$

Velocidad

$$\frac{dt}{d\lambda} = \frac{-g_{\phi\phi} - b g_{t\phi}}{g_{\phi\phi} g_{tt} - g_{t\phi}^2}$$

$$\frac{d\phi}{d\lambda} = \frac{b g_{tt} + b g_{t\phi}}{g_{\phi\phi} g_{tt} - g_{t\phi}^2}$$

"Fuerza"

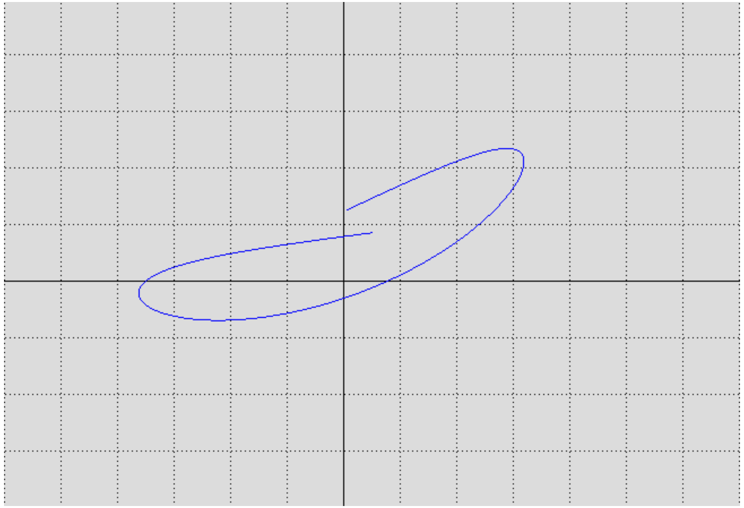
$$\frac{d^2 t}{d\lambda^2} = -2\Gamma^t_{tr} \frac{dt}{d\lambda} \frac{dr}{d\lambda}$$

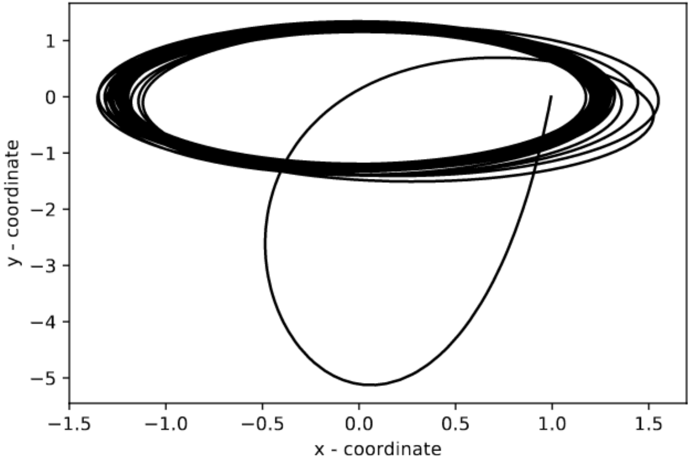
$$\frac{d^2 r}{d\lambda^2} = -\Gamma^r_{tt} \left(\frac{dt}{d\lambda}\right)^2 - \Gamma^r_{rr} \left(\frac{dr}{d\lambda}\right)^2 - \Gamma^r_{\theta\theta} \left(\frac{d\theta}{d\lambda}\right)^2 - \Gamma^r_{\phi\phi} \left(\frac{d\phi}{d\lambda}\right)^2$$

$$\frac{d^2 \theta}{d\lambda^2} = -2\Gamma^\theta_{r\theta} \frac{dr}{d\lambda} \frac{d\theta}{d\lambda} - \Gamma^\theta_{\phi\phi} \left(\frac{d\theta}{d\lambda}\right)^2$$

$$\frac{d^2 \phi}{d\lambda^2} = -2\Gamma^\phi_{r\phi} \frac{dr}{d\lambda} \frac{d\phi}{d\lambda} - 2\Gamma^\phi_{\theta\phi} \frac{d\phi}{d\lambda} \frac{d\theta}{d\lambda}$$

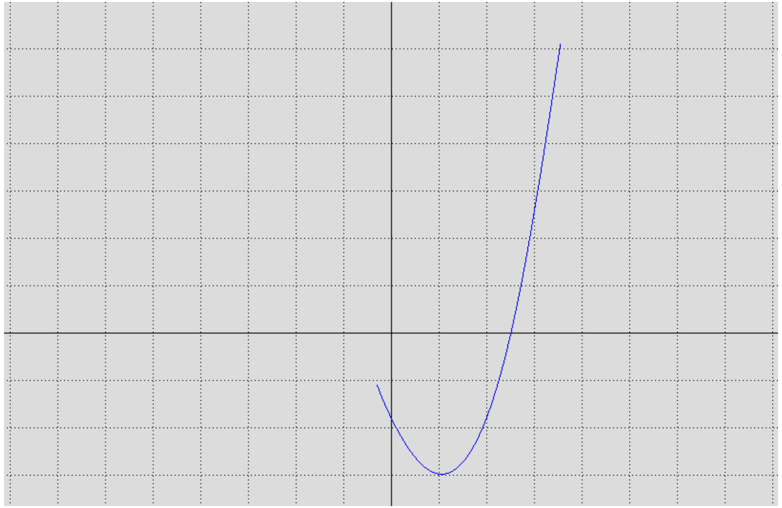
Dinámica Molecular

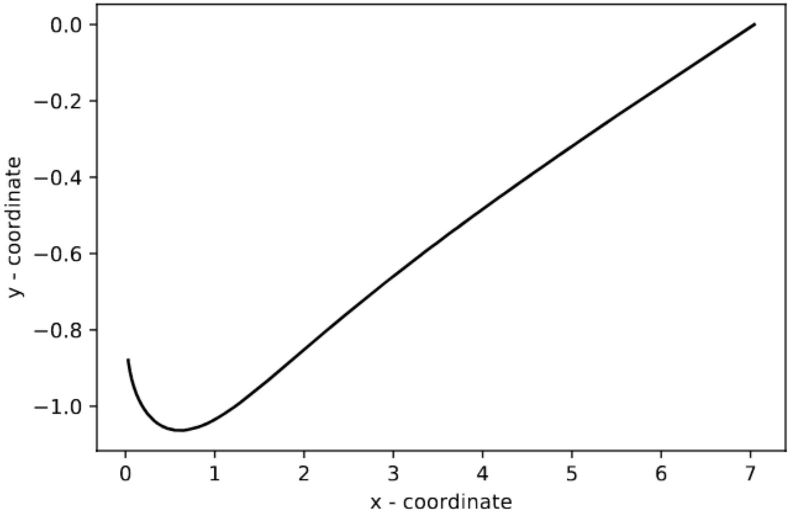




$$\theta = 0$$

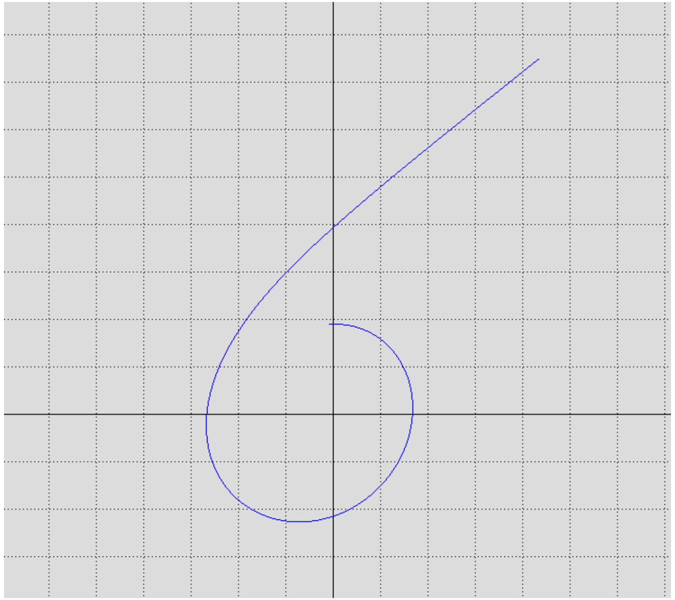
Dinámica Molecular

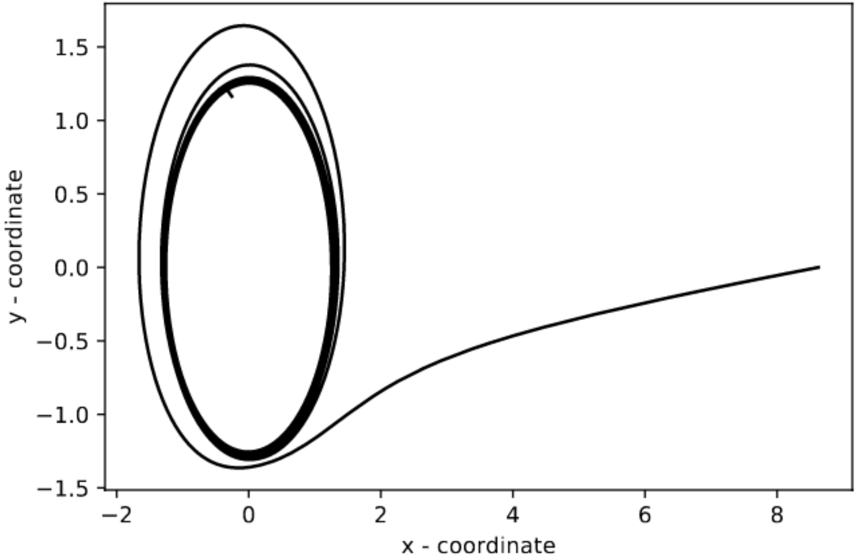




$$\theta = \frac{\pi}{4}$$

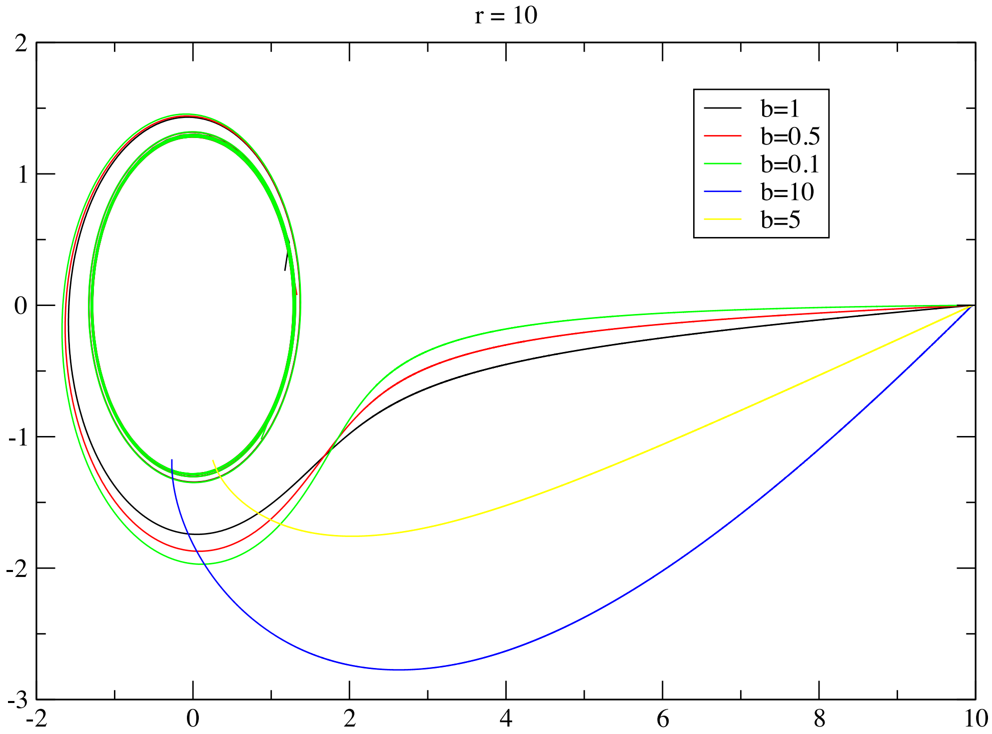
Dinámica Molecular





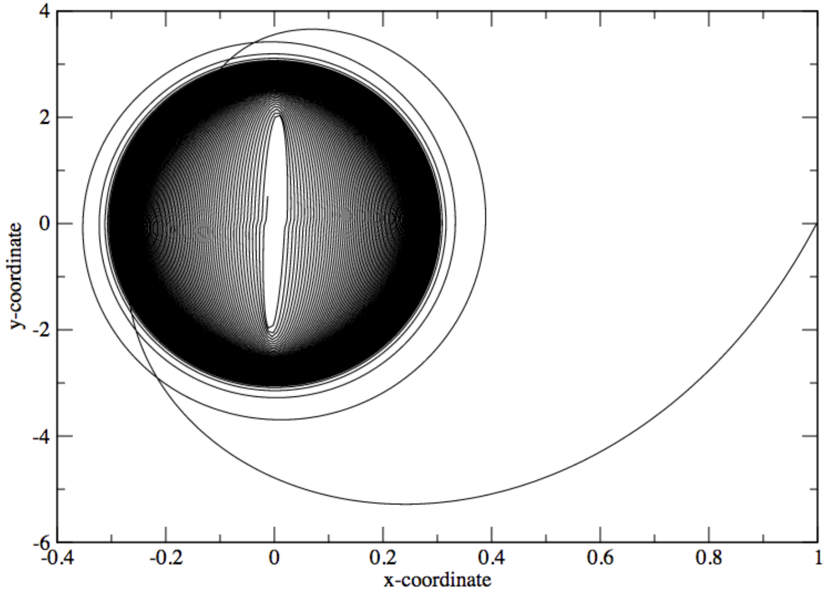
$$\theta = \frac{\pi}{3}$$

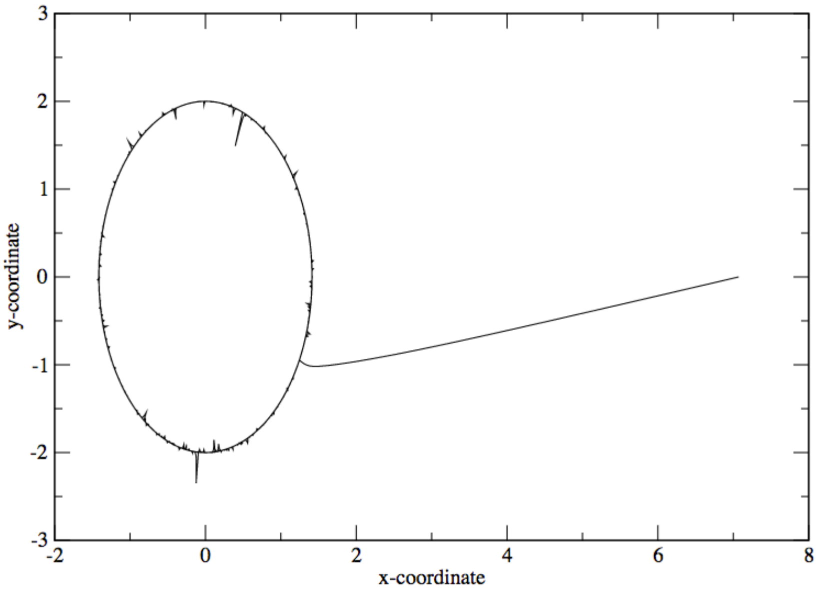
Dinámica Molecular



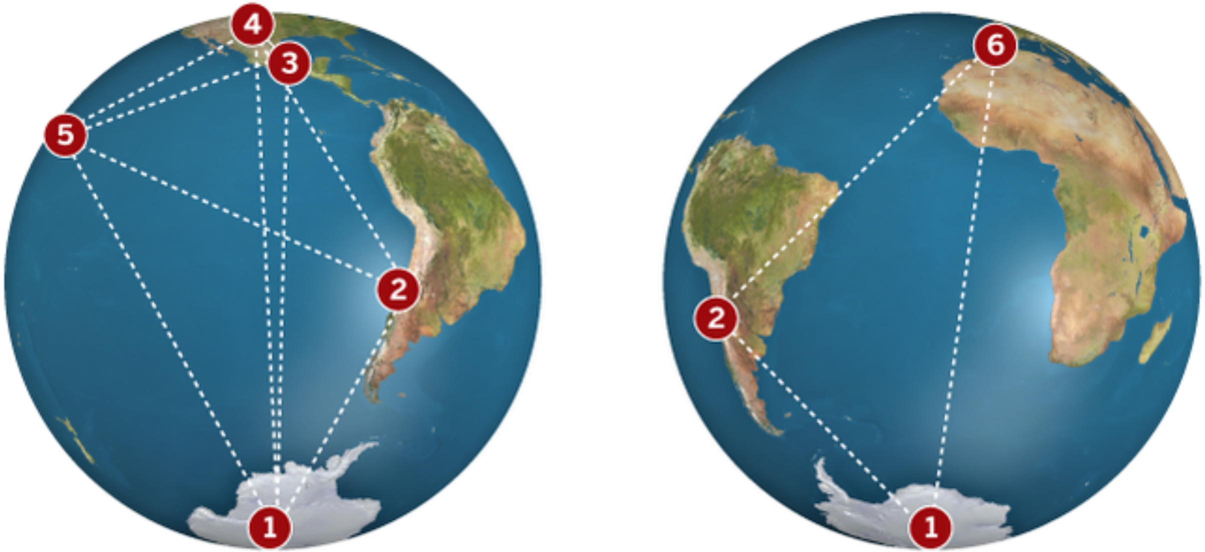
Dinámica Molecular

$$ds^2 = -\left(1 - \frac{r_s}{r}\right) c^2 dt^2 + \frac{1}{1 - r_s/r} dr^2 + r^2 \left(d\vartheta^2 + \sin^2\vartheta d\varphi^2\right),$$



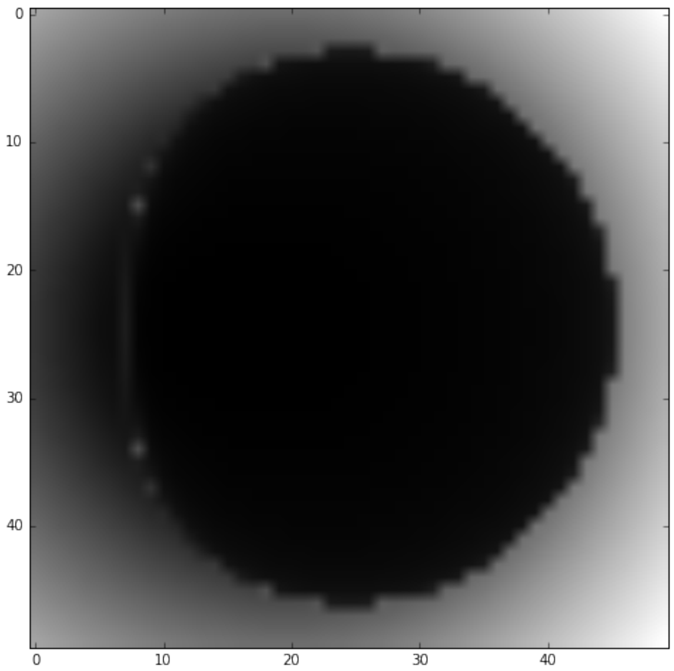


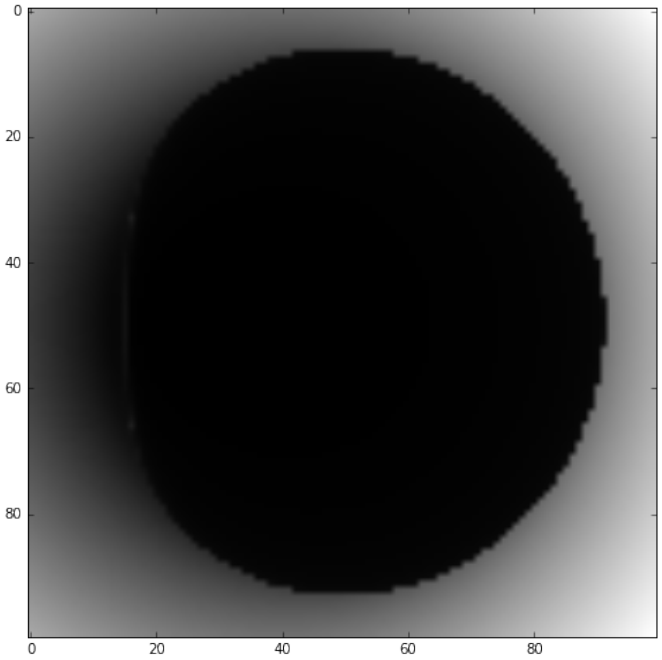
Dinámica Molecular



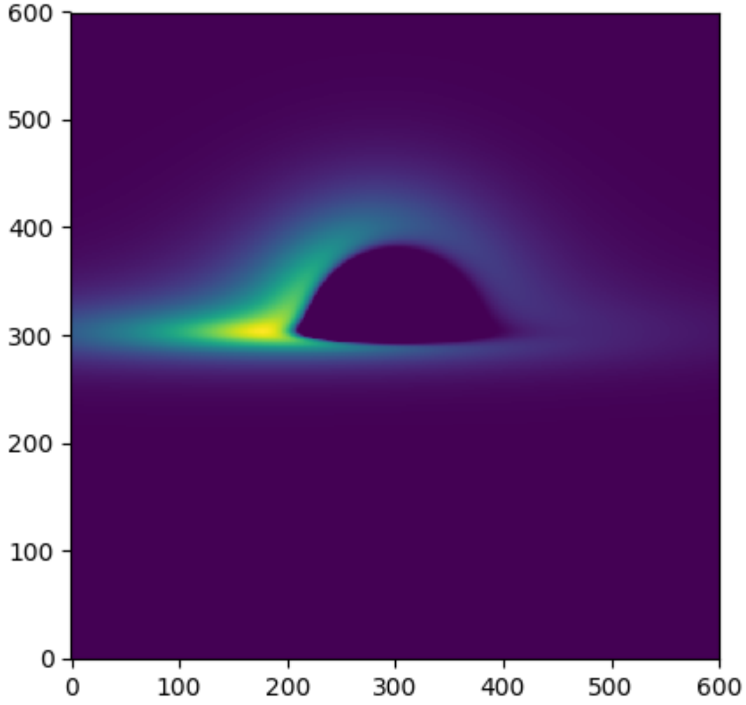
1. South Pole Telescope **2.** Atacama Large Millimeter/submillimeter Array and Atacama Pathfinder Experiment (Chile) **3.** Large Millimeter Telescope (Mexico) **4.** Submillimeter Telescope (Arizona) **5.** James Clerk Maxwell Telescope and Submillimeter Array (Hawaii) **6.** IRAM 30-meter (Spain)

Geokerr





GRTrans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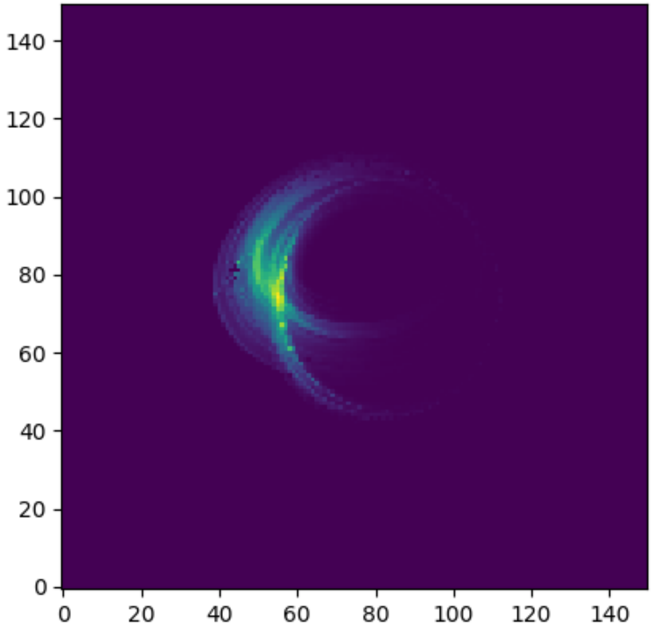
140
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140

What is Numerical Relativity?

Field of study which tries to solve Einstein's field equations using numerical techniques and complex codes. M. Alcubierre

$$\Gamma^{\sigma}_{\lambda\mu} = \frac{1}{2} g^{\nu\sigma} \left(\partial_\lambda g_{\mu\nu} + \partial_\mu g_{\lambda\nu} - \partial_\nu g_{\mu\lambda} \right)$$

$$8\pi T_{\alpha\beta} = \sum_{\delta} \left[\frac{\partial \Gamma^{\delta}_{\alpha\beta}}{\partial x^{\delta}} - \frac{\partial \Gamma^{\delta}_{\delta\beta}}{\partial x^{\alpha}} + \sum_{\gamma} \left(\Gamma^{\delta}_{\alpha\beta} \Gamma^{\gamma}_{\delta\gamma} - \Gamma^{\delta}_{\gamma\beta} \Gamma^{\gamma}_{\delta\alpha} \right) \right] - \frac{1}{2} g_{\alpha\beta} \sum_{\delta} \sum_{\gamma} \left\{ g^{\delta\gamma} \sum_{\mu} \left[\frac{\partial \Gamma^{\mu}_{\delta\gamma}}{\partial x^{\mu}} - \frac{\partial \Gamma^{\mu}_{\mu\gamma}}{\partial x^{\delta}} + \sum_{\gamma} \left(\Gamma^{\mu}_{\delta\gamma} \Gamma^{\nu}_{\mu\nu} - \Gamma^{\mu}_{\nu\gamma} \Gamma^{\nu}_{\mu\delta} \right) \right] \right\}$$

3+1 Formalism

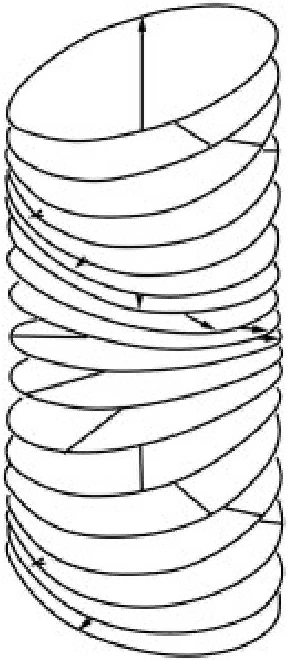
$$\forall t \in \mathbb{R},\ \Sigma_t := \{p \in \mathcal{M},\ \hat{t}(p) = t\}$$

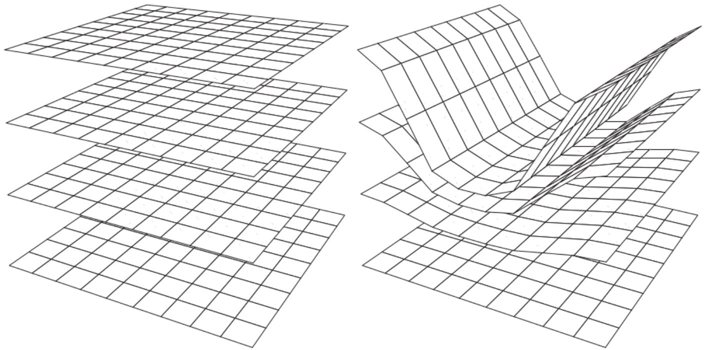
Scalar field on $\mathcal{M}$

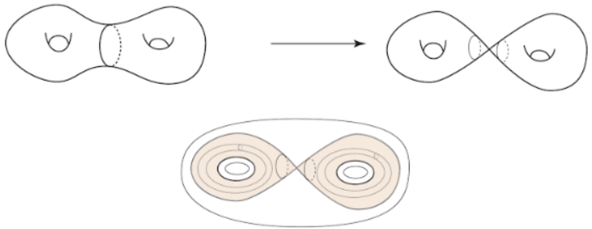
$$\Sigma_t \cap \Sigma_{t'} = \emptyset$$

$$t \neq t'$$

$$\mathcal{M} = \cup_{t\in\mathbb{R}} \Sigma_t$$







Main Variables

$$\nabla g_{\mu\nu} = 0 \longrightarrow \boxed{D\gamma_{ij} = 0}$$

$$\forall\, \vec{v} \in \mathcal{T}_p,\ (D_i D_j - D_j D_i)\, v^k = R^k_{lij} v^l$$

Intrinsic Curvature

$$g \to \gamma$$

3-Metric

$$R = \gamma^{ij} R^k_{ikj} = \gamma^{ij} R_{ij}$$

Gaussian Curvature

$$\vec{K} : T_p(\Sigma) \times T_p(\Sigma) \longrightarrow \mathbb{R}$$

$$(\vec{u}, \vec{v}) \longrightarrow -\vec{u} \cdot \nabla_{\vec{v}} \hat{n}$$

Extrinsic Curvature

Relationship between 4D and 3D

$$DT = \vec{\gamma}^{*}\nabla T$$

$$\forall\ (\vec{u},\vec{v})\ \epsilon\, T(\Sigma)\times T(\Sigma)\,,\quad D_{\vec{u}}\vec{v} = \nabla_{\vec{u}}\vec{v} + K(\vec{u},\vec{v})\,\hat{n}$$

$$\gamma^{\rho}_{\alpha}\gamma^{\sigma}_{\beta}\gamma^{\gamma}_{\lambda}\ {}^{4}R^{\lambda}_{\mu\rho\sigma} = R^{\gamma}_{\mu\alpha\beta} - K_{\alpha\mu}K^{\gamma}_{\beta} + K_{\beta\mu}K^{\gamma}_{\alpha}$$

$${}^{4}R + 2\ {}^{4}R_{\mu\nu}n^{\mu}n^{\nu} = R + K^{2} - K_{ij}K^{ij}$$

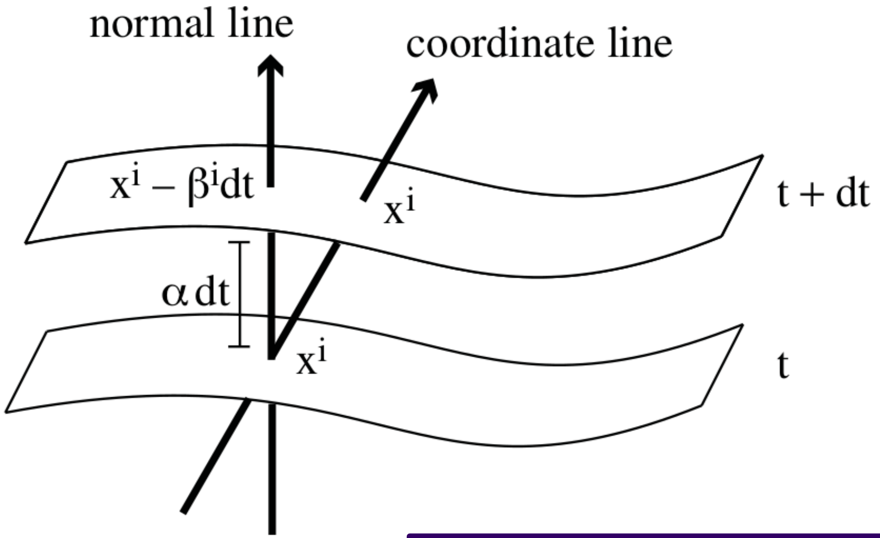
Second Gauss relation and scalar Gauss relation

Codazzi Relation and contracted Codazzi Relation

$$\gamma^{\gamma}_{\rho}n^{\sigma}\gamma^{\mu}_{\alpha}\gamma^{\nu}_{\beta}\quad {}^{4}R^{\rho}_{\sigma\mu\nu} = D_{\beta}K^{\gamma}_{\alpha} - D_{\alpha}K^{\gamma}_{\beta}$$

$$\gamma^{\mu}_{\rho}n^{\nu}\quad {}^{4}R_{\mu\nu} = D_{\alpha}K - D_{\mu}K^{\mu}_{\alpha}$$

Coordinate Evolution of a Foliation



$$\hat{n} := \left(\pm\vec{\nabla}t \cdot \vec{\nabla}t\right)^{-1/2} \vec{\nabla}t = \alpha\vec{\nabla}t$$

Lapse Function

$$\vec{m} := \alpha\hat{n}$$

Normal evolution vector

$$\partial_t = \vec{m} + \vec{\beta}$$

Shift vector

$$\boxed{\mathcal{L}_{\vec{m}}\gamma_{\nu\beta} = -2\alpha K_{\nu\beta}}$$

3+1 Einstein Equations

$$\left(\frac{\partial}{\partial t} - \mathcal{L}_{\vec{\beta}}\right)\gamma_{ij} = -2\alpha K_{ij}$$

$$R + K^2 - K_{ij}K^{ij} = 16\pi E$$

$$D_j K^j{}_i - D_i K = 8\pi p_i$$

$$\left(\frac{\partial}{\partial t} - \mathcal{L}_{\vec{\beta}}\right) K_{ij} = -D_i D_j \alpha + \alpha\left[R_{ij} + K K_{ij} - 2K_{ij}K^k{}_j + 4\pi\left((S-E)\gamma_{ij} - 2S_{ij}\right)\right]$$

$$S = T + E$$

Lattice Boltzmann Model

$$\frac{\partial f}{\partial t} + \zeta_\beta \frac{\partial f}{\partial x_\beta} + \frac{F_\beta}{\rho} \frac{\partial f}{\partial \zeta_\beta} = \Omega(f) = \frac{df}{dt}$$

$\zeta_\beta \rightarrow$ Particle Velocity

$\Omega(f) \rightarrow$ Collision Operator

$f_i(\vec{x}, t) \longrightarrow$ Discrete-Velocity distribution function

$\{\vec{c}_i, \omega_i\} \longrightarrow$ Set numbers of Velocities

Moments of f_i

$$\boxed{\begin{aligned} \rho(\vec{x}, t) &= \Sigma_i f_i(\vec{x}, t) \\ \rho\vec{u}(\vec{x}, t) &= \Sigma_i \vec{c}_i f_i(\vec{x}, t) \end{aligned}}$$

$$c_s^2 = \frac{1}{3}\left(\frac{\Delta x}{\Delta t}\right)^2 \longrightarrow$$ Isothermal models of speed of sound

Lattice Boltzmann Model

$$f_i(\vec{x} + \vec{c}_i \Delta t, t + \Delta T) = f_i(\vec{x}, t) + \Omega_i(\vec{x}, t)$$

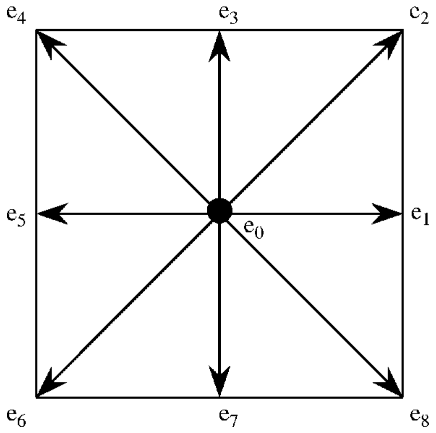
$$\Omega_i(f) = \frac{f_i - f_i^{equ}}{\tau} \Delta t$$

$\longrightarrow$ Bhatnagar-Gross-Krook (BGK) Operator

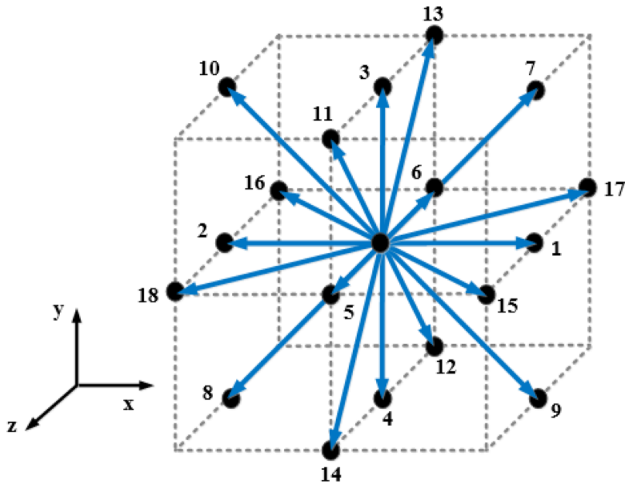
$\tau \longrightarrow$ Relaxation Time

$$f_i^{equ}(\vec{x}, t) = \omega_i \rho \left(1 + \frac{\vec{u} \cdot \vec{c}_i}{c_s^2} + \frac{(\vec{u} \cdot \vec{c}_i)^2}{2c_s^4} - \frac{\vec{u} \cdot \vec{u}}{2c_s^2} \right)$$

Lattice Boltzmann Model



D2Q9 Lattice



D3Q19 Lattice

$$f_i(\vec{x}+\vec{c}_i\Delta t, t+\Delta T) = -f_i(\vec{x},t) + \frac{f_i - f_i^{equ}}{\tau}\Delta t + S_i(\vec{x},t)$$

Lattice Boltzmann for Numerical Relativity

Expansion of Flat Universe

$$ds^2 = -dt^2 + a(t)^2\left(\frac{dr^2}{1-kr^2} + r^2 d\theta^2 + r^2 sin^2\theta d\phi^2\right)$$

$$ds^2 = -dt^2 + a(t)^2(dx^2 + dy^2 + dz^2)$$

Gauge Wave

$$ds^2 = -Hdt^2 + Hdx^2 + dy^2 + dz^2$$

$$H = H(x-t) = 1 - Asin\left(\frac{2\pi(x-t)}{d}\right)$$

$$A = 10^{-3};\ d = 1$$

Linear Wave

$$ds^2 = -dt^2 + dx^2 + (1+b)dy^2 + (1-b)dz^2$$

$$b = Asin\left(\frac{2\pi(x-t)}{d}\right)$$

$$A = 10^{-5};\ d = 1$$

Numerical Implementation: Bona- Massó Formulation

$$\partial_t u + \partial_k F^k\_u = S\_u$$

Einstein's field equations as a set of conservation equations

$$u = (\alpha, \gamma_{ij}, K_{ij}, A_i, D_{rij}, V_i)$$

$$A_k = \partial_k \ln(\alpha),\ {B^i}_k = \frac{1}{2}\partial_k \beta^i,\ D_{kij} = \frac{1}{2}\partial_k \gamma_{ij} \qquad s_{ij} = \frac{B_{ij} + B_{ji}}{\alpha}, V_i = {D_{ir}}^r - {D^r}_{ri}$$

Set of partial differential linear equations in time and space

Fluxes and Sources

Fluxes

$$F^k_{-}\gamma_{ij} = 0,$$
$$F^k_{-}\alpha = 0,$$
$$F^k_{-}A_i = \alpha Q,$$
$$F^k_{-}V_i = 0,$$
$$F^k_{-}D_{lij} = \alpha K_{ij},$$
$$F^k_{-}K_{ij} = \alpha \Big[D^k{}_{ij} - \frac{1}{2}V^k\gamma_{ij} + \frac{1}{2}\delta^k{}_i(A_j + 2V_j - D^r{}_{jr}) + \frac{1}{2}\delta^k{}_j(A_i + 2V_i - D^r{}_{ir}) \Big]$$

Sources

$$S_{-}\gamma_{ij} = -2\alpha K_{ij},$$
$$S_{-}\alpha = -\alpha^2 Q,$$
$$S_{-}A_i = 0,$$
$$S_{-}V_i = \alpha\left[\alpha G^0{}_i + A_r(K^r{}_i - tr(K)\delta^r{}_i)\right., \\ +K^r{}_s(D^s{}_{ir} - 2D^s{}_{ri}) - K^r{}_i(D^s{}_{rs} - 2D^s{}_{sr})]$$
$$S_{-}D_{lij} = 0,$$
$$S_{-}K_{ij} = \alpha \Big[-{}^{(4)}R_{ij} - 2K_i{}^kK_{kj} + tr(K)K_{ij} \\ - \Gamma^k{}_{ri}\Gamma^r{}_{kj} + \Gamma^k{}_{kr}\Gamma^r{}_{ij} + 2D^r{}_{ik}D^k{}_{rj} \\ + 2D^r{}_{jk}D^k{}_{ri} - V^kD_{kij} \\ + A_i(V_j - \frac{1}{2}D^k{}_{jk}) + A_j(V_i - \frac{1}{2}D^k{}_{ik}) \\ - (2D^k{}_{kr} - A_r)(D^r{}_{ij} + D^r{}_{ji}) \\ + \frac{1}{4}\gamma_{ij}\left(-D^{rs}_k\Gamma^k{}_{rs} + D^r{}_{kr}D^{ks}{}_s - 2V^kA_k\right. \\ + tr(K^2) - (tr(K))^2 + 2\alpha^2G^{00})\Big]$$

QUESTIONS?

рахмат
danke
謝謝
ngiyabonga
شكراً جزيلاً
Баярлалаа
mersi
barka
welalin
tack
spas
teşekkür ederim
mahalo
tapadh leat
спасибо
faafetai lava
vinaka
спасибі
blagodaram
dank je
misaotra
matondo
paldies
grazzi
gracias
хвала
asante
manana
obrigada
tenki
kiitos
dankie
dhanyavad
nanni
nandri
thank you
bedankt
enkosi
bayarlalaa
gracie
hvala
mauruuru
köszönöm
akun
dankon
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taiku
go raibh maith agat
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obrigado
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ありがとう
grazie
arigatô
takk
dakujem
trugarez
mèsi
didi madloba
najis tuke
kam sah hamnida
rahmat
terima kasih
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diolch
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shukriya
merce
мерси
তোমাকে ধন্যবাদ
감사합니다
xièxie
ευχαριστώ
merci