

## Homework 2

Remember you are allowed to discuss with classmates (or an AI tool), but that you need to tell me who/what you discussed with + the final submitted writeup should be your own work.

- (1) Let  $R$  be a ring and let  $f_1, \dots, f_t$  be a regular sequence.

- (a) Prove the following lemma:

**Lemma 1** (Colon Capturing). *For all  $0 \leq i < t$ , we have*

$$\langle f_1, \dots, f_i \rangle : \langle f_{i+1} \rangle = \langle f_1, \dots, f_i \rangle,$$

*where when  $i = 0$  the left side of the colon is the ideal generated by NO elements, i.e., the zero ideal.*

- (b) Using the above lemma and other facts about regular sequences, prove the following proposition:

**Proposition 1.** *Assume that  $R$  has  $\text{char } p > 0$ . Let  $I = \langle f_1, \dots, f_t \rangle$ . Then*

$$I^{[p^e]} : I = I^{[p^e]} + \langle (f_1 \cdots f_t)^{p^e - 1} \rangle.$$

- (2) (a) Let  $S$  be an  $F$ -finite regular local ring, let  $\{F_* b_i\}_{i=1}^t$  be basis for  $F_* S$ , and let  $\Phi \in \text{Hom}_S(F_* S, S)$  be the generating map. Suppose that  $g \in S$  can be written as  $g = \sum_i a_i^p b_i$  for some  $a_i \in S$ . Show that

$$\Phi(F_*(gS)) = \langle a_1, \dots, a_t \rangle.$$

- (b) The *Frobenius root* of an ideal  $I$  (written  $I^{[1/p]}$ ) is the smallest ideal  $J$  such that  $I \subset J^{[p]}$ . Prove that in the setup of part (a), we have

$$\langle g \rangle^{[1/p]} = \langle a_1, \dots, a_t \rangle.$$

- (3) Let  $S = k[x_1, \dots, x_n]$  and let  $f$  be a homogeneous polynomial of degree  $d$ . Using Fedder's criterion<sup>1</sup> prove that if  $d > n$ , then  $S/f$  is not  $F$ -split.

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<sup>1</sup>Recall that Fedder also applies for homogeneous ideals in a (standard) graded ring, taking  $\mathfrak{m}$  = the homogeneous maximal ideal

- (4) Let  $k$  be a char  $p > 0$  field and let  $R = k[[x, y, z]]/\langle x^2 + y^3 + z^7 \rangle$ . Prove that  $R$  is never  $F$ -split in any characteristic.

- (5) Let  $X$  be a  $4 \times 4$  matrix of indeterminates, and let  $S = \mathbb{Z}/3[X]$  be the polynomial ring over these indeterminates (so, a 16 variable polynomial ring). Let  $I_2$  be the ideal of  $2 \times 2$  minors. Using Macaulay2 and Fedder's criterion, verify that  $S/I_2$  is  $F$ -split.<sup>2</sup>

*Allowed Functions:* The only thing from `TestIdeals` you are allowed to use is the `frobenius` function (so, no `isFPure!`). However, you are free to use ANY other functions available in base Macaulay2 or any other packages.<sup>3</sup>

*Submission:* Turn in your M2 code, HCC .submit file (if you used one), AND the output to your code. Format flexible: can be a screenshot if you did it in interactive mode, or copy-pasting the file contents in a verbatim environment into tex, or whatever works best for you.

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<sup>2</sup>For me on the HCC, this check took about 1 hour, and used about 5gb of memory. If you want to double-check your code for errors, first try the much-faster  $p = 2$  version (which is also  $F$ -split!)

<sup>3</sup>Explore the documentation... you might find something useful, like `minors`, or how to quickly make a ring with 16 variables!