Functions

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Math 122: Calculus for Business Administration and Social Sciences

OUTLINE

1. 1.1: FUNCTIONS
   - Graphs

Outline

Definition

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Notation: A function named f that takes as input the *independent variable*, x, and outputs the *dependent variable*, y, is written as

$$y = f(x).$$

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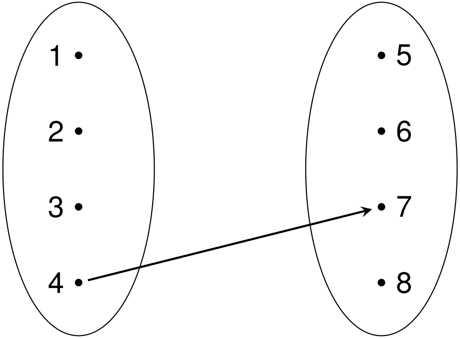

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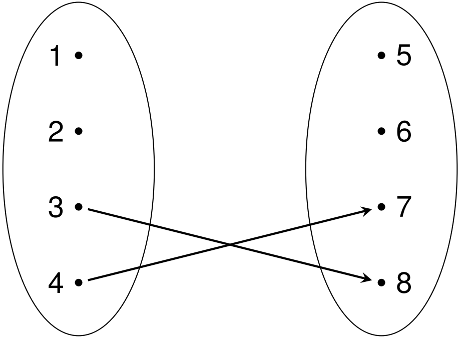

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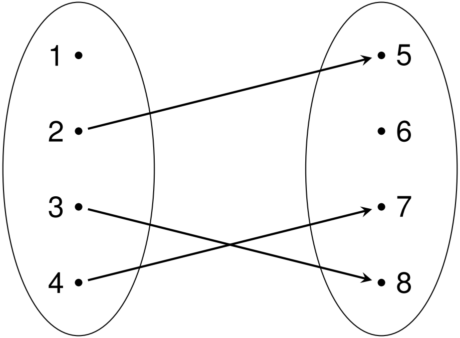

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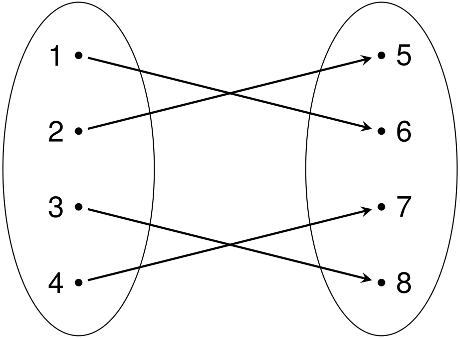

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- The domain of f is the set of all real numbers, $\mathbb{R}$.
- The range of f is the set of all non-negative real numbers,

$$\{x \in \mathbb{R} \mid 0 \leq x\}.$$

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The following depicts a non-function.

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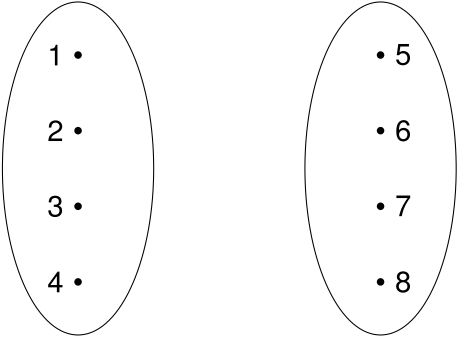

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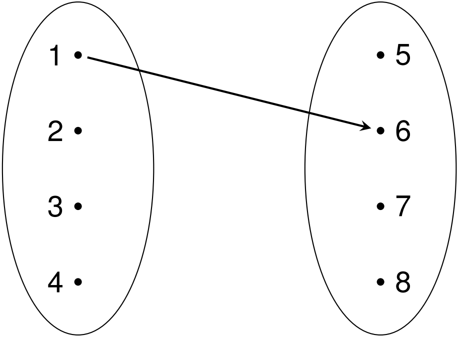

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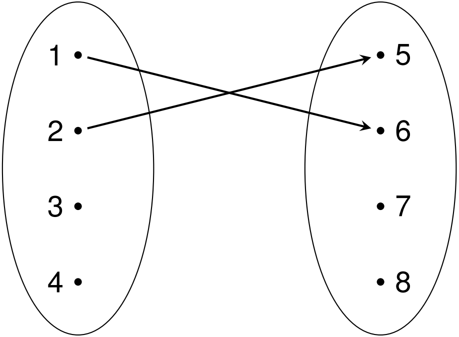

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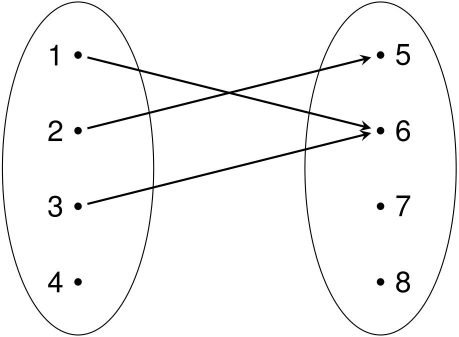

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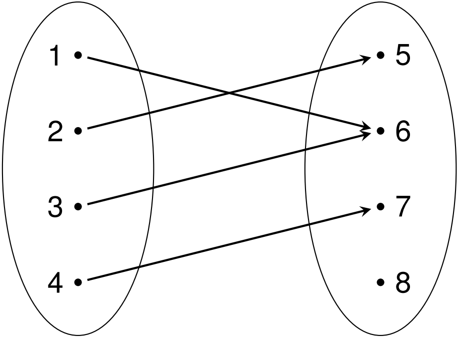

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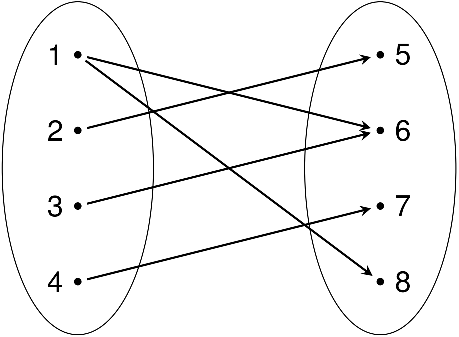

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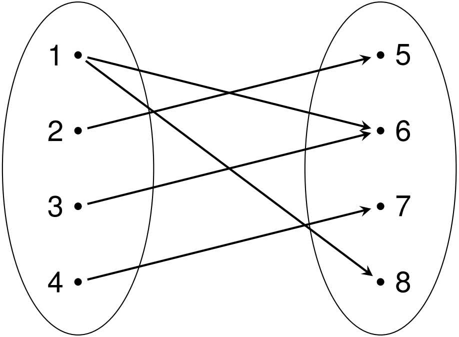

The value $f(1)$ is not well-defined because it requires a choice: it could be either 6 or 8.

CARTESIAN PLANE

Recall that the *Cartesian plane* is the set of all pairs

$$\mathbb{R}^2 = \{(x, y) \mid x \in \mathbb{R}, y \in \mathbb{R}\}.$$

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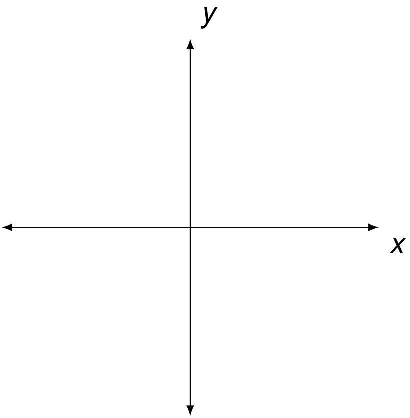
It can be depicted as



Graph of a Function

Definition 2

The graph of a real-valued function, f, with domain $D \subseteq \mathbb{R}$ is the set of pairs

$$\{(x, f(x)) \mid x \in D\} \subseteq \mathbb{R}^2.$$

It can be drawn on the Cartesian plane.

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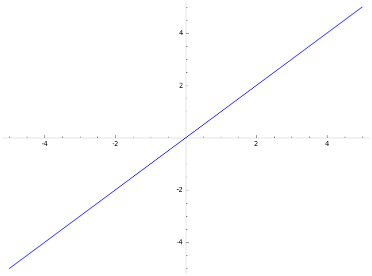

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Increasing/Decreasing Functions

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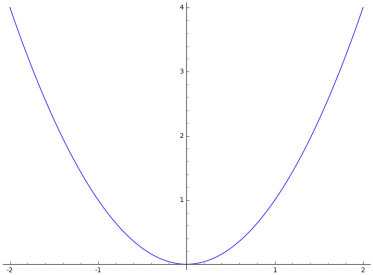
- *decreasing on [a,b]* if $f(x_2) < f(x_1)$ whenever $a \leq x_1 < x_2 \leq b$.

We say that f is increasing/decreasing if it is increasing/decreasing on its entire domain.

Example



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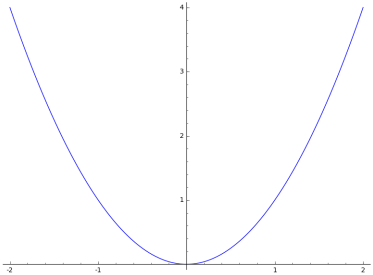

- Increasing on:
- Decreasing on:

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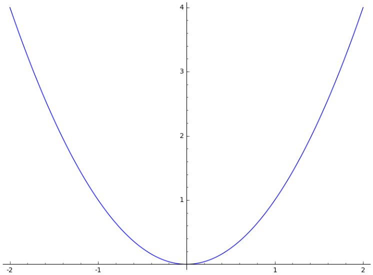

- Increasing on: $(0, \infty)$
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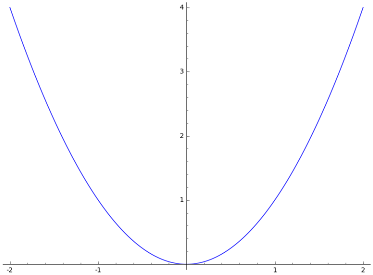
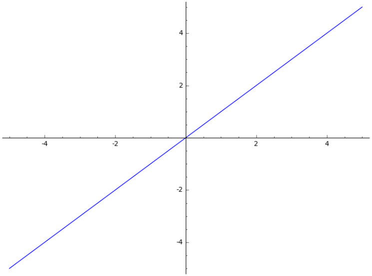


- Increasing on: $(0, \infty)$
- Decreasing on: $(-\infty, 0)$

Example

Increasing



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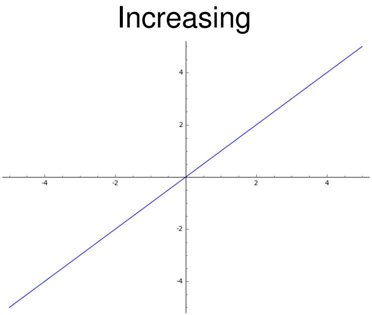

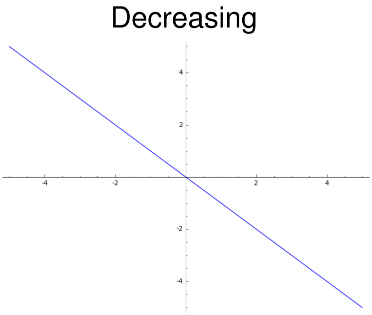

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- The *y-intercept* is the point $(0, f(0))$ on the graph.

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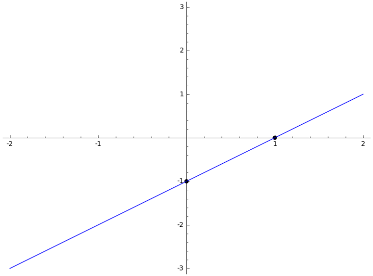

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- The number m is called the *slope* of the f,
- The number b is the y-intercept,
- This form is usually called the *Slope-Intercept Form* of a line.

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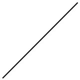

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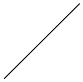
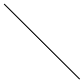


- Decreasing ($m < 0$):



GRAPH OF A LINEAR FUNCTION

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- Increasing ($0 < m$):
- Decreasing ($m < 0$):
- Horizontal ($m = 0$):

Point-Slope Form

Definition 6

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the equation of the line through (x_0, y_0) with slope m is

$$y - y_0 = m(x - x_0).$$

Two Points Determine a Line

Given two points, (x_0, y_0) and (x_1, y_1), the slope of the line passing through them is

$$m = \frac{y_0 - y_1}{x_0 - x_1} = \frac{y_1 - y_0}{x_1 - x_0}.$$

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To see these are the same line, put them both into Slope-Intercept Form.

Two Points Determine a Line (cont.)

$$y = mx - \frac{y_0 - y_1}{x_0 - x_1}x_0 + y_0$$

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Two Points Determine a Line (Cont.)

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y &= mx - \frac{y_0 - y_1}{x_0 - x_1} x_0 + y_0 \\
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Difference Quotients

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Let f be a function. Given x_0, x_1 in the domain of f, the *difference quotient* is

$$\frac{f(x_1) - f(x_0)}{x_1 - x_0} = \frac{f(x_0) - f(x_1)}{x_0 - x_1}$$

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This is just the slope of the line through $(x_0, f(x_0))$ and $(x_1, f(x_1))$. This line is usually called the *Secant Line*.

Let $f(x) = mx + b$.

Difference Quotients for Linear Functions

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Difference Quotients for Linear Functions

Let $f(x) = mx + b$. Given x_0 and x_1:

$$\frac{f(x_1) - f(x_0)}{x_1 - x_0} = \frac{mx_1 + b - (mx_0 + b)}{x_1 - x_0}$$

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\end{aligned}$$

Hence for a linear function, the difference quotient is just the slope.

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$$\frac{f(-1) - f(2)}{-1 - 2} = \frac{(-1)^2 - 2^2}{-3}$$

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$$\frac{f(-1) - f(2)}{-1 - 2} = \frac{(-1)^2 - 2^2}{-3} = \frac{1 - 4}{-3}$$

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Functions

Farman

1.1: Functions

Graphs

1.2: Linear Functions

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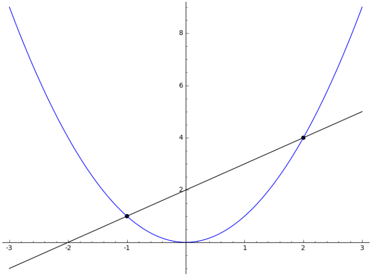
This is the slope of the secant line:



Difference Quotients for Non-Linear Functions (cont.)

For $x_0 = 0$, $x_1 = 2$:

Functions
Farman
1.1: Functions
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For $x_0 = 0$, $x_1 = 2$:

$$\frac{f(0) - f(2)}{0 - 2}$$

Difference Quotients for Non-Linear Functions (cont.)

For $x_0 = 0$, $x_1 = 2$:

$$\frac{f(0) - f(2)}{0 - 2} = \frac{0 - 4}{-2}$$

Difference Quotients for Non-Linear Functions (Cont.)

For $x_0 = 0$, $x_1 = 2$:

$$\frac{f(0) - f(2)}{0 - 2} = \frac{0 - 4}{-2} = \frac{4}{2} = 2.$$

Difference Quotients for Non-Linear Functions (Cont.)

For $x_0 = 0$, $x_1 = 2$:

$$\frac{f(0) - f(2)}{0 - 2} = \frac{0 - 4}{-2} = \frac{4}{2} = 2.$$

This is the slope of the secant line: