

#### CIMPA COURSE @ IMBM

### THE INVERSE JACOBIAN PROBLEM

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# MOTIVATION The situation for elliptic curves

explicit & numerical 
$$\{E: y^2 = x^3 + Ax^2 + B\} / \cong_{E(\mathbb{C}) \cong \mathbb{C}/(\mathbb{Z} + \tau \mathbb{Z})} \tau \in \mathbb{C}/\operatorname{Sp}(2, \mathbb{Z})$$
 
$$\{j - \text{invariant}\}$$

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This gives us isomorphism class of the curve  $y^2 = x^3 - x$ .

THE GENERALIZATION:

GENUS-g CURVES

#### **DOES IT MAKE SENSE?**

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$$\dim H^0(\omega_C) = g$$
,  $\dim H_1(C, \mathbb{Z}) = 2g$ ,

so we define

$$\alpha: C(\mathbb{C}) \to \mathbb{C}^g / \Lambda$$

$$Q \mapsto \left( \int_P^Q \omega_1, \dots, \int_P^Q \omega_g \right)$$

with

$$\Lambda = \left\{ \left( \int_{\gamma} \omega_1, \dots, \int_{\gamma} \omega_g \right) : \gamma \in H_1(C, \mathbb{Z}) \right\}.$$

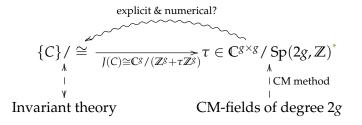
#### DOES IT MAKE SENSE?

With the right basis we obtain

$$\Lambda \simeq \mathbb{Z}^g + \tau \mathbb{Z}^g$$

for  $\tau \in \mathbb{H}_g = \{M \in \mathbb{C}^{g \times g} \colon M = M^T, \operatorname{Im}(M) > 0\}$  a period matrix of C. We define  $J(C) = \mathbb{C}^g / \Lambda$  the Jacobian of the curve

#### **CONSTRUCTION OF CM CURVES**



### STATE OF THE ART

- g = 2 All curves are hyperelliptic. Done.
- g = 3 Hyperelliptic curves: Done.

Picard curves  $y^3 = x(x-1)(x-\lambda)(x-\mu)$ : We will see this next.

Plane quartics: Done.

g = 6 Superelliptic curves:  $y^5 = x(x-1)(x-\lambda)(x-\mu)$ : Follows the same principle as Picard curves.

#### **PRELIMINARIES**

The Riemann theta functions

#### **Definition**

The Riemann theta function is the function  $\theta: \mathbb{C}^g \times \mathbb{H}_g \to \mathbb{C}$  given by  $\theta(z, \Omega) = \sum_{g} \exp(\pi i n^t \Omega n + 2\pi i n^t z)$ 

$$\theta(z,\Omega) = \sum_{n \in \mathbb{Z}^g} \exp(\pi i n^t \Omega n + 2\pi i n^t z).$$

#### **PRELIMINARIES**

#### The Riemann theta functions

#### Theorem (Riemann's Vanishing theorem)

Let *C* be a curve over  $\mathbb{C}$  of genus g, let J(C) be the Jacobian of *C* with period matrix  $\Omega \in \mathbb{H}_g$  and let  $\alpha$  be an Abel-Jacobi map.

There is an element  $\Delta \in J(C)$ , called a Riemann constant with respect to  $\alpha$ , such that the function  $\theta(\cdot, \Omega)$  vanishes at  $z \in \mathbb{C}^g$  if and only if there exist  $Q_1, \ldots, Q_{g-1} \in C$  that satisfy

$$z \equiv \alpha(Q_1 + \dots + Q_{g-1}) - \Delta \mod \Omega \mathbb{Z}^g + \mathbb{Z}^g$$
.

#### THE FORMULA

#### Theorem (Siegel)

Let *C* be a curve of genus *g* over  $\mathbb C$  with distinguished point  $P \in C$  and let  $\omega$  be a basis of  $H^0(\omega_C)$  that gives the period matrix  $\Omega \in \mathbb H_g$ . Let  $\phi$  be a function on *C* with  $\operatorname{div}(\phi) = A_1 + \cdots + A_m - B_1 - \cdots - B_m$ .

Let  $\Delta$  be the Riemann constant with respect to P, and choose paths from the base point P to  $A_i$  and  $B_i$  that satisfy

$$\sum_{i=1}^m \int_P^{A_i} \omega = \sum_{i=1}^m \int_P^{B_i} \omega.$$

Then, given an effective non-special divisor  $D = Q_1 + \cdots + Q_g$  that satisfies  $Q_j \notin \{A_i, B_i : 1 \le i \le m\}$ , one has

$$\phi(P_1)\dots\phi(P_g) = E \prod_{i=1}^m \frac{\theta(\sum_{j=1}^g \int_P^{Q_j} \omega - \int_P^{A_i} \omega - \Delta, \Omega)}{\theta(\sum_{j=1}^g \int_P^{Q_j} \omega - \int_P^{B_i} \omega - \Delta, \Omega)},$$

where  $E \in \mathbb{C}^{\times}$  is independent of D, and the integrals from P to  $Q_j$  take the same paths both in the numerator and the denominator.

A Picard curve *C* over *C* is given by

$$y^3 = x(x-1)(x-\lambda)(x-\mu)$$

A Picard curve C over  $\mathbb{C}$  is given by

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Then  $\rho(P_t) = P_t$ .

#### THE FORMULA FOR PICARD CURVES

#### **Theorem**

Let C be a Picard curve and let  $\omega$  be a basis of  $H^0(\omega_C)$  that gives the period matrix  $\Omega \in \mathbb{H}_g$ . Let  $\phi = x$  be a function on C with  $\operatorname{div}(\phi) = 3P_0 - 3P_\infty$ . Let  $\Delta$  be the Riemann constant with respect to  $P_\infty$ , and choose paths  $a_i, b_i$  from the base point  $P_\infty$  to  $P_0$  and  $P_\infty$  that satisfy

$$\sum_{i=1}^{3} \int_{a_i} \omega = \sum_{i=1}^{3} \int_{b_i} \omega.$$

Then, given an effective non-special divisor  $D=Q_1+Q_2+Q_3$  that satisfies  $Q_j \notin \{P_0, P_\infty\}$ , one has

$$x(Q_1)x(Q_2)x(Q_3) = E \prod_{i=1}^{3} \frac{\theta(\sum_{j=1}^{3} \int_{P_{\infty}}^{Q_j} \omega - \int_{a_i} \omega - \Delta, \Omega)}{\theta(\sum_{j=1}^{3} \int_{P_{\infty}}^{Q_j} \omega - \int_{b_i} \omega - \Delta, \Omega)},$$
(\*)

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$$x(Q_1)x(Q_2)x(Q_3) = E \prod_{i=1}^{3} \frac{\theta(\sum_{j=1}^{3} \int_{P_{\infty}}^{Q_j} \omega - \int_{a_i} \omega - \Delta, \Omega)}{\theta(\sum_{j=1}^{3} \int_{P_{\infty}}^{Q_j} \omega - \int_{b_i} \omega - \Delta, \Omega)}, \tag{*}$$

where  $E \in \mathbb{C}^{\times}$  is independent of D, and the integrals from  $P_{\infty}$  to  $Q_j$  take the same paths both in the numerator and the denominator.

We take the quotient of the expression (\*) for  $D_1 = P_1 + 2P_{\lambda}$ ,  $D_2 = 2P_1 + P_{\lambda}$ .

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$$\qquad \qquad \bullet \ \rho(P_t) = P_t \to \alpha(P_t) \in J(C)[1-\rho_*] \subset J(C)[3].$$

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We have the following results:

- $\blacktriangleright \langle \alpha(P_t) \rangle = J(C)[1 \rho_*].$
- $\begin{array}{c} \bullet \ \operatorname{div}(y) = P_0 + P_1 + P_\lambda + P_\mu 4P_\infty \to \\ \alpha(P_0) + \alpha(P_1) + \alpha(P_\lambda) + \alpha(P_\mu) = 0. \end{array}$

#### Reminder: Riemann Vanishing theorem

$$\theta(x,\Omega) = 0 \Leftrightarrow \exists P_1, \dots, P_{g-1} \in C : x \equiv \alpha(P_1) + \dots + \alpha(P_{g-1}) - \Delta$$

## THE ALGORITHM Main idea

The bijection  $\underline{\cdot}: J(C) \to \mathbb{R}^{2g}/\mathbb{Z}^{2g}$  given by

$$\Omega x_1 + x_2 \mapsto (x_1, x_2)$$

maps the *m*-torsion of J(C) to  $\frac{1}{m}\mathbb{Z}^{2g}/\mathbb{Z}^{2g}$ .

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We define

$$\Theta_3 := \left\{ x \in \frac{1}{3} \mathbb{Z}^6 / \mathbb{Z}^6 : \theta[x + \underline{\Delta}](\Omega) = 0 \right\}.$$

Then  $\alpha(\mathcal{B})$  and  $-\alpha(\mathcal{B})$  are the only subsets  $\mathcal{T} \subset J(C)$  of four elements such that:

- (i) the sum  $\sum_{x \in \mathcal{T}} x$  is zero,
- (ii)  $\mathcal{T}$  is a set of generators of  $J(C)[1-\rho_*]$ , and
- (iii) the set  $\mathcal{O}(\mathcal{T}) := \{ \sum_{x \in \mathcal{T}} a_x x : a \in \mathbb{Z}^4_{>0}, \sum_{x \in \mathcal{T}} a_x \le 2 \}$  satisfies

$$\mathcal{O}(\mathcal{T}) = \Theta_3$$
.

#### FINAL THEOREM

#### **Theorem**

Let *C* be a Picard curve over ℂ given by

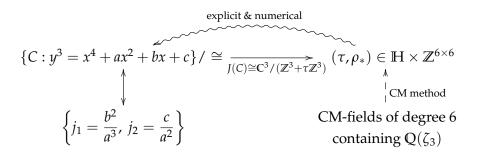
$$y^3 = x(x-1)(x-\lambda)(x-\mu),$$

let  $\Omega \in \mathbb{H}_3$  be a period matrix of the Jacobian J(C), let  $\alpha$  be the Abel-Jacobi map with base point (0:1:0), and let  $\Delta$  be the Riemann constant with respect to  $\alpha$ . Let  $P_t = (t,0)$  for  $t \in \{0,1,\lambda,\mu\}$  and let  $\eta \in \{\lambda,\mu\}$ . Then we have

$$\eta = \varepsilon_{\eta} \left( \frac{\theta [\tilde{P}_{1} + 2\tilde{P}_{\eta} - \tilde{P}_{0} - \tilde{\Delta}](\Omega)}{\theta [2\tilde{P}_{1} + \tilde{P}_{\eta} - \tilde{P}_{0} - \tilde{\Delta}](\Omega)} \right)^{3}, \tag{1}$$

with  $\varepsilon_{\eta} = \exp(6\pi i((\tilde{P}_{\eta} - \tilde{P}_{1})_{1}(\tilde{P}_{0})_{2} + (\tilde{P}_{1} + 2\tilde{P}_{\eta} - \tilde{\Delta})_{1}(2\tilde{\Delta} - 3(\tilde{P}_{1} + \tilde{P}_{\eta}))_{2})).$ 

#### **CONSTRUCTION OF CM PICARD CURVES**



#### RELATED PROBLEMS

#### Riemann-Schottky problem

Describe the image of  $J: M_g \to A_g$ . (More on this on Friday)

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How to prove that a curve has CM by a certain field?

#### **EXAMPLES**

For  $K = \mathbb{Q}(\zeta_3, v)$  with  $v^3 - 21v - 28 = 0$ ,  $y^3 = x^4 - 2 \cdot 3^2 \cdot 5^2 \cdot 7^2 x^2 + 2^9 \cdot 7^2 \cdot 71 x - 3^2 \cdot 5 \cdot 7^3 \cdot 2621$ ,

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,

• for 
$$K = \mathbb{Q}(\zeta_5, v)$$
 with  $v^3 - v^2 - 2v + 1 = 0$ ,  

$$y^5 = x^4 - 7x^2 + 7x.$$



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