2.13. Знайти асимптотичний час виконання програми у явному вигляді

а) $T(n) = \begin{cases} O(1), & n = 0 \\ T(n-1) + O(1), & n \geq 1 \end{cases}$

$$T(n) = T(n-1) + O(1) \leq T(n-1) + C \leq T(n-2) + 2C \leq$$

$$\leq T(n-3) + 3C \leq \ldots \leq O(1) + nC = O(n)$$

в) $T(n) = \begin{cases} O(1), & n \leq a,\ a > 1 \\ T(n-a) + O(1), & n > a \end{cases}$

$$T(n) \leq T(n-a) + C \leq T(n-2a) + 2C \leq \ldots \leq T(n-ka) + kC \leq$$

$$\leq \left\{ \begin{array}{l} 0 \leq n-ka \leq a \\ k \geq \frac{n-a}{a} \\ k \leq \frac{n}{a} \end{array} \right\} \leq O(1) + \frac{n}{a} C = O(n)$$

c) $T(n) = \begin{cases} O(1) & , n = 0 \\ a\,T(n-1) + O(1) & , n \geq 1,\ a > 1 \end{cases}$

$$T(n) \leq aT(n-1) + C \leq a(aT(n-2) + C) + C = a^2 T(n-2) + C(1+a) \leq$$
$$\leq a^2(aT(n-3) + C) + C(1+a) = a^3 T(n-3) + C(1+a+a^2) \leq \ldots \leq$$
$$\leq a^n T(n-n) + C \cdot \sum_{i=0}^{n-1} a^i = a^n \cdot O(1) + C \cdot \frac{a^n - 1}{a-1} =$$
$$= O(a^n)$$

d) $T(n) = \begin{cases} O(1) & , n \leq a,\ a > 1 \\ aT(n-a) + O(1) & , n > a \end{cases}$

$$T(n) \leq aT(n-a) + C \leq a(aT(n-2a) + C) + C \leq$$
$$\leq a^2(aT(n-3a) + C) + C(1+a) \leq \ldots \leq$$
$$\leq a^k T(n-ka) + C(1 + a + \ldots + a^{k-1}) \leq \left\{ \begin{array}{l} 0 < n - ka \leq a \\ k \leq \frac{n}{a} \\ \frac{n}{a} - 1 \leq k \leq \frac{n}{a} \end{array} \right\} \leq$$
$$\leq a^k \cdot O(1) + C \cdot \frac{a^k - 1}{a-1} = a^{n/a} \cdot O(1) + C \cdot \frac{a^{n/a} - 1}{a-1} =$$
$$= O(a^n)$$

e) $T(n) = \begin{cases} O(1), & n = 0 \\ T(n-1) + O(n), & n \geq 1 \end{cases}$

$$T(n) \leq T(n-1) + Cn \leq T(n-2) + 2Cn \leq \ldots \leq T(0) + n \cdot C \cdot n =$$
$$= O(1) + Cn^2 = O(n^2)$$

f) $T(n) = \begin{cases} O(1) & n \leq a,\ a > 1 \\ T(n-a) + O(n) & , n > a \end{cases}$

$$T(n) \leq T(n-a) + Cn \leq T(n-2a) + 2Cn \leq T(n-3a) + 3Cn \leq \ldots \leq$$
$$\leq T(n-ka) + k \cdot Cn \leq \left\{ \begin{array}{l} 0 < n - ka \leq a \\ \Downarrow \\ \frac{n}{a} - 1 \leq k \leq \frac{n}{a} \end{array} \right\} \leq O(1) + \frac{Cn^2}{a} = O(n^2)$$

g) $T(n) = \begin{cases} O(1) & n = 1 \\ aT([n/a]) + O(1) & n \geq 2,\ a \geq 2 \end{cases}$

$$n = a^m \Rightarrow m = \log_a n$$
$$T(n) \leq aT([n/a]) + C = aT(a^{m-1}) + C \leq a(aT(a^{m-2}) + C) +$$
$$+ C = a^2 T(a^{m-2}) + C(1+a) \leq \ldots \leq a^m \cdot O(1) + C \cdot \frac{a^m - 1}{a-1} =$$
$$= n \cdot O(1) + C \cdot \frac{n-1}{a-1} = O(n)$$

h) $T(n) = \begin{cases} O(1), & n = 1 \\ aT([n/a]) + O(n) & , n \geq 2,\ a \geq 2 \end{cases}$

$$n = a^m \Rightarrow m = \log_a n$$
$$T(n) \leq aT(a^{m-1}) + Cn \leq a^2 T(a^{m-2}) + Cn(1+a) \leq \ldots \leq a^m \cdot O(1) +$$
$$+ Cn \cdot \frac{a^m - 1}{a-1} = n \cdot O(1) + Cn \cdot \frac{n-1}{a-1} = O(n^2).$$