Review (Ch5)

5.3 The Fundamental Theorem of Calculus

Def'n: ① A function f is continuous at a number a if

$$\lim_{x\to a} f(x) = f(a).$$

② A function f is continuous on an interval I, if for every $a \in I$

$$\lim_{x\to a} f(x) = f(a).$$

Rmk: If I is closed (i.e. $I=[a,b]$), then we require

$$\lim_{x\to a^+} f(x) = f(a)$$

and

$$\lim_{x\to b^-} f(x) = f(b).$$

Thm If f is differentiable at a point a, then f is continuous at a.

Pf: We need to show

$$\lim_{x\to a} f(x) = f(a)$$

and this is the same as $\lim_{x\to a} f(x)-f(a) = 0$.
We observe that

$$f(x)-f(a) = \frac{f(x)-f(a)}{x-a}(x-a)$$

so

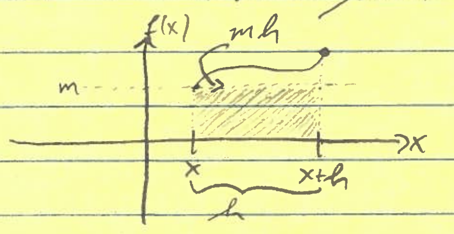
$$\begin{aligned}\lim_{x\to a}\left(f(x)-f(a)\right) &= \lim_{x\to a}\left(\frac{f(x)-f(a)}{x-a}(x-a)\right)\\ &= \left(\lim_{x\to a}\frac{f(x)-f(a)}{x-a}\right)\left(\lim_{x\to a}(x-a)\right)\\ &= f'(x)\cdot 0\\ &= 0. \quad \blacksquare\end{aligned}$$

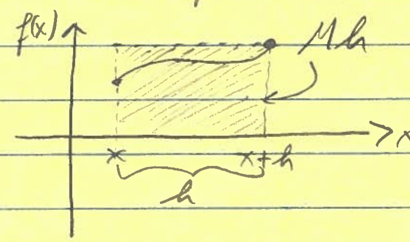
Thm (Fundamental Theorem of Calculus, Part I): If f is a continuous function on $[a,b]$, then the function
$$g(x) = \int_a^x f(t)\,dt, \quad x \in [a,b]$$
is continuous on $[a,b]$ and differentiable on (a,b), and $g'(x) = f(x)$.

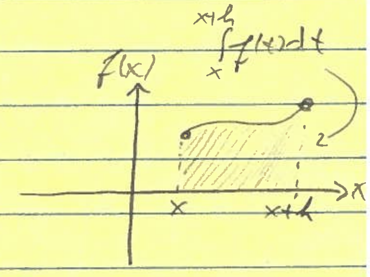
Pf: Let $x, x+h \in (a,b)$ be given. We observe that
$$g(x+h) - g(x) = \int_a^x f(t)\,dt + \int_x^{x+h} f(t)\,dt - \int_a^x f(t)\,dt = \int_x^{x+h} f(t)\,dt.$$
So long as $h \neq 0$,
$$\frac{1}{h}\big(g(x+h) - g(x)\big) = \frac{1}{h}\int_x^{x+h} f(t)\,dt.$$

Recall: (Extreme Value Theorem): If f is continuous on a closed interval, $[x, x+h]$, then for some number $u \in [x, x+h]$ f attains its minimum, $m = f(u)$, and for some number $v \in [x, x+h]$, f attains its maximum value, $M = f(v)$.

Then we see that $mh \le \int_x^{x+h} f(t)\,dt \le Mh$







Since $h \neq 0$, we have for $h > 0$
$$f(u) = m \le \int_x^{x+h} f(t)\,dt \le M = f(v)$$

Rmk: When $h < 0$, we have a mirror image. Flip everything around.

We now let $h \to 0$, and so since $x+h \to x$, $u \to x$ and $v \to x$, and thus
$$\lim_{h\to 0} f(u) = \lim_{u \to x} f(u) = f(x) \text{ and } \lim_{h \to 0} f(v) = \lim_{v \to x} f(v) = f(x).$$
follows from continuity of f.

Recall: (Squeeze Thm): If $f(x) \le g(x) \le h(x)$ when x is near a (except possibly at a), and

$$\lim_{x \to a} f(x) = \lim_{x \to a} h(x) = L,$$

then

$$\lim_{x \to a} g(x) = L.$$

Apply the squeeze theorem to

$$m = f(u) \le \frac{g(x+h) - g(x)}{h} \le f(v) = M$$

to see that $g'(x) = f(x)$. ▨

Rmk: The function g is not necessarily differentiable at a or at b. We only know that

$$\lim_{h \to 0^+} \frac{g(a+h) - g(a)}{h} \quad \text{and} \quad \lim_{h \to 0^-} \frac{g(b+h) - g(b)}{h}$$

exist. These are, however, enough to get left- and right-continuity at a and b.

$$\lim_{x \to a^+} g(x) - g(a) = \lim_{x \to a^+} \frac{g(x) - g(a)}{x-a}(x-a) = 0$$

and

$$\lim_{x \to b^-} g(x) - g(b) = \lim_{x \to b^-} \frac{g(x) - g(b)}{x-b}(x-b) = 0.$$

E.g.: Find $\frac{d}{dx} \int_1^{x^4} \sec(t)\,dt$

$g(x) = \int_1^{x^4} \sec(t)\,dt, \quad u = x^4$

$\Rightarrow g(u) = \int_1^{u} \sec(u)\,dt$

$$\frac{d}{dx} g(u) = g'(u)\frac{du}{dx} = g'(u)\,4x^3 = \sec(u)\,4x^3 = \boxed{4x^3 \sec(x^4).}$$

Thm If f is continuous on $[a,b]$ and F is any function such that $F'(x) = f(x)$, then
$$\int_a^b f(x)\,dx = F(b) - F(a).$$

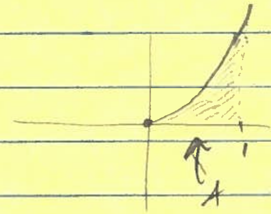
Pf: Let $g(x) = \int_a^x f(t)\,dt$. By FTC Part I, $g'(x) = f(x)$ and
$$g(b) - g(a) = \int_a^b f(t)\,dt - \int_a^a f(t)\,dt = \int_a^b f(t)\,dt.$$
If F is any other antiderivative for f, then
$$(F-g)'(x) = F'(x) - g'(x) = f(x) - f(x) = 0$$
and so by Theorem 4.2.6, $F = g(x) + C$, for some constant C. Therefore
$$\begin{aligned} F(b) - F(a) &= g(b) + C - g(a) - C \\ &= g(b) - g(a) \\ &= \int_a^b f(t)\,dt. \ \blacksquare \end{aligned}$$

Thm (Mean-Value Theorem): Let f be a function continuous on $[a,b]$ and differentiable on (a,b). There exists some $c \in (a,b)$ such that
$$f'(c) = \frac{f(b) - f(a)}{b-a}$$

Thm 4.2.6 If $f'(x) = 0$ holds for all $x \in (a,b)$, then f is constant on (a,b).

Pf: Let $(x_1, x_2) \subseteq (a,b)$ be given. Apply the Mean-Value Theorem on $[x_1, x_2]$ to find $x_1 < c < x_2$ such that
$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} = f'(c)$$
so that $f(x_2) - f(x_1) = f'(c)(x_2 - x_1) = 0$ implies $f(x_1) = f(x_2)$, as desired $\blacksquare$

Eg. Find the area under the parabola $y=x^2$ from 0 to 1.



$$A = \int_0^1 x^2\,dx = \tfrac{1}{3}x^3\Big|_0^1 = \tfrac{1}{3}(1-0) = \tfrac{1}{3}.$$

5.4 The Substitution Rule

If $u=g(x)$ is a differentiable function whose range is an interval I and f is continuous on I, then

$$\int f(g(x))g'(x)\,dx = \int f(u)\,du.$$

Pf: If $F'=f$, then by the FTC

$$\int F'(g(x))g'(x)\,dx = F(g(x))+C$$

because, by the chain rule,

$$\frac{d}{dx}F(g(x)) = F'(g(x))\,g'(x).$$

Similarly

$$\begin{aligned}\int F'(u)\,du &= F(u)+C\\ &= F(g(x))+C\\ &= \int F'(g(x))g'(x)\,dx\end{aligned}$$

and thus

$$\int f(u)\,du = \int F'(u)\,du = \int F'(g(x))\,g'(x)\,dx = \int f(g(x))g'(x)\,dx. \blacksquare$$

Eg. ① Find $\int x^3\cos(x^4+2)\,dx$

$u = x^4+2 \Rightarrow du = 4x^3\,dx \Rightarrow \tfrac{1}{4}du = x^3\,dx$

$\Rightarrow \int \cos(x^4+2)x^3\,dx = \tfrac{1}{4}\int\cos(u)\,du = \tfrac{1}{4}\sin(x^4+2)+C.$

(2) $\int \tan(x)dx$

$\tan(x) = \sin(x)/\cos(x)$, $\quad \frac{d}{dx}\cos(x) = -\sin(x)$

$u = \cos(x) \Rightarrow du = -\sin(x)dx \Rightarrow -du = \sin(x)dx$

$$\Rightarrow \int \tan(x)dx = -\int \frac{du}{u} =$$
$$= -\ln|u| + C$$
$$= \ln\left|\frac{1}{u}\right| + C$$
$$= \ln\left|\frac{1}{\cos(x)}\right| + C$$
$$= \ln|\sec(x)| + C.$$

Definite Integrals

Thm If g' is continuous on $[a,b]$ and f is continuous on the range of $u = g(x)$, then

$$\int_a^b f(g(x))g'(x)dx = \int_{g(a)}^{g(b)} f(u)du.$$

Pf: Suppose $F' = f$. Then

$$\frac{d}{dx}F(g(x)) = F'(g(x))g'(x) = f(g(x))g'(x)$$

and so by the F.T.C.

$$\int_a^b f(g(x))g'(x)dx = F(g(b)) - F(g(a))$$
$$= \int_{g(a)}^{g(b)} F'(u)du$$
$$= \int_{g(a)}^{g(b)} f(u)du. \quad \blacksquare$$

Eg.: (1) $\int_0^4 \sqrt{2x+1}\, dx$ $\qquad u = 2x+1,\ du = 2dx \Rightarrow \frac{1}{2}du = dx$

$u(0) = 2(0)+1 = 1,\ u(4) = 2(4)+1 = 9$

$$\int_0^4 \sqrt{2x+1}\, dx = \frac{1}{2}\int_1^9 \sqrt{u}\, du = \frac{1}{2}\int_1^9 u^{1/2} = \frac{1}{3}u^{3/2}\Big|_1^9 = 1/3(27-1) = 1/3(26)$$

(2) $\int_1^e \frac{\ln x}{x}dx$ $\qquad u = \ln(x) \Rightarrow du = \frac{dx}{x}$

$u(e) = \ln(e) = 1$
$u(1) = \ln(1) = 0$

$$\int_1^e \frac{\ln(x)}{x}dx = \int_0^1 u\,du = \frac{1}{2}u^2\Big|_0^1 = 1/2.$$

$F' = f$, $\int_a^b f(x)dx = F(b) - F(a)$

$-\int_b^a f(x)dx = -(F(a) - F(b)) = F(b) - F(a) = \int_a^b f(x)dx$

Recall A function f is odd if $f(x) = -f(x)$
is even if $f(-x) = f(x)$.

Prop: Suppose f is continuous on $[-a, a]$.
① If f is even, then $\int_{-a}^a f(x)dx = 2\int_0^a f(x)dx$,
② If f is odd, then $\int_{-a}^a f(x)dx = 0$.

Pf: First write

$$\int_{-a}^a f(x)dx = \int_{-a}^0 f(x)dx + \int_0^a f(x)dx$$

$$= -\int_0^{-a} f(x)dx + \int_0^a f(x)dx.$$

$du = -dx$

Let $u = -x$, so that $-du = dx$, $u(-a) = -(-a) = a$.
Then

$$-\int_0^{-a} f(x)dx = -\int_0^a f(-u)(-du) = \int_0^a f(-u)du$$

If f is even, then

$$\int_{-a}^a f(x)dx = \int_0^a f(-u)du + \int_0^a f(x)dx = \int_0^a f(u)du + \int_0^a f(x)dx = 2\int_0^a f(x)dx.$$

If f is odd, then

$$\int_{-a}^a f(x)dx = -\int_0^a f(u)du + \int_0^a f(x)dx = 0. \quad ■$$