Functions

Blake Farman [1]

[1]University of South Carolina, Columbia, SC USA

Math 122: Calculus for Business Administration and Social Sciences

Outline

1. 1.1: Functions
   - Graphs

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   - Graphs
2. 1.2: Linear Functions

Definition

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Notation: A function named f that takes as input the *independent variable*, x, and outputs the *dependent variable*, y, is written as

$$y = f(x).$$

Example (Discrete Function)

Given any two sets we can define a function.

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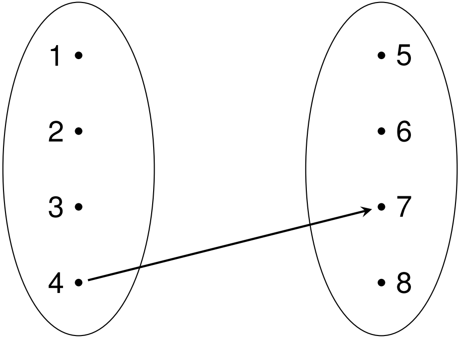

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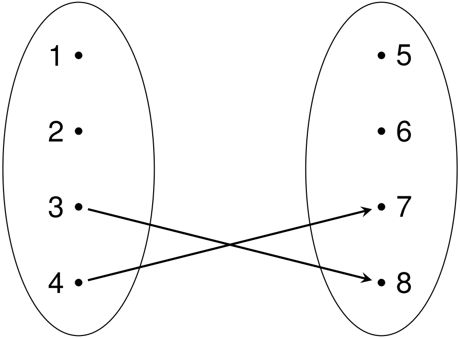

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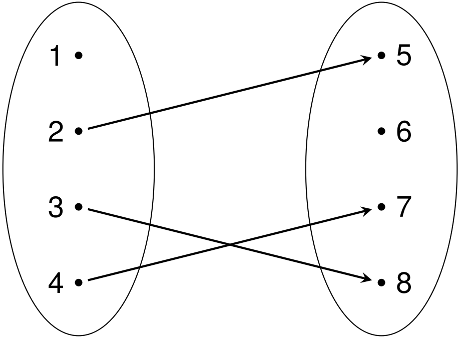

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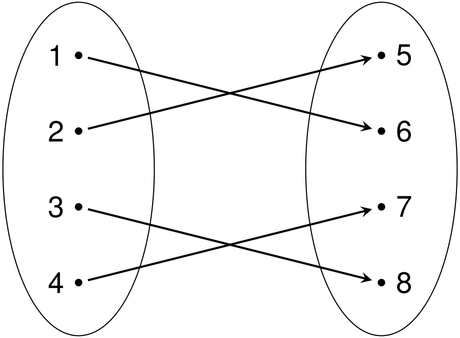

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- The domain of f is the set of all real numbers, $\mathbb{R}$.
- The range of f is the set of all non-negative real numbers,
$$\{x \in \mathbb{R} \mid 0 \leq x\}.$$

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The following depicts a non-function.

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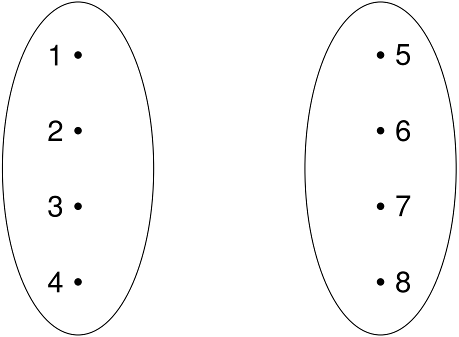

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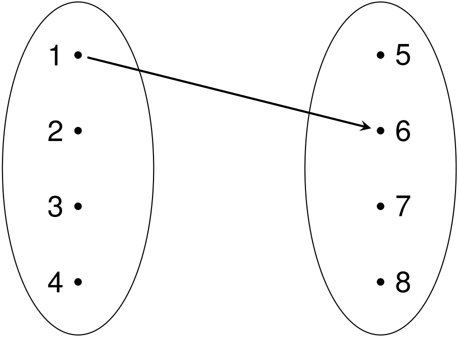

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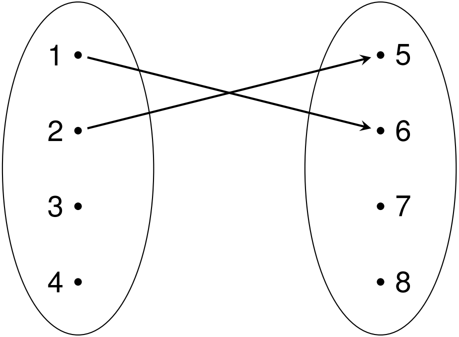

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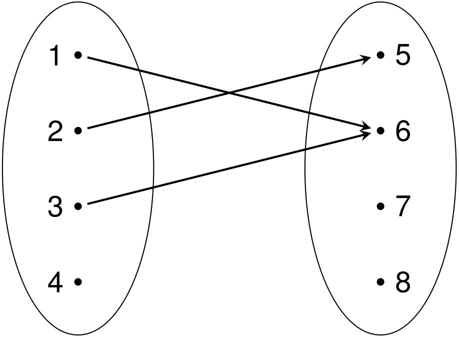

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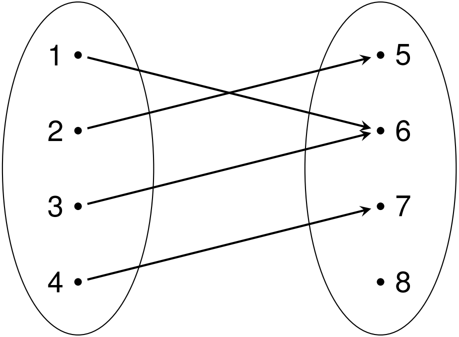

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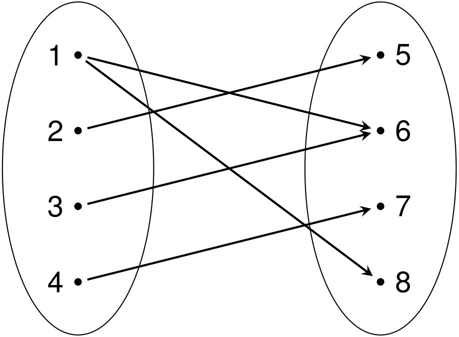

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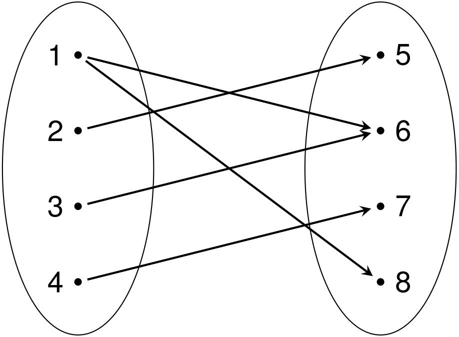

The value $f(1)$ is not well-defined because it requires a choice: it could be either 6 or 8.

Cartesian Plane

Recall that the *Cartesian plane* is the set of all pairs

$$\mathbb{R}^2 = \{(x, y) \mid x \in \mathbb{R}, y \in \mathbb{R}\}.$$

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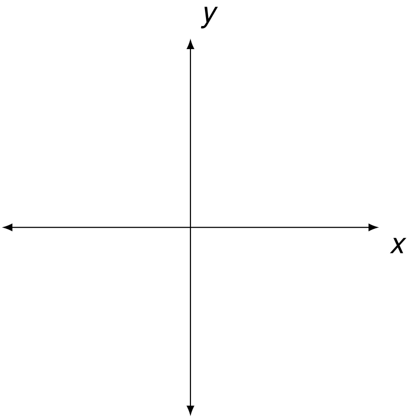

Graph of a Function

Definition 2

The graph of a real-valued function, f, with domain $D \subseteq \mathbb{R}$ is the set of pairs

$$\{(x, f(x)) \mid x \in D\} \subseteq \mathbb{R}^2.$$

It can be drawn on the Cartesian plane.

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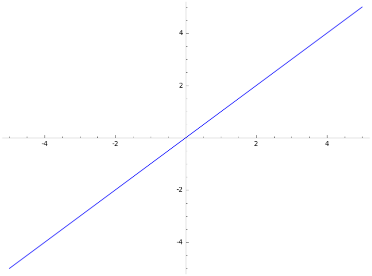

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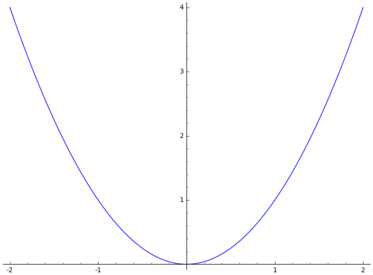
We say that f is increasing/decreasing if it is increasing/decreasing on its entire domain.

Example



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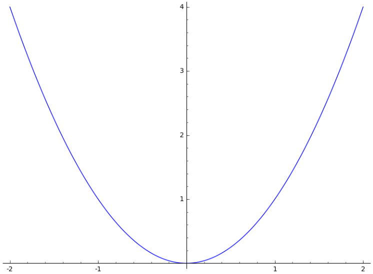

- Increasing on:
- Decreasing on:

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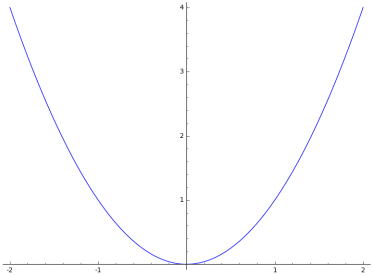

- Increasing on: $(0, \infty)$
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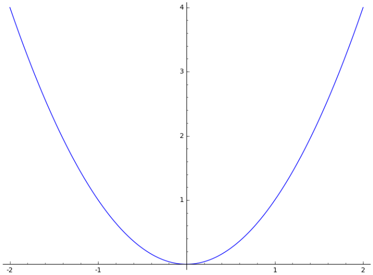

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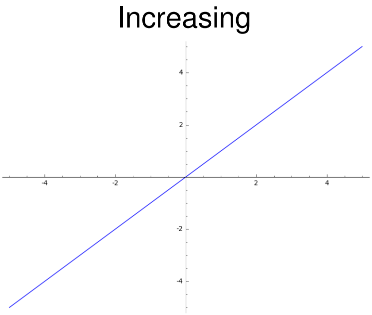

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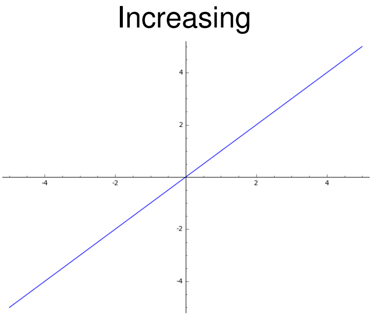

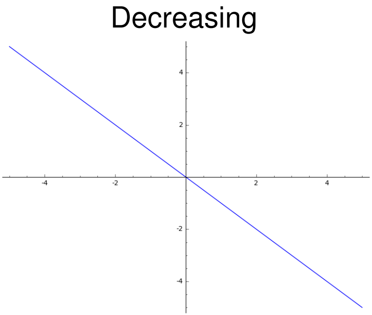

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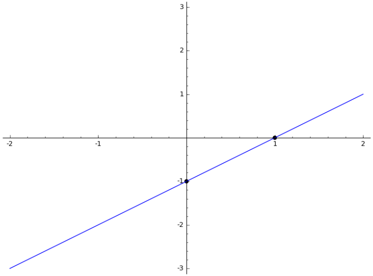

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- Linear functions have domain and range $\mathbb{R}$,
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- This form is usually called the *Slope-Intercept Form* of a line.

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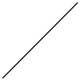

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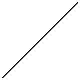

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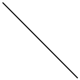

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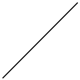

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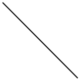

- Horizontal ($m = 0$):

POINT-SLOPE FORM

DEFINITION 6

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- a point, (x_0, y_0),
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the equation of the line through (x_0, y_0) with slope m is

$$y - y_0 = m(x - x_0).$$

Two Points Determine a Line

Given two points, (x_0, y_0) and (x_1, y_1), the slope of the line passing through them is

$$m = \frac{y_0 - y_1}{x_0 - x_1} = \frac{y_1 - y_0}{x_1 - x_0}.$$

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To see these are the same line, put them both into Slope-Intercept Form.

Two Points Determine a Line (Cont.)

$$y = mx - \frac{y_0 - y_1}{x_0 - x_1}x_0 + y_0$$

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Two Points Determine a Line (Cont.)

$$\begin{aligned} y &= mx - \frac{y_0 - y_1}{x_0 - x_1} x_0 + y_0 \\ &= mx + \frac{(y_1 - y_0)x_0 + (x_0 - x_1)y_0}{x_0 - x_1} \end{aligned}$$

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DEFINITION 7

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Difference Quotients

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This is just the slope of the line through $(x_0, f(x_0))$ and $(x_1, f(x_1))$. This line is usually called the *Secant Line*.

Let $f(x) = mx + b$.

Difference Quotients for Linear Functions

Let $f(x) = mx + b$. Given x_0 and x_1:

DIFFERENCE QUOTIENTS FOR LINEAR FUNCTIONS

Let $f(x) = mx + b$. Given x_0 and x_1:

$$\frac{f(x_1) - f(x_0)}{x_1 - x_0} = \frac{mx_1 + b - (mx_0 + b)}{x_1 - x_0}$$

Difference Quotients for Linear Functions

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&= \frac{m(x_1 - x_0)}{x_1 - x_0} \\
&= m
\end{aligned}$$

Hence for a linear function, the difference quotient is just the slope.

Let $f(x) = x^2$. For $x_0 = -1$, $x_1 = 2$:

Functions

Farman

1.1: Functions

Graphs

1.2: Linear Functions

Let $f(x) = x^2$. For $x_0 = -1$, $x_1 = 2$:

$$\frac{f(-1) - f(2)}{-1 - 2}$$

Difference Quotients for Non-Linear Functions

Let $f(x) = x^2$. For $x_0 = -1$, $x_1 = 2$:

$$\frac{f(-1) - f(2)}{-1 - 2} = \frac{(-1)^2 - 2^2}{-3}$$

Functions

Farman

1.1: Functions

Graphs

1.2: Linear Functions

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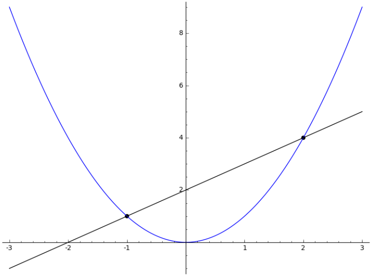
This is the slope of the secant line:



Difference Quotients for Non-Linear Functions (cont.)

For $x_0 = 0$, $x_1 = 2$:

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Difference Quotients for Non-Linear Functions (Cont.)

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