# Exactness of SOS relaxations in copositive programming

Research Project Report - "Modelling seminar project" course

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## Abstract

For our research project, we studied the Parrilo relaxations for certifying copositivy of a given matrix of size $6 \times 6$. In this small project report, we first present quickly the framework of Parrilo relaxations with its hypotheses and the usual notations, along with a sum-up of the sums of squares literature. Our main contribution is to certify the exactness of Parrilo relaxations on unit diagonal copositive matrices for $n = 6$. We first present the framework of Parrilo relaxations quickly while recapitulating notations. In section 2, we present the computational approach taken to check the exactness of Parrilo relaxations and the results obtained. Then, we present the structure of the sums-of-squares decomposition obtained numerically and explain it in section 3. In section 4, we prove the 1-Parrilo cone to be a tractable approximation of the unit-diagonal copositive cone for $n = 6$.

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If needed, see on-line at `https://github.com/sofianetanji/copositive-matrices` for additional resources (slides, code, figures, complete bibliography etc), open-sourced under the GNU GPL v3 License.

# Outline



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**Project Advisor:** Roland Hildebrand (DAO team, Laboratoire Jean Kuntzmann)
**Course:** *"Modeling seminar and projects"*, by Emmanuel Maître, in 2020,
**Master** 2 **program:** Master of Science in Industrial and Applied Mathematics (MSIAM) at Grenoble INP - Ensimag.

# 1 Presentation and motivation

Here are detailed the main results on copositive matrices and its applications along with common approximations of the copositive cone.

## 1.1 The copositive cone

## 1.2 Parrilo cones and sums of squares

## 1.3 Quick overview of the goals of this project

The next sections aim to prove that every matrix $A \in \mathcal{COP}^6$ with unit diagonal is in $\mathcal{K}_6^1$.

# 2 Numerical certificate of exactness

This section presents the computational approach taken to have a numerical certificate. We detail how we obtained the special copositive matrices and how we checked existence of a sums of squares decomposition.

## 2.1 Generation of random instances

We focused on generating random instances of the extreme rays of $\mathcal{COP}^6$. This is motivated by [1] which provides a complete classification of the extreme rays and by the strong result that an inner approximation of a convex cone is exact if and only if it contains all extreme rays. Namely, checking the exactness of Parrilo relaxations on the extreme rays is equivalent to checking it on the whole $\mathcal{COP}^6$ cone.

1. Pick a family of special matrices.
2. Generate random angles $(\phi_i)_i$.
3. Generate the special matrix $A$ parametrized by the angles.

**Figure 1:** RANDOM GENERATION OF A SPECIAL MATRIX (randomgen.py)

The family of special matrices are all defined in [1] and corresponds to stratums 13.1, 13.2, 16, 17, 19.
The number of operations (*"flops"*) necessary for this algorithm is about :

$$T_{\text{RandomSpecialInstance}}(n_\phi) = n_\phi^2 + 7n_\phi + 48 = \mathcal{O}(n_\phi^2). \tag{1}$$

where $n_\phi$ is the number of angles needed to parametrize one special matrix, namely 5, 6 or 7 in our case. We provide in appendix the analytical expression of the parametrization of each special matrix. The corresponding code is in `src/randomgen.py`.

## 2.2 Certify a matrix is in the Parrilo cone

If each special matrix $A$ built using the previous algorithm in Figure 1 belongs to a Parrilo cone, we will have the numerical certificate we are searching for. Therefore, if for a given $r$, we are able to check if a given matrix belongs to the Parrilo cone $\mathcal{K}_6^r$, we will have our numerical certificate. We explain below how this was implemented.

## 2.3 Implementation of the semi-definite program and results

## 2.4 Solving the SDP : issues tackled

We quickly present the issues faced to solve the SDP.

# 3 Structure of the SOS decomposition

In this section, we detail the structure of the SOS decompositions obtained numerically for each family, discuss about the monomials participating to the decomposition for each family and what this tells us on the structure of the problem.

### 3.1 Family 1

### 3.2 Family 2

### 3.3 Family 3

### 3.4 Family 4

### 3.5 Family 5

## 4 Analytical certificate of exactness

We now present the main contribution of our project, an analytical certificate of exactness of Parrilo relaxations in the case $n = 6$.

### 4.1 SOS decomposition as a function of angles

### 4.2 SOS decomposition and the kernels of the special matrices

### 4.3 Certificate of exactness for each family

### 4.4 Questions still not answered

Hopefully, we'll be able to delete this subsection ☺

## 5 Conclusion

## A Special matrices

# B Introduction

## B.1 Copositive matrices

Write a proper introduction, usefulness of copositive matrices, what do they represent, what are we going to do, the layout of the report

**Definition B.1.** *A matrix $A \in \mathcal{M}_{n\times n}(\mathbb{R})$ is a* ***copositive*** *matrix if $\forall x \in \mathbb{R}^n_+, x^T Ax \geq 0$. The set of copositive matrices*

$$\mathcal{COP}^n = \{A \in \mathcal{M}_{n\times n}\mathbb{R} |\ A \text{ is copositive}\}$$

*is called the* ***copositive cone****.*

Let us define

$$\begin{aligned} \mathcal{S}^n_+ &: \text{ the set of positive semi-definite matrices} \\ \mathcal{N}_n &: \text{ the set of element-wise non-negative symmetric matrices} \end{aligned}$$

fill the blank spaces with comments to create a smooth flow

**Theorem B.2** (Dianonda)**.** *For $n \leq 4$*

$$\mathcal{S}^n_+ + \mathcal{N}_n = \mathcal{COP}^n$$

**Theorem B.3** (A. Horn)**.** *For $n \geq 5$*

$$\mathcal{S}^n_+ + \mathcal{N}_n \subsetneq \mathcal{COP}^n$$

**Theorem B.4.** *If $A \in \mathcal{COP}^5$ and $A$ has a unit diagonal then $A \in \mathcal{K}^1_5$.*

The copositive cone $\mathcal{COP}^n$ is invariant under the action of the multiplicative group $\mathbb{R}^n_{++}, A \mapsto DAD$ where $D = diag(d), d \in \mathbb{R}^n_{++}$. Thus $A \in \mathcal{COP}^n$ with a non unit positive diagonal can be scaled to a copositive matrix with unit diagonal by setting $d = diagA$ and applying

$$\tilde{A} = diag\left(d^{-\frac{1}{2}}\right) Adiag\left(d^{-\frac{1}{2}}\right).$$

Put the references

Consequently for $n = 5$ a work-around strategy to check if a matrix $A$ is copositive would be to scale it to have its diagonal a unit vector and then check if it belongs to $\mathcal{K}^1_5$.
A matrix $A$ belongs to the $r$-th Parrilo cone $\mathcal{K}^r_n$ if the polynomial of degree $2r + 4$

$$\left(\sum_{k=1}^{n} x_k^r\right)^r \sum_{K,l=1}^{n} A_{k,l} x_k^2 x_l^2$$

can be represented as a SOS of polynomials of degree $2 + r$.
The goal of the project is to check the exactness of the Parrilo relaxations on unit diagonal copositive matrices in the $n = 6$ case.

### B.2 Sum of Squares

Let $p : \mathbb{R}^n \to \mathbb{R}$ be any polynomial of degree $2d$. If $p(x) \geq 0, \forall x \in \mathbb{R}^n$, the question is whether can the polynomial be represented as a sum of squares (SOS), i.e. can be decomposed as

$$p(x) = \sum_{n=1}^{m} q_j^2(x)$$

where $q_j(x)$ is a homogeneous polynomial of degree $d$, $i = 1, \ldots, m$.
Let $\mathbf{x} = \{x^\alpha\}_{|\alpha|=d}$ the sequence of all monomials of degree $d$,

$$x^\alpha = \prod_{i=1}^{n} x_i^{\alpha_i}$$

with $\alpha = (\alpha_i)_{i=1,\ldots,d} \in \mathbb{N}^n$ and $|\alpha| = \sum_{i=1}^n \alpha_i$ is the degree .

**Proposition B.5.** *A polynomial $p(x)$ is a SOS if and only if there exists a PSD matrix $C$ such that $p(x) = \mathbf{x}^T C\mathbf{x}$.*

*Proof.* We have

$$q_j(x) = c_j^T \mathbf{x} = \sum_{|\alpha|=d} c_{j,\alpha}^T x^\alpha \tag{2}$$

where $c_j = \{c_{j,\alpha}\}_{|\alpha|=d}$ is the vector of coefficients of $q_j$. The component $c_j^T\mathbf{x}$ is a scalar so

$$\left(c_j^T \mathbf{x}\right)^T = \mathbf{x}^T c_j = c_j^T \mathbf{x} \tag{3}$$

and as a result

$$\begin{aligned}
\sum_{j=1}^{m} q_j(x)^2 &= \sum_{j=1}^{m} \left(c_j^T \mathbf{x}\right)^2 = \sum_{j=1}^{m} c_j^T \mathbf{x} c_j^T \mathbf{x} \\
&= \sum_{j=1}^{m} \mathbf{x}^T c_j c_j^T \mathbf{x} = \mathbf{x}^T \left(\sum_{j=1}^{m} c_j c_j^T\right) \mathbf{x} \\
&= \mathbf{x}^T C \mathbf{x}
\end{aligned}$$

where $C = \sum_{j=1}^m c_j c_j^T$ is a PSD matrix.
On the other hand any semi-definite matrix $C \succeq 0$ can be written as the sum $C = \sum_{j=1}^m c_j c_j^T$ and equivalently $\mathbf{x}^T C\mathbf{x}$ is a SOS. □

Verifying that $p(x)$ is a SOS is equivalent to solving the SDP

$$\min_{t,C} t : p(x) = \mathbf{x}^T C\mathbf{x} \text{ and } C + tI \succeq 0 \tag{4}$$

If we write

$$p(x) = \sum_{|\gamma|=2d} a_\gamma x^\gamma$$

then the equality constraint $p(x) = \mathbf{x}^T C\mathbf{x}$ can be divided into simpler constraints on the entries of $C$ by comparing the coeficients at the powers of $x$ on both sides, i.e.

$$a_\gamma = \sum_{\alpha+\beta=\gamma} = C_{\alpha,\beta}$$

*Example* B.6. We take a polynomial $p$ of degree 2

$$p(x) = 9x_1^2 + 4x_2^2 + 12x_1x_2$$

that can be written as the square of a homogeneous matrix $q(x) = 3x_1 + 2x_2$

$$p(x) = (q(x))^2 .$$

We shall find the matrix $C = \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix}$ such that

$$p(x) = \mathbf{x}^T C\mathbf{x}$$

We distribute

$$\begin{aligned}
\mathbf{x}^T C\mathbf{x} = (x_1, x_2) \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \\
&= (c_{11}x_1 + c_{21}x_2, c_{12}x_1 + c_{22}x_2) \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \\
&= c_{11}x_1^2 + (c_{21} + c_{12})\, x_1x_2 + c_{22}x_2^2
\end{aligned}$$

The component $c_{11}$ corresponds to $x_1$ the first component of both $\mathbf{x}$ and $\mathbf{x}^T$ so we can note $c_{11} = c_{(1,0),(1,0)}$. Also $c_{12}$ corresponds to the second component of $\mathbf{x}$ and to the first component of $\mathbf{x}^T$ so we can write it as $c_{12} = c_{(1,0),(0,1)}$. Equivalently we get $c_{21} = c_{(0,1),(1,0)}$ and $c_{22} = c_{(0,1),(0,1)}$. So

$$\mathbf{x}^T C\mathbf{x} = c_{(1,0),(1,0)}x_1^2 + \left(c_{(0,1),(1,0)} + c_{(1,0),(0,1)}\right) x_1x_2 + c_{22} = c_{(0,1),(0,1)}x_2^2$$

We get the system

$$\begin{cases}
a_{(2,0)} = \sum_{\alpha+\beta=(2,0)} C_{\alpha\beta} = c_{(1,0),(1,0)} = 9 \\
a_{(1,1)} = \sum_{\alpha+\beta=(1,1)} C_{\alpha\beta} = c_{(1,0),(0,1)} + c_{(0,1),(1,0)} = 4 \\
a_{(0,2)} = \sum_{\alpha+\beta=(0,2)} C_{\alpha\beta} = c_{(0,1),(0,1)} = 12
\end{cases}$$

And a solution would be

$$C = \begin{pmatrix} 9 & 6 \\ 6 & 4 \end{pmatrix}$$

### B.3 Sum of Squares

Let p be temporary quadratic function $p(x, y, z)$

$$p(x, y, z) = x^4 + y^4 + z^4 - 2x^2yz - -2xy^2z - -2xyz^2$$
$$X = (x^2, y^2, z^2, yz, xz, xy)^T$$

**please complete this part**

$$\max_{t,c} t : C - tI \geq 0 \tag{5}$$
$$a_r = \sum_{\alpha+\beta=r} c_{\alpha\beta} \forall r \tag{6}$$

If $t \geq 0$ then p is SOS representable if $t < 0$ then p is not SOS
$A \in S^n$. A is copositive if $X^T AX \geq 0 \forall X \in \mathbb{R}^n_+$
Question Is A Co-positive?

$$\text{A copositive } \Leftrightarrow P_A = \sum_{i,j=1}^{n} A_{ij} X_i^2 X_j^2 \geq 0$$

## C Semi-definite programming

### C.1 Generation of instances of the special matrices

## References

[1] Andrey Afonin, Roland Hildebrand, and Peter Dickinson. The extreme rays of the $6\times 6$ copositive cone. 2020.