Data thinning for convolution-closed distributions

Anna Neufeld

June, 2023

We often wish that we had access to two independent datasets, drawn from the same data-generating mechanism

	Feature 1	Feature 2
Obs. 1	12	6
Obs. 2	31	8
Obs. 3	11	31
Obs. 4	22	34

	Feature 1	Feature 2
Obs. 1	9	3
Obs. 2	33	9
Obs. 3	12	29
Obs. 4	25	41

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Train

	Feature 1	Feature 2
Obs. 1	12	6
Obs. 2	31	8
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Test

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Obs. 1	9	3
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Obs. 3	12	29
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Fit model.

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Validate model.

As we typically have access to only one dataset, a common solution is sample splitting

Original data

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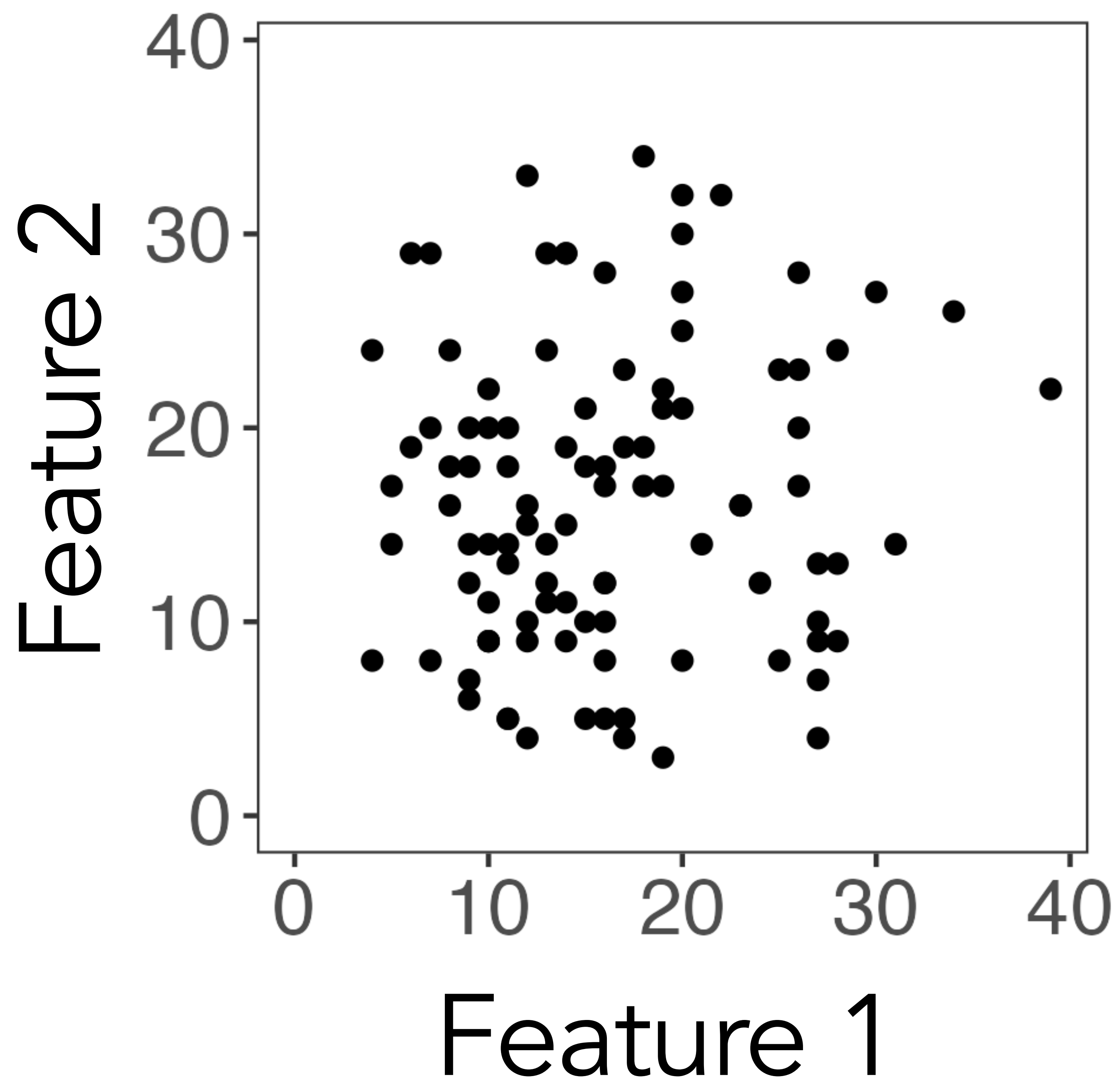
Validate model.

Outline

1. **Motivation: sample splitting doesn't always work**
2. Poisson thinning
3. Data thinning
4. Application to changepoint detection
5. Generalized data thinning

Example: how many clusters are in this data?



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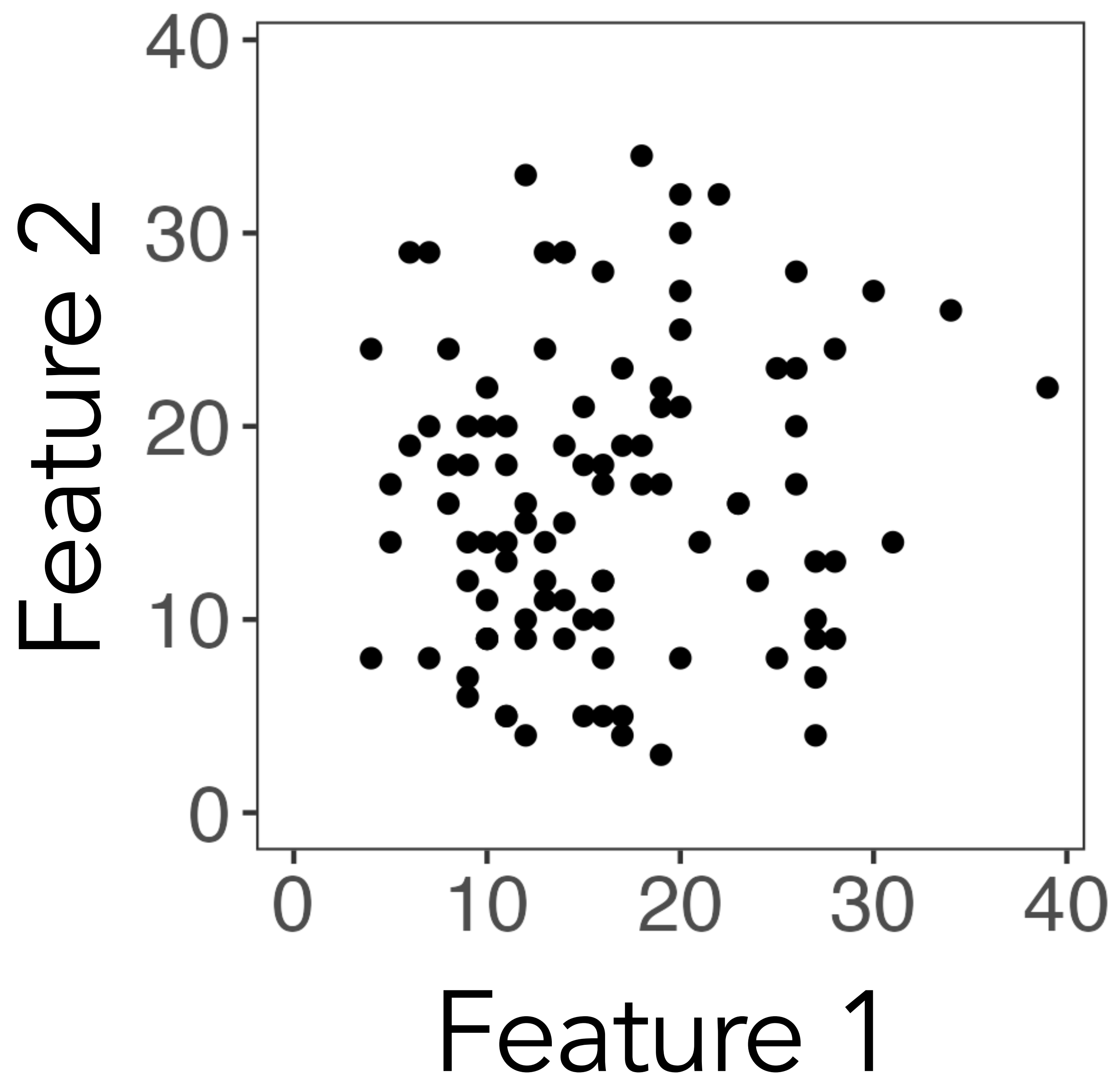

For several values of k:

Step 1: fit a model with k clusters.

Step 2: evaluate model using a loss function.

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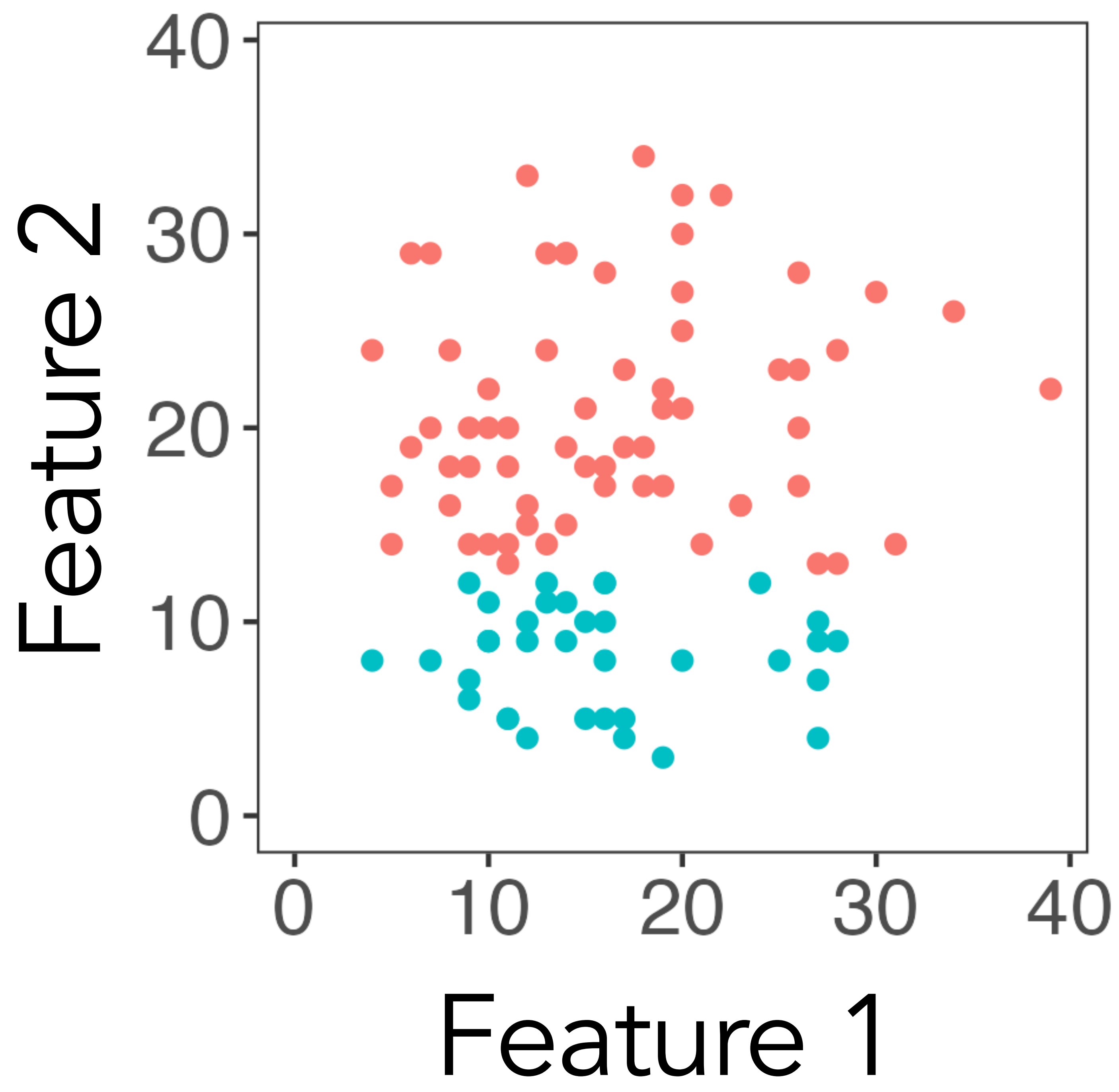

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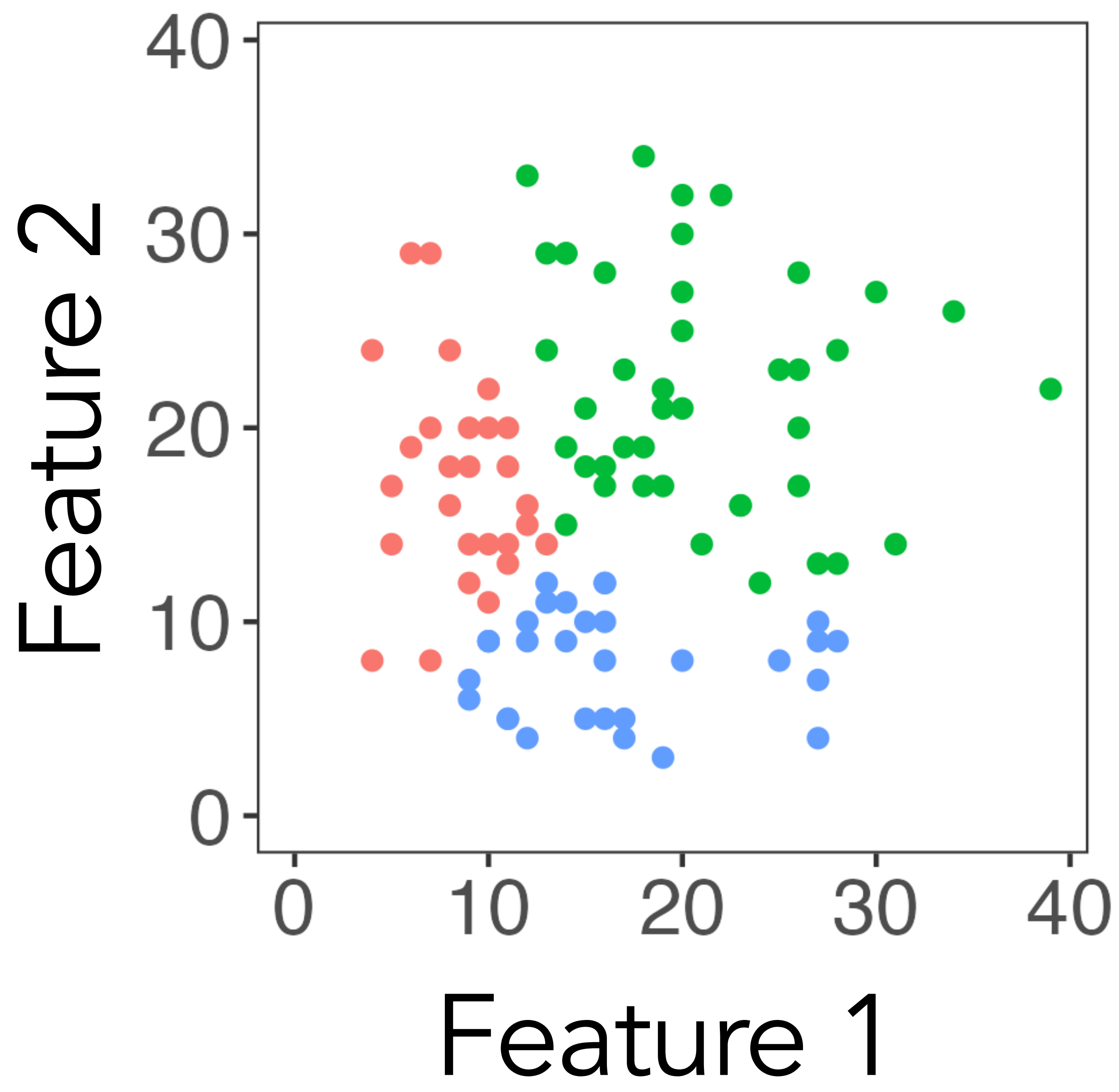

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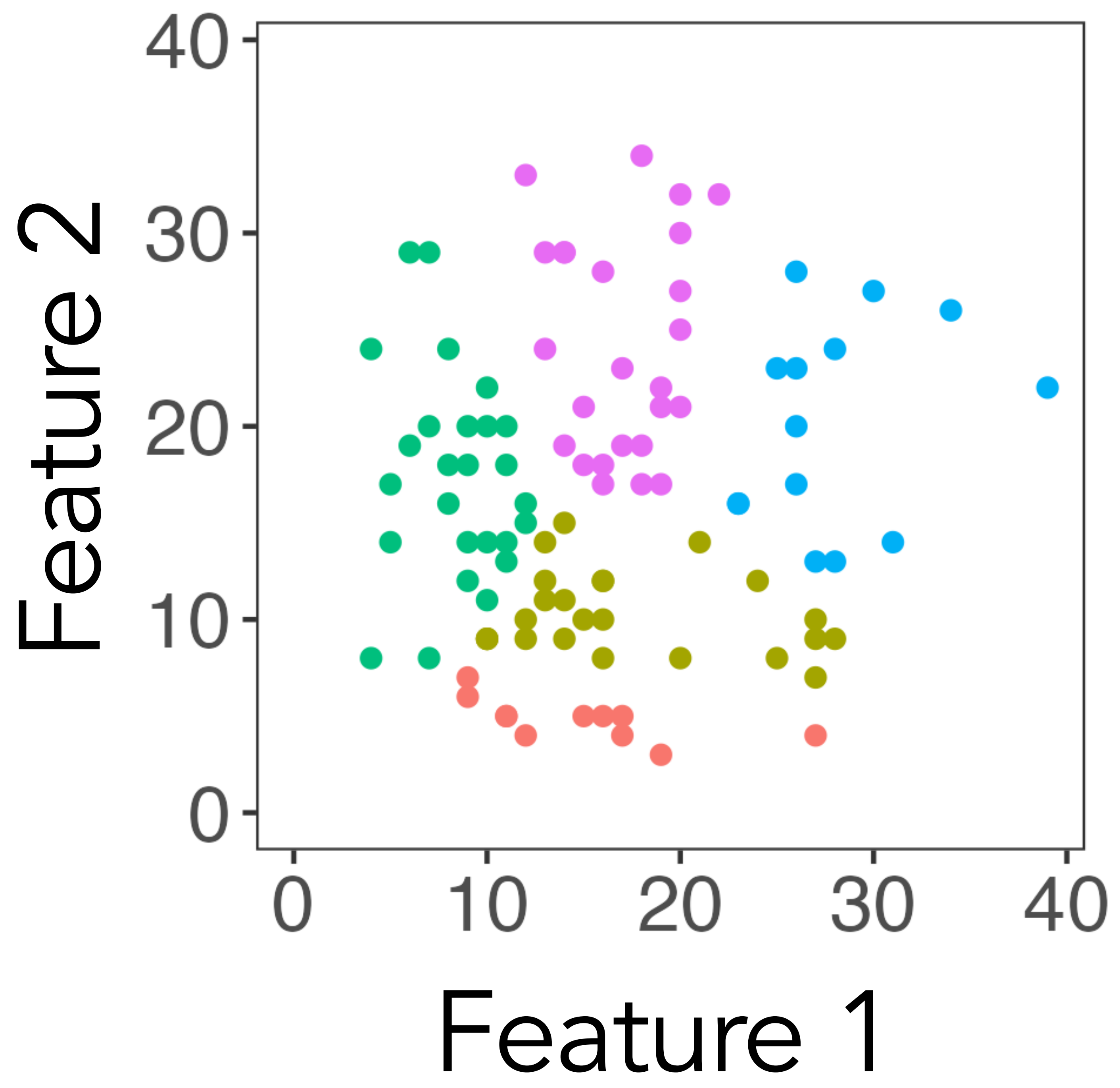

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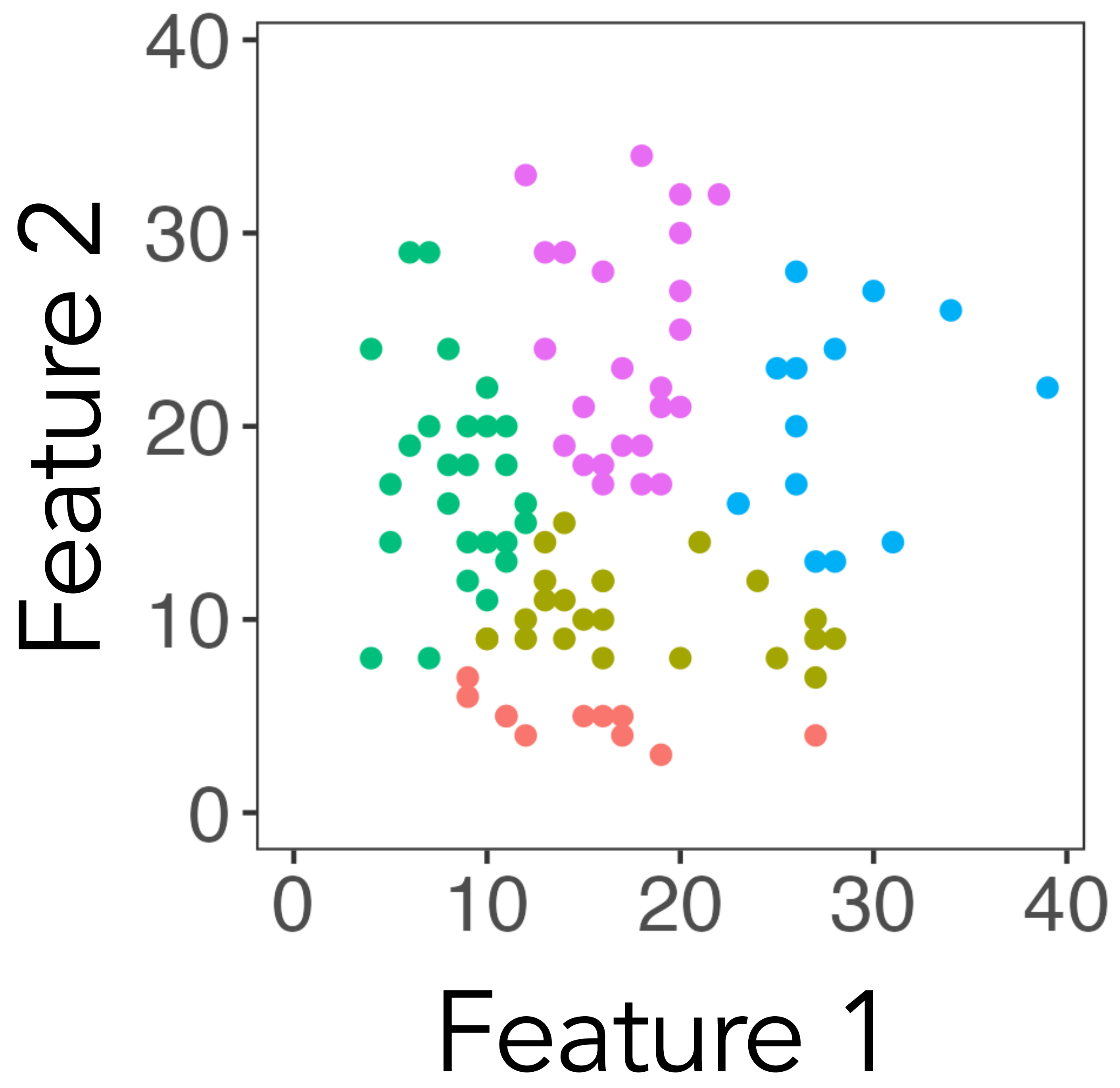

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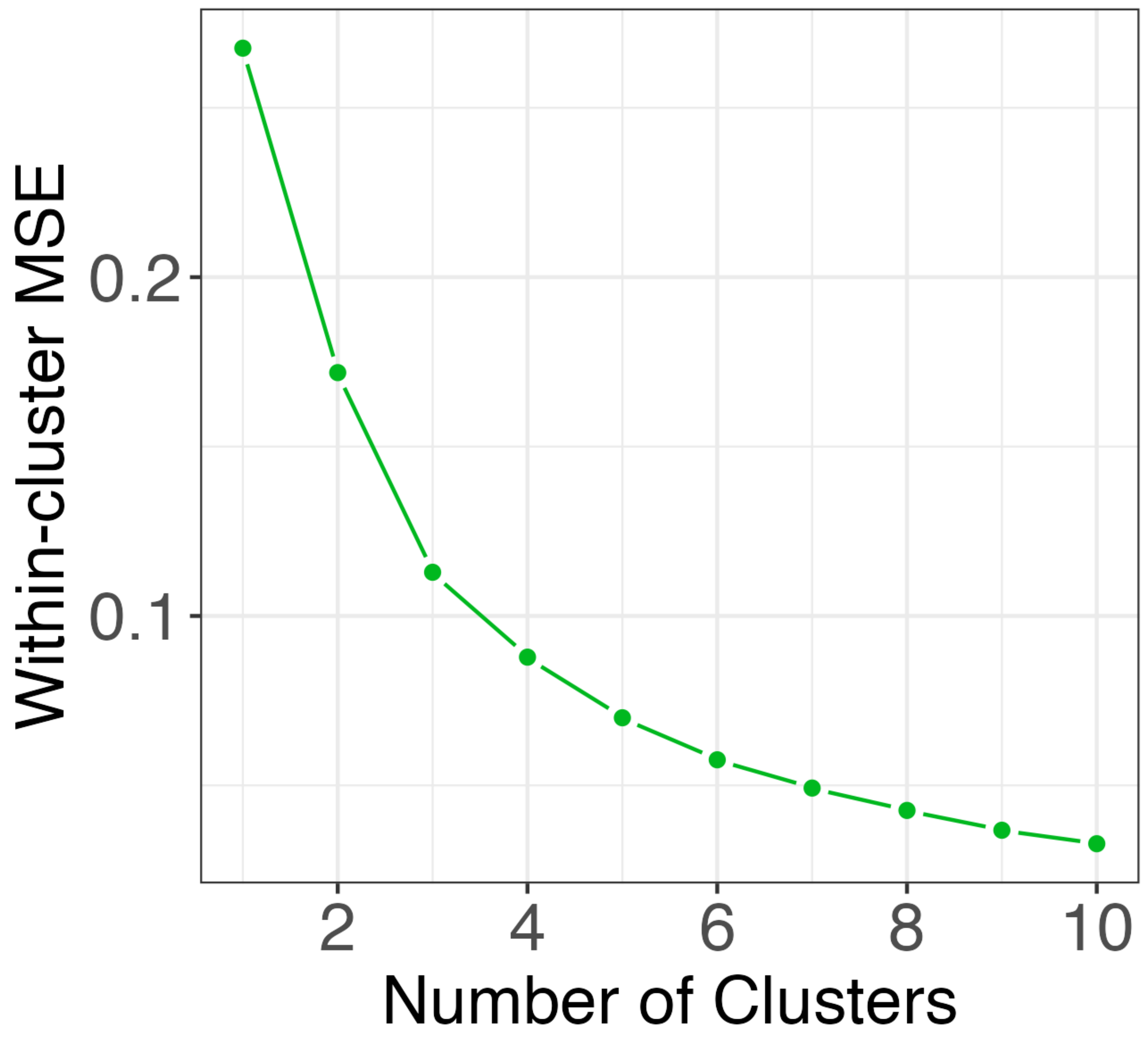
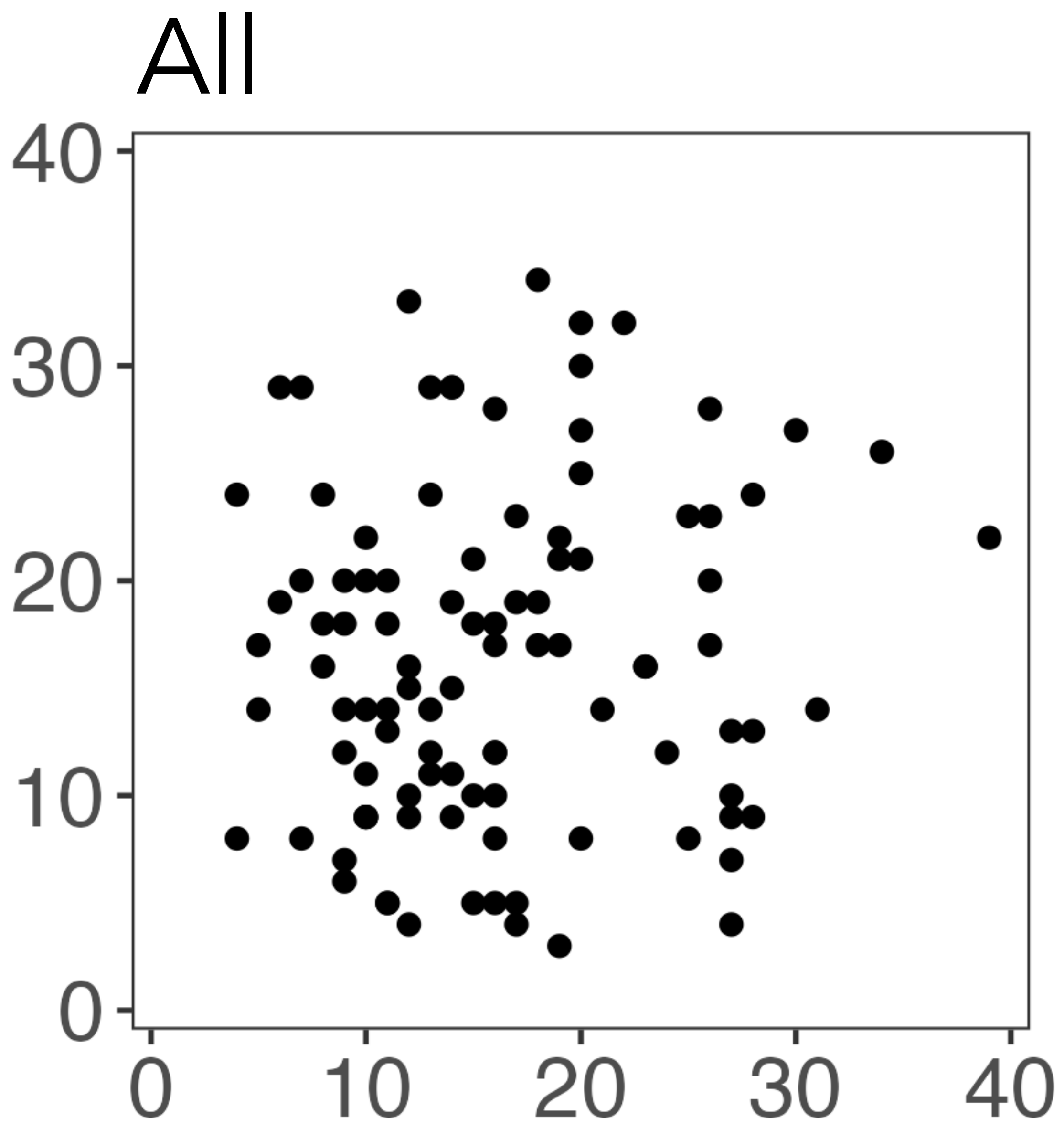


Sample splitting cannot be used to validate the results of clustering



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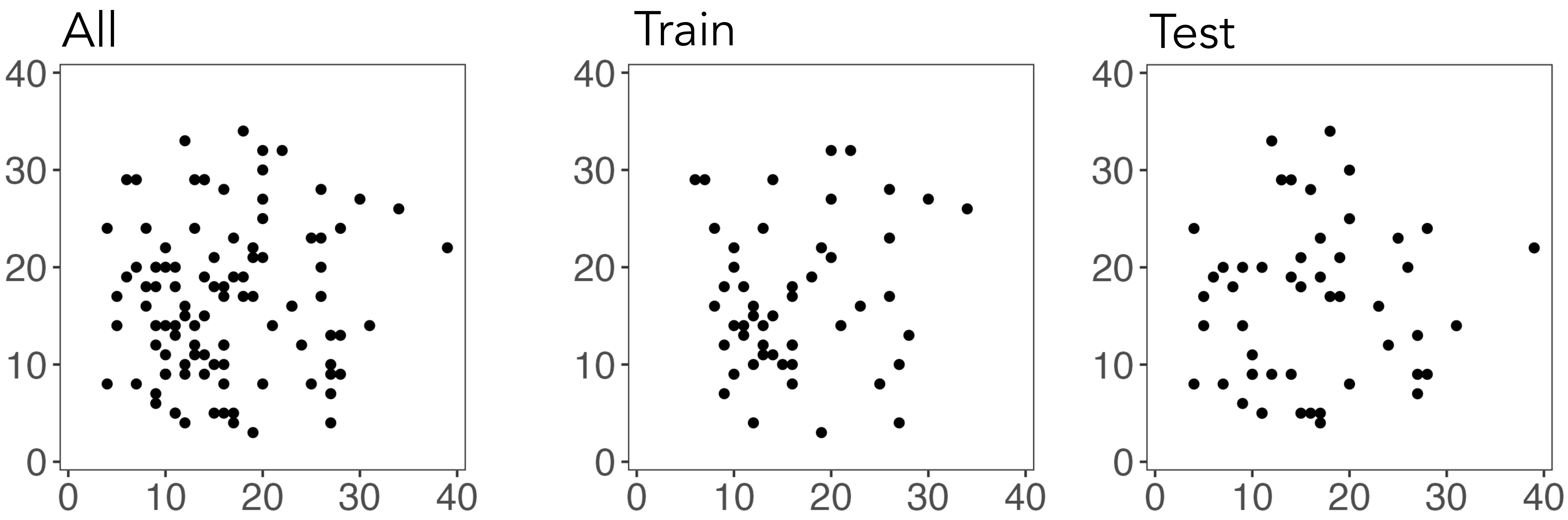

Step 1: split observations into train/test.

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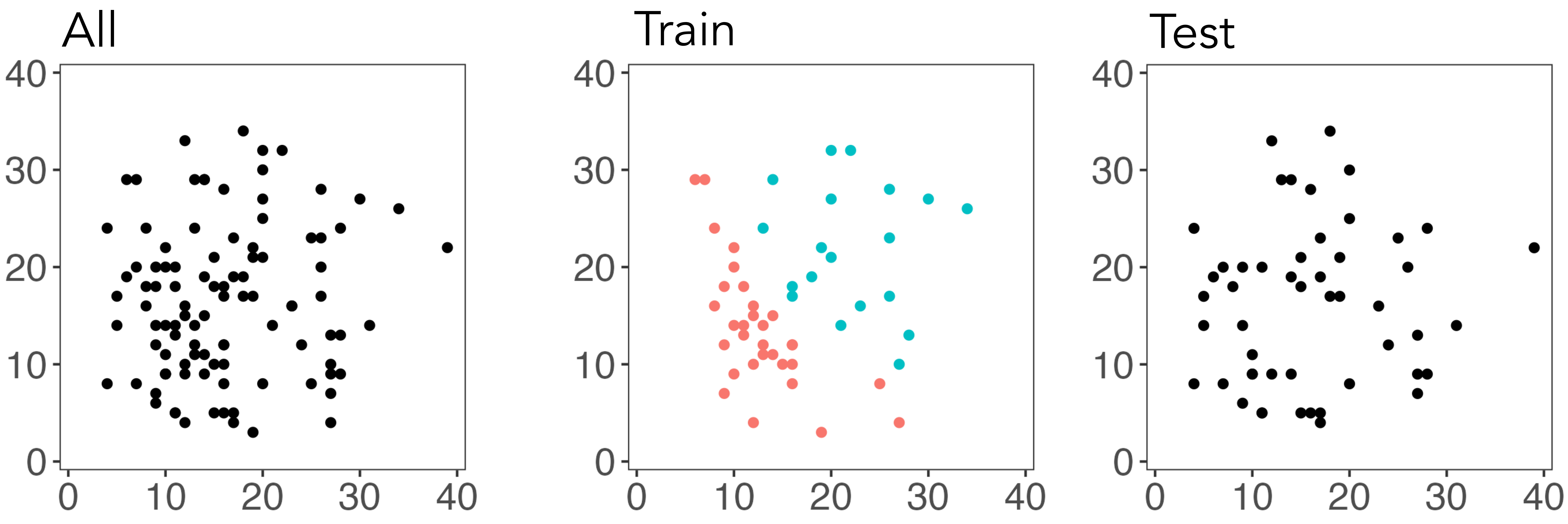

Step 1: split observations into train/test.

Step 2: cluster the training set.

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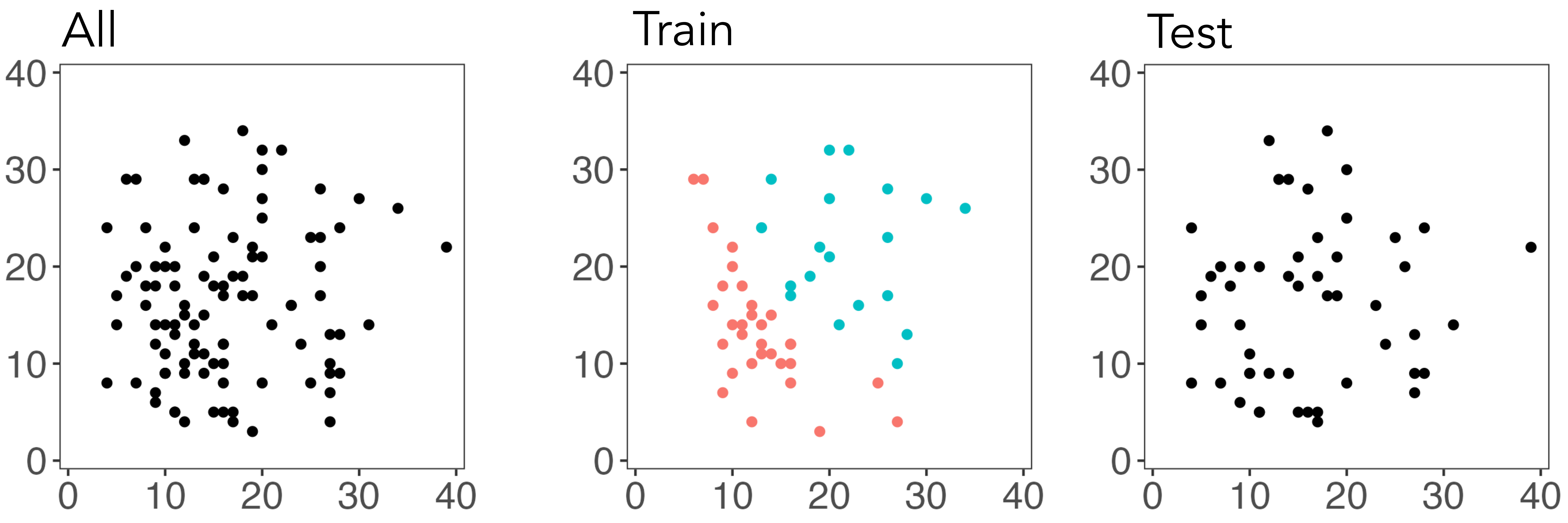

Step 1: split observations into train/test.

Step 2: cluster the training set.

Step 3: evaluate clusters using test set.

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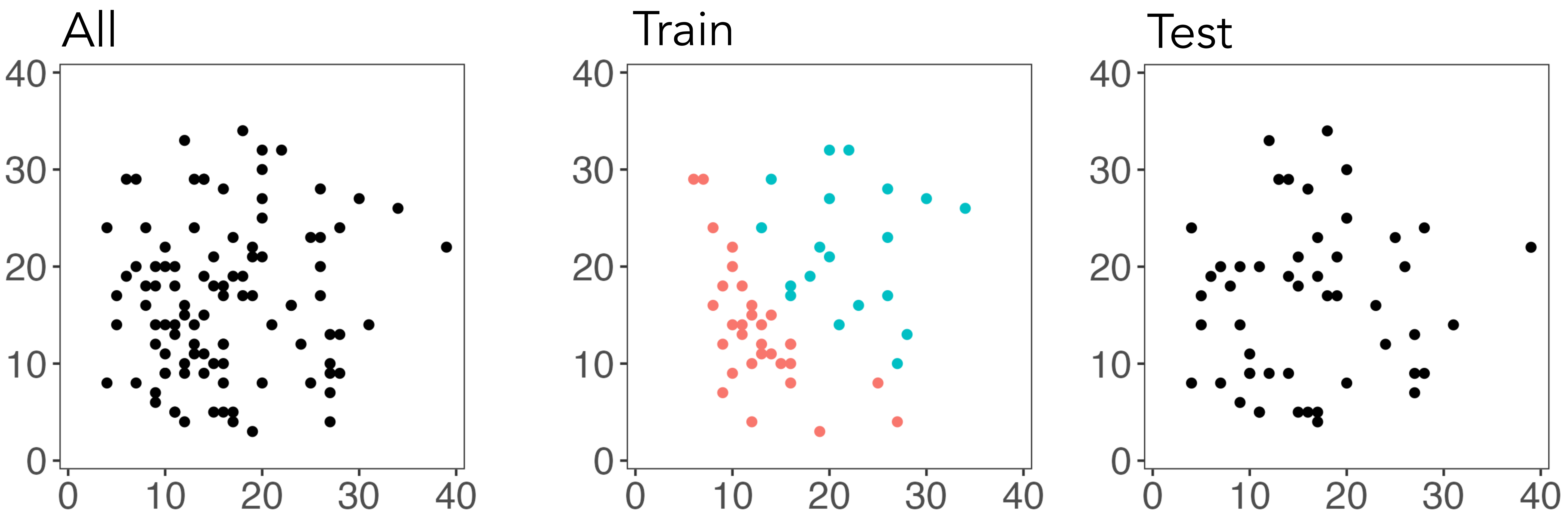

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Step 2.5: assign labels to observations in test set.

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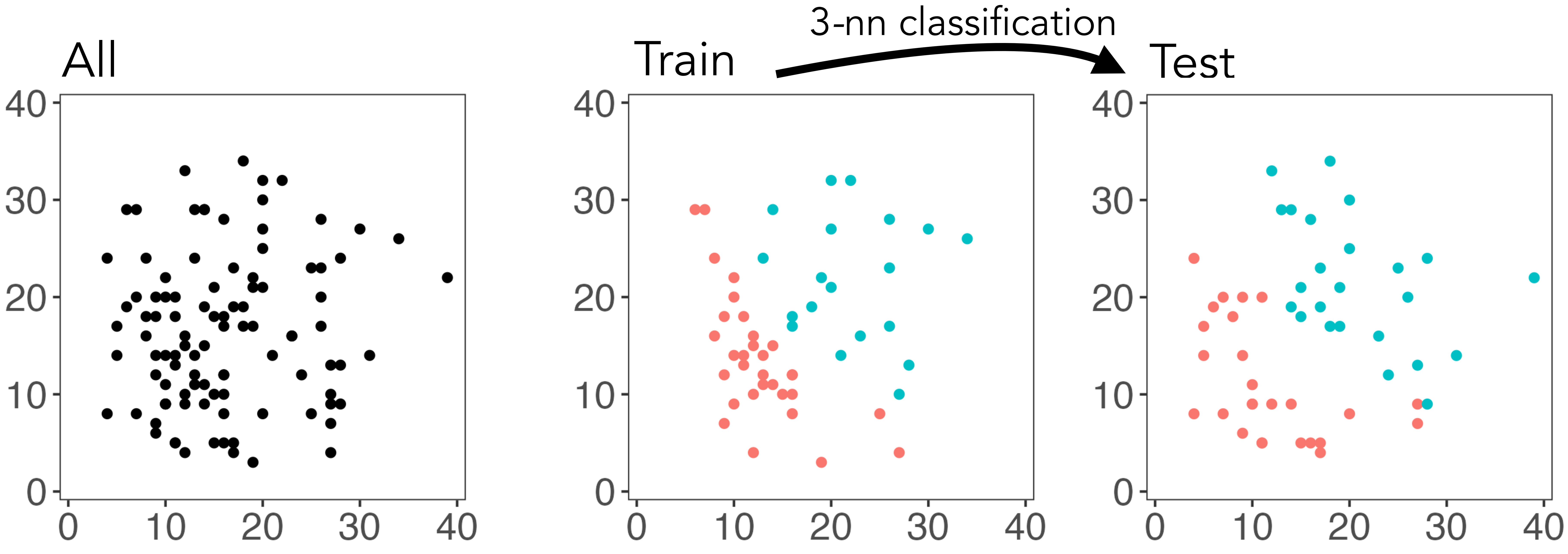

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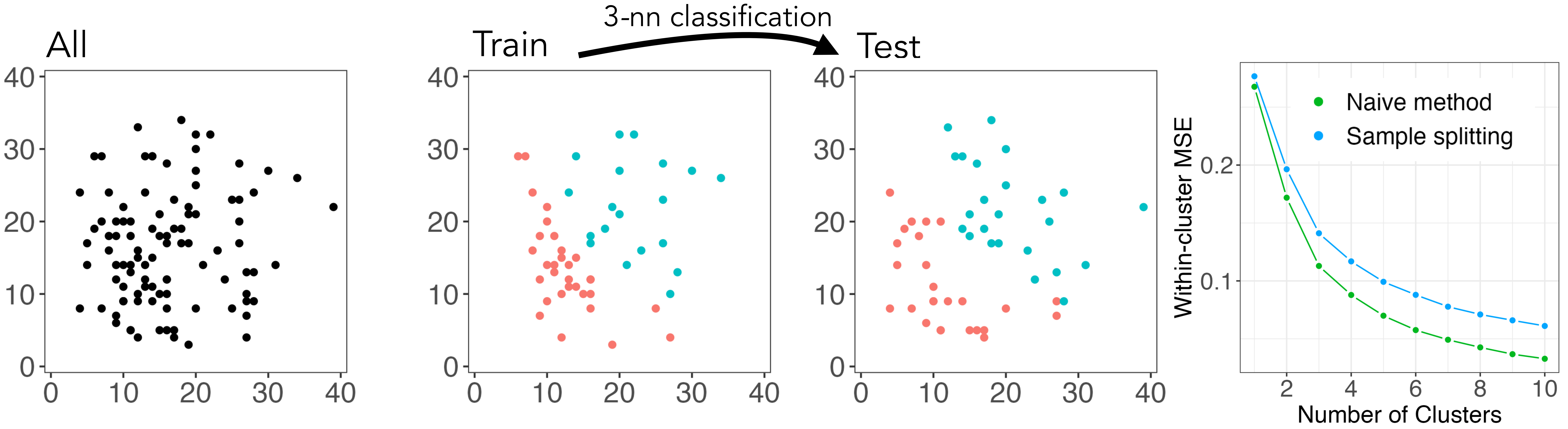

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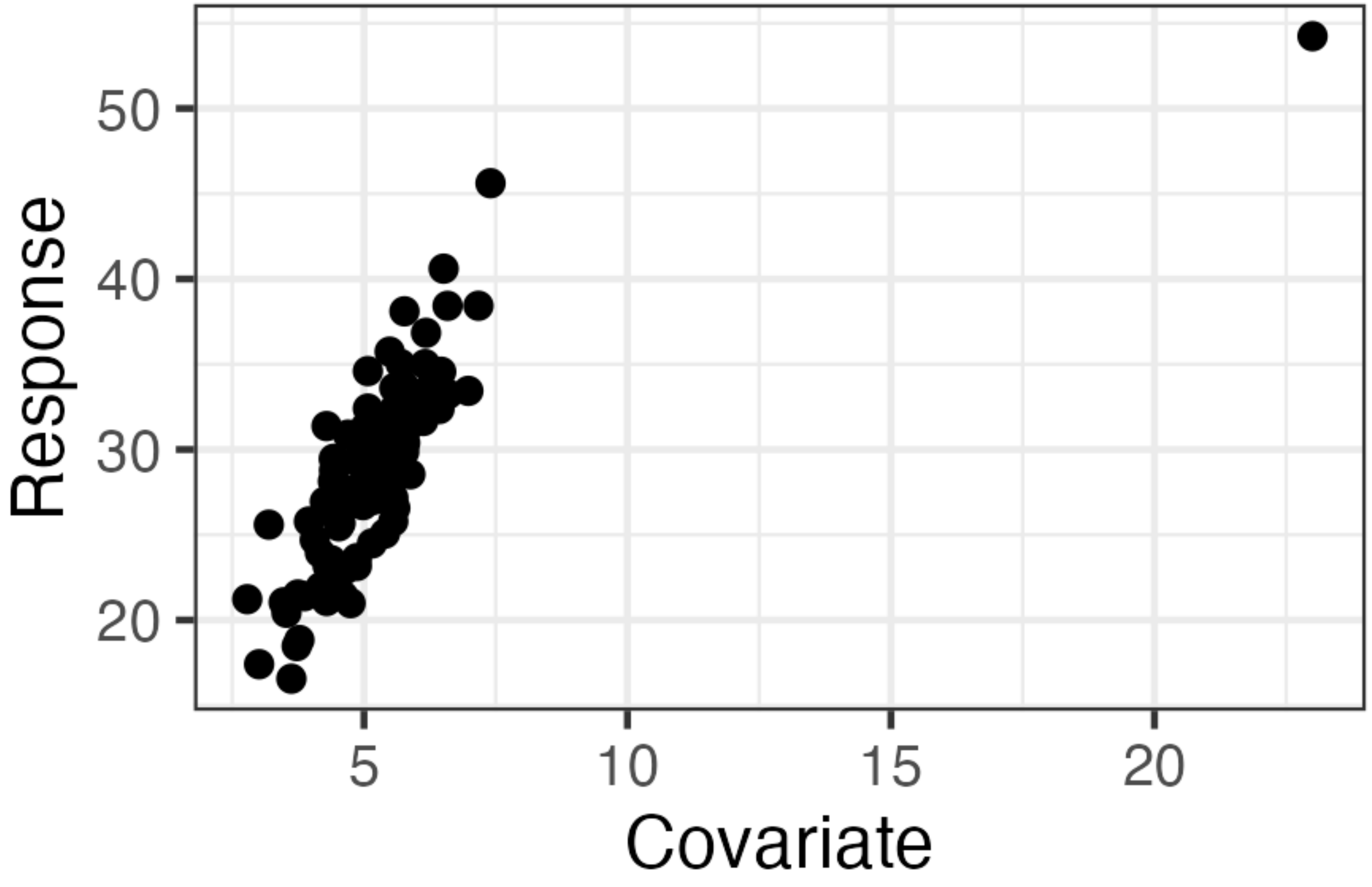
Other situations in which sample splitting is not a good option

1. Fixed-X regression settings.

2. Non-IID data.

3. Data with outliers or influential points.



Outline

1. Motivation: sample splitting doesn’t always work
2. **Poisson thinning**
3. Data thinning
4. Application to changepoint detection
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Poisson thinning

X

	Feature 1	Feature 2
Obs. 1	18	6
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Obs. 3	11	31
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	Feature 1	Feature 2
Obs. 1	18	6
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Obs. 3	11	31
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$X^{(1)}$

	Feature 1	Feature 2
Obs. 1	14	1
Obs. 2	10	6
Obs. 3	5	17
Obs. 4	6	25

$X^{(2)}$

	Feature 1	Feature 2
Obs. 1	4	5
Obs. 2	21	2
Obs. 3	6	14
Obs. 4	16	9

Poisson thinning

X

	Feature 1	Feature 2
Obs. 1	18	6
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X_{ij}

$X^{(1)}$

	Feature 1	Feature 2
Obs. 1	14	1
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$X^{(2)}$

	Feature 1	Feature 2
Obs. 1	[illegible]	5
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Poisson thinning

X

	Feature 1	Fea[illegible]
Obs. 1	18	[illegible]
Obs. 2	31	8
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X_{ij}

$$X_{ij}^{(1)} \mid X_{ij} = x_{ij} \sim \text{Binomial}(x_{ij}, \epsilon)$$

$X^{(1)}$

	[illegible]ture 1	Feature 2
[illegible]	14	1
[illegible]	[illegible]	6
Obs. 3	5	17
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$$X_{ij}^{(2)} := X_{ij} - X_{ij}^{(1)}$$

$X^{(2)}$

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If $X_{ij} \sim \text{Poisson}(\Lambda_{ij})$, then:

1. $X^{(1)}_{ij} \sim \text{Poisson}(\epsilon\Lambda_{ij})$
2. $X^{(2)}_{ij} \sim \text{Poisson}((1-\epsilon)\Lambda_{ij})$
3. $X^{(1)}_{ij} \perp\!\!\!\perp X^{(2)}_{ij}$

A very well-known result.

Poisson thinning

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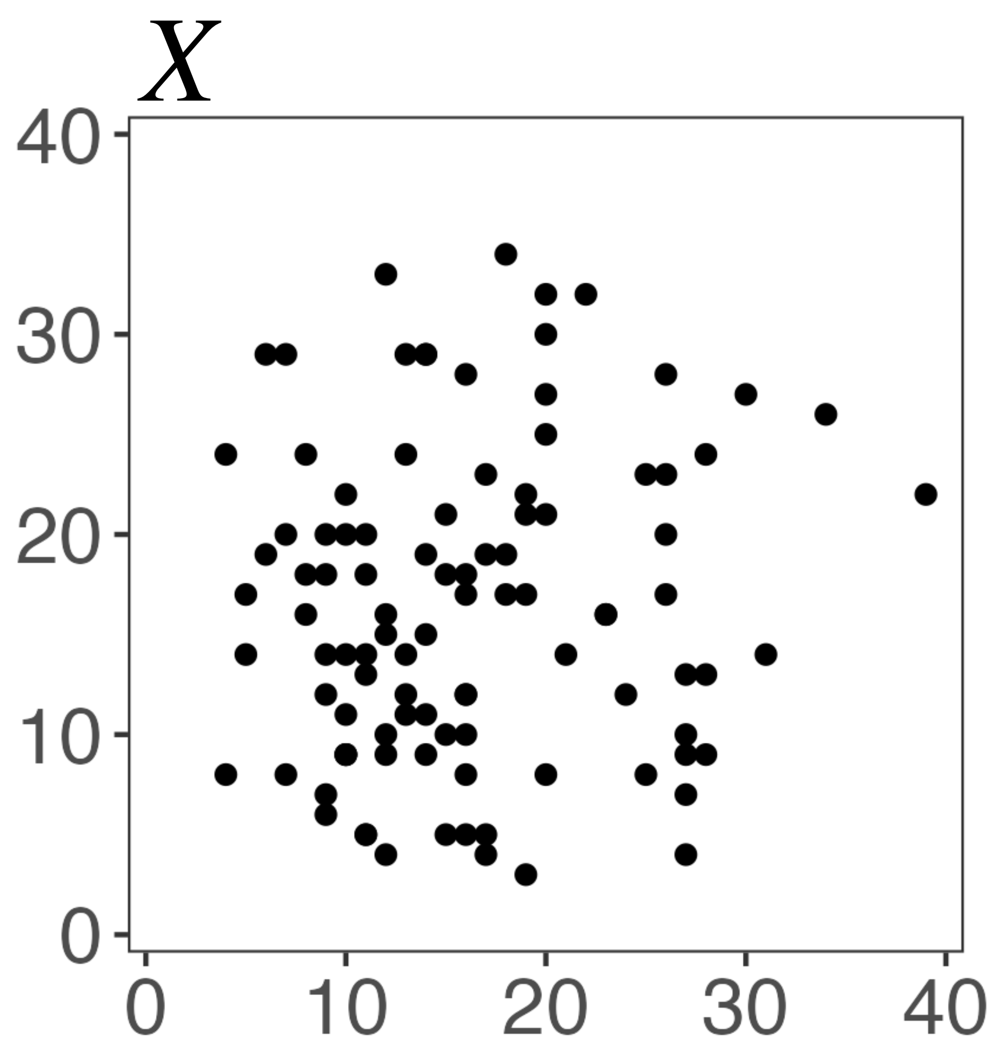
Thinning avoids the pitfall of sample splitting on our motivating example



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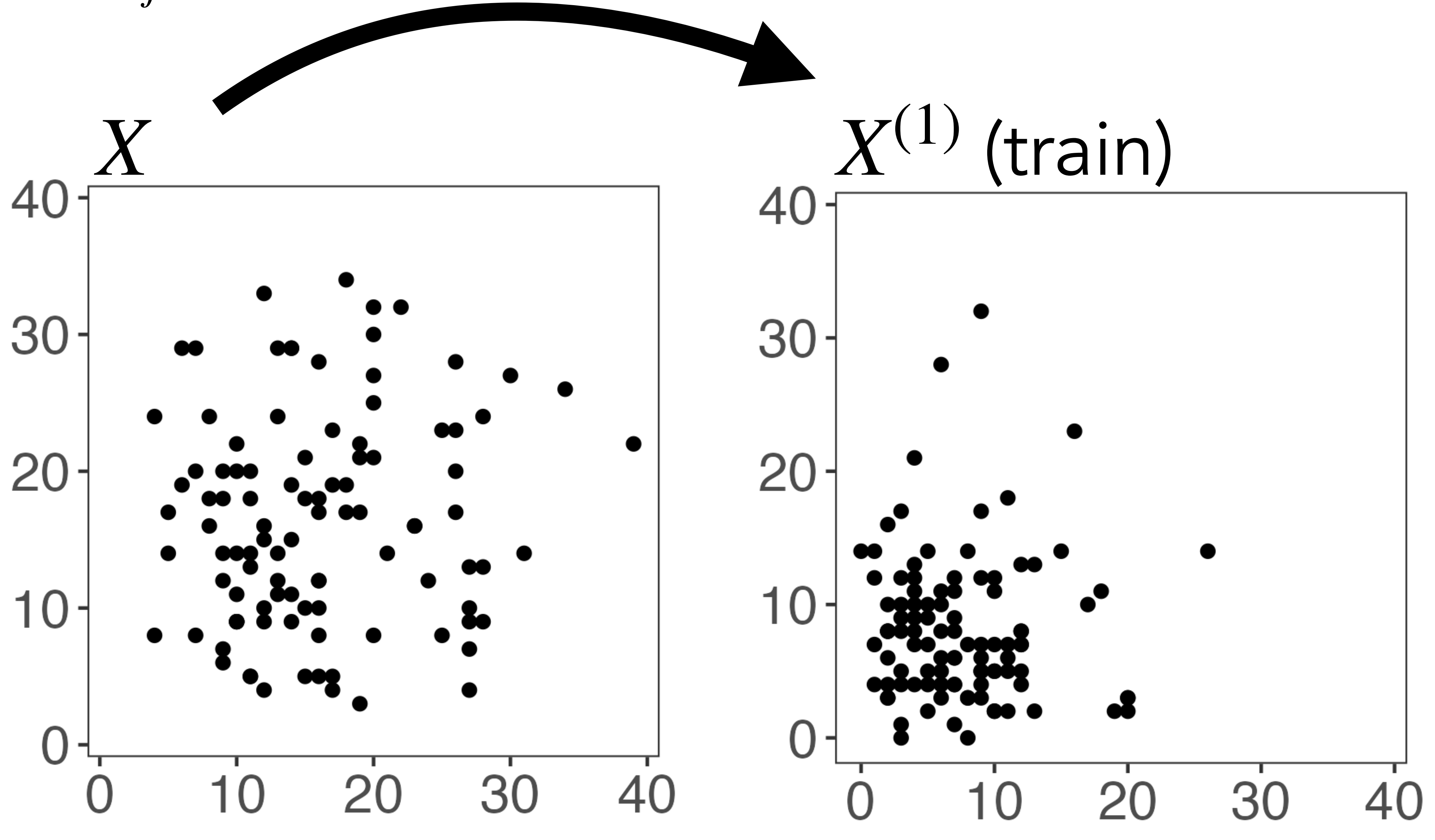

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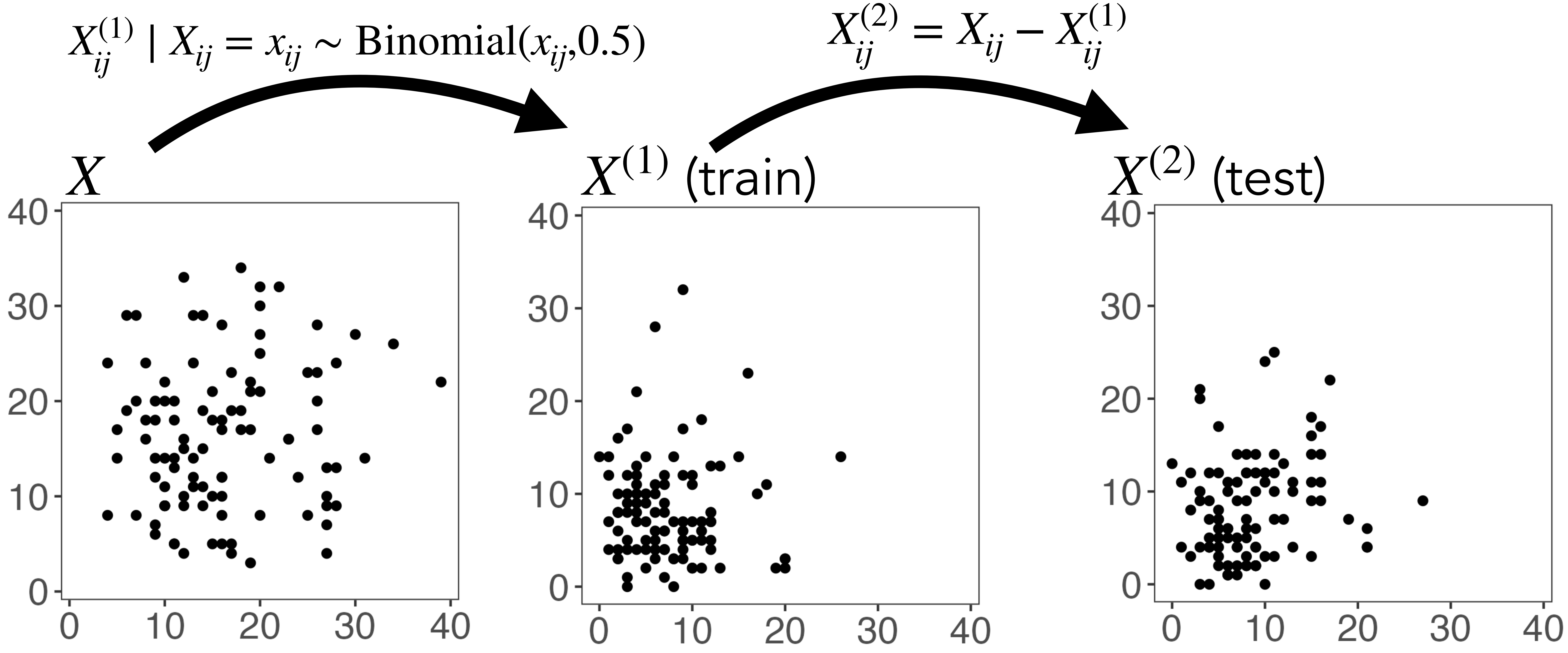

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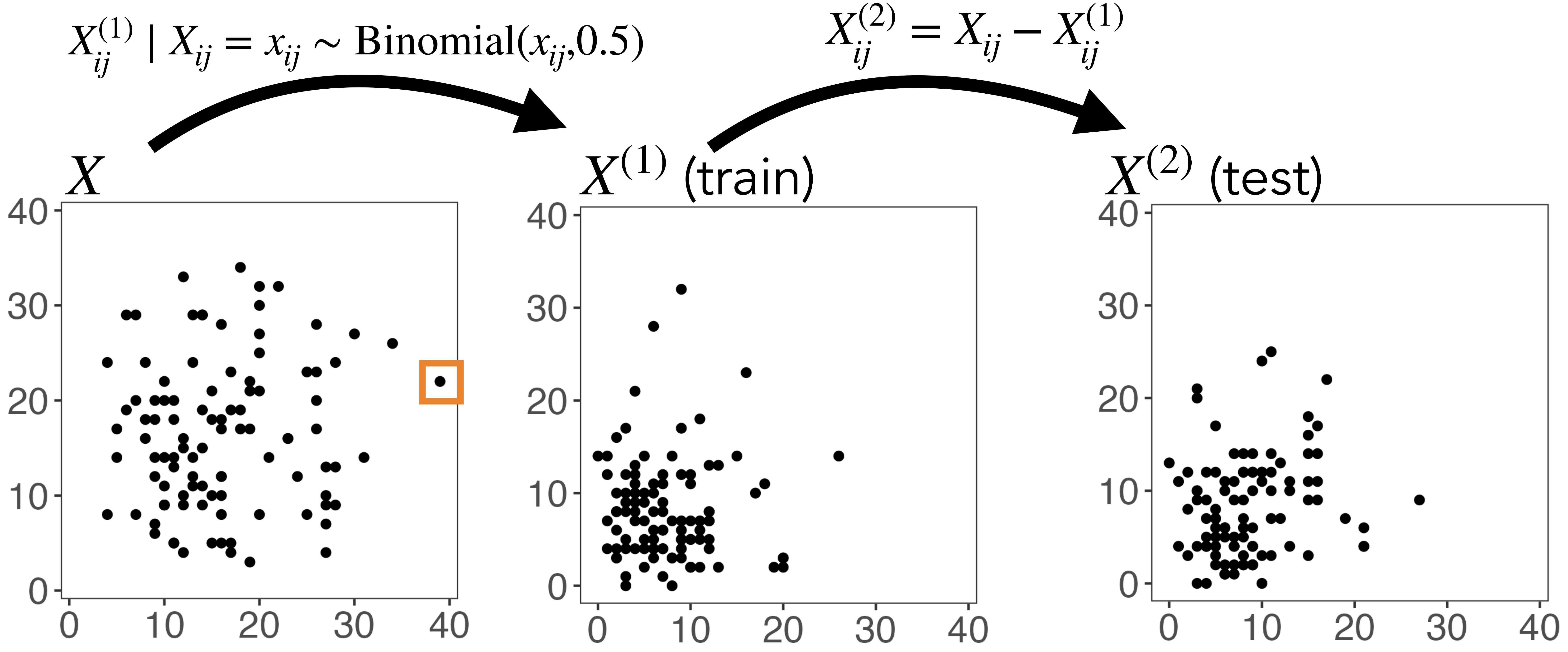

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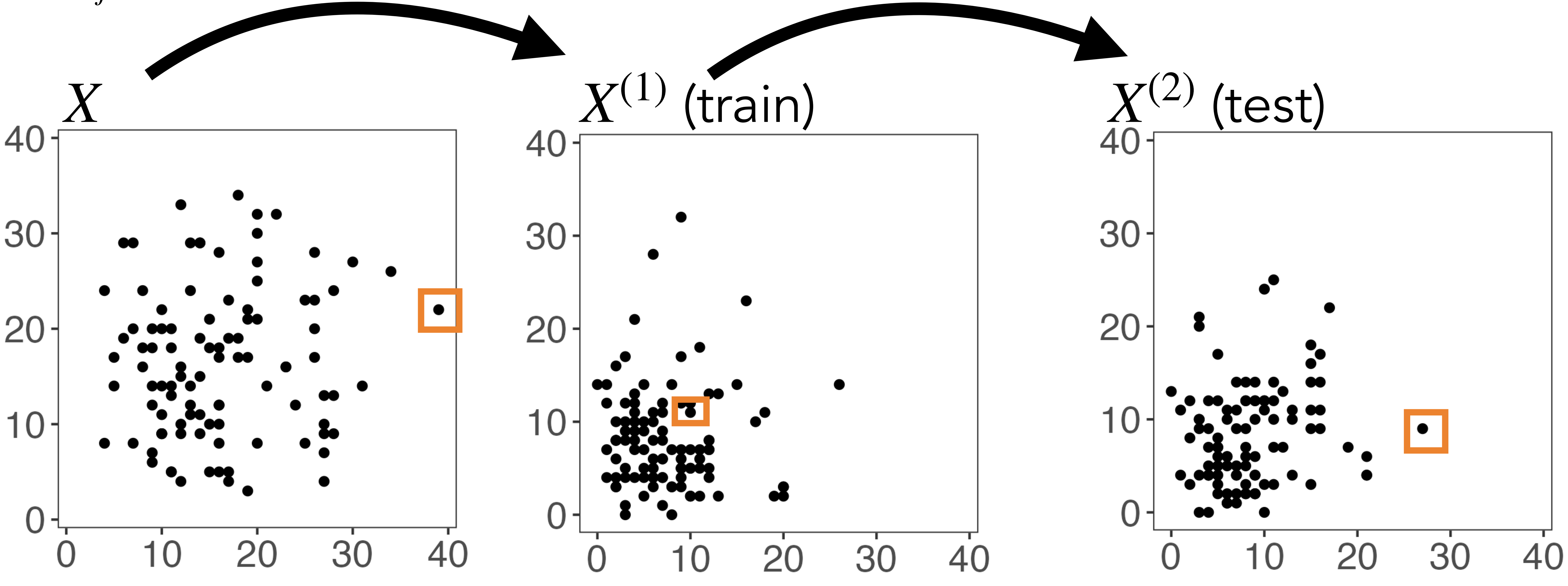

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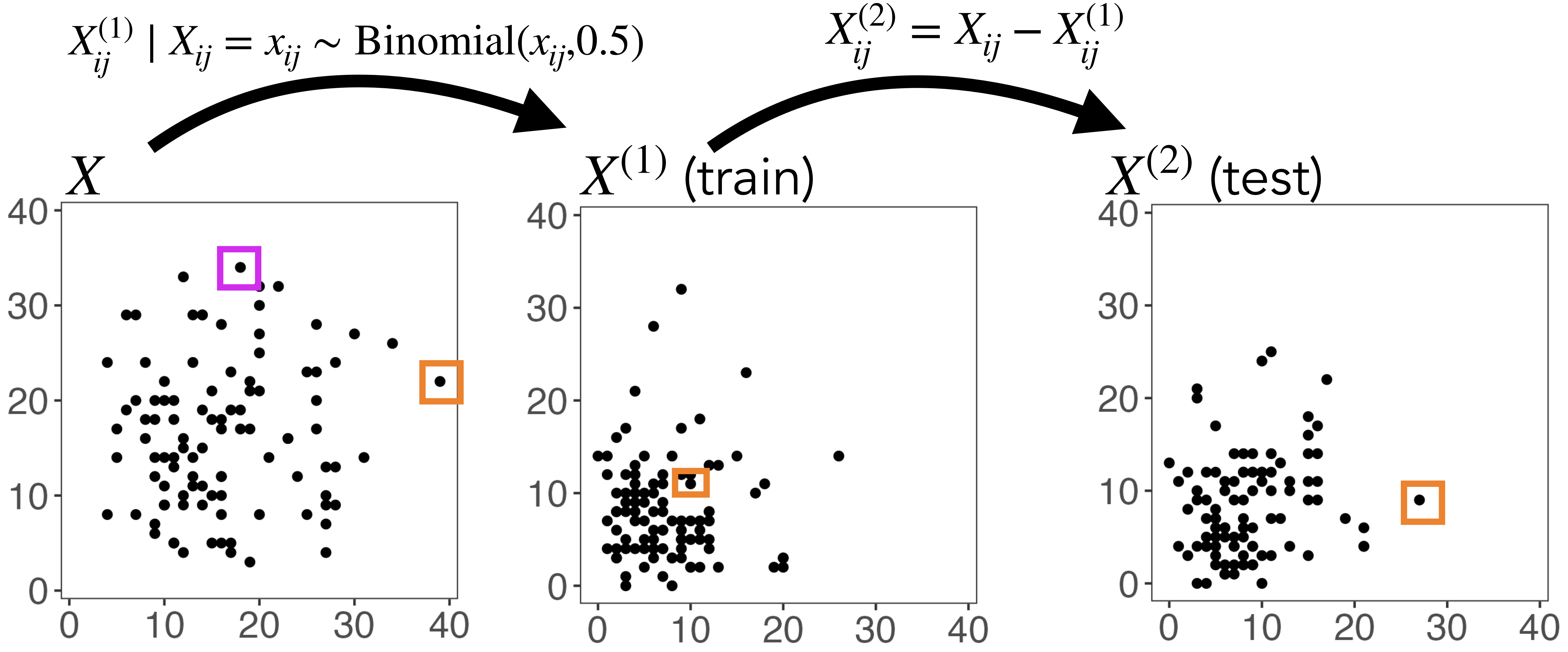

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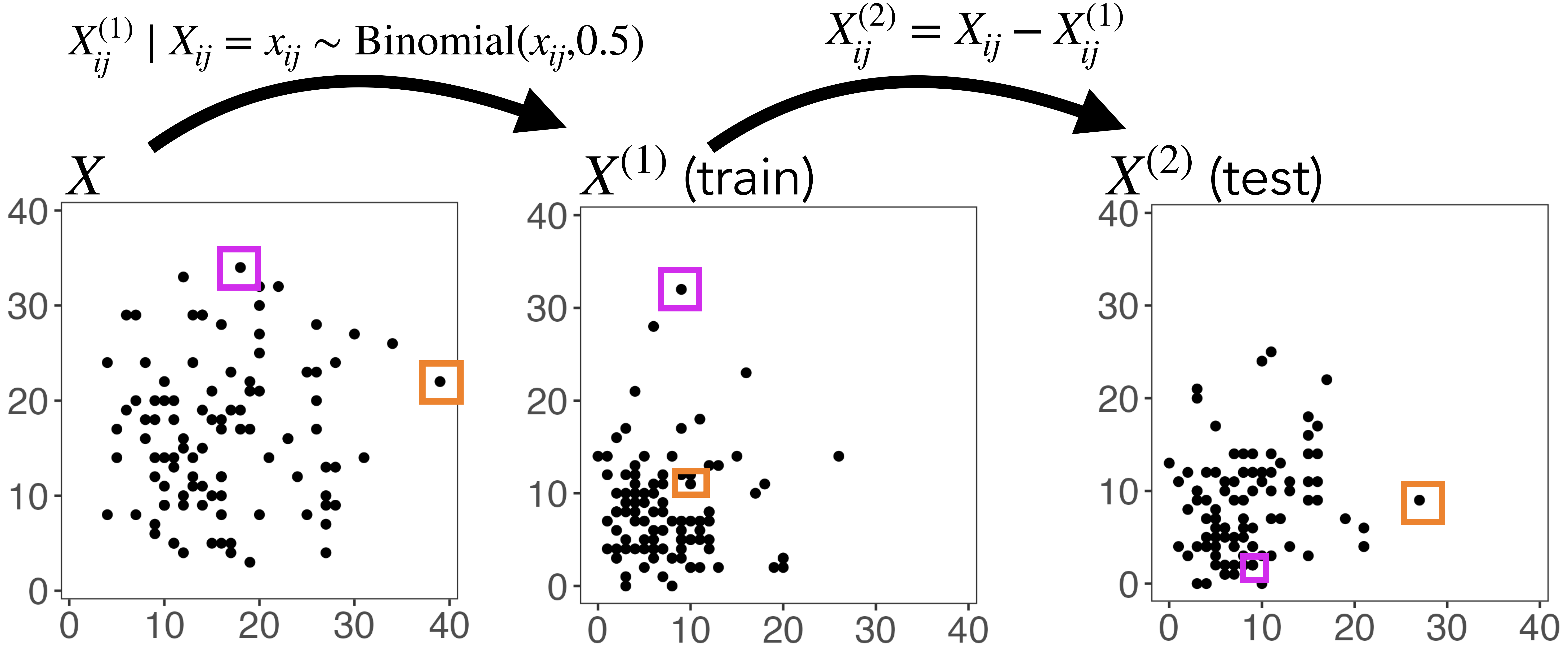

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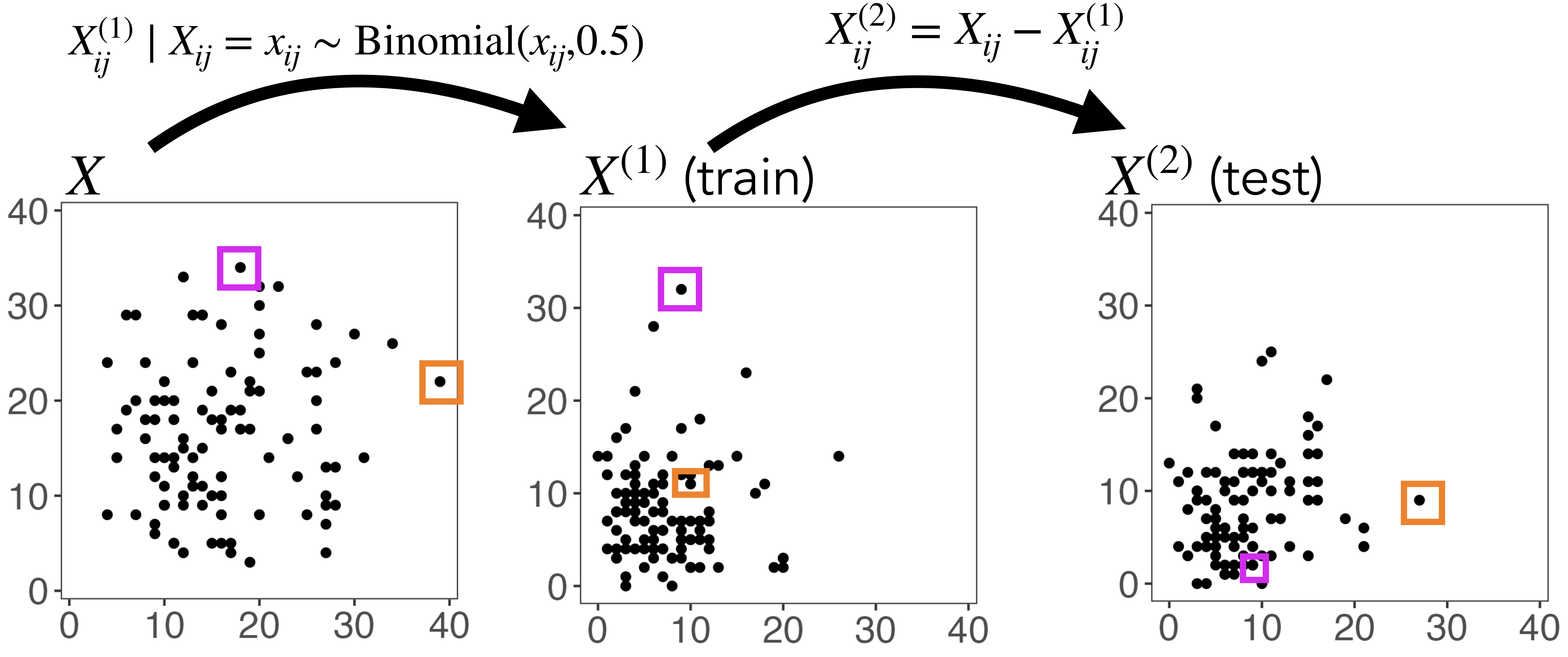

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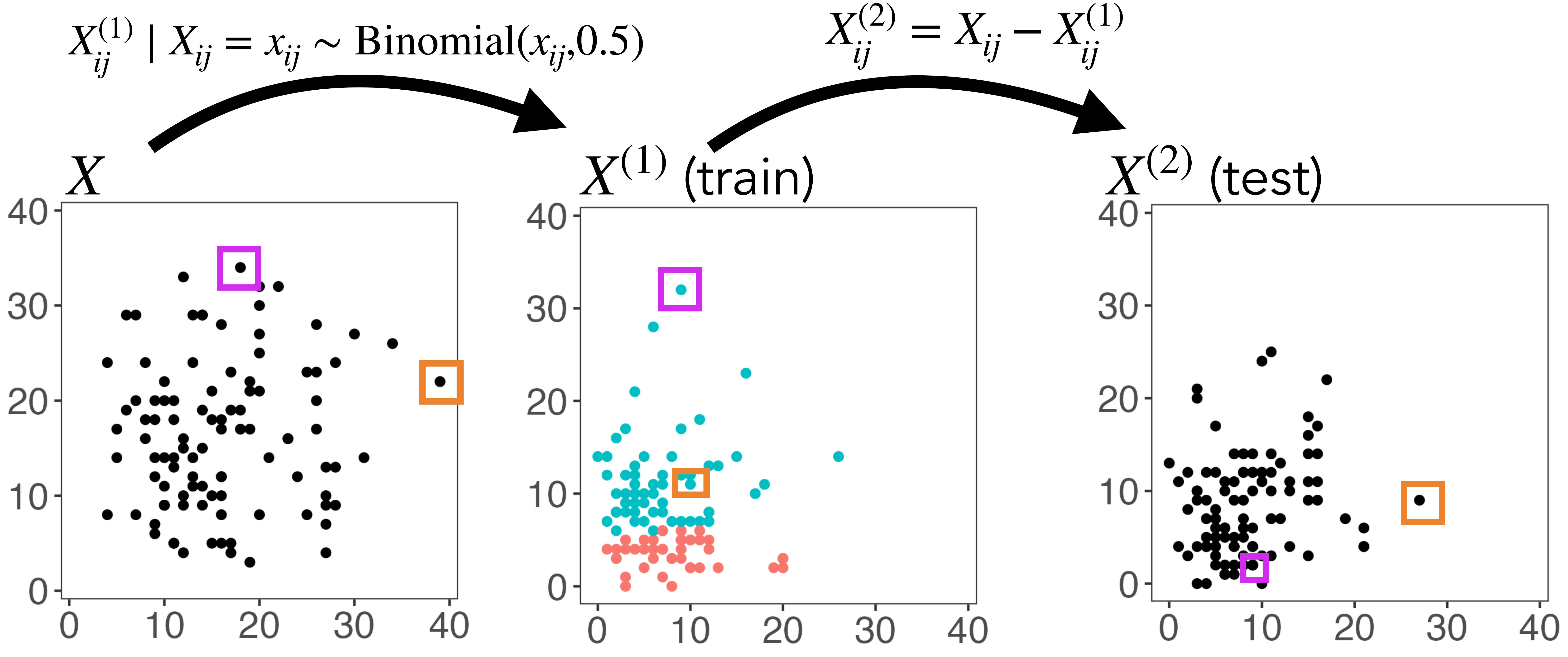

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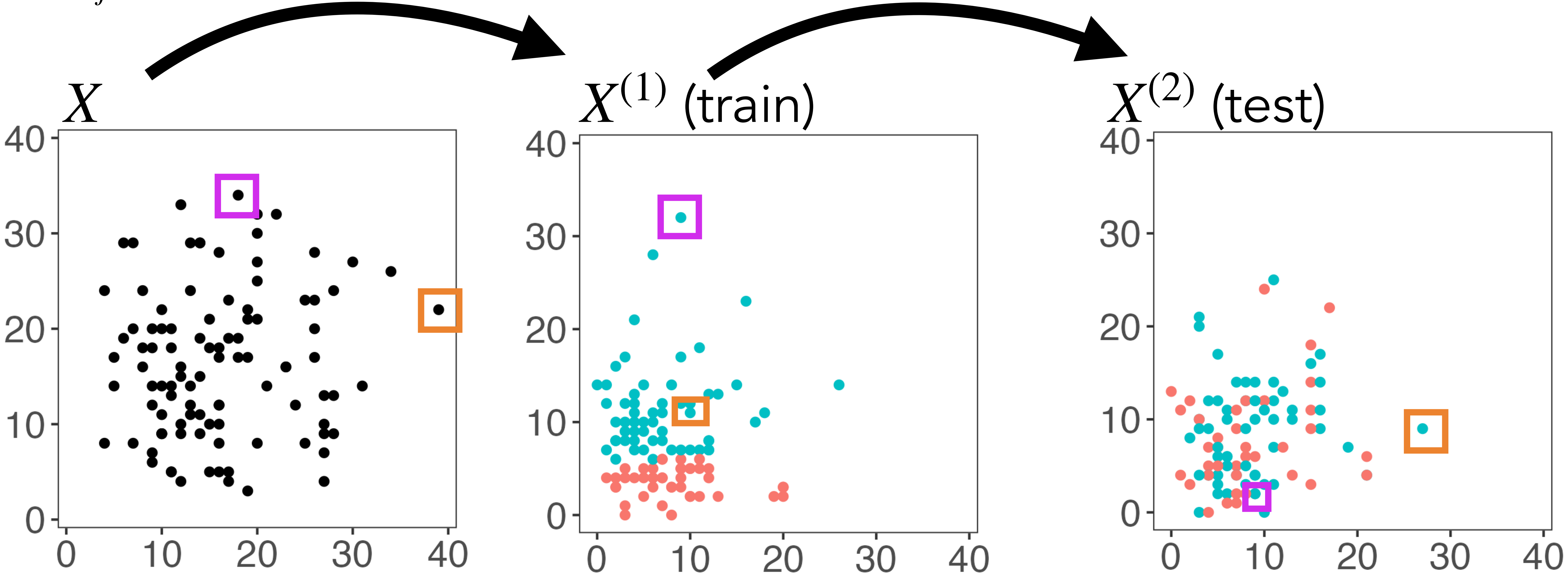

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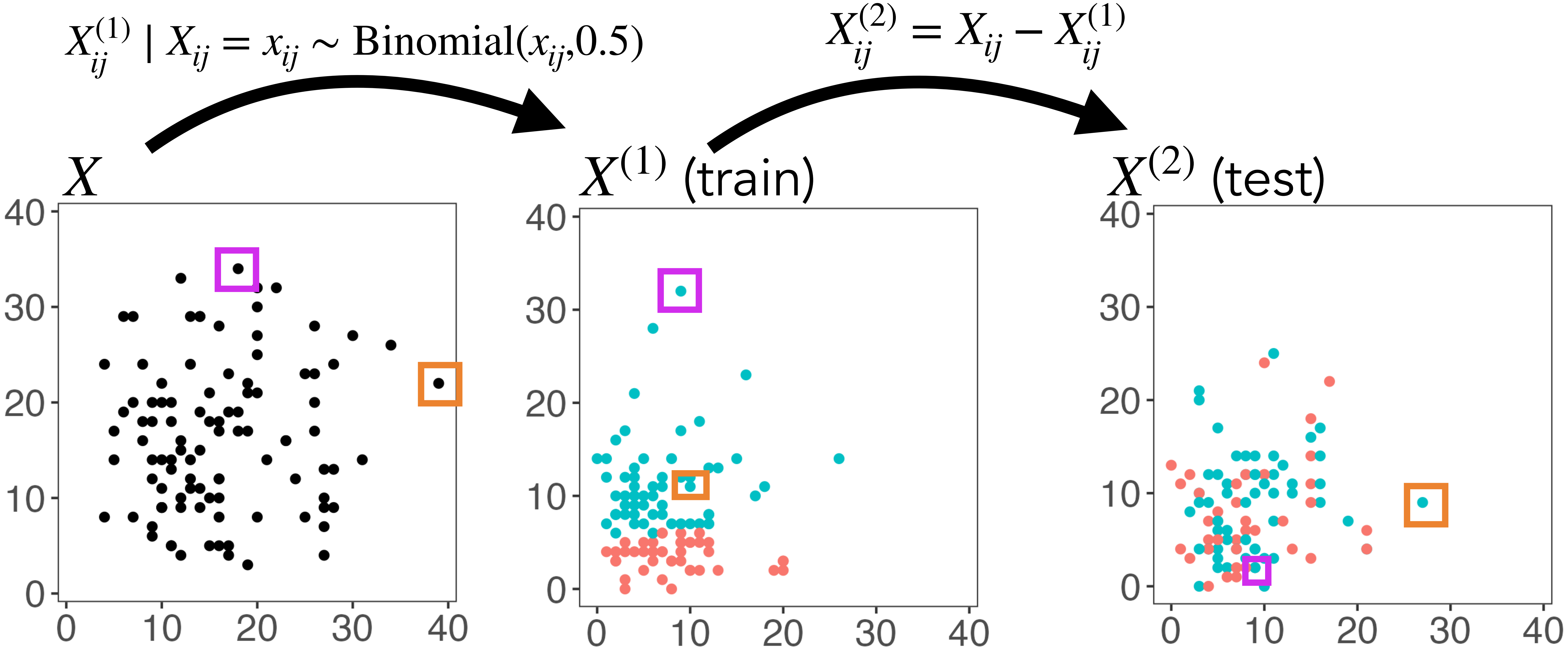

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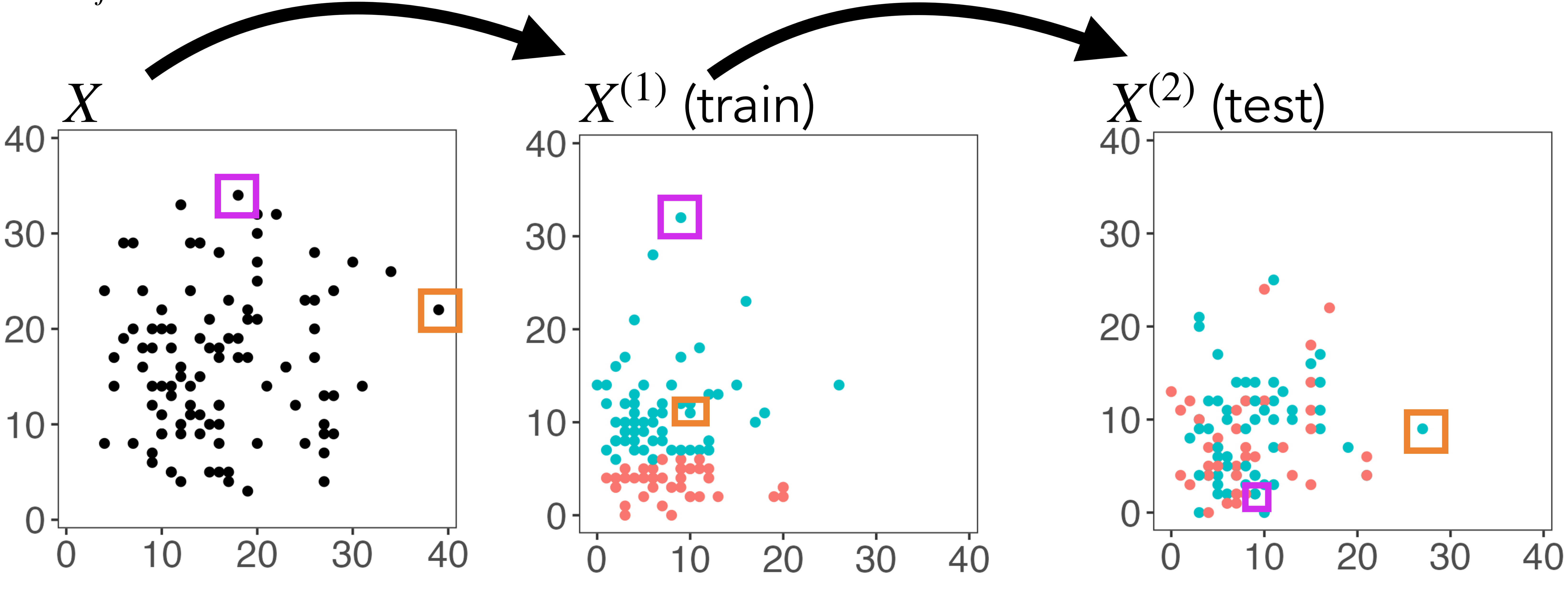

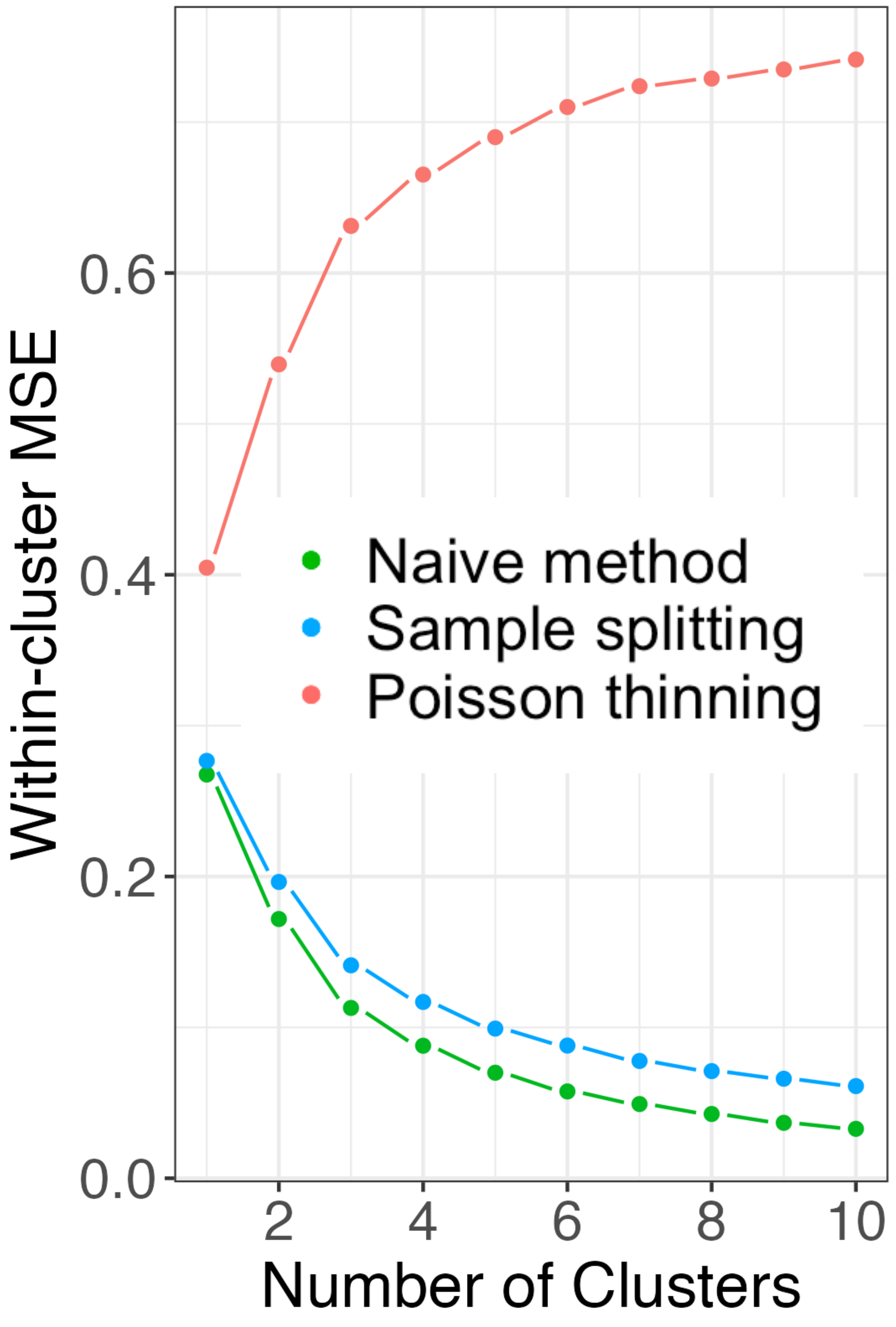

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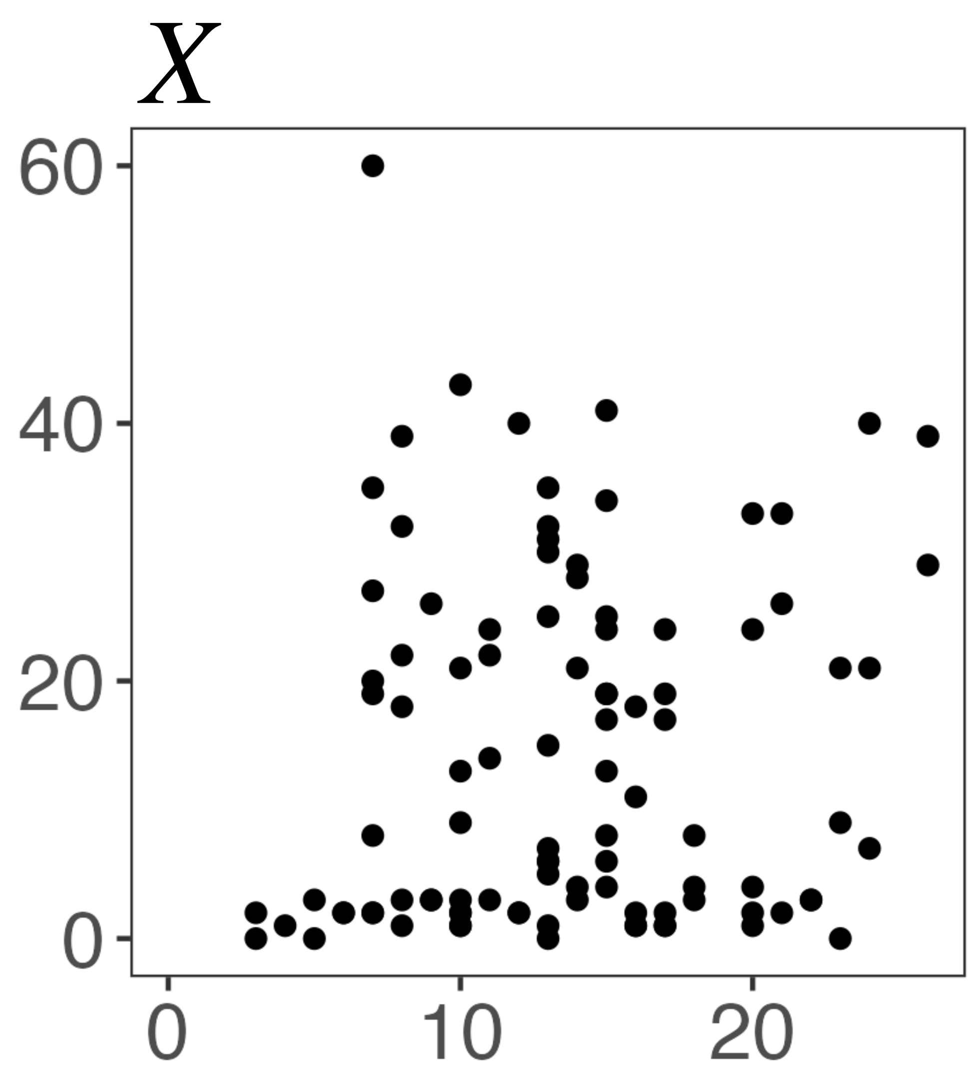

$$X_{i2} \sim \begin{cases} \text{Poisson}(3) & \text{if } i \leq 50 \\ \text{Poisson}(25) & \text{if } i > 50 \end{cases}$$

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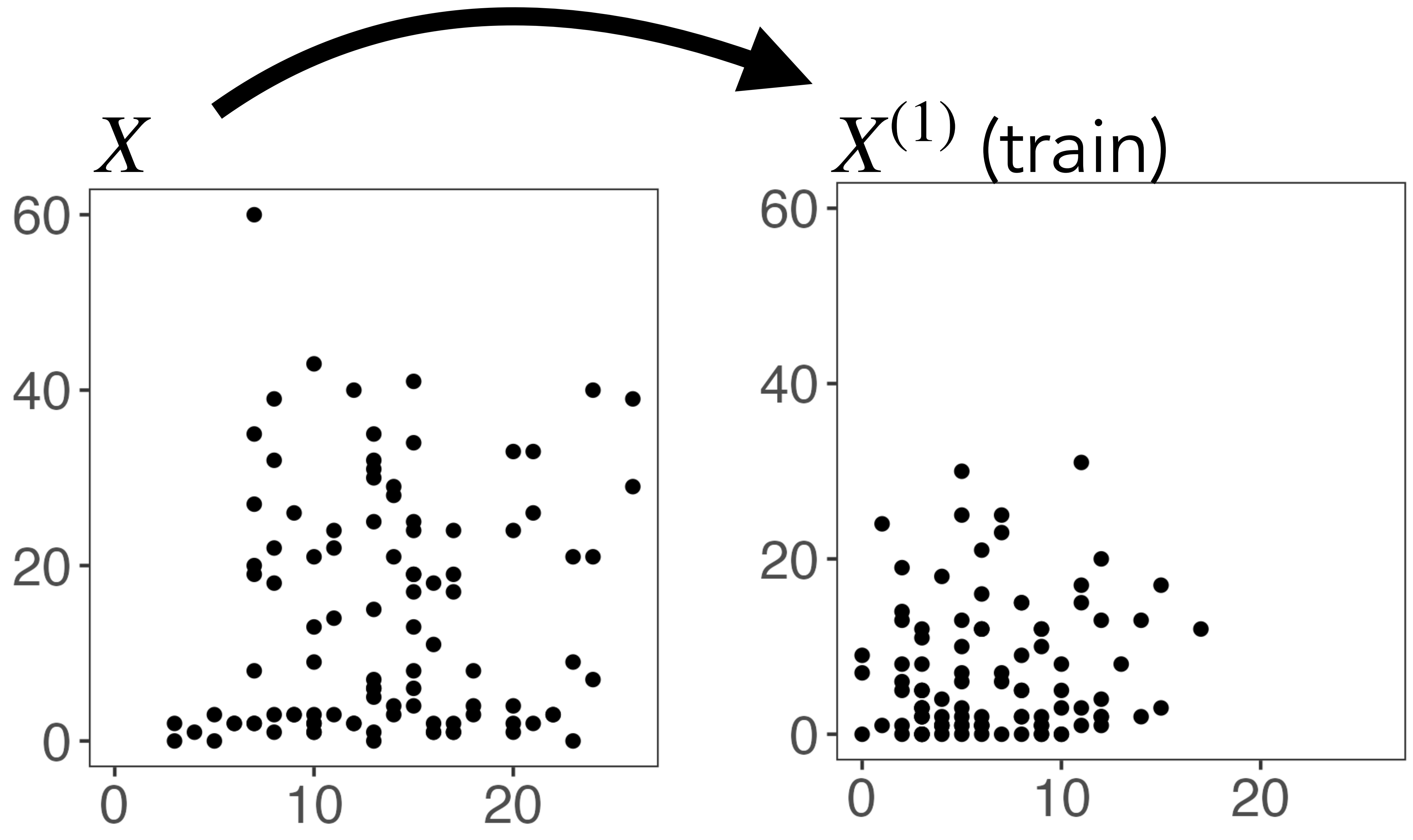

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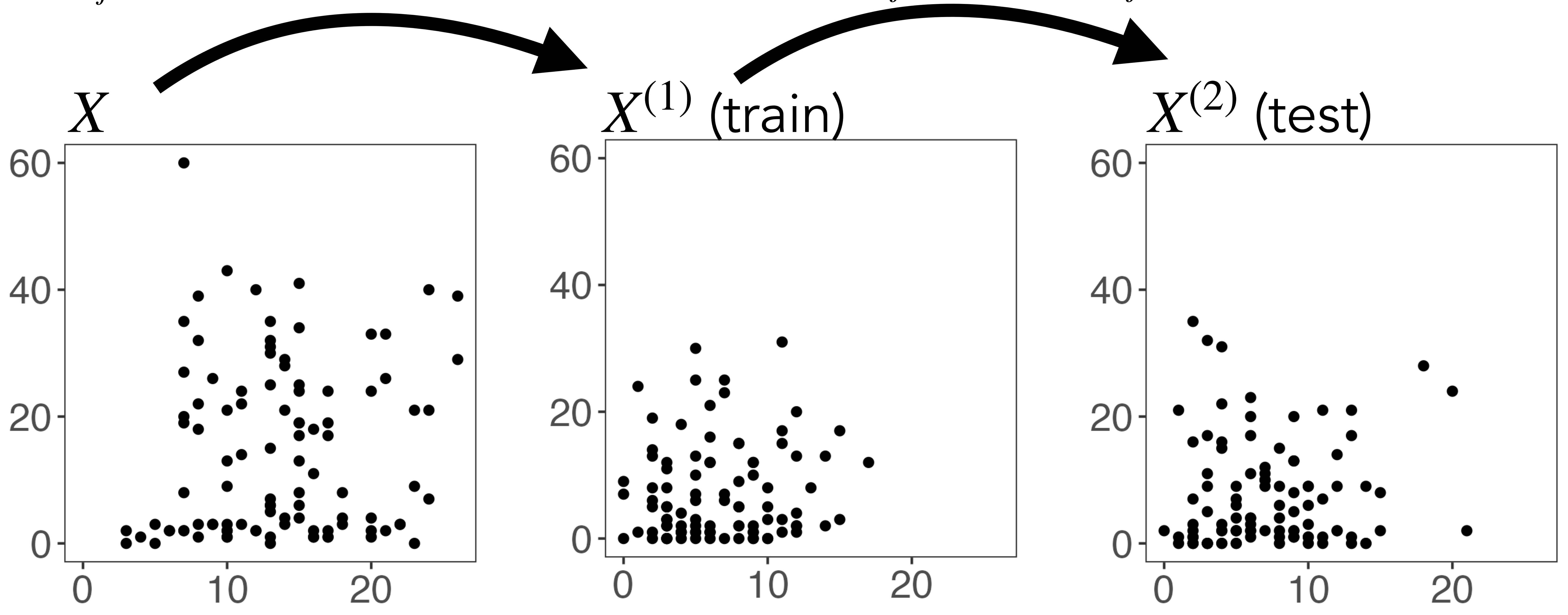

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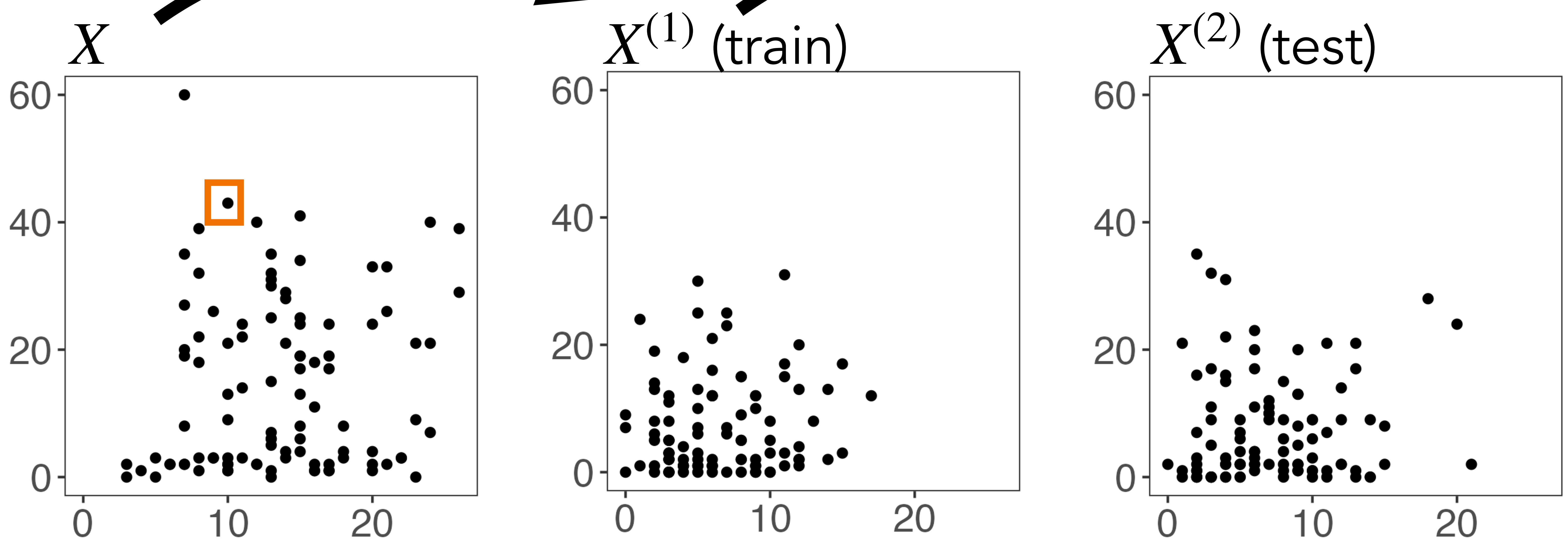
$X^{(2)}_{ij} = X_{ij} - X^{(1)}_{ij}$



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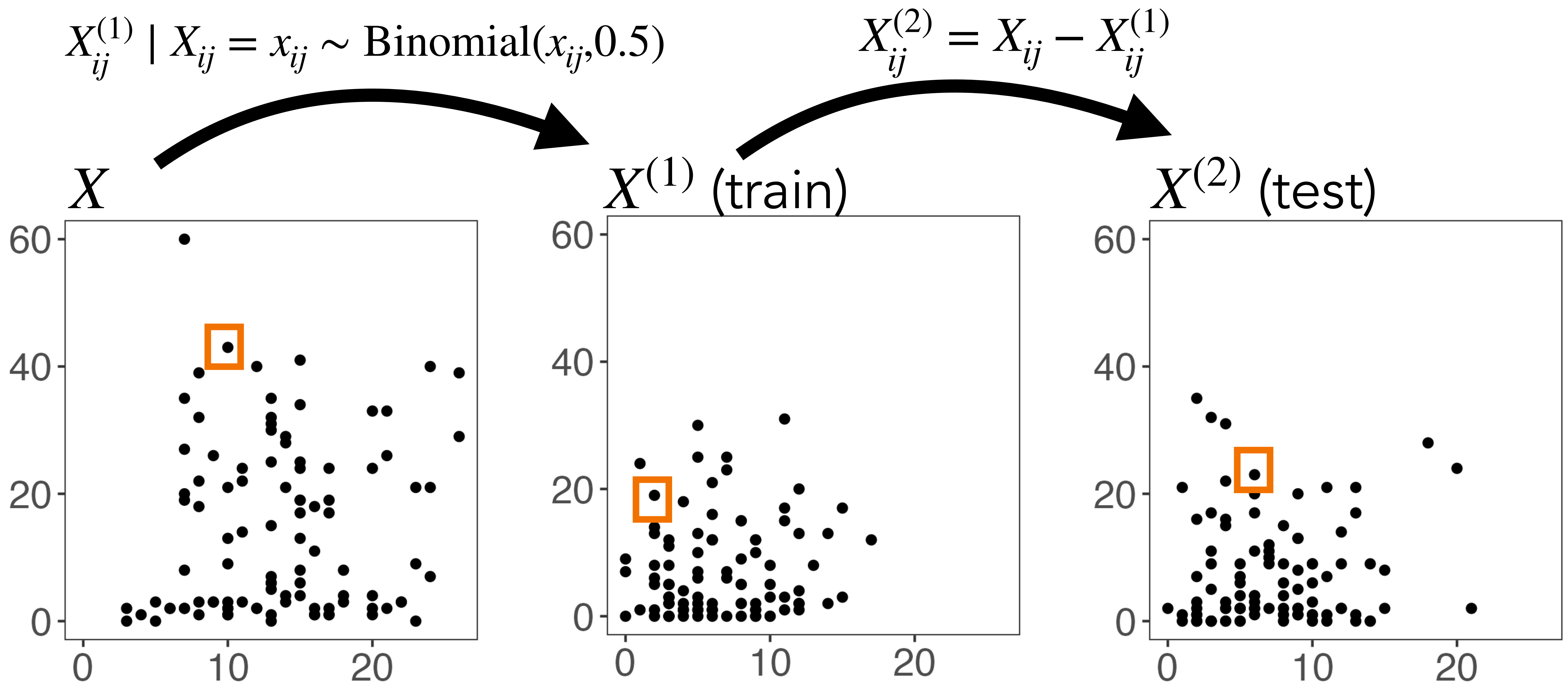

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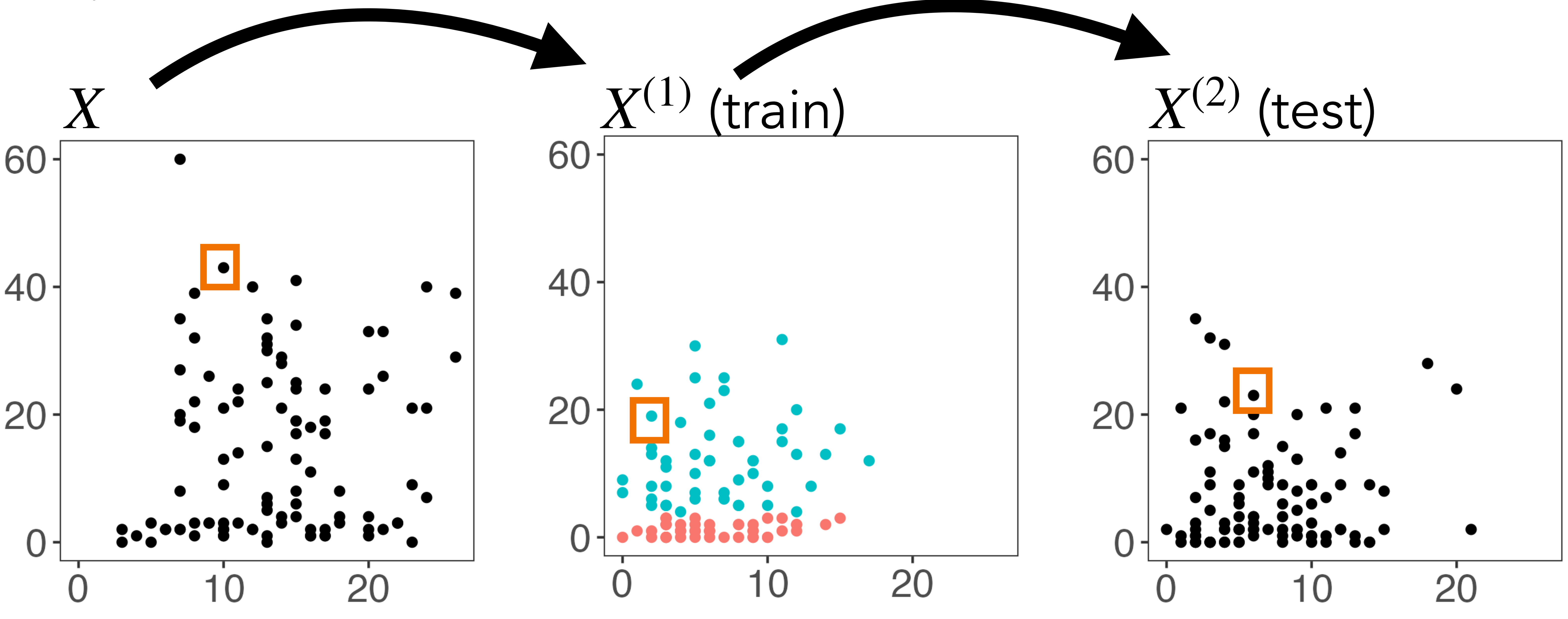

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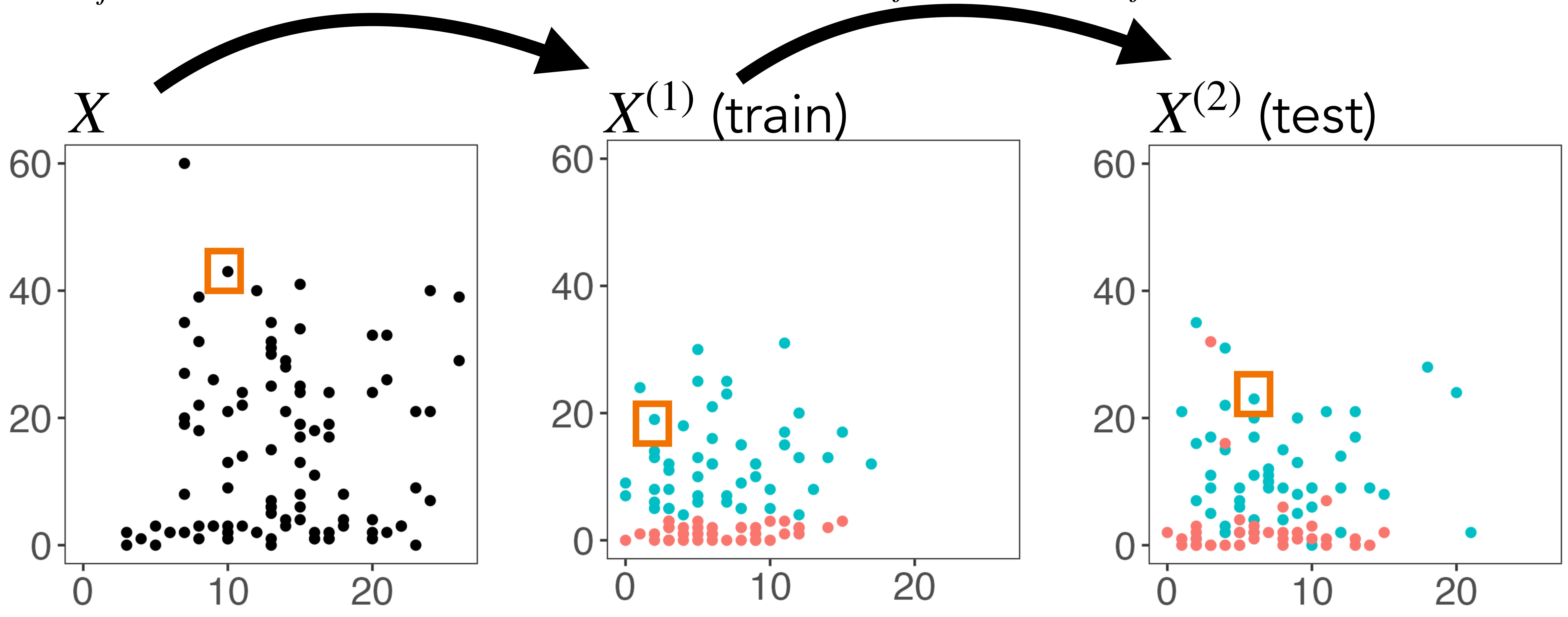
$X_{ij}^{(2)} = X_{ij} - X_{ij}^{(1)}$



$$X_{i2} \sim \begin{cases} \text{Poisson}(3) & \text{if } i \leq 50 \\ \text{Poisson}(25) & \text{if } i > 50 \end{cases}$$

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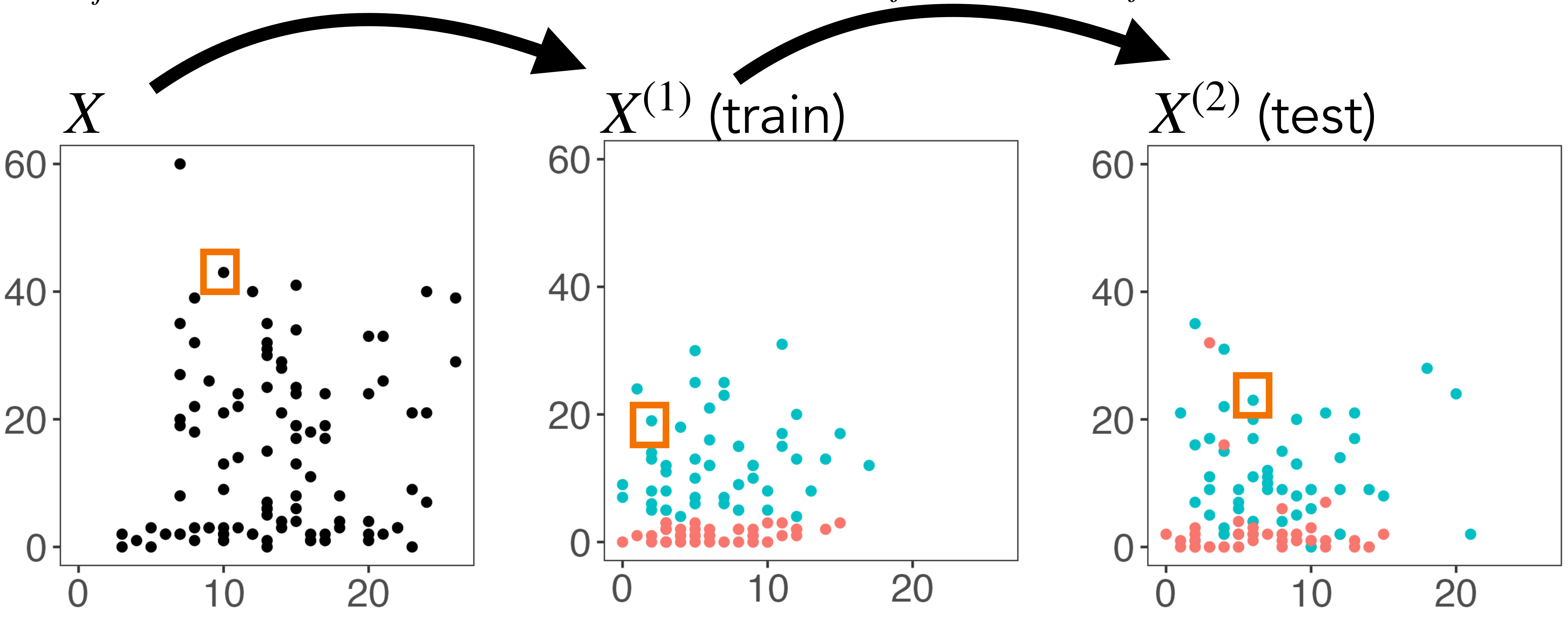

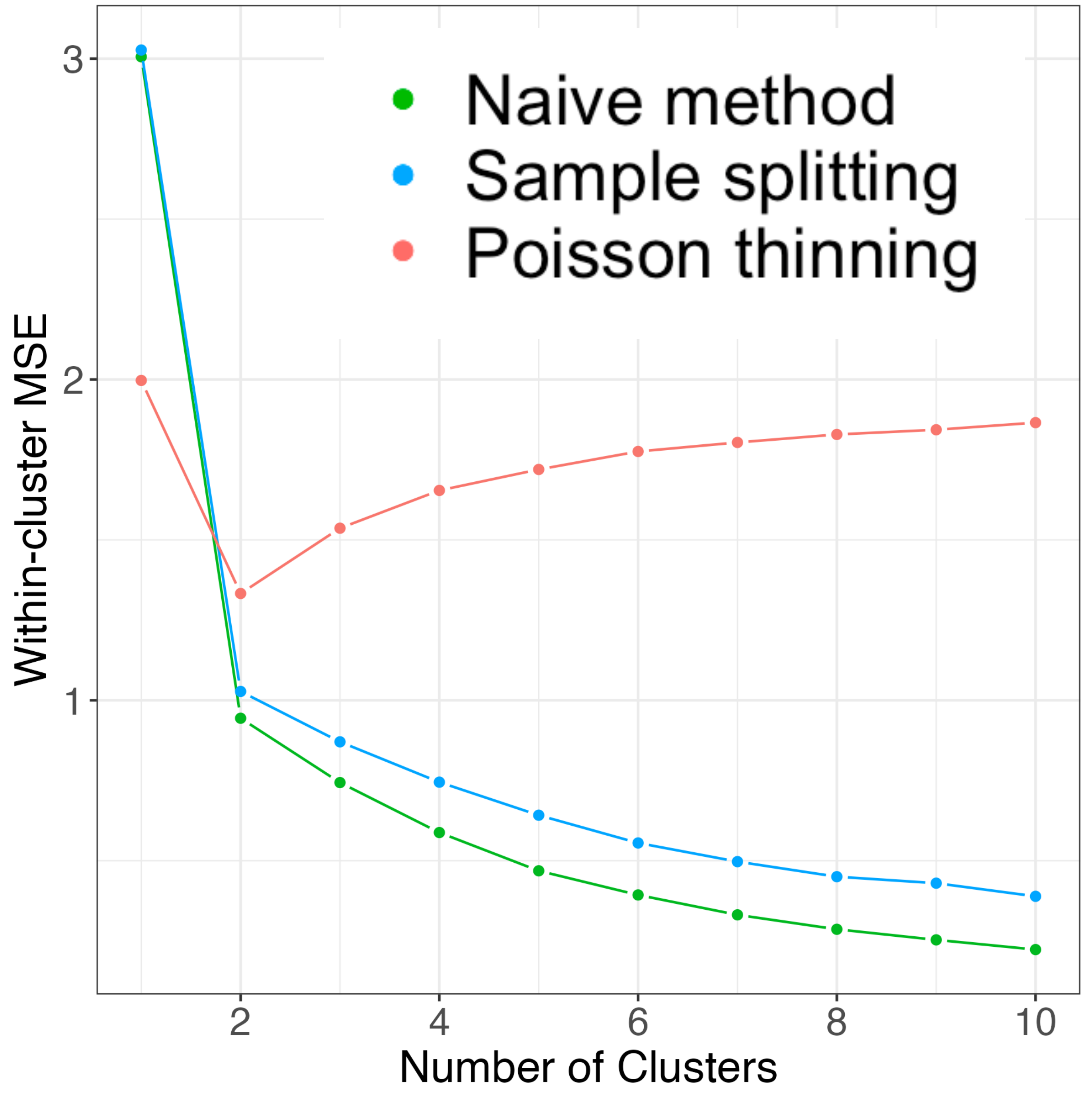

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Outline

1. Motivation: sample splitting doesn't always work
2. Poisson thinning
3. **Data thinning**
4. Application to changepoint detection
5. Generalized data thinning

Data thinning

Goal: split a single observation X into $X^{(1)}$ and $X^{(2)}$ such that:

(1) $X^{(1)}$ and $X^{(2)}$ have the same distribution as X, up to a parameter scaling.

(2) $X^{(1)} \perp\!\!\!\perp X^{(2)}$.

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TIME SERIES MODELS WITH UNIVARIATE MARGINS IN THE CONVOLUTION-CLOSED INFINITELY DIVISIBLE CLASS

HARRY JOE,* *University of British Columbia*

Convolution-closed distributions

A family of distributions F_λ is "convolution-closed" in parameter λ if

- $X' \sim F_{\lambda_1}$
- $X'' \sim F_{\lambda_2}$
- $X' \perp\!\!\!\perp X''$

together imply that $X' + X'' \sim F_{\lambda_1+\lambda_2}$.

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Distribution	Convolution-closed in:
$X \sim \text{Poisson}(\lambda)$	λ
$X \sim \text{N}(\mu, \sigma^2)$	(μ, σ^2)
$X \sim \text{NegativeBinomial}(\mu, b)$	(μ, b)
$X \sim \text{Gamma}(\alpha, \beta)$	α, if β is fixed
$X \sim \text{Binomial}(r, p)$	r, if p is fixed
$X \sim \text{N}_k\,(\mu, \Sigma)$.	(μ, Σ).
$X \sim \text{Multinomial}_k\,(r, p)$	r, if p is fixed
$X \sim \text{Wishart}_p\,(n, \Sigma)$	n, if p and Σ are fixed.

Data thinning for convolution-closed distributions

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$$\boxed{\text{We observe realization } x \text{ from } X \sim F_\lambda.}$$

Data thinning for convolution-closed distributions

We know x could have arisen as $x' + x''$, where
$X' \sim F_{\epsilon\lambda}, \;\; X'' \sim F_{(1-\epsilon)\lambda}, \;\; X' \perp\!\!\!\perp X''$.

We observe realization x from $X \sim F_\lambda$.

Data thinning for convolution-closed distributions

We know x could have arisen as $x' + x''$, where $X' \sim F_{\epsilon\lambda},\ X'' \sim F_{(1-\epsilon)\lambda},\ X' \perp\!\!\!\perp X''$.

$\Downarrow$

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If we had observed x' and x'', we would have satisfied our goal of data thinning!

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Can we work backwards to recover x' and x''?

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Let $G_{\epsilon,x}$ be the conditional distribution of $X' \mid X = x$.

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Theorem:
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Data thinning for the Poisson distribution

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The conditional distribution of $X' \mid X = x$ is $\text{Binomial}(x, \epsilon)$.

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Draw $X^{(1)}$ from $\text{Binomial}(x, \epsilon)$. Let $X^{(2)} := X - X^{(1)}$.

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Can we work backwards to recover x' and x''?

The conditional distribution of $X' \mid X = x$ is $\text{Binomial}(x, \epsilon)$.

We have recovered Poisson thinning!

Data thinning recipe for the negative binomial distribution

Data thinning recipe for the negative binomial distribution

We observe realization x from $X \sim \text{NB}(\mu, b)$.

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The conditional distribution of $X' \mid X = x$ is $\mathrm{BetaBinomial}(x, \epsilon b, (1-\epsilon)b)$.

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Can we work backwards to recover x' and x''?

Draw $X^{(1)}$ from $\mathrm{BetaBinomial}(x, \epsilon\tilde{b}, (1-\epsilon)\tilde{b})$. Let $X^{(2)} := X - X^{(1)}$.

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The conditional distribution of $X' \mid X = x$ is $\text{BetaBinomial}(x, \epsilon b, (1-\epsilon)b)$.

$$\text{Cov}\left(X^{(1)}, X^{(2)}\right) = \epsilon(1-\epsilon)\frac{\mu^2}{b}\left(1 - \frac{b+1}{\tilde{b}+1}\right).$$

This is a new result!

For many common distributions, the distribution $G_{\epsilon,x}$ has a simple form

Distribution of X:	Draw $X^{(1)} \mid X = x$ from $G_{\epsilon,x}$, where $G_{\epsilon,x}$ is:	Distribution of $X^{(1)}$:	Distribution of $X^{(2)}$, where $X^{(2)} = X - X^{(1)}$:
Poisson(λ)	Binomial(x, ϵ)	Poisson$(\epsilon\lambda)$	Poisson$((1-\epsilon)\lambda)$

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Related work on Poisson thinning:

- Sarkar and Stephens, 2021, Nature Genetics.
- Chen et al., 2021, arXiv:2108.03336
- Leiner et al., 2021, arXiv:2112.11079.
- Neufeld et al., 2022, Biostatistics.
- Oliveira, Lei, and Tibshirani, 2022, arXiv:2212.01943.

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$\mathrm{Poisson}(\lambda)$	$\mathrm{Binomial}(x, \epsilon)$	$\mathrm{Poisson}(\epsilon\lambda)$	$\mathrm{Poisson}((1-\epsilon)\lambda)$
$\mathrm{N}(\mu, \sigma^2)$	$\mathrm{N}(\epsilon x, \epsilon(1-\epsilon)\sigma^2)$	$\mathrm{N}(\epsilon\mu, \epsilon\sigma^2)$	$\mathrm{N}((1-\epsilon)\mu, (1-\epsilon)\sigma^2)$

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Related work on Gaussian thinning:

- Tian and Taylor, 2018, Annals of Statistics.
- Tian, 2020, Annals of Statistics.
- Rasines and Young, 2022, Biometrika.
- Leiner et al., 2022, arXiv:2112.11079.
- Oliveira, Lei, and Tibshirani, 2022, arXiv:2111.09447.

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$\text{NegativeBinomial}(\mu, b)$	$\text{BetaBinomial}(x, \epsilon b, (1-\epsilon)b)$.	$\text{NegativeBinomial}(\epsilon\mu, \epsilon b)$	$\text{NegativeBinomial}((1-\epsilon)\mu, (1-\epsilon)b)$

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$\mathrm{Binomial}(r, p)$	$\mathrm{Hypergeometric}(\epsilon r, (1-\epsilon)r, x)$.	$\mathrm{Binomial}(\epsilon r, p)$	$\mathrm{Binomial}((1-\epsilon)r, p)$
$\mathrm{Gamma}(\alpha, \beta)$	$x \cdot \mathrm{Beta}\,(\epsilon\alpha, (1-\epsilon)\alpha)$.	$\mathrm{Gamma}(\epsilon\alpha, \beta)$	$\mathrm{Gamma}((1-\epsilon)\alpha, \beta)$
$\mathrm{Exponential}(\lambda)$	$x \cdot \mathrm{Beta}(\epsilon, (1-\epsilon))$.	$\mathrm{Gamma}(\epsilon, \lambda)$	$\mathrm{Gamma}(1-\epsilon, \lambda)$
$\mathrm{N}_k(\mu, \Sigma)$	$\mathrm{N}(\epsilon x, \epsilon(1-\epsilon)\Sigma)$.	$\mathrm{N}_k(\epsilon\mu, \epsilon\Sigma)$	$\mathrm{N}_k((1-\epsilon)\mu, (1-\epsilon)\Sigma)$
$\mathrm{Multinomial}_k(r, p)$	$\mathrm{MultivarHypergeom}(x_1, \ldots, x_K, \epsilon r)$	$\mathrm{Multinom}_k(\epsilon r, p)$	$\mathrm{Multinomial}_k((1-\epsilon)r, p)$
$\mathrm{Wishart}_p(n, \Sigma)$.	$x^{1/2} Z x^{1/2}$, where . $Z \sim \mathrm{MatrixBeta}_p(\epsilon n/2, (1-\epsilon)n/2)$	$\mathrm{Wishart}_p(\epsilon n, \Sigma)$	$\mathrm{Wishart}_p((1-\epsilon)n, \Sigma)$

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Goal: split a single observation X into $\left(X^{(1)}, \ldots, X^{(M)}\right)$ such that:

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$\text{N}(\mu, \sigma^2)$	$\text{N}_M(\epsilon\mu, \sigma^2 \text{diag}(\epsilon) - \sigma^2 \epsilon\epsilon^T)$.	$\text{N}(\epsilon_m \mu, \epsilon_m \sigma^2)$
$\text{NegativeBinomial}(\mu, b)$	$\text{DirichletMultinomial}(x, \epsilon_1 b, \ldots, \epsilon_M b)$.	$\text{NegativeBinomial}(\epsilon_m \mu, \epsilon_m b)$
$\text{Gamma}(\alpha, \beta)$	$x \cdot \text{Dirichlet}\,(\epsilon_1 \alpha, \ldots, \epsilon_M \alpha)$	$\text{Gamma}(\epsilon_m \alpha, \beta)$
$\text{Exponential}(\lambda)$	$x \cdot \text{Dirichlet}\,(\epsilon_1, \ldots, \epsilon_M)$	$\text{Gamma}(\epsilon_m, \lambda)$
$\text{Binomial}(r, p)$	$\text{MultivariateHypergeometric}(\epsilon_1 r, \ldots, \epsilon_M r, x)$.	$\text{Binomial}(\epsilon_m r, p)$

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Distribution of X	Draw $(X^{(1)}, \ldots, X^{(M)}) \mid X = x$ from:	Distribution of $X^{(m)}$
$\mathrm{Poisson}(\lambda)$	$\mathrm{Multinomial}(x, \epsilon_1, \ldots, \epsilon_M)$	$\mathrm{Poisson}(\epsilon_m \lambda)$
$\mathrm{N}(\mu, \sigma^2)$	$\mathrm{N}_M(\epsilon\mu, \sigma^2 \mathrm{diag}(\epsilon) - \sigma^2 \epsilon\epsilon^T)$.	$\mathrm{N}(\epsilon_m \mu, \epsilon_m \sigma^2)$
$\mathrm{NegativeBinomial}(\mu, b)$	$\mathrm{DirichletMultinomial}(x, \epsilon_1 b, \ldots, \epsilon_M b)$.	$\mathrm{NegativeBinomial}(\epsilon_m \mu, \epsilon_m b)$
$\mathrm{Gamma}(\alpha, \beta)$	$x \cdot \mathrm{Dirichlet}\,(\epsilon_1 \alpha, \ldots, \epsilon_M \alpha)$	$\mathrm{Gamma}(\epsilon_m \alpha, \beta)$
$\mathrm{Exponential}(\lambda)$	$x \cdot \mathrm{Dirichlet}\,(\epsilon_1, \ldots, \epsilon_M)$	$\mathrm{Gamma}(\epsilon_m, \lambda)$
$\mathrm{Binomial}(r, p)$	$\mathrm{MultivariateHypergeometric}(\epsilon_1 r, \ldots, \epsilon_M r, x)$.	$\mathrm{Binomial}(\epsilon_m r, p)$

Outline

1. Motivation: sample splitting doesn't always work
2. Poisson thinning
3. Data thinning
4. **Application to changepoint detection**
5. Generalized data thinning

Application: changepoint detection

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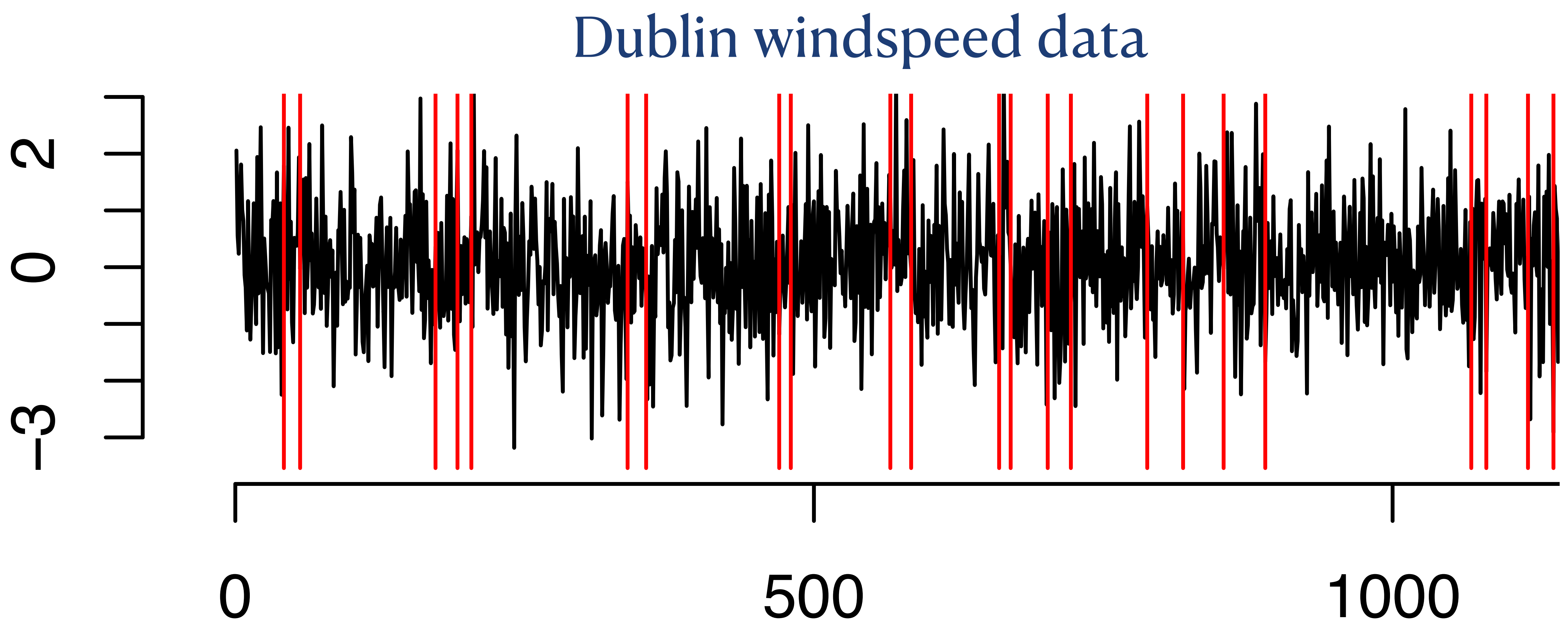

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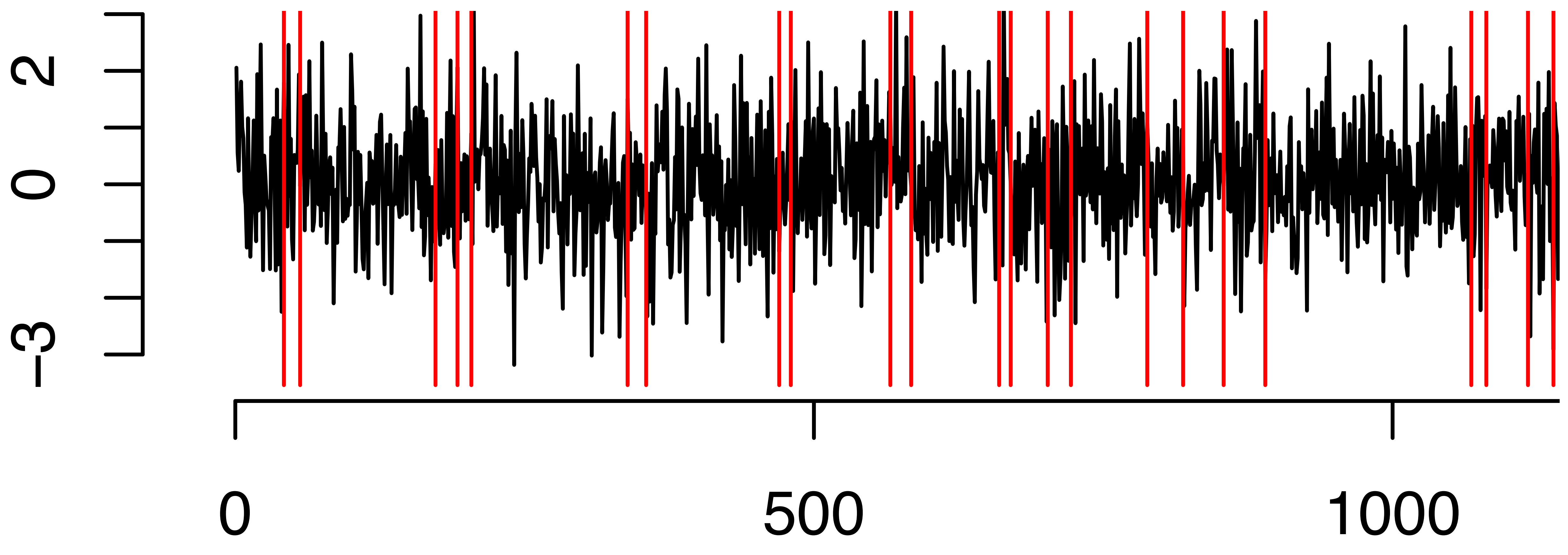
Goal: Can we validate these estimated changepoints?

Dublin windspeed data



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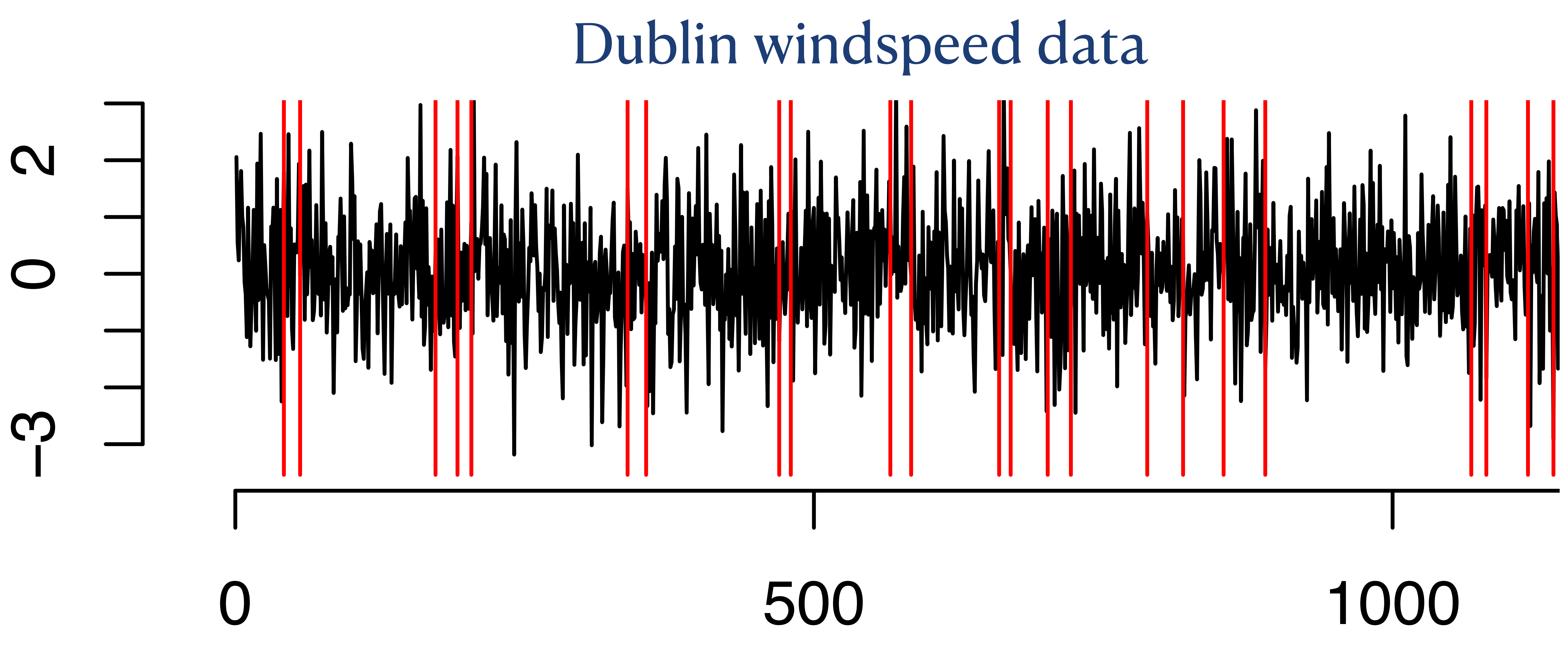

Idea: thin the data, such that we can estimate changepoints on training set and evaluate significance on test set.

We do not know how to thin a Gaussian with unknown variance

Distribution of X:	Draw $X^{(1)} \mid X = x$ from $G_{\epsilon,x}$, where $G_{\epsilon,x}$ is:	Distribution of $X^{(1)}$:	Distribution of $X^{(2)}$, where $X^{(2)} = X - X^{(1)}$:
$\text{Poisson}(\lambda)$	$\text{Binomial}(x, \epsilon)$	$\text{Poisson}(\epsilon\lambda)$	$\text{Poisson}((1-\epsilon)\lambda)$
$\text{N}(\mu, \sigma^2)$	$\text{N}(\epsilon x, \epsilon(1-\epsilon)\sigma^2)$	$\text{N}(\epsilon\mu, \epsilon\sigma^2)$	$\text{N}((1-\epsilon)\mu, (1-\epsilon)\sigma^2)$
$\text{NegativeBinomial}(\mu, b)$	$\text{BetaBinomial}(x, \epsilon b, (1-\epsilon)b)$.	$\text{NegativeBinomial}(\epsilon\mu, \epsilon b)$	$\text{NegativeBinomial}((1-\epsilon)\mu, (1-\epsilon)b)$
$\text{Binomial}(r, p)$	$\text{Hypergeometric}(\epsilon r, (1-\epsilon)r, x)$.	$\text{Binomial}(\epsilon r, p)$	$\text{Binomial}((1-\epsilon)r, p)$
$\text{Gamma}(\alpha, \beta)$	$x \cdot \text{Beta}\,(\epsilon\alpha, (1-\epsilon)\alpha)$.	$\text{Gamma}(\epsilon\alpha, \beta)$	$\text{Gamma}((1-\epsilon)\alpha, \beta)$
$\text{Exponential}(\lambda)$	$x \cdot \text{Beta}(\epsilon, (1-\epsilon))$.	$\text{Gamma}(\epsilon, \lambda)$	$\text{Gamma}(1-\epsilon, \lambda)$
$\text{N}_k(\mu, \Sigma)$	$\text{N}(\epsilon x, \epsilon(1-\epsilon)\Sigma)$.	$\text{N}_k(\epsilon\mu, \epsilon\Sigma)$	$\text{N}_k((1-\epsilon)\mu, (1-\epsilon)\Sigma)$
$\text{Multinomial}_k(r, p)$	$\text{MultivarHypergeom}(x_1, \ldots, x_K, \epsilon r)$	$\text{Multinom}_k(\epsilon r, p)$	$\text{Multinomial}_k((1-\epsilon)r, p)$
$\text{Wishart}_p(n, \Sigma)$.	$x^{1/2} Z x^{1/2}$, where . $Z \sim \text{MatrixBeta}_p(\epsilon n/2, (1-\epsilon)n/2)$	$\text{Wishart}_p(\epsilon n, \Sigma)$	$\text{Wishart}_p((1-\epsilon)n, \Sigma)$

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$\boxed{\mathrm{N}(\mu, \sigma^2)}$	$\boxed{\mathrm{N}(\epsilon x, \epsilon(1-\epsilon)\sigma^2)}$	$\mathrm{N}(\epsilon\mu, \epsilon\sigma^2)$	[illegible]
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$\mathrm{Exponential}(\lambda)$	$x \cdot \mathrm{Beta}(\epsilon, (1-\epsilon))$.	$\mathrm{Gamma}(\epsilon, \lambda)$	[illegible]
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Fact: if $X_i \sim N(0, \theta_i)$, then

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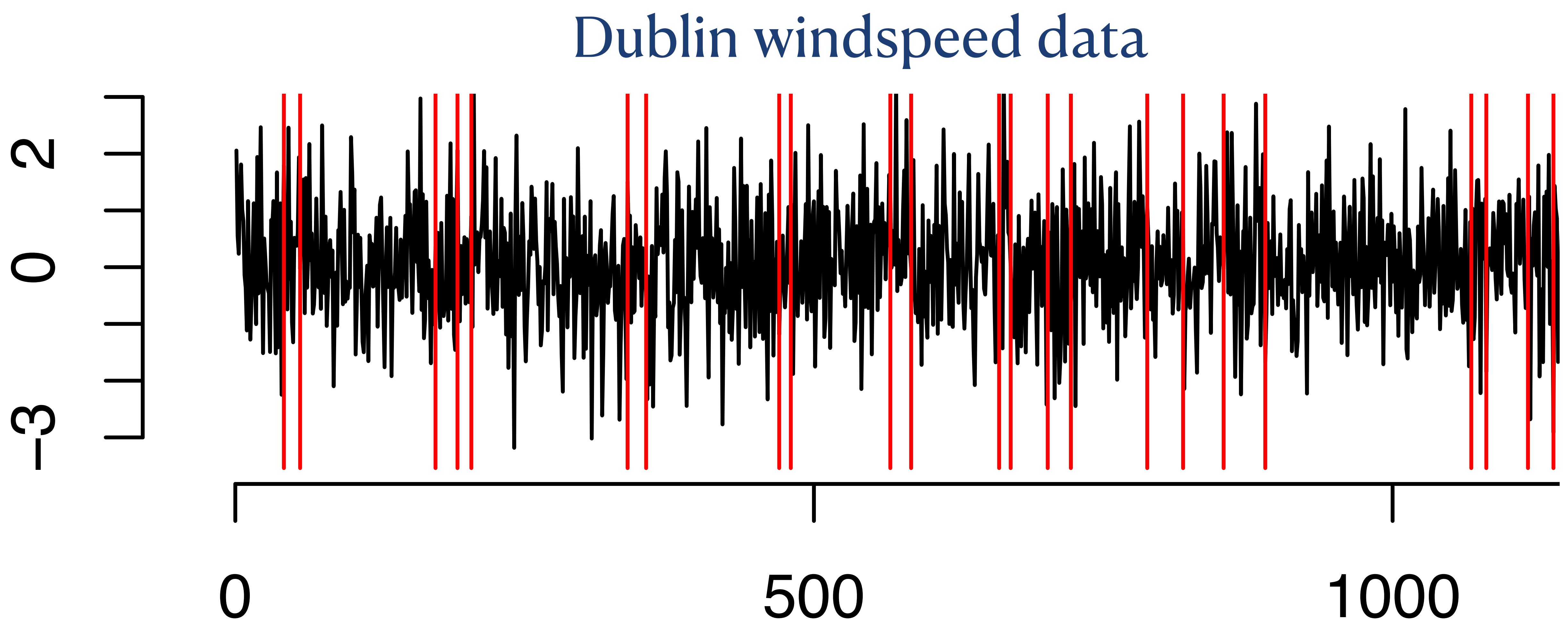

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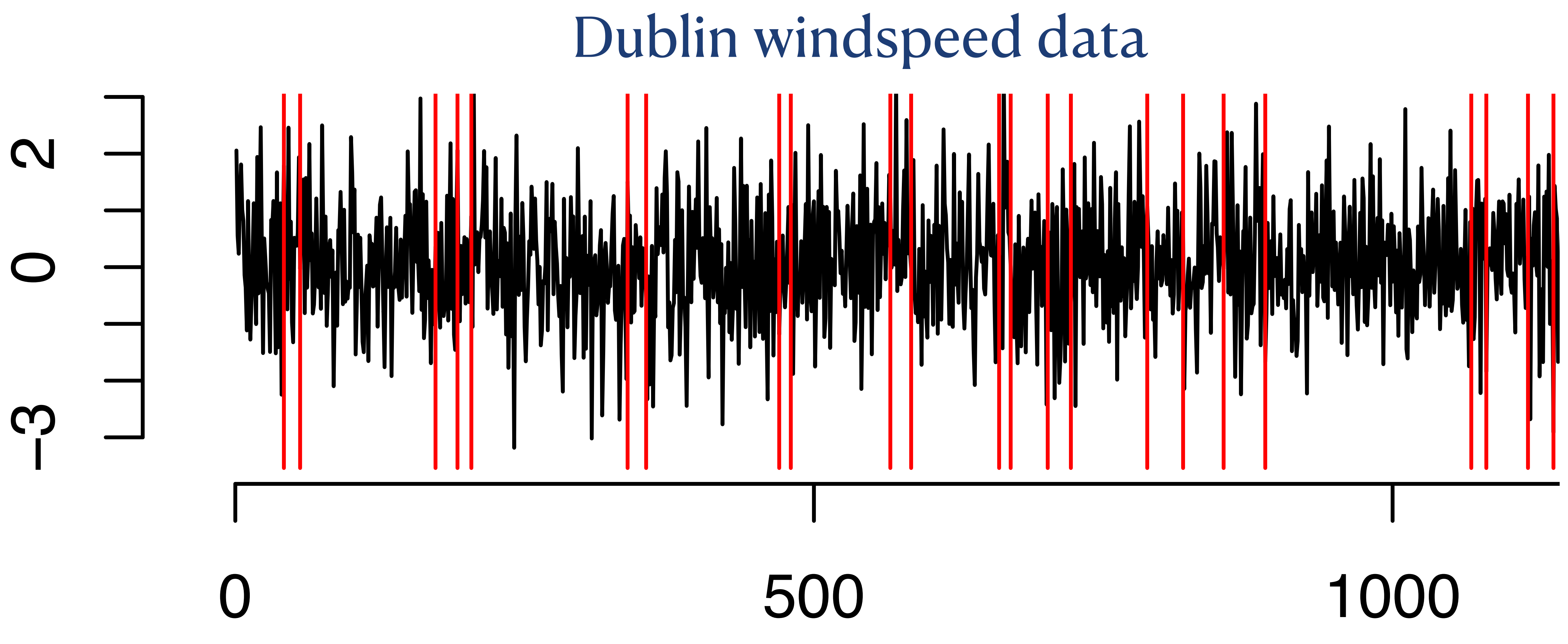


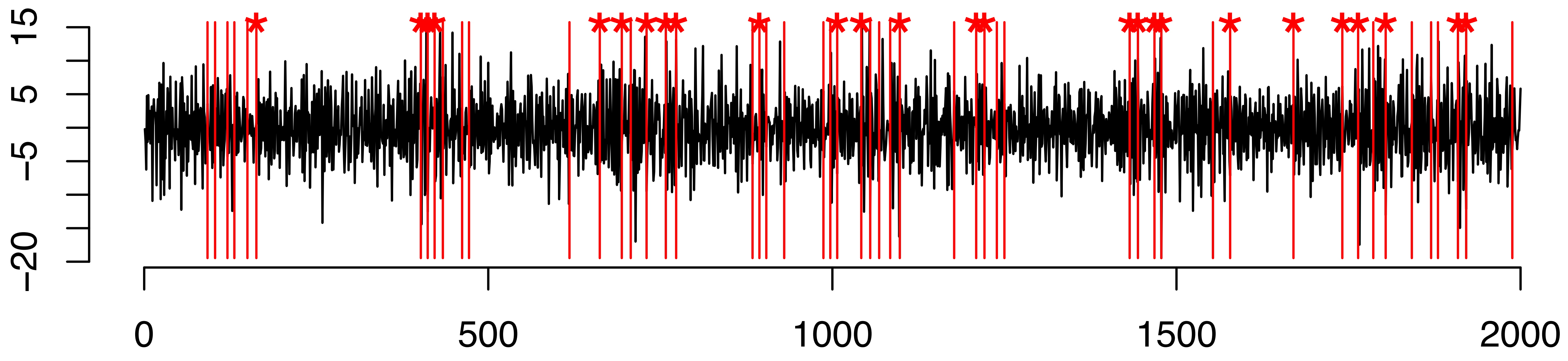
Define $Z_i := X_i^2$. Note that $Z_i \sim$ Gamma $\left(0.5, 0.5/\theta_i^2\right)$.

Naive method: Estimate and test changepoints on Z_i.

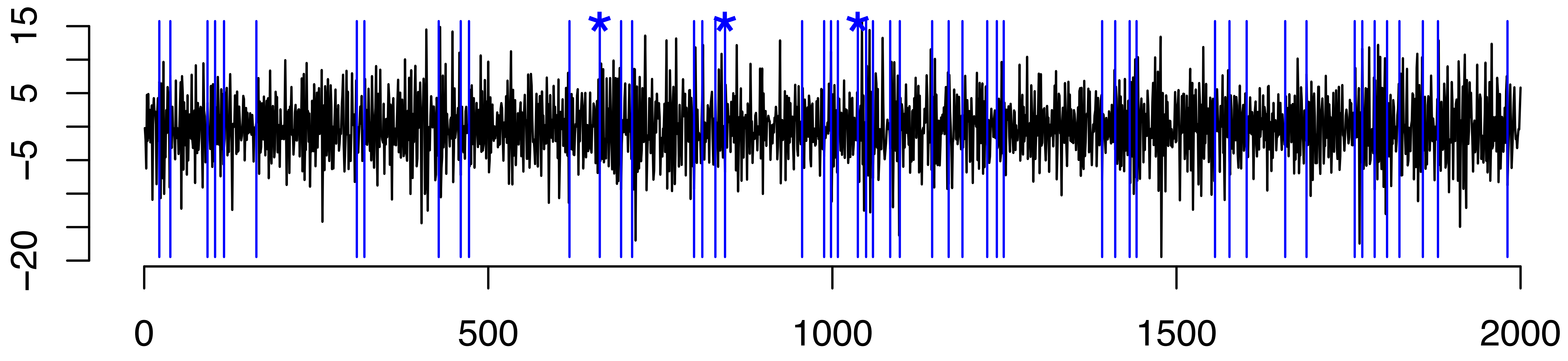
Thinning: Thin Z_i into $Z_i^{(1)} \sim$ Gamma $\left(0.25, 0.5/\theta_i^2\right)$ and $Z_i^{(2)} \sim$ Gamma $\left(0.25, 0.5/\theta_i^2\right)$.
Identify changepoints using $Z^{(1)}$, and test using $Z^{(2)}$.

Dublin wind speed data

Naive: 55 changepoints, 26 p-value < 0.05



Data thinning: 53 changepoints, 3 p-value < 0.05



Outline

1. Motivation: sample splitting doesn't always work
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Revisiting the goals of data thinning

Goal: split a single observation X into $X^{(1)}$ and $X^{(2)}$ such that:

(1) $X^{(1)}$ and $X^{(2)}$ have the same distribution as X, up to a parameter scaling.

(2) $X^{(1)} \perp\!\!\!\perp X^{(2)}$.

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Generalized thinning with non-additive decompositions

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We observe realization x from $X \sim P_\theta$.

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We know x could have arisen as $T(x', x'')$, where
$X' \sim Q_\theta^1,\ \ X'' \sim Q_\theta^2,\ \ X' \perp\!\!\!\perp X''.$

$\Downarrow$

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Can we work backwards to recover x' and x''?

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Let $G_{x,\theta}$ be the conditional distribution of $(X', X'') \mid X = x$.

Generalized thinning with non-additive decompositions

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We observe realization x from $X \sim P_\theta$.

↓

Draw $\left(X^{(1)}, X^{(2)}\right)$ from $G_{x,\theta}$.

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Theorem:
$X^{(1)} \sim Q_\theta^1, \quad X^{(2)} \sim Q_\theta^2, \quad X^{(1)} \perp\!\!\!\perp X^{(2)}$.

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Can we work backwards to recover x' and x''?

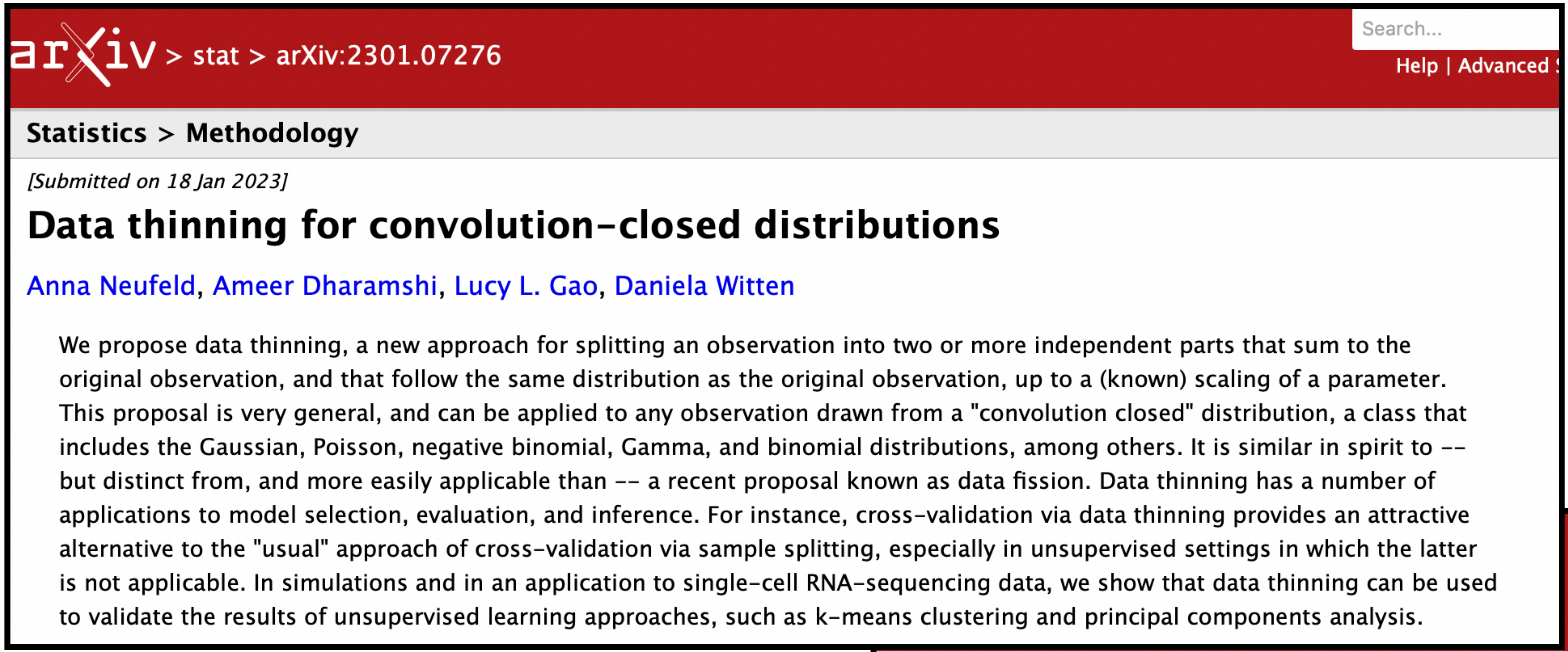
Let $G_{x,\theta}$ be the conditional distribution of $(X', X'') \mid X = x$.

Key idea: If $X = T(X', X'')$ is sufficient for θ in the joint of (X', X''), then $G_{x,\theta}$ does not depend on θ.

The list of distributions we can thin is extensive

Family	Distribution P_θ, where $X \sim P_\theta$.	Distribution $Q_\theta^{(k)}$ where $X^{(k)} \overset{ind.}{\sim} Q_\theta^{(k)}$.	Sufficient statistic T (sufficient for θ)
Natural exponential family (in parameter θ)	$N(\theta, \sigma^2)$	$\mathrm{N}(\epsilon_k\theta, \epsilon_k\sigma^2)$	$\sum_{k=1}^K X^{(k)}$
	$\text{Poisson}(\theta)$	$\text{Poisson}(\epsilon_k\theta)$	
	$\text{NegBin}(r, \theta)$	$\text{NegBin}(\epsilon_k r, \theta)$	
	$\text{Binomial}(r, \theta)$	$\text{Binomial}(\epsilon_k r, \theta)$	
	$\text{Gamma}(\alpha, \theta)$	$\text{Gamma}(\epsilon_k\alpha, \theta)$	
	$\mathrm{N}_p(\boldsymbol{\theta}, \Sigma)$	$N_p(\epsilon_k\boldsymbol{\theta}, \epsilon_k\Sigma)$	
	$\text{Multinomial}_p(r, \boldsymbol{\theta})$	$\text{Multinomial}_p(\epsilon_k r, \boldsymbol{\theta})$	
	$\text{Gamma}(K/2, \theta)$	$N(0, \frac{1}{2\theta})$	$\sum_{k=1}^K \left(X^{(k)}\right)^2$
	$\text{Gamma}(K, \theta)$	$\text{Weibull}(\theta^{-\frac{1}{\nu}}, \nu)$	$\sum_{k=1}^K \left(X^{(k)}\right)^\nu$
General exponential family (in parameter θ)	$\text{Beta}(\theta, \beta)$	$\text{Beta}\left(\frac{1}{K}\theta + \frac{k-1}{K}, \frac{1}{K}\beta\right)$	$\left(\Pi_{k=1}^K X^{(k)}\right)^{1/K}$
	$\text{Beta}(\alpha, \theta)$	$\text{Beta}\left(\frac{1}{K}\alpha, \frac{1}{K}\theta + \frac{k-1}{K}\right)$	$\left(\Pi_{k=1}^K \left(1 - X^{(k)}\right)\right)^{1/K}$
	$\text{Gamma}(\theta, \beta)$	$\text{Gamma}(\frac{1}{K}\theta + \frac{k-1}{K}, \frac{1}{K}\beta)$	$\left(\Pi_{k=1}^K X^{(k)}\right)^{1/K}$
	$\text{Weibull}(\theta, \nu)$	$\text{Gamma}(\frac{1}{K}, \theta^{-\nu})$	$\left(\sum_{k=1}^K X^{(k)}\right)^{1/\nu}$
	$\text{Pareto}(\nu, \theta)$	$\text{Gamma}(\frac{1}{K}, \theta)$	$\nu \times \text{Exp}\left(\sum_{k=1}^K X^{(k)}\right)$
	$\mathrm{N}(0,\theta)$	$\text{Gamma}(\frac{1}{2K}, \frac{1}{2\theta})$	$X^2 = \sum_{k=1}^K X^{(k)}$
	$\mathrm{N}_K(\theta_1 1_K, \theta_2 I_K)$	$N(\theta_1, \theta_2)$	sample mean and variance
Truncated support family	$\text{Unif}(0, \theta)$	$\theta \cdot \text{Beta}(\frac{1}{K}, 1)$	$\max\left(X^{(1)}, \ldots, X^{(K)}\right)$
	$\theta \cdot \text{Beta}(\alpha, 1)$	$\theta \cdot \text{Beta}(\frac{\alpha}{K}, 1)$	
	$\theta + \text{Exp}(\lambda)$	$\theta + \text{Exp}(\lambda/\mathrm{K})$	$\min\left(X^{(1)}, \ldots, X^{(K)}\right)$
Non-parametric	F^n	F^{n_k}	$\text{sort}(X^{(1)}, \ldots, X^{(K)})$

References


arXiv > stat > arXiv:2301.07276

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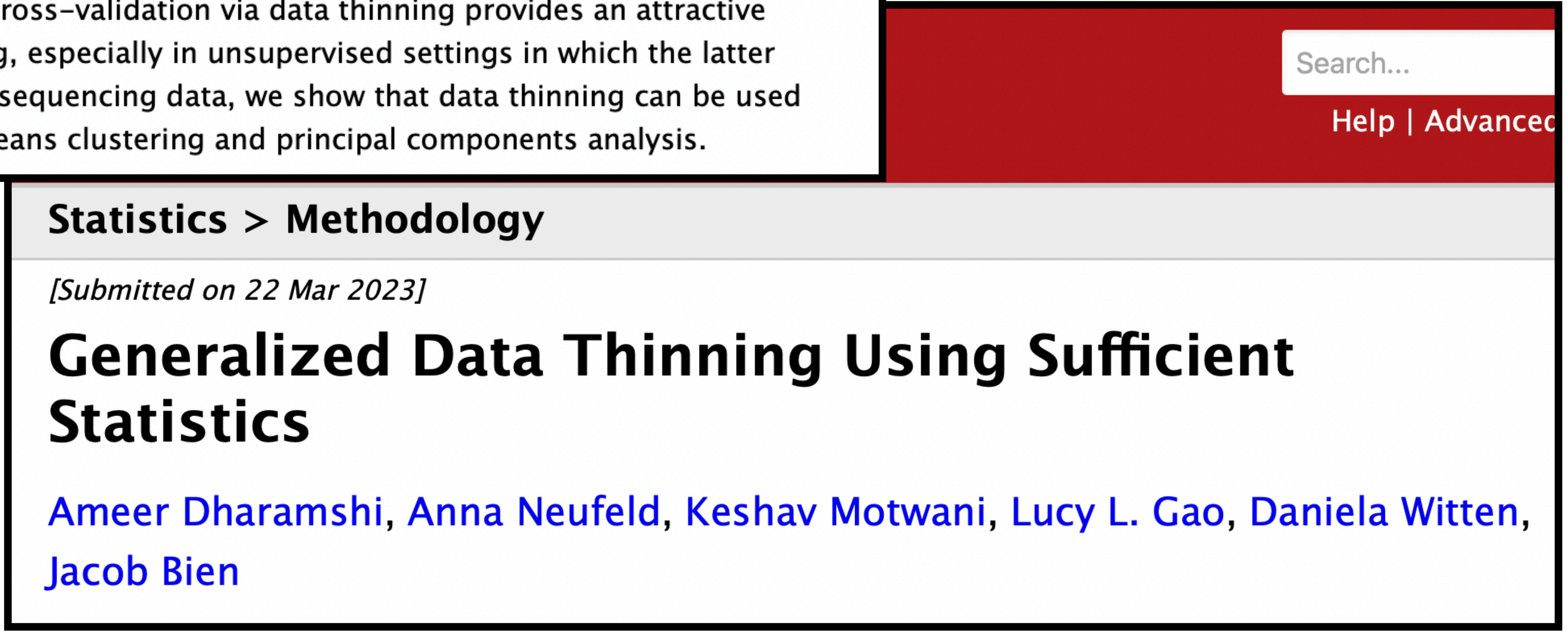
Statistics > Methodology

[Submitted on 18 Jan 2023]

Data thinning for convolution-closed distributions

Anna Neufeld, Ameer Dharamshi, Lucy L. Gao, Daniela Witten

We propose data thinning, a new approach for splitting an observation into two or more independent parts that sum to the original observation, and that follow the same distribution as the original observation, up to a (known) scaling of a parameter. This proposal is very general, and can be applied to any observation drawn from a "convolution closed" distribution, a class that includes the Gaussian, Poisson, negative binomial, Gamma, and binomial distributions, among others. It is similar in spirit to -- but distinct from, and more easily applicable than -- a recent proposal known as data fission. Data thinning has a number of applications to model selection, evaluation, and inference. For instance, cross-validation via data thinning provides an attractive alternative to the "usual" approach of cross-validation via sample splitting, especially in unsupervised settings in which the latter is not applicable. In simulations and in an application to single-cell RNA-sequencing data, we show that data thinning can be used to validate the results of unsupervised learning approaches, such as k-means clustering and principal components analysis.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[Submitted on 22 Mar 2023]

Generalized Data Thinning Using Sufficient Statistics

Ameer Dharamshi, Anna Neufeld, Keshav Motwani, Lucy L. Gao, Daniela Witten, Jacob Bien

Acknowledgements

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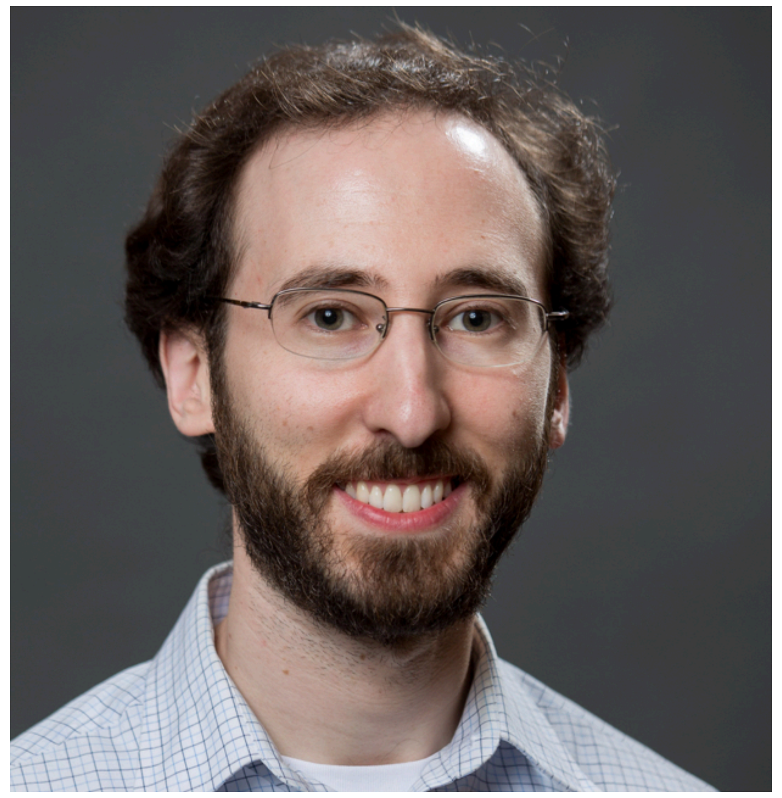
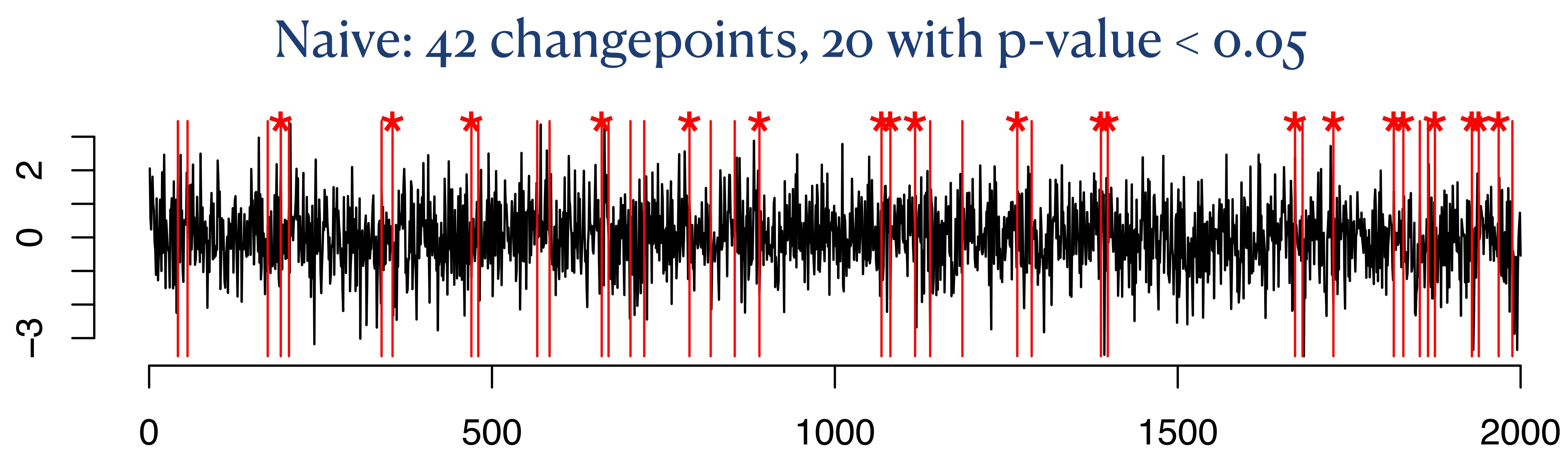


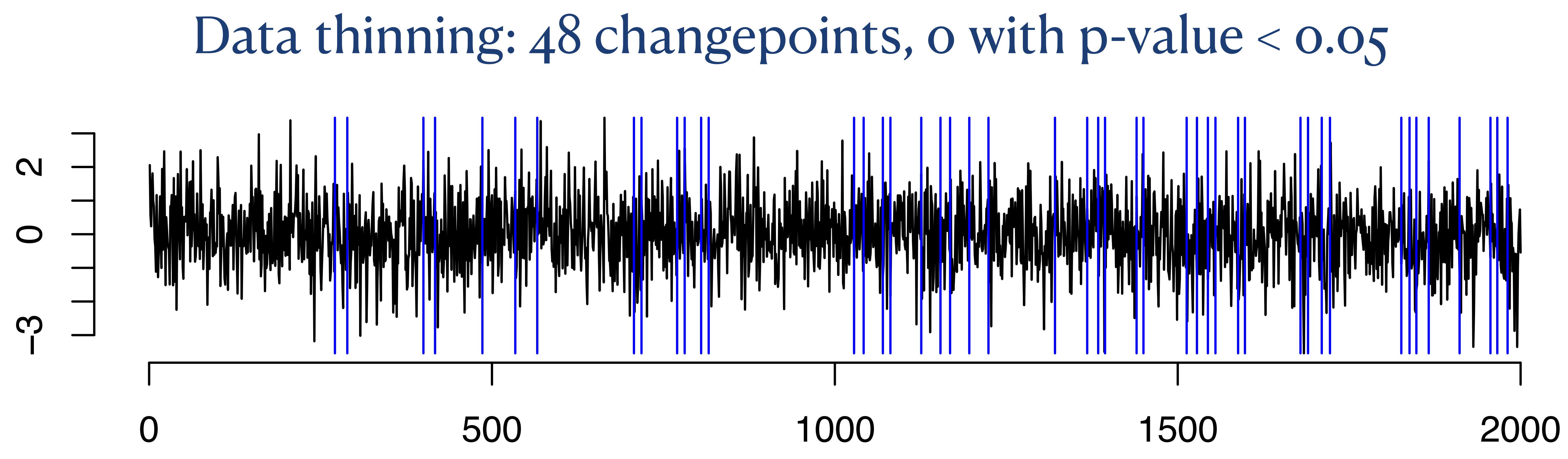
Jacob Bien
University of Southern California

Keshav Motwani
University of Washington

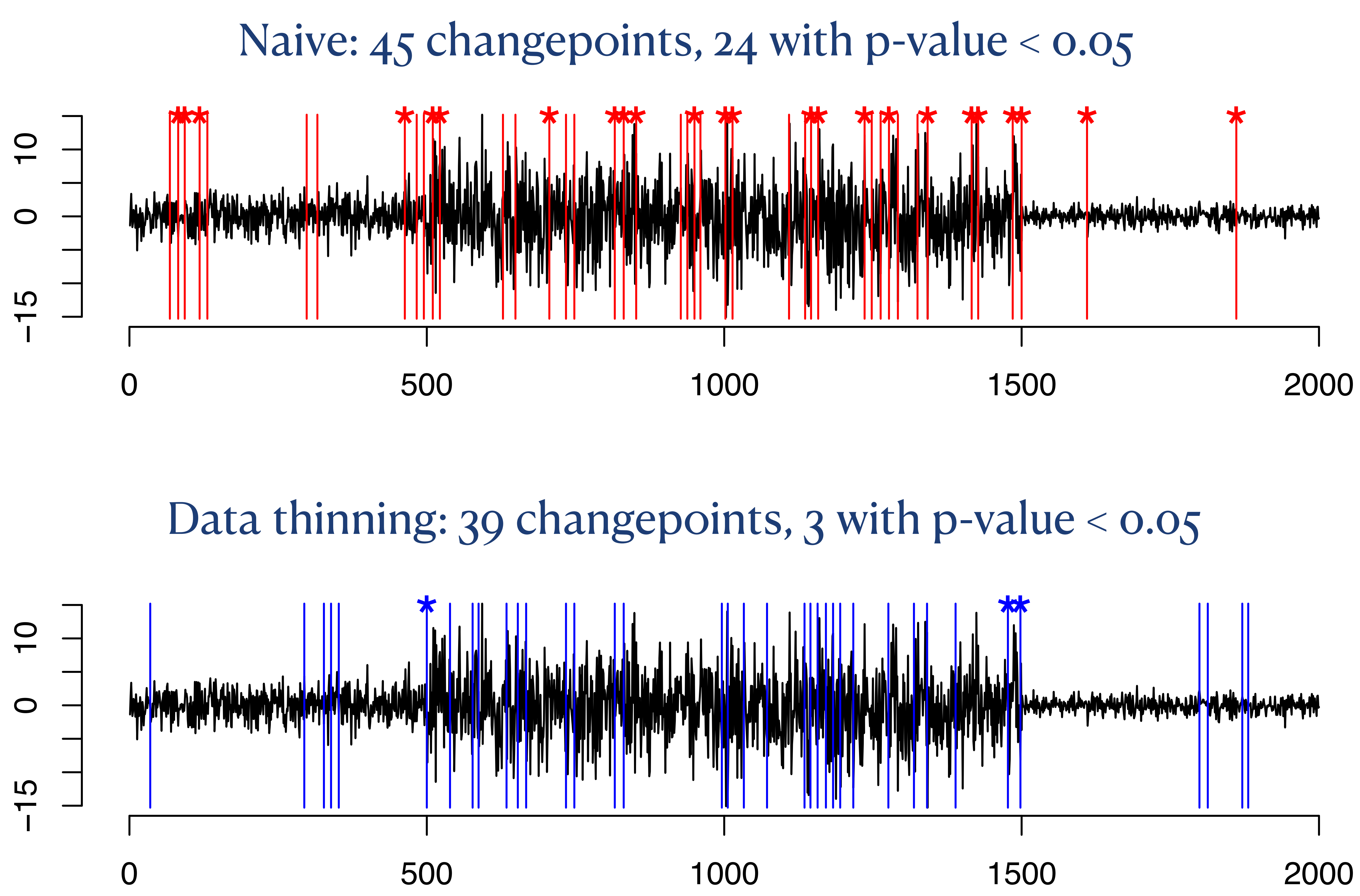
Questions?

Simulated data with no changepoints





Simulated data with two true changepoints



Are the results of our data analysis stable?