Modeling Belief Propagation in Social Networks

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1 Introduction

Diffusion through social networks has been studied in the context of spreading diseases, computer viruses, rumors, information, and more. Researchers across a multitude of fields such as physics, mathematics, computer science, and social sciences have contributed to the literature on this topic [1] [2] [3]. With the rise of social media consumption, a more recent area of interest pertaining to diffusion through networks has been understanding the dissemination of misinformation in a social network. In the short period of time since social media has become the predominant source of news consumption for the average American, we have already observed a plethora of cases where misinformation has caused real world harm. Notably, there have been examples where false stories have lead to changes in the stock market, potential influences on elections, as well as an increase in vaccine hesitancy, mob lynchings, and other types of widespread panic or criminal activities [4] [5] [6].

In thinking about how to model the diffusion of misinformation, we found a glaring issue in the current methods that are out there on diffusion models. Specifically, current, popular models in the literature represent adoption or infection in binary terms: either yes/infected or no/not-infected. However, we know from work in cognitive science and psychology journals that on tough topics people's belief can often not be captured by a binary variable because peoples' ideologies live on a spectrum [7] [8]. It would make more sense to categorize someone as thinking that there is 20% (or some other percentage) likelihood in a story being true than just a 0% or 100% belief. We want to fill in the gap in research by providing a model that does capture the nuance of uncertainty in belief as living on a spectrum. Thus, the model we propose was initially aimed to capture the evolution of the distribution of beliefs of actors within a network.

After developing our work, we realized that our model's applicability goes much beyond modeling uncertainty via a spectrum. For instance, we could consider how the distribution of the centralities across people in a social network changes as more people enter a network or we could analyze the speed and progression of polarization in congress by considering how congresspeople are distributed in terms of their rate of cooperation with coworkers across the aisle. In general, the belief propagation model we introduce is widely applicable to many disciplines.

Note: *Throughout the description of the model, however, we will be referring to things in terms of beliefs. This choice is intentional to contextualize the model and make things easier to understand, but we want to de-emphasize how important it is to think of the model in terms of beliefs. Any kind of distribution dependent on the network that* *evolves through time is compatible with the model.*

In this project proposal, we will first finish motivating and contextualizing our work with a literature review in this section. Then we will move on to describing the model in detail in Section 2. Followed by Section 3, where we will cover future research directions via conjectures and proposed questions. We end with Section 4 on how to use the model in a proposed empirical analysis and finish with a brief conclusion on this work in Section 5.

1.1 Connection to Traditional Models (Bass, SIR, and DeGroot)

While we are proposing a new model, we still want to build on top of existing literature. We start by highlighting how work improves upon the Bass model, SIR model, and DeGroot model.

The first mathematical model of diffusion of new products was proposed by Bass [9]. The basic idea is that a consumer is either not adopting or adopting, which makes sense in terms of what the visible outcome is on the market but fails to capture the gradual evolution that might be needed to push someone into adopting. Another aspect that the Bass model fails to capture is that adopters can "recover" from influencing other people. That brings us to the next model, which is the SIR model. This model characterizes the population as being composed of susceptible individuals, infected individuals, and recovered individuals [10]. Differential equations are used to describe how individuals transition between these different groups, and from the model it is possible to identify threshold quantities that determine whether an epidemic occurs or whether the disease simply dies out. Our model is similar to the SIR model in the idea that it captures the idea that people can go in the direction of being more infected to less infected (i.e. recovery) and our model could help identify certain thresholds for what causes misinformation to go viral.

In general, our model will mirror both the Bass and SIR model in the way that we describe macro level changes in the levels of infection (i.e. belief) in the population. This format differs from the DeGroot model [11], which instead simulates how individual people's beliefs change on a micro level based on their neighbors beliefs. A stochastic matrix is used to model the trust that each agent places on the current belief of the other agents in forming his or her belief for the next period. Therefore, to use the DeGroot model, you must have a sense of how much belief every person places on every other person in the network. The most basic version of our model will not require such a granular understanding of the network, and this allows for bigger picture analysis of how general structure of networks might influence the spread of information. Where our proposed model does emulate the DeGroot model is in the idea that opinion formation is represented by a probability that lies in the interval [0,1]. The probability can be thought of as the probability that an agent thinks a given statement is true or the likelihood that the individual might engage in a given activity, etc.

As described above, our model is combining aspects of the DeGroot model with aspects of the Bass and SIR model in order to create a more flexible model of how belief propagates through a network over time.

1.2 Relation to Work on Newer Models

The approach we take in our paper is highly motivated by Markov chains. Particularly, we assume a version of the Markov property, which translates into our setting as the assumption that the belief distribution at time t is dependent only on the distribution at time $t-1$. This property is a popular assumption in literature when dealing with the evolution of networks. For instance, it is a critical assumption in Paranamana et al. with their model which considers belief propagation to be a process similar to the DeGroot model [12]. This assumption also comes in handy for Shi, Chen, et al., who consider the evolution of the degree distribution of networks by the preferential attachment mechanism [13].

One diverging point between our model and other models in the space of belief propagation over networks is that many other models are focused on actors rather than the network as a whole. Actor-centric models usually update the beliefs of actors based on their local connections like in the DeGroot model or, on the flip side, on general network-wide characteristics such as in Shi, Proutiere, et al., but in large networks this is generally hard to keep track of [11] [14]. Our model takes a macroscopic point of view for belief evolution. The benefit of an approach like this is that it is easier to talk about network-wide effects as well as general properties of certain kinds of belief propagation. The trade-off is that we cannot formulate phenomena on the actor-specific level.

Our approach is also deterministic as opposed to probabilistic, in order to make the model as simplistic as possible so as to be useful for empirical analysis. We see probabilistic models such as [15] to be extensions of our model and helpful theoretical examples of how to take this work further. However, one of the priorities of our model is to be helpful for analyzing various processes in real life, and so we prioritize operability over greater flexibility.

2 Model

We start introducing our model with some basic definitions.

Definition 1. A **belief distribution** at time $t \in \mathbb{R}_{\geq 0}$ is an integrable function $\psi_t : [0,1] \to \mathbb{R}_{\geq 0}$ such that $\int_0^1 \psi_t(x)dx = 1$. A **cumulative belief distribution** of a belief density distribution ψ_t is the function $\Psi_t(x) = \int_0^x \psi_t(x)dx$.

These belief distributions will be the central object of study in our model, and we will be concerned with how they vary based on t.

Definition 2. A sequence of time-indexed belief distributions $\{\psi_t\}_{t\geq 0}$ is called a **belief propagation model** or **BPM**.

BPMs are extremely general and can capture many different processes. However, we want to be able to say interesting things about certain kinds of BPMs, so we need to introduce some assumptions. We specifically consider BPMs where $\psi_t(x)$ is dependent on ψ_{t-1}. In particular, we postulate that it is the case with certain kinds of processes that actors with a certain belief level at one point in time are more likely to have held certain belief levels just one time step before. The concept of a heat map captures this idea by introducing a function that relates the two distributions.

Definition 3. The **heat map** at time t for a BPM $\{\psi_t\}_{t\geq 0}$ is a function $\omega_t : [0,1] \times [0,1] \to \mathbb{R}_{\geq 0}$ such that

$$\psi_t(x) = \frac{\int_0^1 \psi_{t-1}\omega_t(x,y)dy}{C} \tag{1}$$

where C is a normalizing constant so that ψ_t integrates to 1, i.e. is a valid belief distribution function.

One nice thing about heat maps is that we can intuitively understand how a heat map works. ω_t is large for values (x, y) where it is very likely for a person to land at belief level x at time t given that they were at belief level y at time $t-1$, and it is small when it is unlikely. For instance, if a people are stubborn in their beliefs, we might expect that ω_t is only large when $|x - y|$ is small. To give another example, if initial beliefs don't matter but believability is high, then ω_t could be independent of y and be increasing in x. Table 1 exhibits some examples of processes with particular dynamical features, the corresponding language in heat maps, and example heat maps that exemplify those features:

Process Feature	**Heat Map Qualities**	**Example Heat Map**
People are stubborn in beliefs; homophily.	ω_t decreasing function of $\lvert x-y\rvert$	$\omega_t(x,y) = e^{-\lvert x-y\rvert}$
Initial beliefs don't matter, and people tend to have high belief.	ω_t is independent of y and is increasing in x	$\omega_t(x,y) = x$
Belief dynamics are characterized by flip-flopping of beliefs.	(Loosely) ω_t is large for x large and y small and vice-versa. ω_t is small otherwise.	$\omega_t(x,y) = \begin{cases} 1 & \text{if } x,y \in [0,\frac{1}{2}) \times [\frac{1}{2},1] \\ 1 & \text{if } x,y \in [\frac{1}{2},1] \times [0,\frac{1}{2}) \\ 0 & \text{otherwise} \end{cases}$
Belief becomes uniform as time goes on.	$\frac{\max_{x,y\in[0,1]^2} \omega_t(x,y)}{\min_{x,y\in[0,1]^2} \omega_t(x,y)} \to 1$	$\omega_t(x,y) = \frac{1}{t}e^{-\lvert x-y\rvert} + \frac{t-1}{t}$

Table 1: Example Heat Maps

Another nice feature of heat maps is that they are visualized graphically. For instance, take ω_t given by the standard multivariate normal distribution:

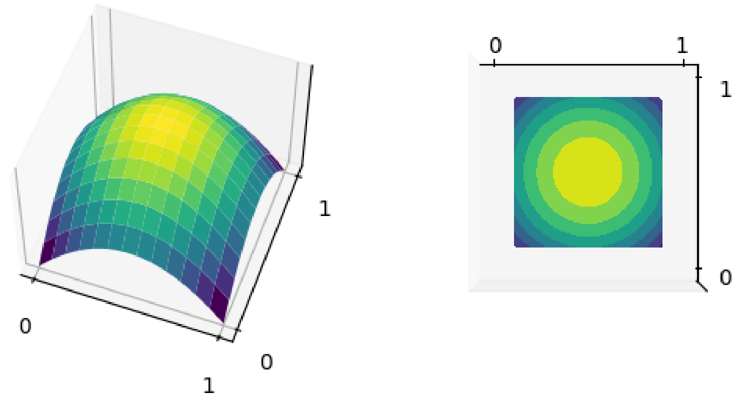

Figure 1: Heat Map for $\omega_t(x, y) = \frac{1}{2\pi} e^{-\left(\frac{x^2+y^2}{2}\right)}$

This makes communicating about heat maps very simple.

The power of heat maps is that an initial belief distribution ψ_0 and a sequence of heat maps $\{\omega_t\}_{t\geq 1}$ can completely specify an entire BPM. This realization is immediately apparent after considering equation 1. As a result, we may refer to $(\psi_0, \{\omega_t\}_{t\geq 1})$ as a BPM. However, we have to be careful about how we define our ω_t's as some will be unsuitable as we highlight in the following remark.

Important Remark: *There is no guarantee that ψ_t is Lebesgue integrable for an arbitrary weight function ω. This motivates a question that we would like to explore more in future study: What kinds of heat maps are valid? For heat maps that aren't valid, what adjustments can be made to the model?*

For now, we do have the following lemma, which can tell us about some heat maps which are valid.

Lemma 2.1. If $\omega_t(x, y) = g_t(x - y)$ for some set of Lebesgue integrable non-negative functions $\{g_t\}_{t\geq 1}$ defined on $\mathbb{R}$ (though they need not be defined nor integrable on this large of a domain), then any BPM with $\{\omega_t\}_{t\geq 1}$ as a heat maps and ψ_0 an initial belief distribution will have ψ_t be belief distributions for all $t > 0$.

Proof. The proof is quite simple and follows directly from the fact that convolutions of two integrable functions are integrable. ψ_0 is already integrable from the assumption. Suppose that ψ_{t-1} is integrable on $[0, 1]$, i.e. $\int_0^1 \psi_{t-1}(x)dx < \infty$. Let f be the extension of ψ_{t-1} to $\mathbb{R}$ by 0, i.e. defining $f(x) = 0$ for $x \notin [0, 1]$ and $f(x) = \psi_{t-1}(x)$ for $x \in [0, 1]$. Note that f is integrable since ψ_{t-1} is integrable. Substituting g_t into 1, we get for some normalizing $0 < C < \infty$ that

$$C\psi_t(x) = \int_0^1 \psi_{t-1}(y)\omega(x, y)dy = \int_0^1 \psi_{t-1}(y)g_t(x - y)dy = \int_{-\infty}^{\infty} f(y)g_t(x - y)dy = (f * g_t)(x)$$

where $*$ indicates convolution on $\mathbb{R}$. Then, as $f * g_t$ is integrable over $\mathbb{R}$, it is integrable over $[0, 1]$ hence $\psi_t = \frac{f * g_t}{C}$ is a valid belief distribution. $\square$

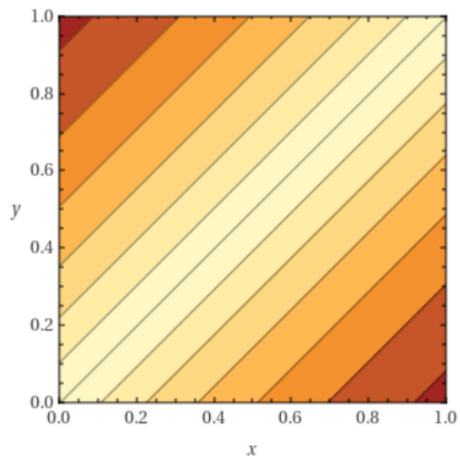

Figure 2: Heat Map for $\omega_t(x,y) = e^{-|x-y|}$

A range of interesting heat maps are included by this lemma. For instance, the heat maps given by $\omega_t(x,y) = e^{-|x-y|}$ from our first example before are valid due to this lemma since $g_t(x) = e^{-|x|}$ is integrable: $\int_{-\infty}^{\infty} g_t(x) = 2$ as can be easily verified. In general, any heat maps covered by lemma 2.1 will have the quality that the weight placed on any point (x, y) will be the same as the weight placed on the point $(x + a, y + a)$ for any a as $g_t(x - y) = g_t(x + a - y - a)$. Thus, these heat maps will look like strips of area with equal density that are all parallel to the line $y = x$.

We now differentiate between certain kinds of BPMs. First, we highlight the potential periodicity of the heat maps.

Definition 4. Periodic belief propagation models $(\psi_0, \{\omega_t\}_{t\geq 1})$ satisfy that for all t, $\omega_t = \omega_{t-p}$ where $p \geq 1$ is called the **period** of the model. In the case of periodic BPMs, the heat maps $\{w_1, w_2, \cdots, w_p\}$ are called the **stages** of the model. BPMs that are not periodic are called **aperiodic**.

Definition 5. Stationary belief propagation models $(\psi_0, \{\omega_t\}_{t\geq 1})$ satisfy that for all t, $\omega_t = \omega$ for some valid heat map ω. Stationary BPMs coincide with periodic BPMs with $p = 1$.

Lastly, we define a particular kind of BPM that we are interested in.

Definition 6. Pointwise stable belief propagation models $(\psi_0, \{\omega_t\}_{t\geq 1})$ satisfy that for all $x \in [0,1]$, $\lim\limits_{t\to\infty} \psi_t(x) = \psi(x)$ for some ψ a belief distribution. In other words, ψ_t converges to ψ pointwise. We call ψ the **limiting belief distribution**.

We can strengthen the notion of convergence to talk about other types of stable BPMs. For instance, we may be interested in L^1-stable BPMs, i.e. $\lim\limits_{t\to\infty} \int_0^1 |\psi_t(x) - \psi(x)| dx = 0$, or uniform stable BPMs, i.e. $\lim\limits_{t\to\infty} \sup\limits_{x\in[0,1]} |\psi_t(x) - \psi(x)| = 0$. Context should determine what kind of stability is important.

We now move onto identifying some potential avenues of further research.

3 Conjectures and Proposed Questions

3.1 Heat Maps

One natural area of study is in the validity of heat maps.

For one, heat maps are a simple yet expressive object. As we remarked in the previous section, one primary question that would be of utmost importance to study is when a sequence of heat maps defines a valid BPM.

Question 3.1. *What conditions on ω_t and ψ_0 are sufficient for the resulting BPM to be valid?*

Conjecture 3.1. *If ψ_0 is continuous and ω_t is continuous for all $t \geq 1$, then ψ_t is a belief distribution for all $t > 0$.*

While we have not yet had the chance to think too deeply about this question, we conjecture that continuity on ψ_0 and ω_t is enough to force ψ_t to be integrable. For one, because the domains of these functions are compact, this would imply that the functions themselves are bounded, so integrability of ψ_t so long as ψ_{t-1} is continuous should not be an issue. However, we would just need to show that ψ_t is continuous. We believe that the continuity of $\omega_t(x, y)$ with respect to x would be enough to extend to continuity of $\psi_t(x)$.

However, not all interesting heat maps that we can imagine fall into the category of continuous heat maps. For instance, the 3rd example on in Table 1 is interesting, but clearly not continuous. Whether or not we can find other conditions on ω_t and ψ_0 that are sufficient for the validity of the generated BPM is not known.

An even more difficult question is to identify what is not just sufficient but also necessary for the validity of the BPM. Such a result would allow us to characterize all valid BPMs and would be extremely helpful in theoretical work with BPMs.

Of course, one way to work around the issue is to discretize BPMs and turn them into finite state processes, however this would unfortunately sacrifice the granularity of the model for ease of mathematical calculation.

3.2 Stable BPMs

Stable BPMs are also another very interesting set of objects in the potential study of BPMs. Stable BPMs are well-behaved in that we know what the belief distributions ψ_t will eventually look like. This could be very insightful for various empirical work as well as makes some very rigid predictions about the steady states of processes that begin with different ψ_0.

Question 3.2. *What kinds of heat maps $\{\omega_t\}_{t\geq 1}$? result in stable BPMs for some ψ_0?*

This question seems a little harder to answer than question 3.1, and so we don't have much idea of how to answer this question as of now. We could begin to come up with a conjecture after doing some simulation of BPMs. Intuitively, though, it feels like a heat map which tends to allocate more weight when x and y are close (people's beliefs don't change much) would lead to stable BPMs. Put into more formal terms, we can offer the following conjecture.

Conjecture 3.2. If the functions $f_{y,t}(x) = \omega_t(x, y)$ are unimodal with mode at $x = y$ for all $t \geq 1$ and $y \in [0, 1]$, then the BPM $(\psi_0, \{\omega_t\}_{t\geq 1})$ is valid for any ψ_0.

We note that the conditions on $f_{y,t}$ may or may not be strong enough. The conjecture may need to be weakened by making $f_{y,t}$ not just unimodal at $x = y$ but also satisfy the condition that enough of the weight is located around $x = y$, i.e. $\int_{x-\epsilon}^{x+\epsilon} f_{y,t}(x)dx \geq C \int_0^1 f_{y,t}(x)dx$ for some $0 < C \leq 1$.

Results in this direction may also feature stable BPMs where the selection of heat maps resulting in stability depend on the choice of initial ψ_0. After all, empirically it may not be the case that any ψ_0 is observable depending on context. Take the case of degree distributions (we disembark from the belief interpretation here). If we happen to believe that scale-free networks are highly present in real life, though we are aware that they very well could be less common than thought, then we may be interested in getting results on stability with ψ_0 a power law.

Another final important thing to note about stable BPMs is that they remind us about important results on stationary distributions in Markov Chain theory. In fact, we may view heat maps like uncountably-infinite dimensional transition matrices and the limiting distribution as some kind of stationary distribution on an infinite state Markov Chain. With this connection, it is conceivable that theorems from Markov Chain theory may import over to the setting of BPMs.

3.3 Additional Questions

In this question, we list some questions that we do not have the space to delve too deeply into.

Question 3.3. *In the case of stable BPMs, are the corresponding limiting distributions unique? Are there infinitely or finitely many if not?*

Question 3.4. *Can we say anything about the properties of limiting distributions based on certain properties of the heat maps?*

Question 3.5. *With periodic BPMs of period p, can we define some notion of stability? Can we get limiting cycles consisting of p distributions rather than limiting distributions?*

Question 3.6. *Does imposing stationarity of a BPM make it easier learn about stability?*

Question 3.7. *Can we introduce probability into the model? How can we formalize probabilistic heat maps and stochastic BPMs?*

While these questions require a lot more theory before we can begin to approach them, we can immediately see how they are interesting questions with relevant conclusions.

In the next section, we disembark from theoretical examination of BPMs, and look at a practical usage case.

4 Proposed Empirical Usage

We see an opportunity to use social media data to fit our proposed model to empirical observations from the real world. The use of empirical data would allow our model to have more applications in helping social media companies understand how harm can be mitigated on their platforms.

A note on practicality: The landscape of researcher access to social media data has been changing recently, as companies like Meta and Twitter have begun restricting access while companies like TikTok have only just begun providing API access (although with very limited access). With these logistical challenges, we will speak in terms of what could have been done with API access similar to the one that Twitter used to support on their platform and a type of access that hopefully will be more widespread in the future. Alternatively, this work can always be done in house by researchers at the social media companies themselves.

4.1 Dataset

Plenty of datasets of the spread of false information based on retweets of the original news post are publicly available for example there is one called the FakeNewsNet [16] and another one used for detecting rumors from microblogs [17]. Additionally, previous researchers have concluded that a simple retweet does not cover the complex methods of information spread, so Murayama et.al. compiled two alternative datasets of fake news items spread on Twitter [6]. We could build a similar dataset echoing their process: first, we would select 10 misinformation items from fact checking sites such as Politifact.com, Snopes.com, or Factcheck.org within a certain time period. Then using the Twitter API, tweets highly relevant to the misinformation stories can be crawled based on the keywords and the URLS. Researchers in a another study also used hashtags to identify tweets of interest, and they were able to we can extract the data fields including the user ID, tweet ID, and domain names of the shared links for every tweet [18]. In addition to noting the users that are posting and retweeting these relevant tweets, we are also interested in recording the users that are liking the tweets.

4.2 Proposed Task Objectives

We propose two example methods of utilizing our model on empirical data.

4.2.1 Evaluation of Prediction Accuracy

First, we can construct a prediction task: For the spread of a misinformation item, we observe a distribution of Twitter actions that captures a distribution of beliefs at three timestamps: t_1, t_2, and t_3. We use the distributions at times t_0 and t_1 to construct a starting heatmap ω_1. Then, we apply ω_1 on the disitbrution at t_1 using our model to create a predicted heatmap, $\hat{\omega_2}$, of what the structure of influence looks like at time t_2. Finally, we can use the distributions at times t_1 and t_2 to construct a actual heatmap of what ω_2 looks like. Comparing our predicted $\hat{\omega_2}$ to the actual ω_2, we can get an evaluation of the prediction accuracy of our model. (See Figure 3.)

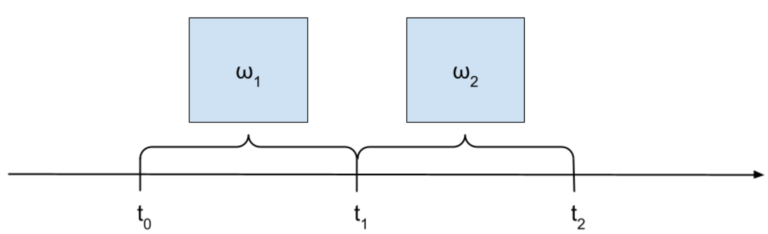

Figure 3: Visualization of use of time series data

4.2.2 Hypothesis Testing on Belief Behaviour

Second, we can use our model to run a hypothesis test on whether the heat maps change over time. The initial setup for this task is the same as the one above. We will want to data that captures the network at three different points in time so that we can build two different heatmaps describing how belief is changing over time. However, now we propose that comparing the similarly of ω_1 and ω_2 (see Figure 3), can inform us on whether the underlying structure stays the same or not. This process can help uncover characteristics of belief propagation that gives insight on what techniques might be most effective. For example, if we find that there is high correlation or high similarity between ω_1 and ω_2 then this suggests that there is stability in the network. Another option is to use statistical testing to evaluate the similarity of ω_1 and ω_2. For instance, we could use a two-dimensional version of the Kolmogorov-Smirnov test, such as that in Justel et al. [19].

4.3 Implementation Method

Both proposed tasks require creating belief distributions and heatmaps from the empirical data. We can create the belief distributions by hypothesizing that directly posting or reposting a tweet that directly supports or denies the claim on the misinformation topic equates to the user having 100% or 0% belief, respectively. Then, for users that like relevant tweets, the number of likes translates into the strength of their belief or disbelief, and a regression could be fitted to determine the specific strengths. Lastly, given that we can collect data on many users, we can calculate the frequencies of each belief strength and construct an empirical model of the cumulative distribution. (A depiction of what the distributions could look like are in the figure below.)

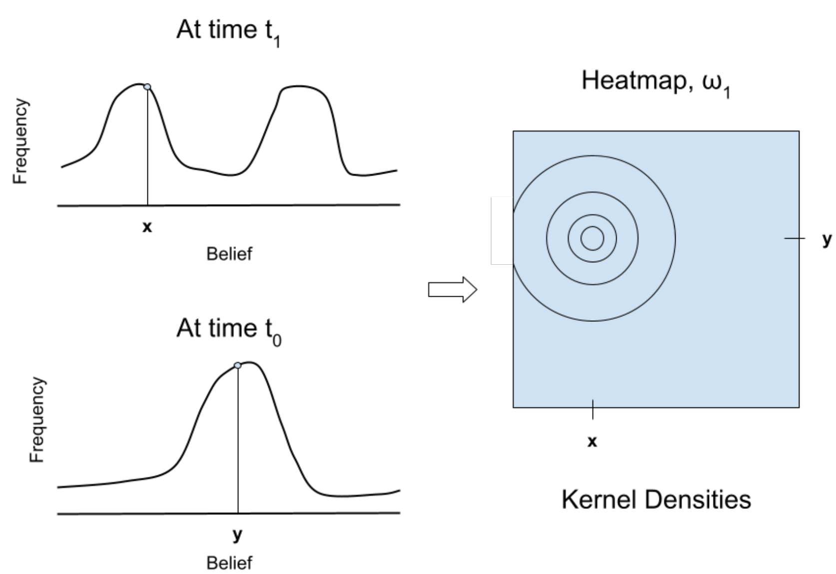

Figure 4: Visualization of heatmap creation

Next, one way that we could fit heatmaps is by sampling users from our dataset, which we call $\{u_1, \ldots u_n\}$ (where n is the number of users we sample), note how their belief changes overtime, which we can write as $\{(x_1, y_1), (x_2, y_2), \ldots (x_n, y_n)\}$, and then estimate the heatmap using those pairs of points through kernel density estimation. (See Figure 4 for a visualization of the process.)

More specifically, we use kernel density estimation (KDE) by first using each of the points (x_i, y_i) to generate a kernel using a Gaussian distribution and fitting the hyperparameters of the KDE with cross-validation. Then, we can apply kernel smoothing to estimate the probability density functions for the heatmap. This method suites our requirements nicely because it returns a heatmap that we conjecture is valid for our model.

5 Conclusion

In this proposal, we introduced the idea of belief propagation models (BPMs) as a potential way to capture the evolution of distributions related to networks. We cover the model in detail and introduce some preliminary definitions of various objects of interest in the study of BPMs. Furthermore, we introduce some avenues of future thought as well as a couple of conjectures. Lastly, we gave an overview of how this model could be used empirically in the setting of tracking and predicting the dynamics of misinformation. Overall, we believe that BPMs have great potential to be a useful idea in network science and provide greater flexibility in modeling how processes can affect populations in a network by looking at distributions rather than statistics of those distributions.

References

[1] Gadi Fibich. Bass-sir model for diffusion of new products in social networks. *Physical Review E*, 94(3), 2016.

[2] Matthew O. Jackson. *Social and Economic Networks*. Princeton University Press, 2011.

[3] Everett M. Rogers. *Diffusion of innovations*. Free Press, 2005.

[4] S Shyam Sundar, Maria D Molina, and Eugene Cho. Seeing is believing: Is video modality more powerful in spreading fake news via online messaging apps? *Journal of Computer-Mediated Communication*, 26(6):301–319, 2021.

[5] Rui Hou, Veronica Perez-Rosas, Stacy Loeb, and Rada Mihalcea. Towards automatic detection of misinformation in online medical videos. *2019 International Conference on Multimodal Interaction*, 2019.

[6] Taichi Murayama, Shoko Wakamiya, Eiji Aramaki, and Ryota Kobayashi. Modeling the spread of fake news on twitter. *PLOS ONE*, 16(4), 2021.

[7] Reneé Weber and Jennifer Crocker. Cognitive processes in the revision of stereotypic beliefs. *Journal of Personality and Social Psychology*, 45(5):961–977, 1983.

[8] Daniel Williams. Socially adaptive belief. *Mind and Language*, 36(3):333–354, 2020.

[9] Frank M. Bass. A new product growth for model consumer durables. *Management Science*, 15(5):215–227, 1969.

[10] William Ogilvy Kermack and A. G. McKendrick. A contribution to the mathematical theory of epidemics. *Proceedings of the Royal Society of London. Series A, Containing Papers of a Mathematical and Physical Character*, 115(772):700–721, 1927.

[11] Morris H. Degroot. Reaching a consensus. *Journal of the American Statistical Association*, 69(345):118–121, 1974.

[12] Pushpi Paranamana, Pei Wang, and Patrick Shafto. Evolution of beliefs in social networks. *Collective Intelligence*, 1(2):1–14, Dec 2022.

[13] Dinghua Shi, Qinghua Chen, and Liming Liu. Markov chain-based numerical method for degree distributions of growing networks. *Physical Review E*, 71(3), 2005.

[14] Guodong Shi, Alexandre Proutiere, Mikael Johansson, John S. Baras, and Karl H. Johansson. The evolution of beliefs over signed social networks. *Operations Research*, 64(3):585–604, Jun 2016.

[15] Alessandro Nordio, Alberto Tarable, Carla-Fabiana Chiasserini, and Emilio Leonardi. Belief dynamics in social networks: A fluid-based analysis. *IEEE Transactions on Network Science and Engineering*, 5(4):276–287, 2018.

[16] Kai Shu, Deepak Mahudeswaran, Suhang Wang, Dongwon Lee, and Huan Liu. Fakenewsnet: A data repository with news content, social context, and spatiotemporal information for studying fake news on social media. *Big Data*, 8(3):171–188, 2020.

[17] Jing Ma, Wei Gao, Prasenjit Mitra, Sejeong Kwon, Bernard J Jansen, Kam-Fai Wong, and Meeyoung Cha. Detecting rumors from microblogs with recurrent neural networks. *Proceedings of the Twenty-Fifth International Joint Conference on Artificial Intelligence*, 2016.

[18] Bao Tran Truong, Oliver Melbourne Allen, and Filippo Menczer. News sharing networks expose information polluters on social media. *Association for the Advancement of Artificial Intelligence*, Jan 2022. arXiv:2202.00094 [cs].

[19] Ana Justel, Daniel Peña, and Rubén Zamar. A multivariate kolmogorov-smirnov test of goodness of fit. *Statistics & Probability Letters*, 35(3):251–259, Oct 1997.