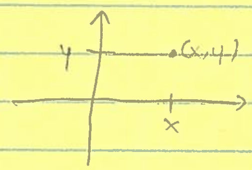
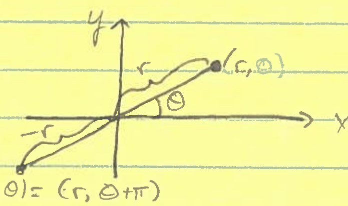
10.3 Polar Coordinates

Euclidean 2-space.

Usually, we denote points in the cartesian plane by (x,y), where x denotes the distance along one axis, and y the distance along another perpendicular axis.



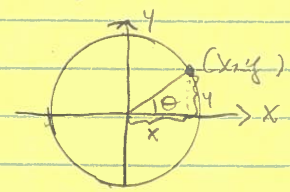
Polar coordinates specify points using a radial distance, r, and an angle, θ. The angle is measured from the traditional x-axis.



Similar to (x,y) coordinates, r is directional.

Why?

Recall basic trig: the unit circle in the plane

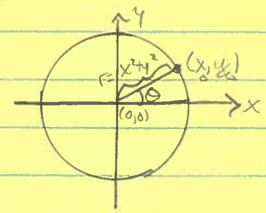


the points on the circle have coordinates (x,y), where $x=\cos(\theta)$ and $y=\sin(\theta)$, and they satisfy $x^2+y^2=1$. In fact, every circle centered at the origin satisfies this property:

$$x^2+y^2=r^2, \quad r \text{ the radius}$$

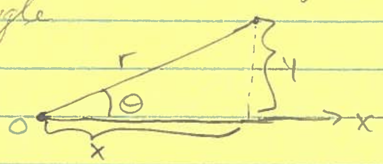
So we may take any pair (x,y) and if we let $r=\sqrt{x^2+y^2}$, then this point lies on the circle of radius r about O.

So, given (x_0, y_0), we have



Hence $(x,y) = (r, \theta)$, where θ is the angle between the line from the origin to (x_0, y_0) (which has slope $m = \frac{y_0 - 0}{x_0 - 0} = y_0/x_0$ when $x \neq 0$, and so is either $x = 0$ or the line $y = \frac{y_0}{x_0}$) and the x-axis.

By the same token, given (r, θ), we have the triangle



$\cos(\theta) = x/r \Rightarrow x = r\cos(\theta)$,

$\sin(\theta) = y/r \Rightarrow y = r\sin(\theta)$.

So we can return to (x,y)-coordinates by $(r, \theta) \mapsto (r\cos(\theta), r\sin(\theta))$.

Rmk: $x^2 + y^2 = (r\cos(\theta))^2 + (r\sin(\theta))^2 = r^2(\cos^2(\theta) + \sin^2(\theta)) = r^2$.

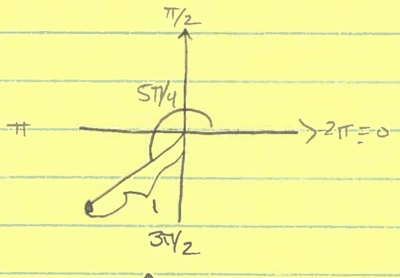
$\tan(\theta) = x/y$.

E.g. ① Plot the points

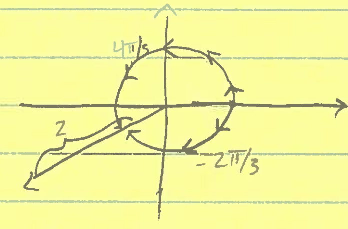
a) $(1, 5\pi/4)$ c) $(2, -2\pi/3)$

b) $(2, 3\pi)$ d) $(-3, 3\pi/4)$

a)

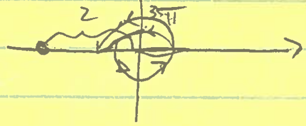


b)

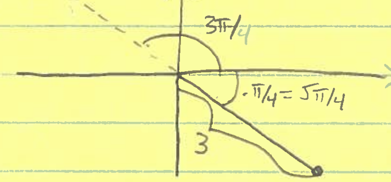


$2\pi - 2\pi/3 = 4\pi/3$

c)



d)



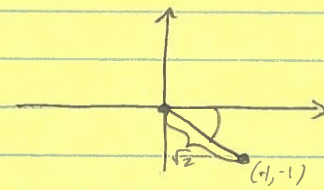
② Convert $(2, \pi/3)$ from polar to cartesian.

$\cos(\pi/3) = \frac{1}{2}$, $\sin(\pi/3) = \frac{\sqrt{3}}{2}$

$(2, \pi/3)$ "=" $(1, \sqrt{3})$.

③ Represent the point $(1,-1)$ in polar coordinates.

$r^2 = 1^2 + (-1)^2 = 2$
$\Rightarrow r = \sqrt{2}$
$\tan(\theta) = -1$
$\Rightarrow \theta = \arctan(-1) = 7\pi/4$.



$(\sqrt{2}, 7\pi/4)$. (also, $(-\sqrt{2}, 3\pi/4)$, $(\sqrt{2}, -\pi/4)$, etc.)

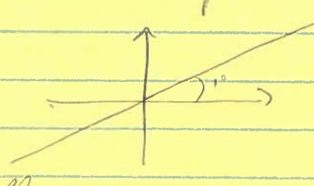
Polar Curves

The graph of a polar equation $r = f(\theta)$, or $F(r,\theta)=0$, consists of all points p that have a representation $(f(\theta), \theta)$.

Eg: ④ What curve is represented by $r = 2$?

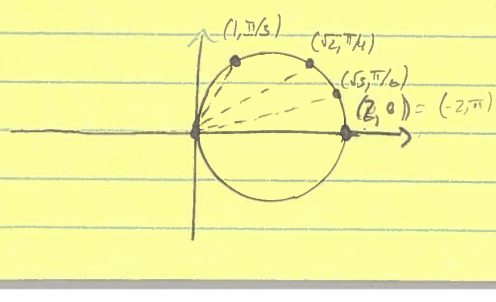
The circle of radius 2 centered at 0.

⑤ Sketch the polar curve $\theta = 1$



$(2\cos(\theta), \theta)$

⑥ a) Sketch $r = 2\cos(\theta)$
b) Find a cartesian equation for this



b) We observe that

$$x = r\cos(\theta) = r\left(\frac{r}{2}\right)$$

so

$$2x = r^2 = x^2 + y^2$$

and we get the equation

$$x^2 + y^2 - 2x = 0.$$

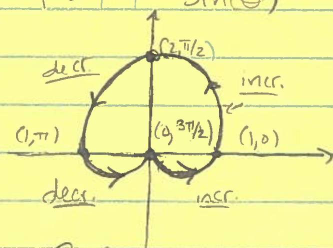
This gives

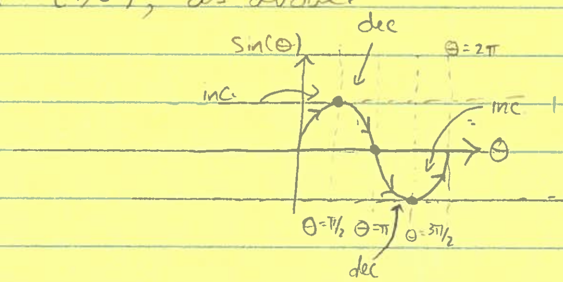
$$(x^2 - 2x + 1) + y^2 - 1 = 0$$

$$\Rightarrow (x-1)^2 + y^2 = 1$$

The circle of radius 1 about (1,0), as above.

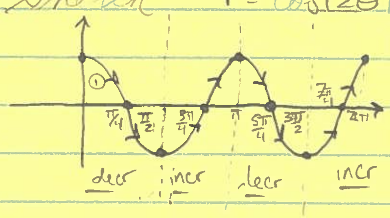
⑦ Sketch $r = 1 + \sin(\theta)$

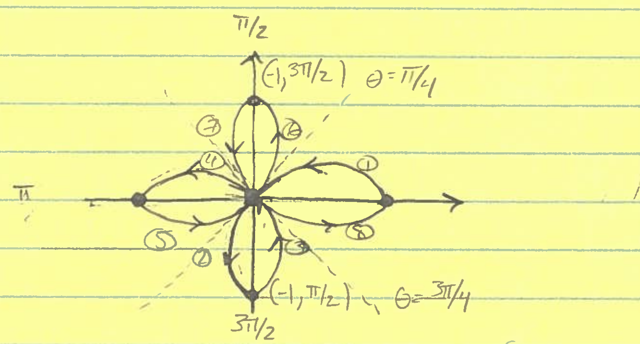




Cardiod (shaped like a heart).

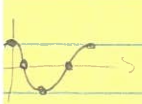
⑧ Sketch $r = \cos(2\theta)$





$(1, \pi/2) \xrightarrow{1} (0, \pi/4) \xrightarrow{2} (-1, \pi/2) \xrightarrow{3} (0, 3\pi/4) \xrightarrow{4} (1, \pi) \xrightarrow{5} (0, 5\pi/4) \xrightarrow{6} (-1, 3\pi/2) \xrightarrow{7} (0, 7\pi/4) \xrightarrow{8} (1, 2\pi)$

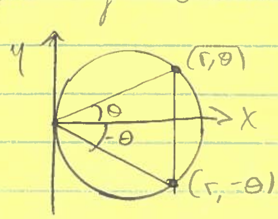
Four leaved rose.



Symmetry

a) If a polar equation is unchanged when θ is replaced by $-\theta$, there is symmetry across the x-axis.

e.g. $r = \cos(\theta)$, since $\cos(-\theta) = \cos(\theta)$.



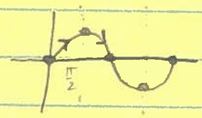
b) If the equation is unchanged when r is replaced by $-r$ or θ by $\theta+\pi$, then there is symmetry about the origin

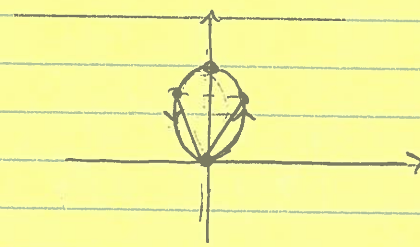
E.g. $r=\cos(2\theta)$ since $\cos(2(\theta+\pi))=\cos(2\theta+2\pi)=\cos(2\theta)$

We can rotate the four-leaved rose $180°$ without changing its appearance

c) If the equation is unchanged when θ is replaced by $\pi-\theta$, the curve is symmetric about the line $\theta=\pi/2$. (y-axis).

E.g. $r=\sin(\theta)$





$$(0,0) \xrightarrow{1} (1,\pi/2) \xrightarrow{2} (0,\pi) \xrightarrow{3} (-1, 3\pi/2) \longrightarrow (0, 2\pi)$$

since $\sin(\pi-\theta)=\sin(\theta)$

Tangents to Polar Curves

Suppose $r=f(\theta)$. Then we have

$$x=r\cos(\theta)=f(\theta)\cos(\theta)$$
$$y=r\sin(\theta)=f(\theta)\sin(\theta)$$

$$* \quad \frac{dy}{dx}=\frac{dy}{d\theta}\Big/\frac{dx}{d\theta}=\frac{\frac{dr}{d\theta}\sin(\theta)+r\cos(\theta)}{\frac{dr}{d\theta}\cos(\theta)-r\sin(\theta)}$$

- Provided $dx/d\theta\neq 0$, we solve $dy/d\theta=0$ to find horizontal tangents
- Provided $dy/d\theta\neq 0$, we solve $dx/d\theta=0$ to find vertical tangents

Rmk: When $r=0$, if $dr/d\theta\neq 0$, then $*$ reduces to $dy/dx=\tan(\theta)$.
E.g. in $r=\cos(2\theta)$, $\theta=\pi/4$ and $\theta=3\pi/4$ are tangent to the origin.