

Homework 2

Remember you are allowed to discuss with classmates (or an AI tool), but that you need to tell me who/what you discussed with + the final submitted writeup should be your own work.

- (1) Let R be a ring and let f_1, \dots, f_t be a regular sequence.

- (a) Prove the following lemma:

Lemma 1 (Colon Capturing). *For all $0 \leq i < t$, we have*

$$\langle f_1, \dots, f_i \rangle : \langle f_{i+1} \rangle = \langle f_1, \dots, f_i \rangle,$$

where when $i = 0$ the left side of the colon is the ideal generated by NO elements, i.e., the zero ideal.

- (b) Using the above lemma and other facts about regular sequences, prove the following proposition:

Proposition 1. *Assume that R has char $p > 0$. Let $I = \langle f_1, \dots, f_t \rangle$. Then*

$$I^{[p^e]} : I = I^{[p^e]} + \langle (f_1 \cdots f_t)^{p^e - 1} \rangle.$$

- (2) (a) Let S be an F -finite regular local ring, let $\{F_*b_i\}_{i=1}^t$ be basis for F_*S , and let $\Phi \in \text{Hom}_S(F_*S, S)$ be the generating map. Suppose that $g \in S$ can be written as $g = \sum_i a_i^p b_i$ for some $a_i \in S$. Show that

$$\Phi(F_*(gS)) = \langle a_1, \dots, a_t \rangle.$$

- (b) The *Frobenius root* of an ideal I (written $I^{[1/p]}$) is the smallest ideal J such that $I \subset J^{[p]}$. Prove that in the setup of part (a), we have

$$\langle g \rangle^{[1/p]} = \langle a_1, \dots, a_t \rangle.$$

- (3) Let $S = k[x_1, \dots, x_n]$ and let f be a homogeneous polynomial of degree d . Using Fedder's criterion¹ prove that if $d > n$, then S/f is not F -split.

¹Recall that Fedder also applies for homogeneous ideals in a (standard) graded ring, taking \mathfrak{m} = the homogeneous maximal ideal

- (4) Let k be a char $p > 0$ field and let $R = k[[x, y, z]]/\langle x^2 + y^3 + z^7 \rangle$. Prove that R is never F -split in any characteristic.

- (5) Let X be a 4×4 matrix of indeterminates, and let $S = \mathbb{Z}/3[X]$ be the polynomial ring over these indeterminates (so, a 16 variable polynomial ring). Let I_2 be the ideal of 2×2 minors. Using Macaulay2 and Fedder's criterion, verify that S/I_2 is F -split.²

Allowed Functions: The only thing from **TestIdeals** you are allowed to use is the **frobenius** function (so, no **isFPure**!). However, you are free to use ANY other functions available in base Macaulay2 or any other packages.³

Submission: Turn in your M2 code, HCC .submit file (if you used one), AND the output to your code. Format flexible: can be a screenshot if you did it in interactive mode, or copy-pasting the file contents in a verbatim environment into tex, or whatever works best for you.

²For me on the HCC, this check took about 1 hour, and used about 5gb of memory. If you want to double-check your code for errors, first try the much-faster $p = 2$ version (which is also F -split!)

³Explore the documentation... you might find something useful, like **minors**, or how to quickly make a ring with 16 variables!