*$a = \alpha$:)

3.) a) $$\frac{du}{dt} = \frac{a}{1+v^n} - u = f(u,v) \qquad (1)$$

$$\frac{dv}{dt} = \frac{a}{1+u^n} - v = g(u,v) \qquad (2)$$

i. u & v
ii. α
iii. n
iv. 1 (both have term $-u$ or $-v$, constant being 1)

→ b & c done w/ code & in plot doc

d) $\frac{du}{dt} = f(u,v)$ @ s.s. $f(u_s,v_s) = g(u_s,v_s) = 0$

$\frac{dv}{dt} = g(u,v)$ linearize: $U = u - u_s$
$V = v - v_s$

$$\frac{du}{dt} = f(u_s,v_s) + \left.\frac{\delta f}{\delta u}\right|_{u_s,v_s} U + \left.\frac{\delta f}{\delta v}\right|_{u_s,v_s} V$$

$$\frac{dv}{dt} = g(u_s,v_s) + \left.\frac{\delta g}{\delta u}\right|_{u_s,v_s} U + \left.\frac{\delta g}{\delta v}\right|_{u_s,v_s} V$$

$$\frac{du}{dt} = f_u U + f_v V \qquad \frac{dv}{dt} = g_u U + g_v V$$

$$\dot{\vec{x}} = \vec{J}\vec{x}, \quad \vec{x} = \begin{pmatrix} U \\ V \end{pmatrix}, \quad \vec{J} = \begin{pmatrix} f_u & f_v \\ g_u & g_v \end{pmatrix}$$

$$f_u = -1 \qquad f_v = -\left.\frac{v^{n-1} n a}{(1+v^n)^2}\right|_{u_s,v_s}$$

$$g_u = -\left.\frac{u^{n-1} n a}{(1+u^n)^2}\right|_{u_s,v_s} \qquad g_v = -1$$

So... $\overleftrightarrow{J} = \begin{array}{c} \\ \begin{pmatrix} -1 & -\frac{v_s^{n-1} n\alpha}{(1+v_s^n)^2} \\ -\frac{u_s^{n-1} n\alpha}{(1+u_s^n)^2} & -1 \end{pmatrix} \end{array}$

(columns u, v; rows f, g)

to find stability criterion...

$tr(\overleftrightarrow{J}) = f_u + g_v = -2 \quad \therefore \; tr(\overleftrightarrow{J}) < 0$

$det(\overleftrightarrow{J}) = f_u g_v - f_v g_u$

$= 1 - \frac{u_s^{n-1} n\alpha}{(1+u_s^n)^2} * \frac{v^{n-1} n\alpha}{(1+v_s^n)^2}$

let $\alpha = 10$

$n_1 = 1$

$n_2 = 2$

@ s.s. ... for $n_1 = 1$

$0 = \frac{10}{1+v_s} - u_s \qquad u_s = \frac{10}{1+v_s}$

$0 = \frac{10}{1+u_s} - v_s \qquad v_s = \frac{10}{1+u_s}$

$u_s = \frac{10}{1+\left(\frac{10}{1+u_s}\right)}$

$u_s = \frac{10(1+u_s)}{1+u_s+10} \qquad u_s + u_s^2 + \cancel{10u_s} = 10 + \cancel{10u_s}$

$u_s^2 + u_s - 10 = 0$

$u_s = 2.702$ or $-.5 + \sqrt{41}/2$

~~-3.702~~ 1

one steady state when $n=1$ ⟹

$v_s = \frac{10}{0.5 + \frac{\sqrt{41}}{2}}$ or 2.702

So... $det(\overleftrightarrow{J})$ when $n=1$, $det(\overleftrightarrow{J}) > 0$ $\therefore$

$= 1 - \frac{10}{(1+u_s)^2} * \frac{10}{(1+v_s)^2} \approx .467$

stable center S.S.

@ S.S.
for $n=2$

$$0 = \frac{10}{1+v_s^2} - u_s \qquad u_s = \frac{10}{1+v_s^2}$$

$$0 = \frac{10}{1+u_s^2} - v_s \qquad v_s = \frac{10}{1+u_s^2}$$

$$u_s = \frac{10}{1+\left(\frac{10}{1+u_s^2}\right)^2}$$

$$u_s\left(1 + \frac{100}{(1+u_s^2)^2}\right) = 10$$

$$u_s + \frac{100u_s}{1+2u_s^2+u_s^4} = 10$$

$$u_s(1+2u_s^2+u_s^4) + 100u_s = 10(1+2u_s^2+u_s^4)$$

$$u_s + 2u_s^3 + u_s^5 + 100u_s = 10 + 20u_s^2 + 10u_s^4$$

$$u_s^5 - 10u_s^4 + 2u_s^3 - 20u_s^2 + 101u_s - 10 = 0$$

three steady states for $n=2$ $(n>1)$ $\Rightarrow$

$$u_s = \underset{①}{2},\ \underset{②}{5-2\sqrt{6}},\ \underset{③}{5+2\sqrt{6}}$$

$$v_s = 2,\ 5+2\sqrt{6},\ 5-2\sqrt{6}$$

* Steady state concentration values will remain in domain $(0,\infty)$

$\det(\overleftrightarrow{J})$ when $n=2$...

① $1 - \left[\frac{2(2)(10)}{(1+2^2)^2}\right]^2 = -1.56$ $\quad \det(\overleftrightarrow{J}) < 0$

$\therefore$ $\boxed{\text{unstable center steady state}}$

② & ③ $.96$ $\quad \det(\overleftrightarrow{J}) > 0 \therefore$ stable

* stable if $tr(\overleftrightarrow{J}) < 0$ & $\det(\overleftrightarrow{J}) > 0$

e) found via MATLAB code

eigenvalues for center S.S. when $n=1$: $\boxed{-1.7298 \ \& \ -0.2702}$

" " " " " $n=2$: $\boxed{-2.6 \ \& \ 0.6}$

When n increased to 2, not all eigenvalues were less than 0 so the center steady state is unstable. n=1 eigenvalues were all <0 ∴ stable.

last explanation may have been hard to read:
when n increased to 2, not all eigenvalues <0, so center S.S. is unstable. when n=1, eigenvalues all <0 ∴ stable.

f)-1 $\frac{dR_i^*}{dt} = K_f L R_i - K_r R_i^*$ (3) $\Rightarrow N_i = \text{const.}$

$\frac{dN_i^*}{dt} = K_f^{ND} N_i D_j - K_r^{ND} N_i^*$ (4)

$\frac{dD_i}{dt} = K_D R_i^* - \gamma_D D_i$ (5)

$\frac{dR_i}{dt} = \frac{\beta^n}{K^n + N_i^{*n}} - \gamma_R R_i$ (6)

Assuming fast eq^m 3-5...

$0 = K_f L R_i - K_r R_i^* \Rightarrow R_i^* = \frac{K_f}{K_r} L R_i$

$0 = K_f^{ND} N_i D_j - K_r^{ND} N_i^* \Rightarrow N_i^* = \frac{K_f^{ND}}{K_r^{ND}} N_i D_j$

$0 = K_D R_i^* - \gamma_D D_i \Rightarrow D_i = \frac{K_D}{\gamma_D} R_i^*$

$D_i = \frac{K_D}{\gamma_D}\left(\frac{K_f}{K_r} L R_i\right)$

$$N_i^* = \underbrace{\frac{K_f^{ND} K_D K_f}{K_r^{ND} \gamma_D K_r}}_{\text{let this be } K_{all}} N_i L R_j$$

general solution: $\frac{dR_i}{dt} = \frac{\beta^n}{K^n + (K_{all} N_i L R_j)^n} - \gamma_R R_i$

putting it in two forms...

$$\frac{dR_1}{dt} = \frac{\beta^n}{K^n + (K_{an} N_1 L R_2)^n} - \gamma_R R_1 = f(R_1, R_2)$$

$$\frac{dR_2}{dt} = \frac{\beta^n}{K^n + (K_{an} N_2 L R_1)^n} - \gamma_R R_2 = g(R_1, R_2)$$

-2

$$\frac{dR_1}{dt} = \frac{\beta^n \;\frac{1}{K^n}}{\left[K^n + (K_{an} N_1 L R_2)^n\right]\frac{1}{K^n}} - \gamma_R R_1$$

$$\frac{1}{K}\left(\frac{dR_1}{dt}\right) = \frac{\left(\frac{\beta}{K}\right)^n}{1 + \left(\frac{K_{an} N_1 L}{K}\right)^n v^n} - \gamma_R R_1$$

$$\frac{du}{dt} = \frac{\left(\frac{\beta}{K}\right)^n}{K\left(1 + \left(\frac{K_{an} N_1 L}{K}\right)^n v^n\right)} - \gamma_R u$$

$$\frac{1}{\gamma_R}\left(\gamma_R \frac{du}{d\tau}\right) = \left(\frac{\left(\beta/K\right)^n}{K\left(1 + \left(\frac{K_{an} N_1 L}{K}\right)^n v^n\right)} - \gamma_R u\right)\frac{1}{\gamma_R}$$

$$\frac{du}{d\tau} = \frac{\left(\frac{\beta}{K}\right)^n}{\gamma_R K\left(1 + \left(\frac{K_{an} N_1 L}{K}\right)^n v^n\right)} - u$$

$$\frac{du}{d\tau} = \frac{\beta^n}{\gamma_R K^{n+1}} \; \frac{1}{1 + \left(\frac{K_{an} N_1 L}{K}\right)^n v^n} - u$$

n dimensionless term

$$\frac{dv}{d\tau} = \frac{\beta^n}{\gamma_R \underline{K}_2^{\,n+1}} \frac{1}{1 + \left(\frac{k_{a11} N_2 L}{\underline{K}_2}\right)^n u^n} - v$$

As the concentration of ligand increases, the production rates for both cells' receptors decreases since L in the denominator slows down/inhibits such rates. Thus, the uniform state should become stable as the rates approach 0.