PROPORTIONALITY AND POWER FUNCTIONS

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Math 122: Calculus for Business Administration and Social Sciences

Outline

1. 1.9: Proportionality and Power Functions
   - Graphs of Power Functions
   - Quadratics

Proportionality

Definition 1

We say that y is *(directly) proportional* to x if there is a non-zero constant k such that

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Remark 1

This is a fancy way of saying the function $y(x) = kx$ is a line through the origin.

Example

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(C) Estimate the heart mass of a horse with body mass of 650 kg.

Example (Cont.)

$$H(650) \approx 0.006(650)$$

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$$H(650) \approx 0.006(650) = 3.9 \text{ kg}.$$

Power Function

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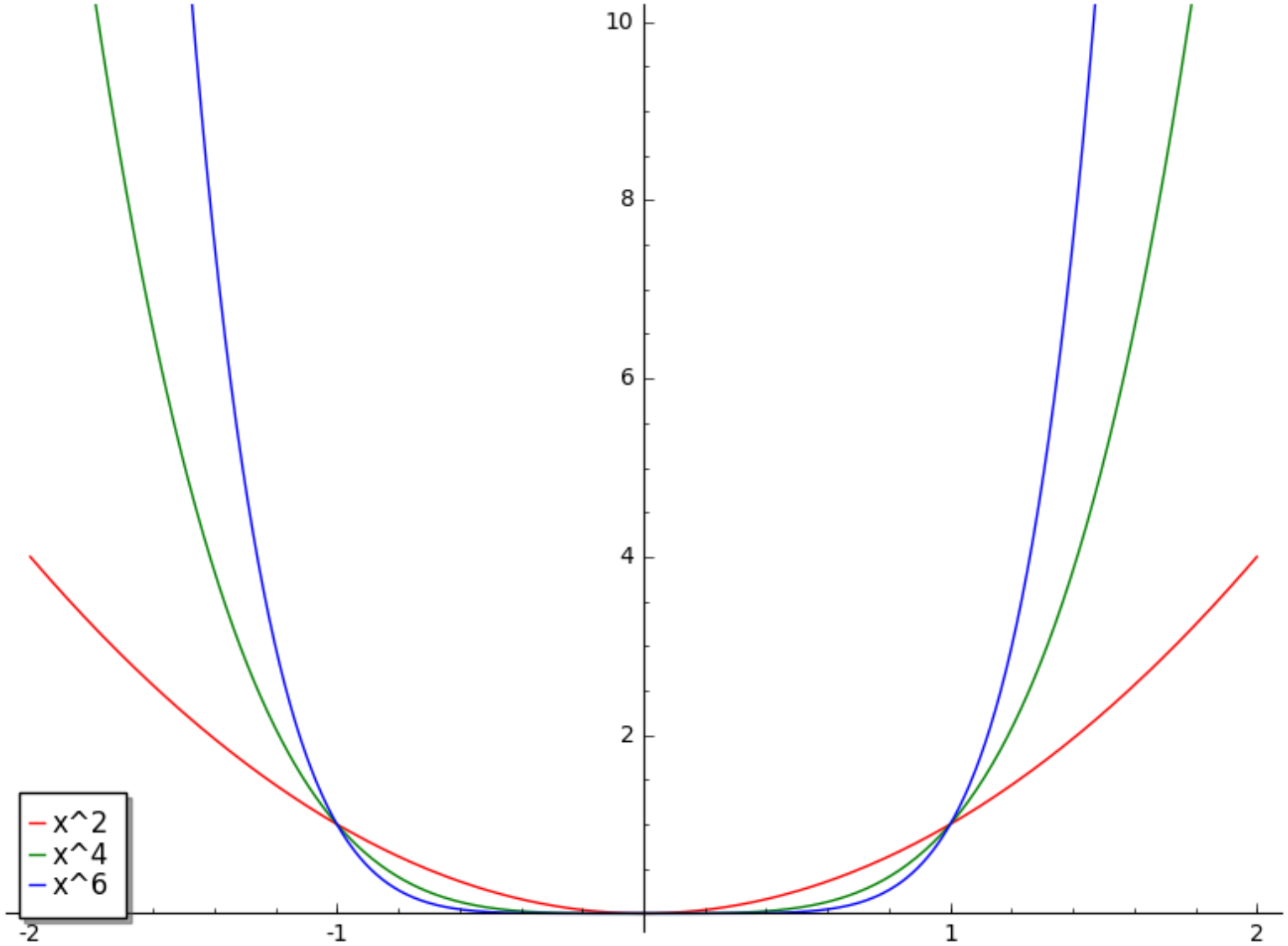
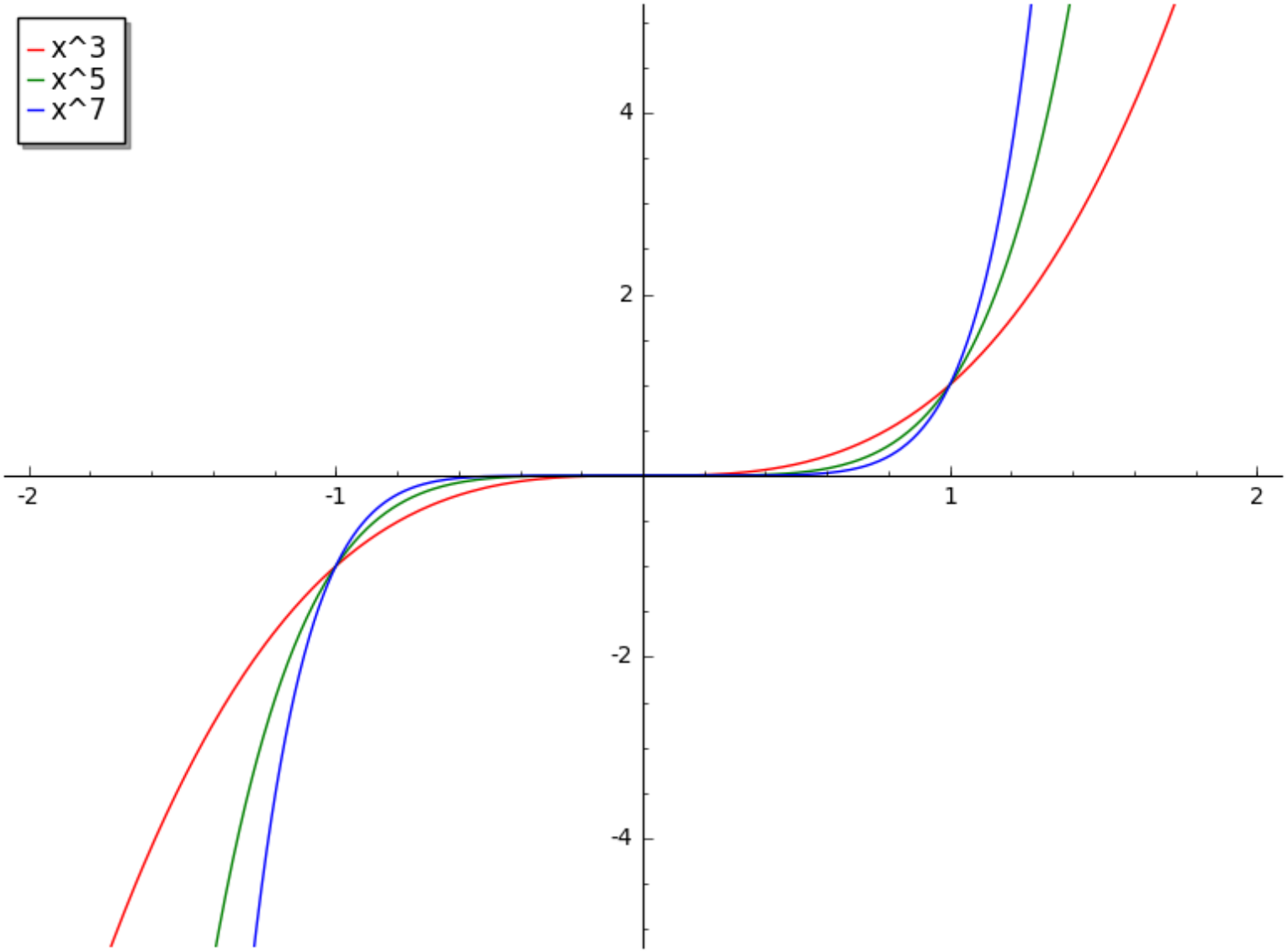
Remark 2

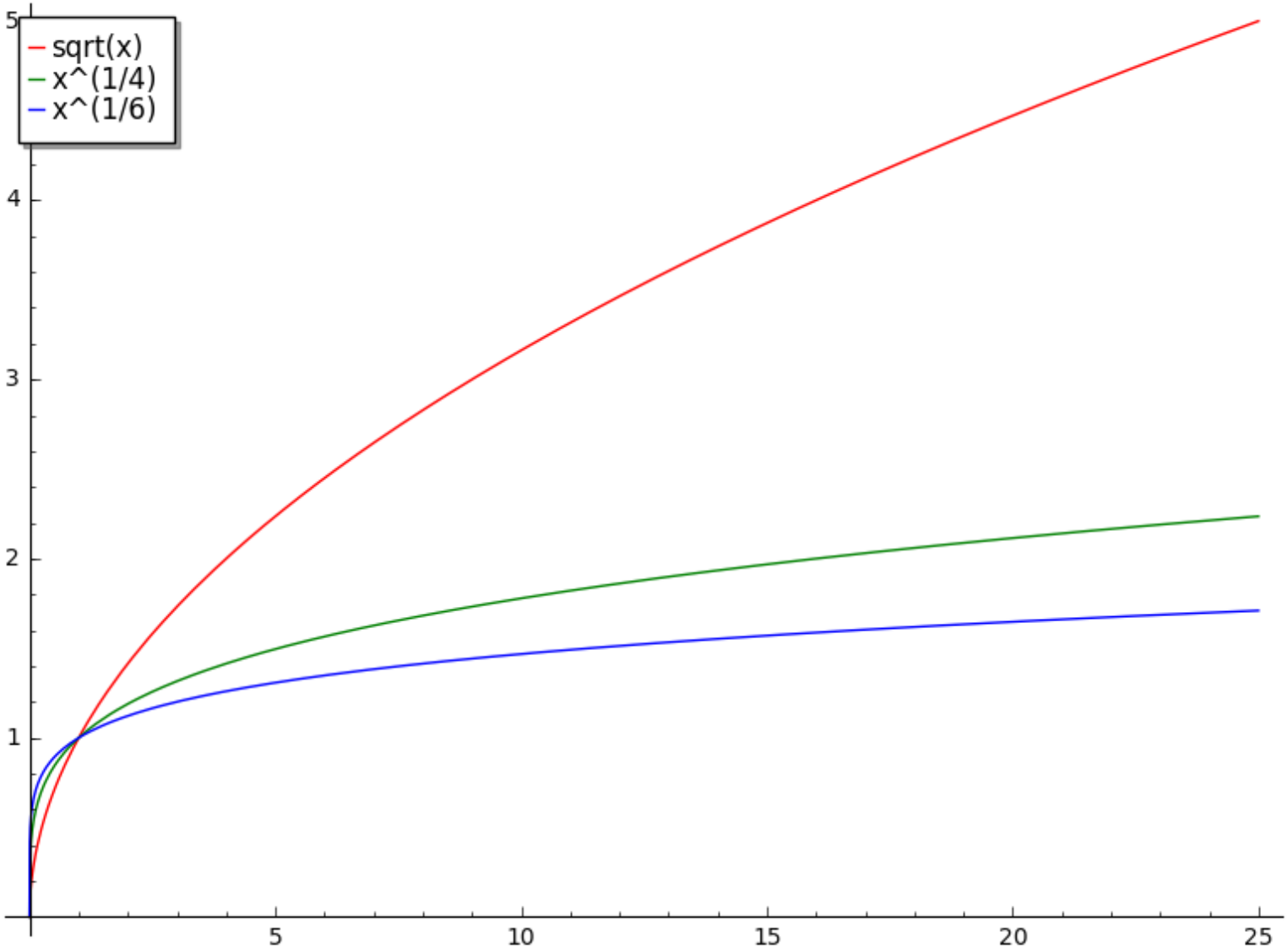
Generally, one calls this a *monomial* when $0 \leq p$.

x^{2n}

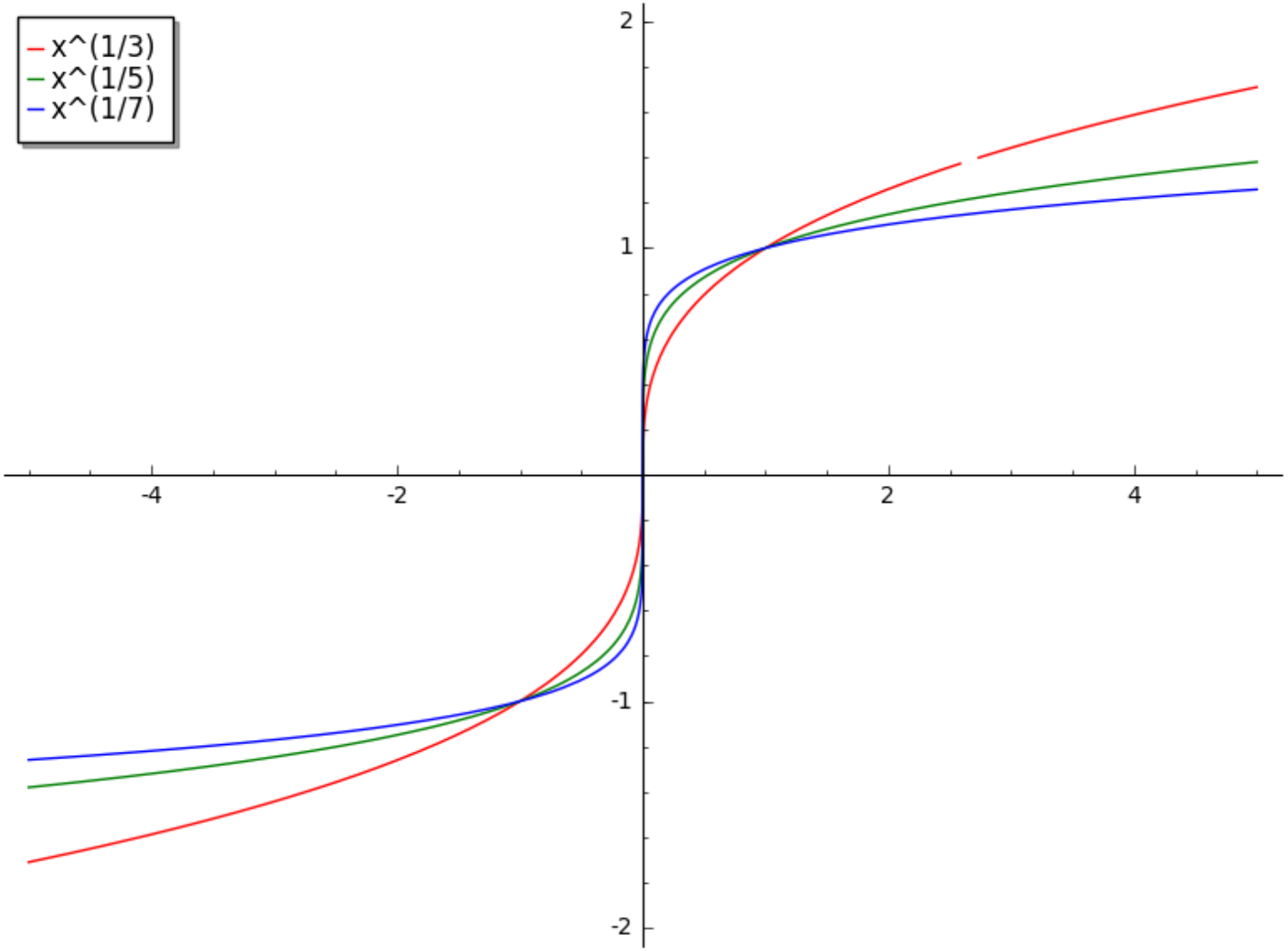


x^{2n+1}

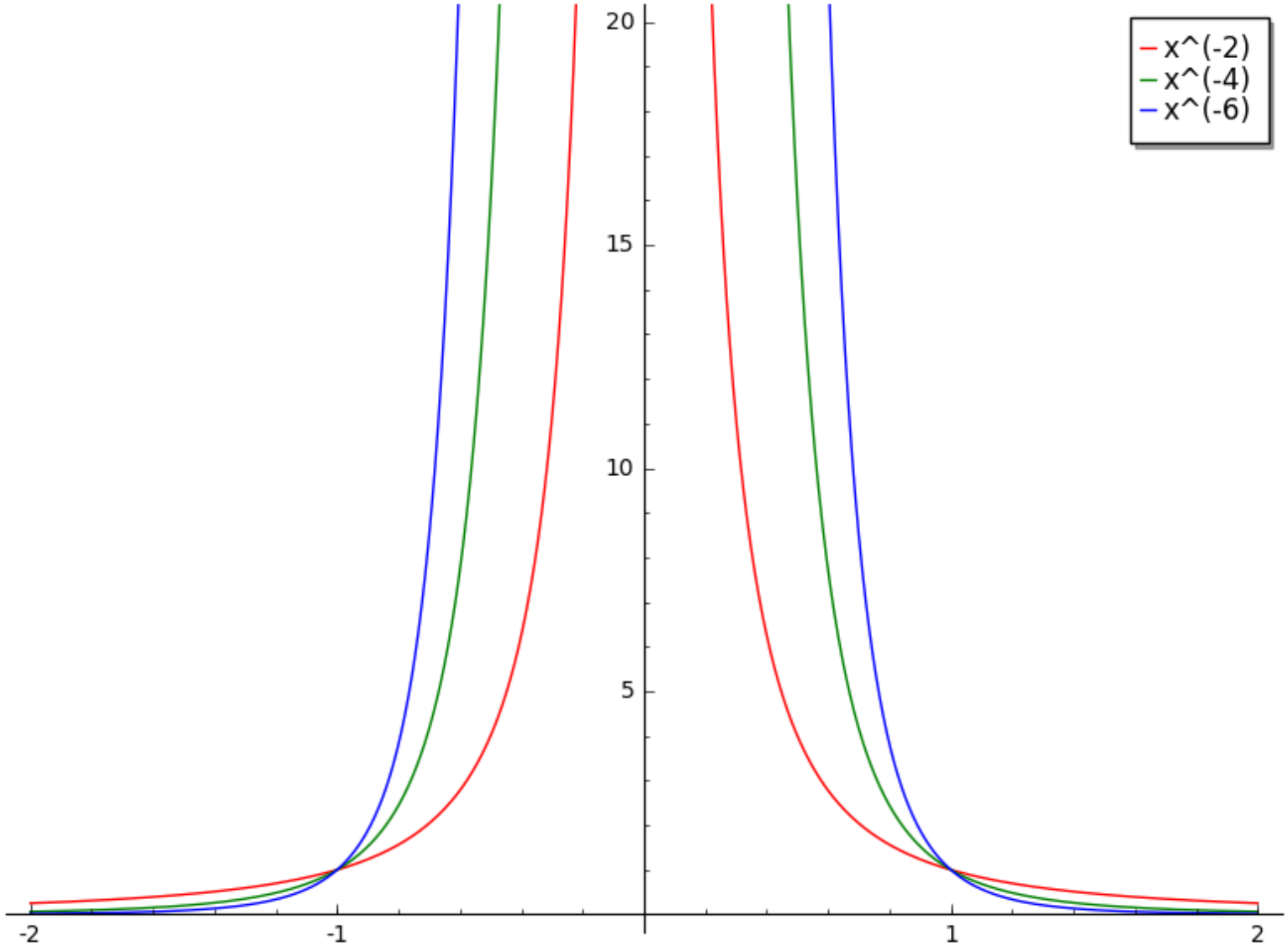



sqrt(x)
x^(1/4)
x^(1/6)
1
2
3
4
5
5
10
15
20
25

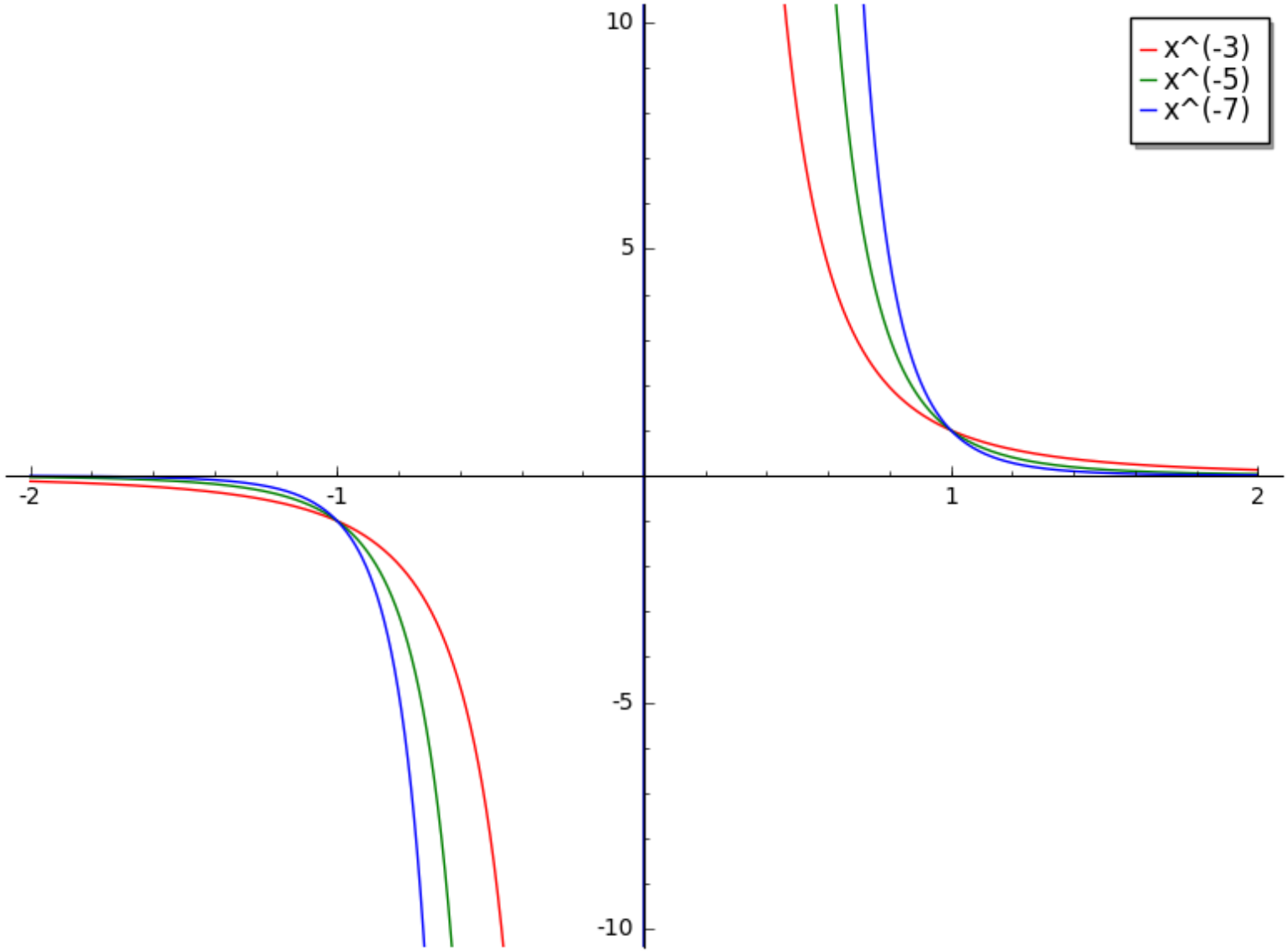
$\sqrt[2n+1]{x}$



x^{-2n}



$x^{-(2n+1)}$



Polynomials

Definition 3

Sums of power functions with non-negative integer exponents

$$a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0, a_n \neq 0$$

are *polynomials of degree n*.

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Remark 3

When the $n = 2$, one calls the polynomial a *quadratic*

Vertex Form

Any quadratic, $f(x) = ax^2 + bx + c$, can be written in vertex form

$$f(x) = a(x - h)^2 + k,$$

where (h, k) is the vertex of the parabola.

Vertex Form (Cont.)

$$ax^2 + bx + c = a\left(x^2 + \frac{b}{a}x\right) + c$$

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$$\begin{aligned} ax^2 + bx + c &= a\left(x^2 + \frac{b}{a}x\right) + c \\ &= a\left(x^2 + \frac{b}{a} + \left(\frac{b}{2a}\right)^2 - \left(\frac{b}{2a}\right)^2\right) + c \end{aligned}$$

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$$
\begin{aligned}
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&= a\left(\left(x + \frac{b}{2a}\right)^2 - \left(\frac{b}{2a}\right)^2\right) + \frac{4ac}{4a} \\
&= a\left(x - \frac{-b}{2a}\right)^2 + \frac{4ac - b^2}{4a}
\end{aligned}
$$

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So $h=-b/2a$ and $k=f(h)=(4ac-b^2)/4a$.

The Quadratic Formula

The solutions to the quadratic equation

$$ax^2 + bx + c = 0$$

are given by

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

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3. Shift horizontally by h.
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5. Translate vertically by k.

Example

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Determine the price that will generate the greatest revenue assuming the number of people attending a concert is a linear function of the price.

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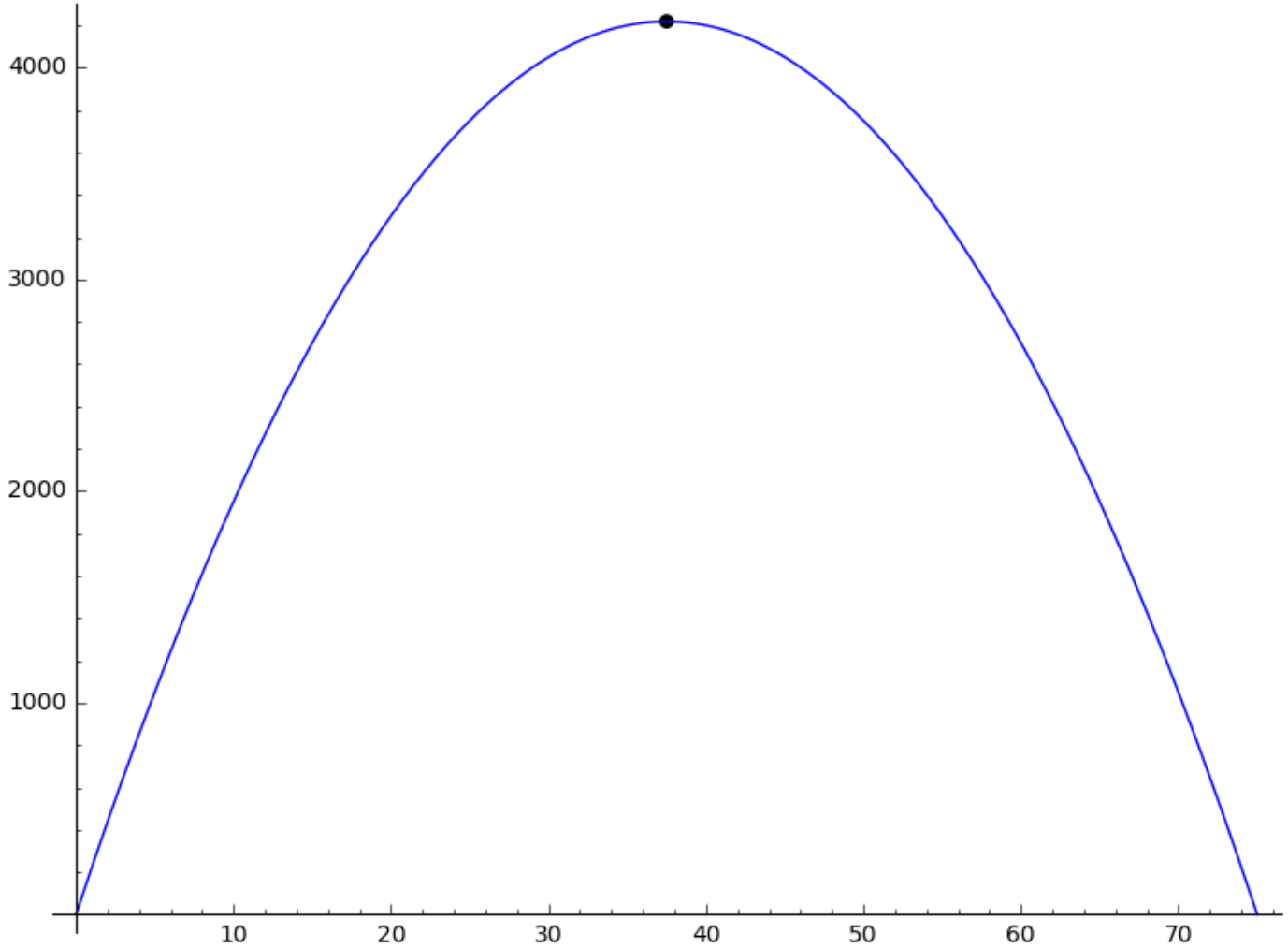
$$\begin{aligned} R(37.5) &= -3(37.5)(37.5 - 75) \\ &= -3(37.5)(-37.5) \\ &= 3(37.5)^2 \\ &= 4,218.75. \end{aligned}$$

It's clear from the graph that this is the maximum revenue: