Homework 2

Remember you are allowed to discuss with classmates (or an AI tool), but that you need to tell me who/what you discussed with + the final submitted writeup should be your own work.

(1) Let R be a ring and let $f_1, \ldots, f_t$ be a regular sequence.

(a) Prove the following lemma:

Lemma 1 (Colon Capturing)**.** *For all* $0 \leq i < t$*, we have*
$$\langle f_1, \ldots, f_i \rangle : \langle f_{i+1} \rangle = \langle f_1, \ldots, f_i \rangle,$$
where when $i = 0$ *the left side of the colon is the ideal generated by NO elements, i.e., the zero ideal.*

(b) Using the above lemma and other facts about regular sequences, prove the following proposition:

Proposition 1. *Assume that* R *has char* $p > 0$*. Let* $I = \langle f_1, \ldots, f_t \rangle$*. Then*
$$I^{[p^e]} : I = I^{[p^e]} + \langle (f_1 \cdots f_t)^{p^e - 1} \rangle.$$

(2) (a) Let S be an F-finite regular local ring, let $\{F_* b_i\}_{i=1}^{t}$ be basis for $F_* S$, and let $\Phi \in \operatorname{Hom}_S(F_* S, S)$ be the generating map. Suppose that $g \in S$ can be written as $g = \sum_i a_i^p b_i$ for some $a_i \in S$. Show that
$$\Phi(F_*(\langle g \rangle)) = \langle a_1, \ldots, a_t \rangle.$$

(b) The *Frobenius root* of an ideal I (written $I^{[1/p]}$) is the smallest ideal J such that $I \subset J^{[p]}$. Prove that in the setup of part (a), we have
$$\langle g \rangle^{[1/p]} = \langle a_1, \ldots, a_t \rangle.$$

(3) Let $S = k[x_1, \ldots, x_n]$ and let f be a non-zero homogeneous polynomial of degree d. Using Fedder's criterion[1] prove that if $d > n$, then S/f is not F-split.

[1] Recall that Fedder also applies for homogeneous ideals in a (standard) graded ring, taking $\mathfrak{m}$ = the homogeneous maximal ideal

(4) Let k be a char $p > 0$ field and let $R = k[[x, y, z]]/\langle x^2 + y^3 + z^7\rangle$. Prove that R is never F-split in any characteristic.

(5) Let X be a 4×4 matrix of indeterminates, and let $S = \mathbb{Z}/3[X]$ be the polynomial ring over these indeterminates (so, a 16 variable polynomial ring). Let I_2 be the ideal of 2×2 minors. Using Macaulay2 and Fedder's criterion, verify that S/I_2 is F-split. [2]

Allowed Functions: The only thing from `TestIdeals` you are allowed to use is the `frobenius` function (so, no isFPure!). However, you are free to use ANY other functions available in base Macaulay2 or any other packages.[3]

Submission: Turn in your M2 code, HCC .submit file (if you used one), AND the output to your code. Format flexible: can be a screenshot if you did it in interactive mode, or copy-pasting the file contents in a verbatim environment into tex, or whatever works best for you.

[2]For me on the HCC, this check took about 1 hour, and used about 5gb of memory. If you want to double-check your code for errors, first try the much-faster $p = 2$ version (which is also F-split!)

[3]Explore the documentation... you might find something useful, like `minors`, or how to quickly make a ring with 16 variables!