

CIMPA COURSE @ IMBM

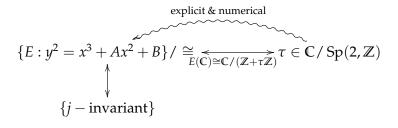
THE INVERSE JACOBIAN PROBLEM

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25th August 2021

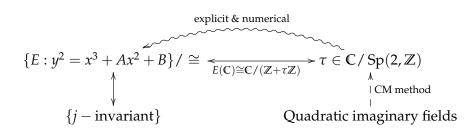
MOTIVATION

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This gives us isomorphism class of the curve $y^2 = x^3 - x$.

THE GENERALIZATION:

GENUS-g CURVES

DOES IT MAKE SENSE?

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$$\dim H^0(\omega_C) = g$$
, $\dim H_1(C, \mathbb{Z}) = 2g$,

so we define

$$\alpha: C(\mathbb{C}) \to \mathbb{C}^g / \Lambda$$

$$Q \mapsto \left(\int_P^Q \omega_1, \dots, \int_P^Q \omega_g \right)$$

with

$$\Lambda = \left\{ \left(\int_{\gamma} \omega_1, \dots, \int_{\gamma} \omega_g \right) : \gamma \in H_1(C, \mathbb{Z}) \right\}.$$

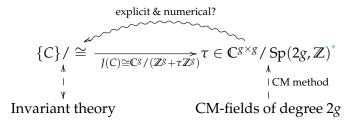
DOES IT MAKE SENSE?

With the right basis we obtain

$$\Lambda \simeq \mathbb{Z}^g + \tau \mathbb{Z}^g$$

for $\tau \in \mathbb{H}_g = \{M \in \mathbb{C}^{g \times g} \colon M = M^T, \operatorname{Im}(M) > 0\}$ a period matrix of C. We define $J(C) = \mathbb{C}^g / \Lambda$ the Jacobian of the curve

CONSTRUCTION OF CM CURVES



STATE OF THE ART

- g = 2 All curves are hyperelliptic. Done.
- g = 3 Hyperelliptic curves: Done.

Picard curves $y^3 = x(x-1)(x-\lambda)(x-\mu)$: We will see this next.

Plane quartics: Done.

g = 6 Superelliptic curves: $y^5 = x(x-1)(x-\lambda)(x-\mu)$: Follows the same principle as Picard curves.

PRELIMINARIES

The Riemann theta functions

Definition

The Riemann theta function is the function $\theta: \mathbb{C}^g \times \mathbb{H}_g \to \mathbb{C}$ given by

$$\theta(z,\Omega) = \sum_{n \in \mathbb{Z}^g} \exp(\pi i n^t \Omega n + 2\pi i n^t z).$$

PRELIMINARIES

The Riemann theta functions

Theorem (Riemann's Vanishing theorem)

Let *C* be a curve over \mathbb{C} of genus *g*, let J(C) be the Jacobian of *C* with period matrix $\Omega \in \mathbb{H}_g$ and let α be an Abel-Jacobi map.

There is an element $\Delta \in J(C)$, called a Riemann constant with respect to α , such that the function $\theta(\cdot, \Omega)$ vanishes at $z \in \mathbb{C}^g$ if and only if there exist $Q_1, \ldots, Q_{g-1} \in C$ that satisfy

$$z \equiv \alpha(Q_1 + \dots + Q_{g-1}) - \Delta \mod \Omega \mathbb{Z}^g + \mathbb{Z}^g$$
.

THE FORMULA

Theorem (Siegel)

Let C be a curve of genus g over C with distinguished point $P \in C$ and let ω be a basis of $H^0(\omega_C)$ that gives the period matrix $\Omega \in \mathbb{H}_g$. Let ϕ be a function on C with $\operatorname{div}(\phi) = A_1 + \cdots + A_m - B_1 - \cdots - B_m$.

Let Δ be the Riemann constant with respect to P, and choose paths from the base point P to A_i and B_i that satisfy

$$\sum_{i=1}^m \int_P^{A_i} \omega = \sum_{i=1}^m \int_P^{B_i} \omega.$$

Then, given an effective non-special divisor $D = Q_1 + \cdots + Q_g$ that satisfies $Q_i \notin \{A_i, B_i : 1 \le i \le m\}$, one has

$$\phi(P_1)\dots\phi(P_g) = E \prod_{i=1}^m \frac{\theta(\sum_{j=1}^g \int_P^{Q_j} \omega - \int_P^{A_i} \omega - \Delta, \Omega)}{\theta(\sum_{j=1}^g \int_P^{Q_j} \omega - \int_P^{B_i} \omega - \Delta, \Omega)},$$

where $E \in \mathbb{C}^{\times}$ is independent of D, and the integrals from P to Q_j take the same paths both in the numerator and the denominator.

A Picard curve *C* over *C* is given by

$$y^3 = x(x-1)(x-\lambda)(x-\mu)$$

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Then $\rho(P_t) = P_t$.

THE FORMULA FOR PICARD CURVES

Theorem

Let C be a Picard curve and let ω be a basis of $H^0(\omega_C)$ that gives the period matrix $\Omega \in \mathbb{H}_g$. Let $\phi = x$ be a function on C with $\operatorname{div}(\phi) = 3P_0 - 3P_\infty$. Let Δ be the Riemann constant with respect to P_∞ , and choose paths a_i, b_i from the base point P_∞ to P_0 and P_∞ that satisfy

$$\sum_{i=1}^{3} \int_{a_i} \omega = \sum_{i=1}^{3} \int_{b_i} \omega.$$

Then, given an effective non-special divisor $D = Q_1 + Q_2 + Q_3$ that satisfies $Q_j \notin \{P_0, P_\infty\}$, one has

$$x(Q_1)x(Q_2)x(Q_3) = E \prod_{i=1}^{3} \frac{\theta(\sum_{j=1}^{3} \int_{P_{\infty}}^{Q_j} \omega - \int_{a_i} \omega - \Delta, \Omega)}{\theta(\sum_{j=1}^{3} \int_{P_{\infty}}^{Q_j} \omega - \int_{b_i} \omega - \Delta, \Omega)},$$
 (*)

where $E \in \mathbb{C}^{\times}$ is independent of D, and the integrals from P_{∞} to Q_j take the same paths both in the numerator and the denominator.

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Then, given an effective non-special divisor $D=Q_1+Q_2+Q_3$ that satisfies $Q_j \notin \{P_0, P_\infty\}$, one has

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where $E \in \mathbb{C}^{\times}$ is independent of D, and the integrals from P_{∞} to Q_j take the same paths both in the numerator and the denominator.

We take the quotient of the expression (*) for $D_1 = P_1 + 2P_{\lambda}$, $D_2 = 2P_1 + P_{\lambda}$.

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- $\blacktriangleright \ \rho(P_t) = P_t \to \alpha(P_t) \in J(C)[1 \rho_*] \subset J(C)[3].$
- $\blacktriangleright \langle \alpha(P_t) \rangle = J(C)[1 \rho_*].$

!! The branch points are the points appearing at the formula.

$$\operatorname{div}(y) = P_0 + P_1 + P_{\lambda} + P_{\mu} - 4P_{\infty} \rightarrow \alpha(P_0) + \alpha(P_1) + \alpha(P_{\lambda}) + \alpha(P_{\mu}) = 0.$$

!! The branch points are the points appearing at the formula.

We have the following results:

- $\blacktriangleright \langle \alpha(P_t) \rangle = J(C)[1 \rho_*].$

Reminder: Riemann Vanishing theorem

$$\theta(x,\Omega) = 0 \Leftrightarrow \exists P_1, \dots, P_{g-1} \in C : x \equiv \alpha(P_1) + \dots + \alpha(P_{g-1}) - \Delta$$

THE ALGORITHM

Main idea

The bijection $\underline{\cdot}:J(C)\to\mathbb{R}^{2g}/\mathbb{Z}^{2g}$ given by

$$\Omega x_1 + x_2 \mapsto (x_1, x_2)$$

maps the *m*-torsion of J(C) to $\frac{1}{m}\mathbb{Z}^{2g}/\mathbb{Z}^{2g}$.

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We define

$$\Theta_3 := \left\{ x \in \frac{1}{3} \mathbb{Z}^6 / \mathbb{Z}^6 : \theta[x + \underline{\Delta}](\Omega) = 0 \right\}.$$

Then $\alpha(\mathcal{B})$ and $-\alpha(\mathcal{B})$ are the only subsets $\mathcal{T} \subset J(C)$ of four elements such that:

- (i) the sum $\sum_{x \in \mathcal{T}} x$ is zero,
- (ii) \mathcal{T} is a set of generators of $J(C)[1-\rho_*]$, and
- (iii) the set $\mathcal{O}(\mathcal{T}) := \{ \sum_{x \in \mathcal{T}} a_x x : a \in \mathbb{Z}^4_{\geq 0}, \sum_{x \in \mathcal{T}} a_x \leq 2 \}$ satisfies

$$\mathcal{O}(\mathcal{T}) = \Theta_3.$$

FINAL THEOREM

Theorem

Let *C* be a Picard curve over ℂ given by

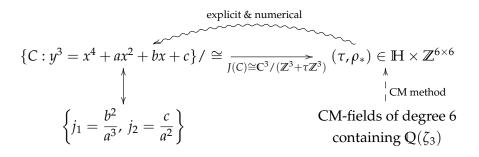
$$y^3 = x(x-1)(x-\lambda)(x-\mu),$$

let $\Omega \in \mathbb{H}_3$ be a period matrix of the Jacobian J(C), let α be the Abel-Jacobi map with base point (0:1:0), and let Δ be the Riemann constant with respect to α . Let $P_t = (t,0)$ for $t \in \{0,1,\lambda,\mu\}$ and let $\eta \in \{\lambda,\mu\}$. Then we have

$$\eta = \varepsilon_{\eta} \left(\frac{\theta [\tilde{P}_{1} + 2\tilde{P}_{\eta} - \tilde{P}_{0} - \tilde{\Delta}](\Omega)}{\theta [2\tilde{P}_{1} + \tilde{P}_{\eta} - \tilde{P}_{0} - \tilde{\Delta}](\Omega)} \right)^{3}, \tag{1}$$

with $\varepsilon_{\eta} = \exp(6\pi i((\tilde{P}_{\eta} - \tilde{P}_1)_1(\tilde{P}_0)_2 + (\tilde{P}_1 + 2\tilde{P}_{\eta} - \tilde{\Delta})_1(2\tilde{\Delta} - 3(\tilde{P}_1 + \tilde{P}_{\eta}))_2)).$

CONSTRUCTION OF CM PICARD CURVES



RELATED PROBLEMS

Riemann-Schottky problem

Describe the image of $J: M_g \to A_g$. (More on this on Friday)

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List all CM-fields whose ring of integers occurs as the endomorphism ring over C of the Jacobian of a curve defined over Q.

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Riemann-Schottky problem

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List all CM-fields whose ring of integers occurs as the endomorphism ring over $\mathbb C$ of the Jacobian of a curve defined over $\mathbb Q$.

How to prove that a curve has CM by a certain field?

EXAMPLES

For $K = \mathbb{Q}(\zeta_3, v)$ with $v^3 - 21v - 28 = 0$, $y^3 = x^4 - 2 \cdot 3^2 \cdot 5^2 \cdot 7^2 x^2 + 2^9 \cdot 7^2 \cdot 71 x - 3^2 \cdot 5 \cdot 7^3 \cdot 2621$,

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,

► for
$$K = \mathbb{Q}(\zeta_5, v)$$
 with $v^3 - v^2 - 2v + 1 = 0$,
$$y^5 = x^4 - 7x^2 + 7x.$$



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