1 Overview

The sequence of events is as follows:

1. The transmitting station computes the data qubit
2. The transmitting station entangles its link qubits A and A'
3. The receiving station entangles its link qubits B and B'
4. The repeater measures the link qubits A' and B'
5. The transmitting station measures the the data qubit and its link qubit $|\Psi\rangle_A$
6. The transmitting station sends its measurement results to the receiving station via a non-quantum communication channel
7. The receiving station selects and applies a gate to its link qubit $|\Psi\rangle_B$, reconstructing the transmitted qubit.
8. The receiving station measures the reconstructed data qubit

2 OPENQASM Quantum Circuit Code

```
OPENQASM 2.0;
include ”qelib1.inc”;

qreg q[5];
creg c[5];

h q[1]; // Endpoint 1 entangles the link qubits A and A’
cx q[1], q[2];
x q[1];
z q[1];

h q[3]; // Endpoint 2 entangles the link qubits B and B’
cx q[3], q[4];
x q[3];
z q[3];


””” + data +
newline ”””

x q[2]; // The repeater measures the link qubits A’ and B’
x q[3];
cx q[2],q[3];
x q[3];
x q[2];
h q[2];
measure q[2]-¿c[2];
measure q[3]-¿c[3];

// Endpoint 1 measures the data qubit with the first link qubit
x q[0];
x q[1];
cx q[0], q[1];
x q[1];
x q[0];
h q[0];
measure q[0]-¿c[0];
measure q[1]-¿c[1];

//Endpoint 2 reconstructs the repeated qubit

// Psi+
if (c==0) z q[4];
if (c==2) y q[4];
if (c==3) x q[4];
```

```
// Psi-
if (c==5) z q[4];
if (c==6) x q[4];
if (c==7) y q[4];

//Phi+
if (c==8) y q[4];
if (c==9) x q[4];
if (c==10) z q[4];

//Phi-
if (c==12) x q[4];
if (c==13) y q[4];
if (c==15) z q[4];

measure q[4]-¿c[4];
```

3 Dirac Notation

$\begin{bmatrix}1\\0\end{bmatrix} \rightarrow |u\rangle$ $\begin{bmatrix}0\\1\end{bmatrix} \rightarrow |d\rangle$

$\begin{bmatrix}1 & 0\end{bmatrix} \rightarrow \langle u|$

$\begin{bmatrix}1\\0\\0\\0\end{bmatrix} \rightarrow |uu\rangle$ $\begin{bmatrix}0\\1\\0\\0\end{bmatrix} \rightarrow |ud\rangle$ $\begin{bmatrix}0\\0\\1\\0\end{bmatrix} \rightarrow |du\rangle$ $\begin{bmatrix}0\\0\\0\\1\end{bmatrix} \rightarrow |dd\rangle$

For n qubits, if the $b_i, i = 0...n-1$ are the states of the qubits left-to-right, then the corresponding state vector has the j-th element equal to one where the binary representation of j is $b_{n-1}, b_{n-2}...b_0$

4 EPR/Bell States

$\Psi^+ = \sqrt{\frac{1}{2}}(|ud\rangle + |du\rangle)$

$\Psi^- = \sqrt{\frac{1}{2}}(|ud\rangle - |du\rangle)$ EPR singlet state

$|\Phi^+\rangle = \sqrt{\frac{1}{2}}(|uu\rangle + |dd\rangle)$

$|\Phi^-\rangle = \sqrt{\frac{1}{2}}(|uu\rangle - |dd\rangle)$

5 Measurement Gate

This is the unitary matrix for the Bell measurement, expressed in the computational basis.

$$U = \begin{bmatrix}\langle\Psi^+| \\ \langle\Psi^-| \\ \langle\Phi^+| \\ \langle\Phi^-|\end{bmatrix} = \begin{bmatrix}0 & 1 & 1 & 0\\ 0 & 1 & -1 & 0\\ 1 & 0 & 0 & 1\\ 1 & 0 & 0 & -1\end{bmatrix}$$

6 Entanglement Swapping

For technical reasons, the sign of the second row of the matrix for the measurement gate, $\langle\Psi^-|$, is inverted in the code. The overall sign of a quantum state does not affect measurement results.

$|\Psi\rangle_{AA'B'B} = \frac{1}{2}(|ud\rangle - |du\rangle)(|ud\rangle - |du\rangle) = \frac{1}{2}[|uddu\rangle - |udud\rangle - |dudu\rangle + |duud\rangle]$

$|uu\rangle = \frac{1}{\sqrt{2}}(|\Phi^+\rangle + |\Phi^-\rangle)$

$|ud\rangle = \frac{1}{\sqrt{2}}(|\Psi^+\rangle + |\Psi^-\rangle)$

$|du\rangle = \frac{1}{\sqrt{2}}(|\Psi^+\rangle - |\Psi^-\rangle)$

$|dd\rangle = \frac{1}{\sqrt{2}}(|\Phi^+\rangle - |\Psi^-\rangle)$

$|\Psi\rangle_{AA'B'B} = \frac{1}{2\sqrt{2}}[|u\rangle\left(|\Phi^+\rangle - |\Phi^-\rangle\right)|u\rangle - |u\rangle\left(|\Psi^+\rangle - |\Psi^-\rangle\right)|d\rangle - |d\rangle\left(|\Psi^+\rangle + |\Psi^-\rangle\right)|u\rangle + |d\rangle\left(|\Phi^+\rangle + |\Phi^-\rangle\right)|d\rangle]$

$|\Psi\rangle_{A'B'AB} = \frac{1}{2\sqrt{2}}[|\Phi^+\rangle|uu\rangle - |\Phi^-\rangle|uu\rangle - |\Psi^+\rangle|ud\rangle + |\Psi^-\rangle|du\rangle + |\Phi^+\rangle|dd\rangle + |\Phi^-\rangle|dd\rangle]$
$= \frac{1}{2\sqrt{2}}[|\Psi^+\rangle(-|ud\rangle - |du\rangle) + |\Psi^-\rangle(|ud\rangle - |du\rangle) + |\Phi^+\rangle(|uu\rangle + |dd\rangle) + |\Phi^-\rangle(-|uu\rangle + |dd\rangle)]$

$= \frac{1}{2}[-|\Psi^+\rangle_{A'B'}|\Psi^+\rangle_{AB} + |\Psi^-\rangle_{A'B'}|\Psi^-\rangle_{AB} + |\Phi^+\rangle_{A'B'}|\Phi^+\rangle_{AB} - |\Phi^-\rangle_{A'B'}|\Phi^-\rangle_{AB}]$

6.1 Endpoint States after Repeater Measurement

The repeater performs a measurement on its entangled states. The possible outcomes and endpoint states are

Measure($A'B'$)	$\|\Psi\rangle_{AB}$
00	$-\|\Psi^+\rangle$
01	$\|\Psi^-\rangle$
10	$\|\Phi^+\rangle$
11	$-\|\Phi^-\rangle$

6.2 Teleportation of data qubit

The state of the data qubit is $\Psi_1 = a|u\rangle + b|d\rangle$

6.2.1 Swapped Entanglement

If $|\Psi\rangle_{AB} = -|\Psi^+\rangle$ then
$|\Psi\rangle_{1AB} = -\frac{1}{\sqrt{2}}(a|u\rangle + b|d\rangle)(|ud\rangle + |du\rangle) = \frac{1}{\sqrt{2}}[-a|uud\rangle - a|udu\rangle - b|dud\rangle - b|ddu\rangle]$
$=\frac{1}{2}[|\Psi^+\rangle(-a|u\rangle + b|d\rangle) + |\Psi^-\rangle(-a|u\rangle - b|d\rangle) + |\Phi^+\rangle(-a|d\rangle + b|u\rangle) + |\Phi^-\rangle(-a|d\rangle - b|u\rangle)]$

If $|\Psi\rangle_{AB} = |\Psi^-\rangle$ then
$|\Psi\rangle_{1AB} = \frac{1}{\sqrt{2}}(a|u\rangle + b|d\rangle)(|ud\rangle - |du\rangle) = a|uud\rangle - a|udu\rangle + b|dud\rangle - b|ddu\rangle$
$= \frac{1}{2}[a(|\Phi^+\rangle + |\Phi^-\rangle)|d\rangle - a(|\Psi^+\rangle - |\Psi^-\rangle)|u\rangle - b(|\Psi^+\rangle|\Psi^-\rangle)|d\rangle + b(|\Phi^+\rangle - |\Phi^-\rangle)|u\rangle]$
$= \frac{1}{2}[|\Psi^+\rangle(-a|u\rangle - b|d\rangle) + |\Psi^-\rangle(a|u\rangle + b|d\rangle) + |\Phi^+\rangle(a|d\rangle + b|u\rangle) + |\Phi^-\rangle(a|d\rangle - b|u\rangle)]$

If $|\Psi\rangle_{AB} = |\Phi^+\rangle$ then
$|\Psi\rangle_{1AB} = \frac{1}{2}[|\Psi^+\rangle(a|d\rangle - b|u\rangle) + |\Psi^-\rangle(a|d\rangle + b|u\rangle) + |\Phi^+\rangle(a|u\rangle - b|d\rangle) + |\Phi^-\rangle(a|u\rangle + b|d\rangle)]$

If $|\Psi\rangle_{AB} = |\Phi^-\rangle$ then
$|\Psi\rangle_{1AB} = \frac{1}{2}[|\Psi^+\rangle(-a|d\rangle - b|u\rangle) + |\Psi^-\rangle(-a|d\rangle + b|u\rangle) + |\Phi^+\rangle(a|u\rangle + b|d\rangle) + |\Phi^-\rangle(a|u\rangle - b|d\rangle)]$

6.2.2 Coding Endpoint Two's Reconstruction of the Teleported Qubit

$\sigma_x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$
$\sigma_y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$
$\sigma_z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

Quantum registers in code:
$1AA'B'B$
01234

Measurement of $A'B'$

A'	B'	b_2	b_3	$\|\Psi\rangle$
0	0	0	0	$-\|\Psi^+\rangle$
0	1	0	1	$\|\Psi^-\rangle$
1	0	1	0	$\|\Phi^+\rangle$
1	1	1	1	$-\|\Phi^-\rangle$

Measurement of $1A$

1	A	b_0	b_1
0	0	0	0
0	1	0	1
1	0	1	0
1	1	1	1

Measurement Combinations

$m(A'B')$	$m(1A)$	Gate	$b_3b_2b_1b_0$	c
00	00	Z	0	0
00	01	none	don't care	
00	10	Y	0001	1
00	11	X	0011	3
01	00	Z	1000	8
01	01	none	don't care	
01	10	X	1001	9
01	11	Y	1011	11
10	00	Y	0100	4
10	01	X	0110	6
10	10	Z	0101	5
10	11	none	don't care	
11	00	X	1100	12
11	01	Y	1110	14
11	10	none	don't care	
11	11	Z	1111	15