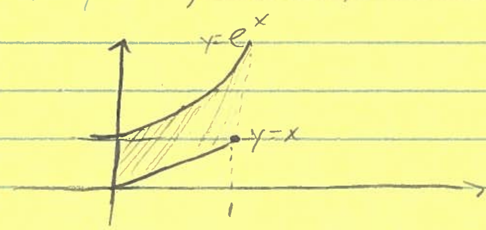
6.1 Areas Between Curves

The area A of the region bounded by the curves $y=f(x)$, $y=g(x)$, $x=a$, $x=b$, where f and g are continuous and $f(x)\geq g(x)$ for all $x\in[a,b]$ is

$$A=\int_a^b (f(x)-g(x))\,dx$$

E.g.: ① $y=e^x$, $y=x$, $x\in(0,1)$

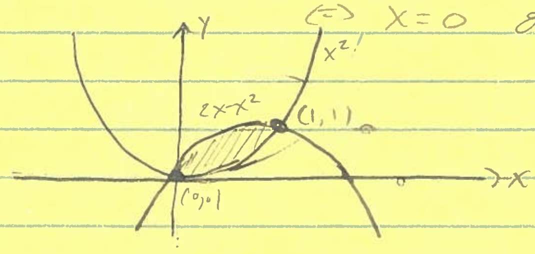


$$\int_0^1 (e^x-x)\,dx=\int_0^1 e^x\,dx-\int_0^1 x\,dx$$
$$=(e-1)-\tfrac{1}{2}(1^2-0^2)$$
$$=e-1-\tfrac{1}{2}$$
$$=\boxed{e-3/2}$$

② $y=x^2$, $y=2x-x^2$

$2x-x^2=0$
$x^2-2x=0$
$x(x-2)=0$

$$x^2=2x-x^2 \iff 2x^2-2x=0$$
$$\iff 2x(x-1)=0$$
$$\iff x=0 \text{ or } x=1$$



$$\int_0^1 ((2x-x^2)-x^2)\,dx$$
$$=2\int_0^1 x\,dx-2\int_0^1 x^2\,dx$$
$$=2(\tfrac{1}{2})x^2\Big|_0^1-2(\tfrac{1}{3})x^3\Big|_0^1$$
$$=1-2/3$$
$$=\boxed{1/3.}$$

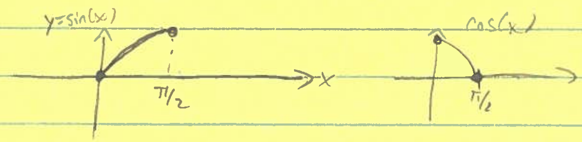
The area between f,g, $x=a$, $x=b$ is

$$\int_a^b |f-g|\,dx$$

Pf: $|f-g|=\begin{cases} f-g & f\geq g,\\ g-f & f\leq g.\end{cases}$ ▨

Eg.: 5 $y=\sin(x)$, $y=\cos(x)$, $x=0$, $x=\pi/2$

$\sin(\pi/4)=\cos(\pi/4)$



$$A = \int_0^{\pi/4} (\cos(x)-\sin(x))\,dx + \int_{\pi/4}^{\pi/2} \sin(x)-\cos(x)\,dx$$

$$= \sin(x)\Big|_0^{\pi/4} + \cos(x)\Big|_0^{\pi/4} - \cos(x)\Big|_{\pi/4}^{\pi/2} - \sin(x)\Big|_{\pi/4}^{\pi/2}$$

$$= \sin(\pi/4)+\cos(\pi/4)-1+\cos(\pi/4)-1+\sin(\pi/4)$$

$$= 4\sin(\pi/4)-2$$

$$= 4\left(\frac{\sqrt{2}}{2}\right)-2$$

$$= \boxed{2\sqrt{2}-2}$$

6 $y=x-1$, $y^2=2x+6$

$x=y+1 \quad x=\frac{y^2-6}{2}=\frac{1}{2}y^2-3$

$$y+1=\tfrac{1}{2}y^2-3 \iff 2y+2=y^2-6$$
$$\iff y^2-2y-8=0$$
$$\iff (y-4)(y+2)=0$$
$$\iff y=2,\ y=4$$

$$\int_2^4 \left((y+1)-\left(\tfrac{1}{2}y^2-3\right)\right)dy = -\frac{1}{2}\int_{-2}^4 y^2\,dy + 4\int_{-2}^4 dy + \int_{-2}^4 y\,dy$$

$$= -\frac{1}{2}\left(\frac{1}{3}\right)y^3\Big|_{-2}^4 + 4y\Big|_{-2}^4 + \frac{1}{2}y^2\Big|_{-2}^4$$

$$= -\frac{1}{6}(64+8)+4(6)+\frac{1}{2}(16-4)$$

$$= -\frac{1}{6}(72)+24+\frac{1}{2}(12)$$

$$= -12+30$$

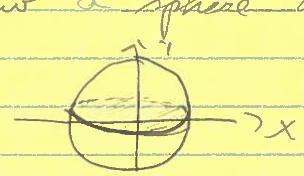
$$= 18.$$

6.2 Volumes

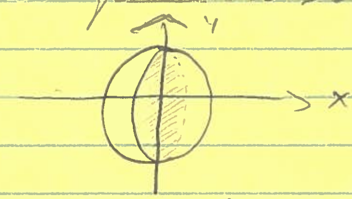
Def^n: Let S be a solid that lies between $x=a$ and $x=b$. If the cross sectional area of S in a plane P_x, through x and perpendicular to the x-axis, is $A(x)$, where A is a continuous function, then the volume of S is

$$V(S) = \lim_{n\to\infty} \sum_{i=1}^{n} A(x_i^*)\,\Delta x = \int_a^b A(x)\,dx.$$

Rmk: ① Read this section for homework see the pictures, etc.
② Draw a sphere with discs for cross sections



or



E.g. ① Show that the volume of a sphere of radius r is $V = \frac{4}{3}\pi r^3$

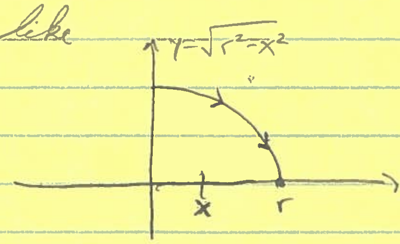
Center the sphere at the origin. Trace the curve

$$x^2+y^2=r^2, \quad x \in [0,r].$$

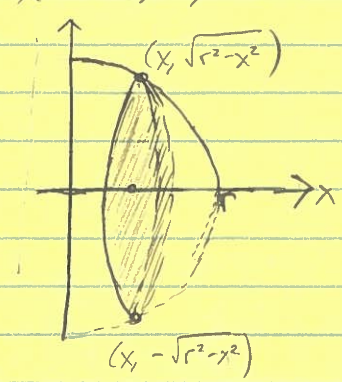
This gives, in the first quadrant,

$$y=\sqrt{r^2-x^2}$$

and looks like



At each point $x \in [0,r]$, we draw a disc of radius $y=\sqrt{r^2-x^2}$ about x



We're adding together these discs of infinitessimal width, and eventually they form by "stacking" them along the y-axis a hemisphere

We measure the "sum" of these discs by using the area function

$$A(x) = \pi y^2 = \pi\left(\sqrt{r^2-x^2}\right)^2 = \pi(r^2-x^2)$$

over finer and finer partitions,

$$V(\text{hemisphere}) = \lim_{n\to\infty} \sum_{i=0}^{n} A(x_i^*)\Delta x$$

$$= \int_0^r A(x)\,dx$$

$$= \pi\int_0^r (r^2-x^2)\,dx$$

$$= \pi\, r^2(r) - \pi\,\tfrac{1}{3}(r^3)$$

$$= \pi\left(r^3 - \tfrac{1}{3}r^3\right)$$

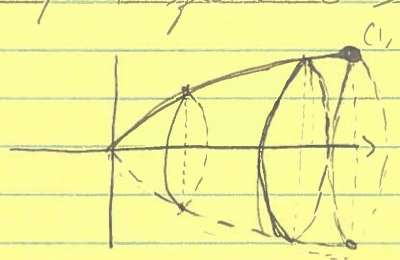
$$= \pi r^3\left(1-\tfrac{1}{3}\right)$$

$$= \tfrac{2}{3}\pi r^3.$$

We get the volume of the whole sphere by counting this one twice,

$$V(S) = 2\int_0^r A(x)\,dx = \tfrac{4}{3}\pi r^3. \quad \blacksquare$$

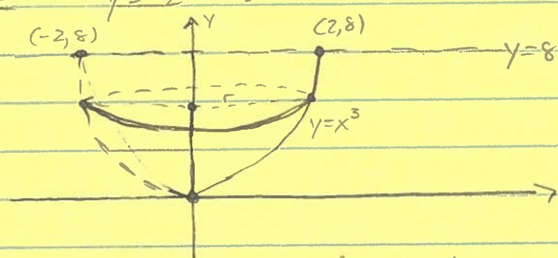
② Find the volume of the solid obtained by rotating about the x-axis the region under the curve $y=\sqrt{x}$ from 0 to 1.



$$V = \int_0^1 \pi\, y(x)^2\,dx = \pi\int_0^1 x\,dx = \pi\cdot\tfrac{1}{2}x^2\Big|_0^1 = \pi/2$$

③ Find the volume of the solid obtained by rotating the region bounded by $y=x^3$, $y=8$, and $x=0$ about the y-axis.

$x=2 \quad y=2^3=8$

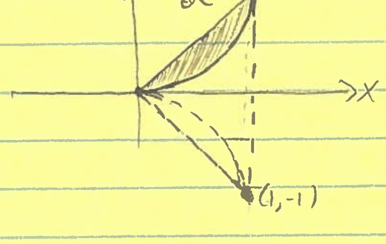


We want discs of radius $r(y)=\sqrt[3]{y}$ for $0\le y\le 8$.

Rmk: This works well because on $0\le y\le 8$, $x=\sqrt[3]{y}$ is a well-defined function.

$$\int_0^8 \pi\, r(y)^2\,dy = \pi\int_0^8 \sqrt[3]{y^2}\,dy = \pi\int_0^8 y^{2/3}\,dy$$
$$= \tfrac{3}{5}\pi\, y^{5/3}\Big|_0^8$$
$$= \tfrac{3}{5}\pi\left(\sqrt[3]{8}\right)^5$$
$$= \tfrac{3}{5}\pi(32)$$
$$= \boxed{96\pi/5}$$

④ The region R enclosed by the curves $y=x$ and $y=x^2$ is rotated about the x-axis. Find the volume of the resulting solid.



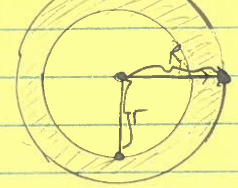
There are two radii to consider here. The large radius,

$R(x)=x$

and the small

$r(x)=x^2$.

The small radius gives a small disc that should be removed from the larger in each cross section



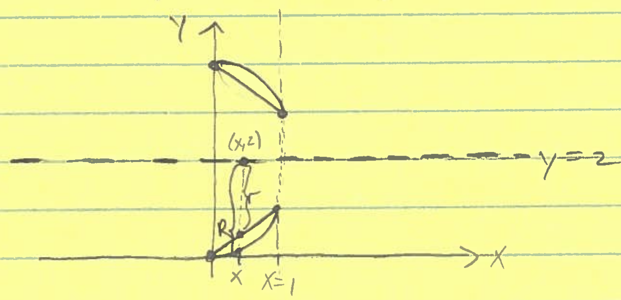
So the area of each cross section is the shaded region

$$A(x) = \pi R(x)^2 - \pi r(x)^2$$
$$= \pi x^2 - \pi x^4$$

Therefore, the volume we desire is

$$V = \int_0^1 A(x)\,dx$$
$$= \pi\left(\int_0^1 x^2\,dx - \int_0^1 x^4\,dx\right)$$
$$= \pi\left(\tfrac{1}{3}(1^2 - 0^2) - \tfrac{1}{5}(1^5 - 0^5)\right)$$
$$= \pi\left(\frac{5-3}{15}\right)$$
$$= \boxed{2\pi/15.}$$

(5) Find the volume of the solid obtained by rotating the previous region about the line $y = 2$.

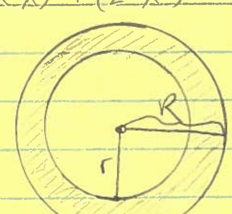


At each point $0 \le x \le 1$, we draw two discs centered about the point $(x,2)$, rather than $(x,0)$. The inner radius is then the distance from $(x,2)$ to (x,x), which is

$$r(x) = \sqrt{(x-x)^2 + (2-x)^2} = \sqrt{0 + (2-x)^2} = 2 - x.$$

The outer radius is the distance from $(x,2)$ to (x,x^2)

$$R(x) = \sqrt{(x-x)^2 + (2-x^2)^2} = 2 - x^2.$$



So the area of the shaded region is

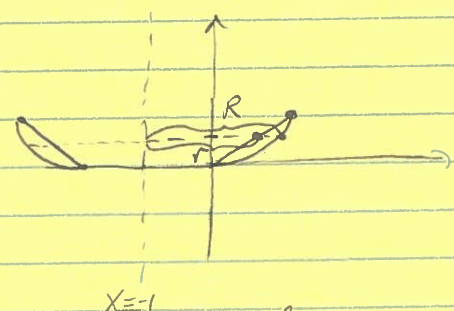
$$A(x) = \pi R(x)^2 - \pi r(x)^2$$
$$= \pi\left((2-x^2)^2 - (2-x)^2\right)$$
$$= \pi\left(4 - 4x^2 + x^4 - (4 - 4x + x^2)\right)$$
$$= \pi\left(\cancel{4-4} - 4x^2 - x^2 + 4x + x^4\right)$$
$$= \pi\left(x^4 - 5x^2 + 4x\right)$$

and the volume of the solid is

$$V = \int_0^1 A(x)\,dx = \pi\left(\int_0^1 x^4\,dx - 5\int_0^1 x^2\,dx + 4\int_0^1 x\,dx\right)$$
$$= \pi\left(\tfrac{1}{5} - \tfrac{5}{3} + 2\right)$$
$$= \pi\left(\tfrac{3}{15} - \tfrac{25}{15} + \tfrac{30}{15}\right)$$
$$= \boxed{\pi\left(\tfrac{8}{15}\right)}.$$

Rmk: These are called solids of revolution

② Find the volume of the solid by revolving around the line $x = -1$.



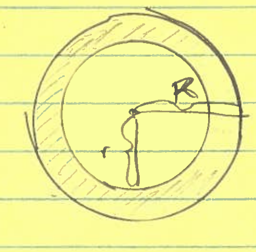
We must translate from integration w.r.t. x to integration w.r.t. to y. The outer radius is given by

$$R(y) = \left|(-1, y) - (\sqrt{y}, y)\right|$$
$$= \left|(-\sqrt{y} - 1, 0)\right|$$
$$= \sqrt{(-\sqrt{y} - 1)^2 + 0^2}$$
$$= \sqrt{y} + 1.$$

The inner radius is

$$r(y) = \left|(-1, y) - (y, y)\right|$$
$$= \left|(-1 - y, 0)\right|$$
$$= \sqrt{(-1-y)^2 + 0^2}$$
$$= \sqrt{(1+y)^2}$$
$$= 1 + y.$$

Then the area of the shaded region



is given by

$$A(y) = \pi\left(R(y)^2 - r(y)^2\right)$$
$$= \pi\left((\sqrt{y}+1)^2 - (y+1)^2\right)$$
$$= \pi\left(y + 2\sqrt{y} + 1 - y^2 - 2y - 1\right)$$
$$= \pi\left(-y^2 - y + 2\sqrt{y}\right)$$

and so

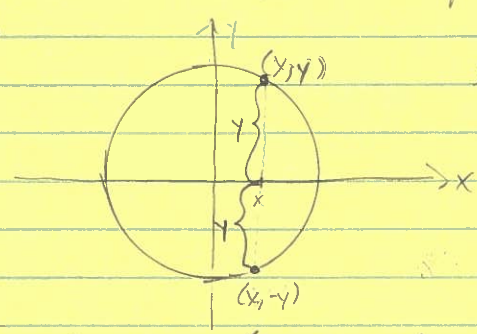
$$V = \int_0^1 A(y)\,dy = \pi\left(-\int_0^1 y^2\,dy - \int_0^1 y\,dy + 2\int_0^1 \sqrt{y}\,dy\right)$$
$$= \pi\left(-\tfrac{1}{3} - \tfrac{1}{2} + \tfrac{4}{3}\right)$$
$$= \boxed{\pi/2}$$

Not Solids of Revolution

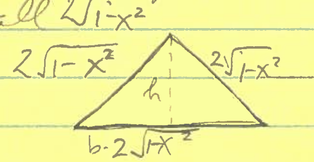
7) Consider a solid with base a circle of unit radius. Parallel cross sections perpendicular to the base are equilateral triangles. Find the volume.

Consider the cross section perpendicular to the x-axis

$x^2+y^2=1 \Rightarrow y=\sqrt{1-x^2}$



The base of the triangle formed along this line has length $2\sqrt{1-x^2}$, by hypothesis its sides are all $2\sqrt{1-x^2}$



We observe that $h^2 + (\sqrt{1-x^2})^2 = (2\sqrt{1-x^2})^2$

$$\Rightarrow h^2 = 4(1-x^2) - (1-x^2)$$
$$= 3(1-x^2)$$
$$\Rightarrow h = \sqrt{3}\,\sqrt{1-x^2}.$$

Then the area of the triangle, as a function of x is

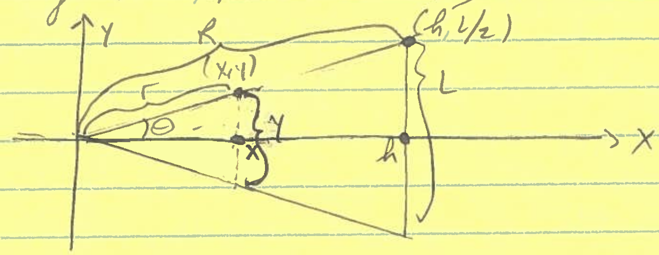
$$A(x) = \tfrac{1}{2}\, b(x)\, h(x)$$
$$= \left(\sqrt{1-x^2}\right)\cdot\sqrt{3}\left(\sqrt{1-x^2}\right)$$
$$= \sqrt{3}\,(1-x^2).$$

~~So~~ we have by symmetry

$$V = \int_{-1}^{1} A(x)\,dx = 2\int_0^1 A(x)\,dx$$
$$= 2\int_0^1 \sqrt{3}\,(1-x^2)\,dx$$
$$= 2\sqrt{3}\left(\int_0^1 dx - \int_0^1 x^2\,dx\right)$$
$$= 2\sqrt{3}\left(1 - \tfrac{1}{3}\right)$$
$$= 2\sqrt{3}\,(2/3)$$
$$= \boxed{4\sqrt{3}/3}$$

(8) Find the volume of a pyramid whose base is a square with side L and whose height is h.

For a given x, consider the cross-section



From trigonometry, we know that

$$x = r\cos(\theta),\quad y = r\sin(\theta)$$
$$h = R\cos(\theta),\quad L/2 = R\sin(\theta)$$

so we have

$$x/h = r\cos(\theta)/R\cos(\theta) = r/R, \text{ and}$$
$$\frac{y}{L} = \frac{y/2}{(L/2)} = r\sin(\theta)/R\sin(\theta) = r/R,$$

so

$$x/h = y/L\,,$$

from which it follows that

$$y(x) = x\,\frac{L}{h}$$

and thus the area of the square cross sections is

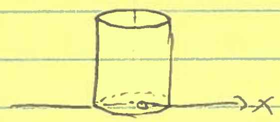
$$A(x) = y(x)^2 = x^2 L^2/h^2.$$

Therefore the volume is, as a function of L and h,

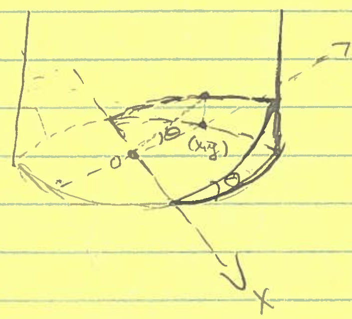
$$V(L,h) = \int_0^h A(x)\,dx = \int_0^h x^2 \frac{L^2}{h^2}\,dx = \frac{L^2}{h^2}\,\frac{1}{3}(h^3) = \boxed{\frac{1}{3}L^2 h}$$

9. A wedge is cut out of a circular cylinder of radius 4 by two planes. One plane is perpendicular to the axis of the cylinder. The other intersects the first at an angle of 30° along a diameter of the cylinder. Find the volume of the wedge.

Place the cylinder so the center of the base is at the origin

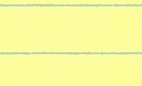


so that the planes meet along the x-axis.



This way, the base of the solid is a semicircle of radius 4; given x, the point on the base of the cylinder, (x,y), satisfies

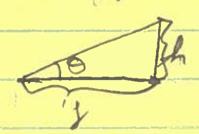
$$x^2+y^2=16 \iff y=\sqrt{16-x^2}.$$



$\tan(\theta) = \frac{h}{y}$

In this way a cross section of the wedge is a triangle with angle $\theta = 30° = \pi/6$, base length $y=\sqrt{16-x^2}$, and height given by

$$h(x) = y(x)\tan(\theta) = \sqrt{16-x^2}\,\tan(\theta)!$$



$\tan(\theta) = h/y \iff y = \tan(\theta).$

So we see that the area of a cross-section is given by

$$A(x) = \tfrac{1}{2}\, y(x)\, h(x)$$
$$= \tfrac{1}{2}\, y(x)^2 \tan(\pi/6)$$
$$= \tfrac{1}{2}\, y(x)^2 \left(\tfrac{1}{\sqrt{3}}\right)$$
$$= \frac{16 - x^2}{2\sqrt{3}}$$

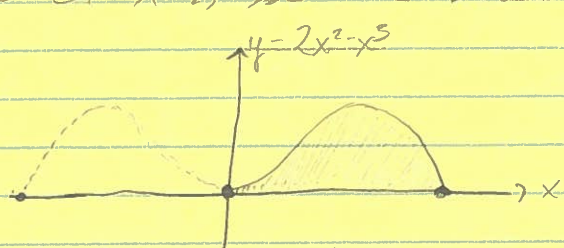
Therefore, the volume of the wedge is, by symmetry,

$$V = \int_{-4}^{4} A(x)\,dx$$
$$= 2\int_{0}^{4} A(x)\,dx$$
$$= 2\int_{0}^{4} \frac{16 - x^2}{2\sqrt{3}}\,dx$$
$$= \frac{1}{\sqrt{3}}\left(\int_0^4 16\,dx - \int_0^4 x^2\,dx\right)$$
$$= \frac{16}{\sqrt{3}}(4-0) - \frac{1}{3\sqrt{3}}(4^3 - 0)$$
$$= \frac{3(64) - 64}{3\sqrt{3}}$$
$$= \boxed{\frac{128}{3\sqrt{3}}}$$

6.3 Volume by Cylindrical Shells

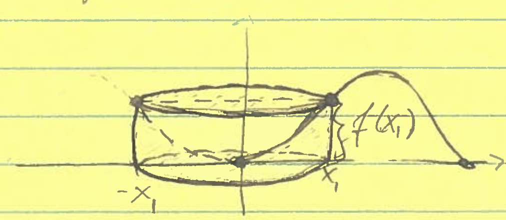
Consider the volume of the solid obtained by rotating about the y-axis the region bounded by $y = 2x^2 - x^3$ and $y = 0$

We know $y = 2x^2 - x^3 = x^2(2-x) = 0$ if and only if $x = 0$ or $x = 2$, so we have the region



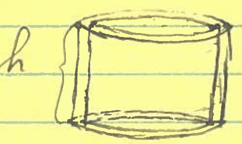
To rotate about the y-axis, we would need to solve for x in terms of y - not easy.

For such a function f, consider the cylinder below



right cylinder
of height h (say $h = f(x_1)$)
radius x_1

Given a second value, $x_2 < x_1$, we can consider the cylinder of radius x_2 and height h as sitting inside the original cylinder



If we remove the second cylinder from the first then we see we have a cylindrical "shell" of volume

$$V = \pi x_1^2 h - \pi x_2^2 h$$
$$= \pi h (x_1 + x_2)(x_1 - x_2)$$

Observe that if we rewrite this as

$$V = 2\pi h \left(\frac{x_1 + x_2}{2}\right)(x_1 - x_2)$$

The value $x_1 - x_2$ is the length of the interval $[x_2, x_1]$, $\frac{1}{2}(x_1 + x_2)$ is the midpoint, and $f(\frac{1}{2}(x_1 + x_2))$ is the function value at that point. We can think of this as the volume of a cylindrical shell with height h, and "thickness" $(x_1 - x_2)$.

If we partition interval (x-axis) where the boundary is defined, say on $[a, b]$, as

$$x_0 = a < x_1 < \cdots < x_n = b$$

and on each interval $[x_i, x_{i+1}]$ construct this shell, we obtain an approximation to the volume of this solid of revolution

$$V \approx \sum_{k=0}^{n-1} 2\pi f\left(\tfrac{1}{2}(x_i + x_{i+1})\right)(x_{i+1} - x_i)$$

If we let the size of the interval go to zero as in the definition of the Riemann integral, then we obtain shells of infinitesimal thickness and the sum ($\Delta x = x_{i+1} - x_i$) becomes an integral.

$$\int_a^b 2\pi x f(x)\,dx = \lim_{n\to\infty} \sum_{k=0}^{n} 2\pi \tfrac{1}{2}(x_i+x_{i+1}) f\left(\tfrac{1}{2}(x_i+x_{i+1})\right)(x_{i+1}-x_i)$$

Rmk: Think of the $2\pi x$ as the circumference of the shell, the $f(x)$ as the height, and dx as the thickness.

Eg: ① $y = 2x^2 - x^3$ and $y=0$

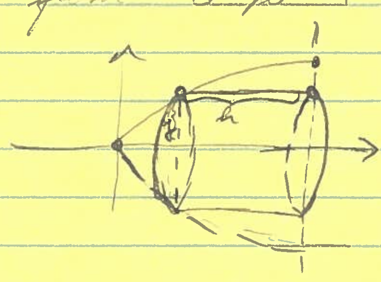
$$V = \int_0^2 2\pi x(2x^2-x^3)\,dx = 2\pi\left(2\int_0^2 x^3\,dx - \int_0^2 x^4\,dx\right)$$
$$= 2\pi\left(2\cdot\tfrac{1}{4}(16) - \tfrac{1}{5}(32)\right)$$
$$= 2\pi\left(8/5\right)$$
$$= \boxed{16\pi/5.}$$

② $y = x$ and $y = x^2$ about the y-axis

Here, the height of the shells will $x - x^2$, so

$$V = \int_0^1 2\pi x(x-x^2)\,dx$$
$$= 2\pi\left(\int_0^1 x^2\,dx - \int_0^1 x^3\,dx\right)$$
$$= 2\pi\left(\tfrac{1}{3} - \tfrac{1}{4}\right)$$
$$= \frac{2\pi}{12} = \boxed{\pi/6.}$$

③ $y = \sqrt{x}$ from 0 to 1 about the x-axis.



$x = y^2 \Leftrightarrow y = \sqrt{x}$ on $[0,1]\times[0,1]$

$h = 1 - x$
$= 1 - y^2$

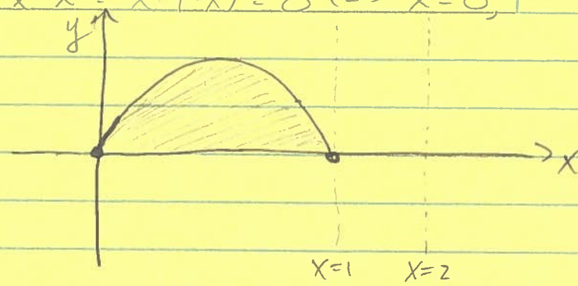
radius: y

thickness: dy

$$\int_0^1 2\pi y(1-y^2)\,dy = 2\pi\left(\int_0^1 y\,dy - \int_0^1 y^3\,dy\right)$$
$$= 2\pi\left(\tfrac{1}{2} - \tfrac{1}{4}\right)$$
$$= 2\pi/4 = \boxed{\pi/2}.$$

4. $y = x - x^2,\ y = 0$ about $x = 2$

$y = x - x^2 = x(1-x) = 0 \iff x = 0, 1$



radius: $2-x$

height: $x - x^2$

$$2\pi \int_0^1 (2-x)(x-x^2)\,dx = 2\pi\left(\int_0^1 (x^3 - 3x^2 + 2x)\,dx\right)$$

$$= 2\pi\left(\int_0^1 x^3 dx - 3\int_0^1 x^2 dx + 2\int_0^1 x\,dx\right)$$

$$= 2\pi\left(\tfrac{1}{4} - 3\left(\tfrac{1}{3}\right) + 2\left(\tfrac{1}{2}\right)\right)$$

$$= 2\pi\left(\tfrac{1}{4} - 1 + 1\right)$$

$$= 2\pi/4$$

$$= \boxed{\pi/2}.$$

6.5 Average Value of a Function

Averages over finite points are easy: f a function, $x_1, \ldots, x_n$ points,

$$\frac{1}{n}\sum_{k=1}^{n} f(x_k).$$

What about for infinitely many data points?

Choose n points $a = x_1 < x_2 < \ldots < x_n = b$ with $x_{i+1} - x_i = \Delta x$ fixed. The average of sample points $x_i^* \in [x_i, x_{i+1}]$ is

$$\frac{1}{n}\sum_{k=1}^{n} f(x_i^*).$$

In particular, we have divided $[a,b]$ into n equal pieces, so

$$n = \frac{b-a}{\Delta x}$$

and thus

$$\frac{1}{n}\sum_{k=1}^{n} f(x_i^*) = \frac{1}{(b-a)}\sum_{k=1}^{n} f(x_i^*)\Delta x$$

By the definition of the Riemann integral, when we send the length of the interval to zero we get the integral

Def^n: $$f_{ave} = \frac{1}{b-a}\int_a^b f(x)dx$$

Eg.: ① Find the average value of $f(x) = 1 + x^2$ on $[-1, 2]$

$$f_{ave} = \left(\frac{1}{2-(-1)}\right)\int_{-1}^{2} 1 + x^2 dx$$

$$= \frac{1}{3}\left(\int_{-1}^{2} dx + \int_{-1}^{2} x^2 dx\right)$$

$$= \frac{1}{3}\left((2-(-1)) + \frac{1}{3}(8-1)\right)$$

$$= \frac{1}{3}(3) + \frac{1}{9}(9)$$

$$= 1 + 1$$

$$= 2.$$

Thm (Mean Value for Integrals): If f is continuous on $[a,b]$, then there exists a number $c \in [a,b]$ such that

$$f(c) = \frac{1}{b-a}\int_a^b f(x)\,dx$$

or

$$f(c)(b-a) = \int_a^b f(x)\,dx.$$

Pf: Exercise.

E.g. ① $f(x) = 1+x^2$, $f_{ave} = 2$ on $[-1,2]$. Find c such that

$$f(c) = 2.$$

$1+x^2 = 2$

$\Rightarrow x^2 = 1$

$\Rightarrow x = \pm 1$

③ Show that the average velocity of a car over a time interval $[t_1,t_2]$ is the same as the average of its velocities during the trip.

Let $s(t)$ be the displacement of the car. Then the average velocity over $[t_1,t_2]$ is

$$\frac{s(t_2)-s(t_1)}{t_2-t_1}.$$

$$v_{ave} = \frac{1}{t_2-t_1}\int_{t_1}^{t_2} v(t)\,dt = \frac{1}{t_2-t_1}\int_{t_1}^{t_2} s'(t)\,dt = \frac{1}{t_2-t_1}\, s(t_2)-s(t_1). \blacksquare$$