Exponential and Logarithmic Functions

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Math 122: Calculus for Business Administration and Social Sciences

Outline

1. 1.5: Exponential Functions

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2. 1.6: Logarithms
   - Inverse Functions
   - Definition
   - Exponential Functions with Base e

Definition

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- The base a is sometimes called the *growth/decay factor*.

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Therefore,

$$P(t) = 14\left(\frac{207}{14}\right)^{\frac{t}{17}} \approx 14(1.172)^t.$$

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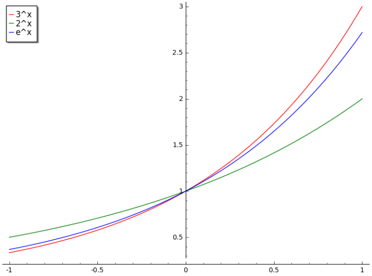
(B) Find the relative growth rate.

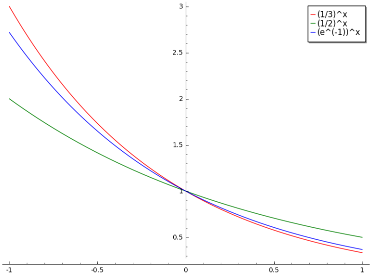
$$r = a - 1 = \sqrt[3]{\frac{91.4}{88.2}} - 1 \approx 0.012$$

Graphs of Exponential Functions

$1 < a$:



$0 < a < 1$:



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A function $f(x)$ has an *inverse* if there exists a function $f^{-1}(x)$ such that

$$f \circ f^{-1}(x) = x \text{ and } f^{-1} \circ f(x) = x.$$

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Theorem 1 (Horizontal Line Test)

If any horizontal line intersects the graph of $f(x)$ in **at most one** *point, then $f(x)$ admits a composition inverse.*

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- One denotes $\log_e(x)$ by $\ln(x)$.

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- $\log_a(x) = \frac{\log_b(x)}{\log_b(a)}$.

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\begin{aligned}
\Rightarrow \ln(3^t) &= \ln(10) \\
\Rightarrow t\ln(3) &= \ln(10) \\
\Rightarrow t &= \frac{\ln(10)}{\ln(3)} (= \log_3(10))
\end{aligned}
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$$\begin{aligned} \Rightarrow e^{3t} &= \frac{12}{5} \\ \Rightarrow \ln(e^{3t}) &= 3t = \ln\left(\frac{12}{5}\right) \\ \Rightarrow t &= \frac{1}{3}\ln\left(\frac{12}{5}\right) \end{aligned}$$

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We call k the *continuous growth/decay rate*.

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Let $a = e^{0.05}$. Then

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$$P(t) = 500(1.06)^t = 500e^{\ln(1.06)t}.$$