# Rates of Change

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Math 122: Calculus for Business Administration and Social Sciences

# Outline

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# Average Rate of Change

## Definition 1

The *average rate of change* of a function $f$ on an interval $[a, b]$ is

$$\frac{f(b) - f(a)}{b - a} = \frac{f(a) - f(b)}{a - b}.$$

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**Remark 1**

This is just the difference quotient from the last section.

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$$\frac{104 - 0}{2 - 0} = \frac{104}{2} = 52 \text{ mph.}$$

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$$\frac{104 - 0}{2 - 0} = \frac{104}{2} = 52 \text{ mph.}$$

## Remark 2

Note that this does not necessarily imply you drove 52 mph the entire time, but rather you averaged 52 mph.

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# Relative Change



Given a quantity, $P$, the *relative change* of the quantity from $P$ to $P'$ is

$$\frac{P' - P}{P}.$$

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$$\frac{4.25-2.25}{2.25}=\frac{2}{2.25}=\frac{2}{\frac{9}{4}}=\frac{8}{9}=0.\overline{8}.$$

# Example



A pair of jeans costs 75.99 normally.

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$$\frac{52.99 - 75.99}{75.99} = \frac{-23}{75.99} \approx -0.303.$$

Hence the price has been reduced by about 30%.

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The number of sales per week for the jeans above is normally 25.

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The number of sales per week for the jeans above is normally 25. During the week the jeans are on sale, the number of weekly sales increases to 45. Find the relative change in weekly sales.

$$\frac{45-25}{25} = \frac{20}{25} = \frac{4}{5}.$$

Hence weekly sales have increased by 80%.

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- the profit from selling $q$ goods by $\pi(q)$.

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- The $\$24,000$ expenditue is called a *fixed cost*.
- The \$7/radio in labour and material is called a *variable cost*.

# Linear Marginal Cost

**Definition 2**

For a linear cost function, the marginal cost is the cost to product one additional unit:

$$\frac{C(q+1)-C(q)}{(q+1)-q} = C(q+1) - C(q).$$

# Linear Marginal Cost



**Definition 2**

For a linear cost function, the marginal cost is the cost to product one additional unit:

$$\frac{C(q+1)-C(q)}{(q+1)-q}=C(q+1)-C(q).$$

**Remark 3**

This is just the slope of the linear cost function.

# Profit

## Definition 3

- Given a revenue and a cost function, the profit function is
$$\pi(q) = R(q) - C(q).$$

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$$\pi(q) = R(q) - C(q).$$
- The *break-even* point is the quantity, $q$, for which
$$\pi(q) = 0$$
holds.

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$$\pi(q) = R(q) - C(q) = 15q - (7q + 24000)$$

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The break-even point is value of $q$ making

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hold. Therefore the break-even point is

$$q = \frac{24000}{8} = 3000.$$

# Marginal Revenue

## Definition 4

The *marginal revenue* for a linear revenue function is the revenue from selling one additional item,

$$\frac{R(q+1)-R(q)}{(q+1)-q} = R(q+1) - R(q).$$

# Marginal Revenue

## Definition 4

The *marginal revenue* for a linear revenue function is the revenue from selling one additional item,

$$\frac{R(q+1)-R(q)}{(q+1)-q} = R(q+1)-R(q).$$

## Remark 4

This is just the slope of the revenue function.

# Marginal Profit

**Definition 5**

The *marginal profit* for linear cost and revenue functions is the profit from selling one additional item

$$\frac{\pi(q+1)-\pi(q)}{(q+1)-q} = \pi(q+1)-\pi(q).$$

# Marginal Profit

## Definition 5

The *marginal profit* for linear cost and revenue functions is the profit from selling one additional item

$$\frac{\pi(q+1)-\pi(q)}{(q+1)-q} = \pi(q+1)-\pi(q).$$

## Remark 5

This is the slope of the revenue function less the slope of the cost function.