CIMPA COURSE @ IMBM

THE INVERSE JACOBIAN PROBLEM

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Motivation

The situation for elliptic curves

$$\begin{tikzcd}
\{E : y^2 = x^3 + Ax^2 + B\}/\cong \arrow[rr, leftrightarrow, "E(\mathbb{C})\cong\mathbb{C}/(\mathbb{Z}+\tau\mathbb{Z})"'] \arrow[d, leftrightarrow] & & \tau \in \mathbb{C}/\mathrm{Sp}(2,\mathbb{Z}) \arrow[ll, squiggly, bend right, "\text{explicit \& numerical}"'] \\
\{j-\text{invariant}\} & &
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\{j - \text{invariant}\} & \text{Quadratic imaginary fields} \arrow[u, dashed, twoheadrightarrow, "\text{CM method}"']
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This gives us isomorphism class of the curve $y^2 = x^3 - x$.

The generalization: Genus-g curves

Does it make sense?

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We have

$$\dim H^0(\omega_C) = g, \quad \dim H_1(C, \mathbb{Z}) = 2g,$$

so we define

$$\begin{aligned} \alpha\colon C(\mathbb{C}) &\to \mathbb{C}^g/\Lambda \\ Q &\mapsto \left(\int_P^Q \omega_1, \ldots, \int_P^Q \omega_g\right) \end{aligned}$$

with

$$\Lambda = \left\{ \left(\int_\gamma \omega_1, \ldots, \int_\gamma \omega_g\right) : \gamma \in H_1(C, \mathbb{Z}) \right\}.$$

Does it make sense?

With the right basis we obtain

$$\Lambda \simeq \mathbb{Z}^g + \tau\mathbb{Z}^g$$

for $\tau \in \mathbb{H}_g = \{M \in \mathbb{C}^{g\times g} \colon M = M^T, \mathrm{Im}(M) > 0\}$ a *period matrix* of C.
We define $J(C) = \mathbb{C}^g / \Lambda$ the *Jacobian of the curve*

Construction of CM curves

$$
\begin{tikzcd}
\{C\}/\cong \arrow[r, "J(C)\cong\mathbb{C}^g/(\mathbb{Z}^g+\tau\mathbb{Z}^g)"'] \arrow[d, dashed, leftrightarrow] & \tau\in\mathbb{C}^{g\times g}/\mathrm{Sp}(2g,\mathbb{Z})^{*} \arrow[l, bend right, squiggly, "\text{explicit \& numerical?}"'] \\
\text{Invariant theory} & \text{CM-fields of degree } 2g \arrow[u, dashed, "\text{CM method}"']
\end{tikzcd}
$$

State of the art

$g = 2$ All curves are hyperelliptic. Done.

$g = 3$ Hyperelliptic curves: Done.

Picard curves $y^3 = x(x-1)(x-\lambda)(x-\mu)$:
We will see this next.

Plane quartics: Done.

$g = 6$ Superelliptic curves: $y^5 = x(x-1)(x-\lambda)(x-\mu)$:
Follows the same principle as Picard curves.

Definition

The Riemann theta function is the function $\theta : \mathbb{C}^g \times \mathbb{H}_g \to \mathbb{C}$ given by

$$\theta(z, \Omega) = \sum_{n \in \mathbb{Z}^g} \exp(\pi i n^t \Omega n + 2\pi i n^t z).$$

Theorem (Riemann's Vanishing theorem)

Let C be a curve over $\mathbb{C}$ of genus g, let $J(C)$ be the Jacobian of C with period matrix $\Omega \in \mathbb{H}_g$ and let α be an Abel-Jacobi map.

There is an element $\Delta \in J(C)$, called a Riemann constant with respect to α, such that the function $\theta(\,\cdot\,, \Omega)$ vanishes at $z \in \mathbb{C}^g$ if and only if there exist $Q_1, \dots, Q_{g-1} \in C$ that satisfy

$$z \equiv \alpha(Q_1 + \cdots + Q_{g-1}) - \Delta \mod \Omega\mathbb{Z}^g + \mathbb{Z}^g.$$

The formula

Theorem (Siegel)

Let C be a curve of genus g over $\mathbb{C}$ with distinguished point $P \in C$ and let ω be a basis of $H^0(\omega_C)$ that gives the period matrix $\Omega \in \mathbb{H}_g$. Let ϕ be a function on C with $\operatorname{div}(\phi) = A_1 + \cdots + A_m - B_1 - \cdots - B_m$.
Let Δ be the Riemann constant with respect to P, and choose paths from the base point P to A_i and B_i that satisfy

$$\sum_{i=1}^{m} \int_P^{A_i} \omega = \sum_{i=1}^{m} \int_P^{B_i} \omega.$$

Then, given an effective non-special divisor $D = Q_1 + \cdots + Q_g$ that satisfies $Q_j \notin \{A_i, B_i \colon 1 \leq i \leq m\}$, one has

$$\phi(P_1) \dots \phi(P_g) = E \prod_{i=1}^{m} \frac{\theta(\sum_{j=1}^{g} \int_P^{Q_j} \omega - \int_P^{A_i} \omega - \Delta, \Omega)}{\theta(\sum_{j=1}^{g} \int_P^{Q_j} \omega - \int_P^{B_i} \omega - \Delta, \Omega)},$$

where $E \in \mathbb{C}^\times$ is independent of D, and the integrals from P to Q_j take the same paths both in the numerator and the denominator.

Picard curves

A Picard curve C over $\mathbb{C}$ is given by

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Then $\rho(P_t) = P_t$.

The formula for Picard curves

Theorem

Let C be a Picard curve and let ω be a basis of $H^0(\omega_C)$ that gives the period matrix $\Omega \in \mathbb{H}_g$. Let $\phi = x$ be a function on C with $\mathrm{div}(\phi) = 3P_0 - 3P_\infty$.
Let Δ be the Riemann constant with respect to P_∞, and choose paths a_i, b_i from the base point P_∞ to P_0 and P_∞ that satisfy

$$\sum_{i=1}^{3} \int_{a_i} \omega = \sum_{i=1}^{3} \int_{b_i} \omega.$$

Then, given an effective non-special divisor $D = Q_1 + Q_2 + Q_3$ that satisfies $Q_j \notin \{P_0, P_\infty\}$, one has

$$x(Q_1)x(Q_2)x(Q_3) = E \prod_{i=1}^{3} \frac{\theta(\sum_{j=1}^{3} \int_{P_\infty}^{Q_j} \omega - \int_{a_i} \omega - \Delta, \Omega)}{\theta(\sum_{j=1}^{3} \int_{P_\infty}^{Q_j} \omega - \int_{b_i} \omega - \Delta, \Omega)}, \qquad (*)$$

where $E \in \mathbb{C}^\times$ is independent of D, and the integrals from P_∞ to Q_j take the same paths both in the numerator and the denominator.

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where $E \in \mathbb{C}^\times$ is independent of D, and the integrals from P_∞ to Q_j take the same paths both in the numerator and the denominator.

We take the quotient of the expression (*) for $D_1 = P_1 + 2P_\lambda$, $D_2 = 2P_1 + P_\lambda$.

Identifying the branch points

Properties

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- $\langle \alpha(P_t) \rangle = J(C)[1-\rho_*]$.
- $\mathrm{div}(y) = P_0 + P_1 + P_\lambda + P_\mu - 4P_\infty \rightarrow$
  $$\alpha(P_0) + \alpha(P_1) + \alpha(P_\lambda) + \alpha(P_\mu) = 0.$$

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- $\langle \alpha(P_t) \rangle = J(C)[1-\rho_*]$.
- $\operatorname{div}(y) = P_0 + P_1 + P_\lambda + P_\mu - 4P_\infty \rightarrow$
  $$\alpha(P_0) + \alpha(P_1) + \alpha(P_\lambda) + \alpha(P_\mu) = 0.$$

Reminder: Riemann Vanishing theorem

$$\theta(x, \Omega) = 0 \Leftrightarrow \exists P_1, \ldots, P_{g-1} \in C : x \equiv \alpha(P_1) + \cdots + \alpha(P_{g-1}) - \Delta$$

The algorithm

Main idea

The bijection $\underline{\cdot} : J(C) \to \mathbb{R}^{2g}/\mathbb{Z}^{2g}$ given by

$$\Omega x_1 + x_2 \mapsto (x_1, x_2)$$

maps the m-torsion of $J(C)$ to $\frac{1}{m}\mathbb{Z}^{2g}/\mathbb{Z}^{2g}$.

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We define

$$\Theta_3 := \left\{ x \in \frac{1}{3}\mathbb{Z}^6/\mathbb{Z}^6 : \theta[x + \underline{\Delta}](\Omega) = 0 \right\}.$$

Then $\alpha(\mathcal{B})$ and $-\alpha(\mathcal{B})$ are the only subsets $\mathcal{T} \subset J(C)$ of four elements such that:

(i) the sum $\sum_{x \in \mathcal{T}} x$ is zero,
(ii) $\mathcal{T}$ is a set of generators of $J(C)[1 - \rho_*]$, and
(iii) the set $\mathcal{O}(\mathcal{T}) := \{\sum_{x \in \mathcal{T}} a_x x : a \in \mathbb{Z}^4_{\geq 0}, \sum_{x \in \mathcal{T}} a_x \leq 2\}$ satisfies

$$\underline{\mathcal{O}(\mathcal{T})} = \Theta_3.$$

Final theorem

Theorem

Let C be a Picard curve over $\mathbb{C}$ given by

$$y^3 = x(x-1)(x-\lambda)(x-\mu),$$

let $\Omega \in \mathbb{H}_3$ be a period matrix of the Jacobian $J(C)$, let α be the Abel-Jacobi map with base point $(0:1:0)$, and let Δ be the Riemann constant with respect to α. Let $P_t = (t, 0)$ for $t \in \{0, 1, \lambda, \mu\}$ and let $\eta \in \{\lambda, \mu\}$. Then we have

$$\eta = \varepsilon_\eta \left(\frac{\theta[\tilde{P}_1 + 2\tilde{P}_\eta - \tilde{P}_0 - \tilde{\Delta}](\Omega)}{\theta[2\tilde{P}_1 + \tilde{P}_\eta - \tilde{P}_0 - \tilde{\Delta}](\Omega)} \right)^3, \tag{1}$$

with $\varepsilon_\eta = \exp(6\pi i((\tilde{P}_\eta - \tilde{P}_1)_1(\tilde{P}_0)_2 + (\tilde{P}_1 + 2\tilde{P}_\eta - \tilde{\Delta})_1(2\tilde{\Delta} - 3(\tilde{P}_1 + \tilde{P}_\eta))_2))$.

Construction of CM Picard curves

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\left\{ j_1 = \frac{b^2}{a^3},\ j_2 = \frac{c}{a^2} \right\} & \begin{array}{c}\text{CM-fields of degree 6}\\ \text{containing } \mathbb{Q}(\zeta_3)\end{array} \arrow[u, dashed, "\text{CM method}"']
\end{tikzcd}
$$

Related problems

Riemann-Schottky problem

Describe the image of $J : M_g \to A_g$. (More on this on Friday)

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How to prove that a curve has CM by a certain field?

Examples

- For $K = \mathbb{Q}(\zeta_3, v)$ with $v^3 - 21v - 28 = 0$,

$$y^3 = x^4 - 2 \cdot 3^2 \cdot 5^2 \cdot 7^2\, x^2 + 2^9 \cdot 7^2 \cdot 71\, x - 3^2 \cdot 5 \cdot 7^3 \cdot 2621,$$

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- for $K = \mathbb{Q}(\zeta_5, v)$ with $v^3 - v^2 - 2v + 1 = 0$,
$$y^5 = x^4 - 7x^2 + 7x.$$

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