CIMPA COURSE @ IMBM

THE INVERSE JACOBIAN PROBLEM

Anna Somoza
Université de Rennes 1

25th August 2021

Motivation

The situation for elliptic curves

$$
\begin{tikzcd}
\{E : y^2 = x^3 + Ax^2 + B\}/\cong \arrow[rr, leftrightarrow, "E(\mathbb{C}) \cong \mathbb{C}/(\mathbb{Z}+\tau\mathbb{Z})"'] \arrow[d, leftrightarrow] & & \tau \in \mathbb{C}/\mathrm{Sp}(2,\mathbb{Z}) \arrow[ll, squiggly, bend right, "\text{explicit \& numerical}"'] \\
\{j-\text{invariant}\} & &
\end{tikzcd}
$$

$$
\begin{tikzcd}
\{E : y^2 = x^3 + Ax^2 + B\}/\cong \arrow[r, leftrightarrow, "E(\mathbb{C})\cong\mathbb{C}/(\mathbb{Z}+\tau\mathbb{Z})"'] \arrow[d, leftrightarrow] & \tau \in \mathbb{C}/\,\mathrm{Sp}(2,\mathbb{Z}) \arrow[l, bend right, squiggly, "\text{explicit \& numerical}"'] \\
\{j - \text{invariant}\} & \text{Quadratic imaginary fields} \arrow[u, dashed, twoheadrightarrow, "\text{CM method}"']
\end{tikzcd}
$$

An example

We take $K = \mathbb{Q}(i)$. Then, the CM method gives us the period $\tau = i$.

An example

We take $K = \mathbb{Q}(i)$. Then, the CM method gives us the period $\tau = i$. If we compute now

$$\theta_{00}(i) = \sum_{n=-\infty}^{\infty} e^{-\pi n^2} \approx 0.91357913815611$$

$$\theta_{10}(i) = \sum_{n=-\infty}^{\infty} e^{-\pi\left(n+\frac{1}{2}\right)^2} \approx 1.08643481121330$$

An example

We take $K = \mathbb{Q}(i)$. Then, the CM method gives us the period $\tau = i$. If we compute now

$$\theta_{00}(i) = \sum_{n=-\infty}^{\infty} e^{-\pi n^2} \approx 0.91357913815611$$

$$\theta_{10}(i) = \sum_{n=-\infty}^{\infty} e^{-\pi\left(n+\frac{1}{2}\right)^2} \approx 1.08643481121330$$

we obtain

$$\lambda(i) = 0.49999999999852$$

An example

We take $K = \mathbb{Q}(i)$. Then, the CM method gives us the period $\tau = i$. If we compute now

$$\theta_{00}(i) = \sum_{n=-\infty}^{\infty} e^{-\pi n^2} \approx 0.91357913815611$$

$$\theta_{10}(i) = \sum_{n=-\infty}^{\infty} e^{-\pi\left(n+\frac{1}{2}\right)^2} \approx 1.08643481121330$$

we obtain

$$\lambda(i) = 0.49999999999999852$$

and

$$j(i) = 1728.0000000000004542.$$

An example

We take $K = \mathbb{Q}(i)$. Then, the CM method gives us the period $\tau = i$. If we compute now

$$\theta_{00}(i) = \sum_{n=-\infty}^{\infty} e^{-\pi n^2} \approx 0.91357913815611$$

$$\theta_{10}(i) = \sum_{n=-\infty}^{\infty} e^{-\pi\left(n+\frac{1}{2}\right)^2} \approx 1.08643481121330$$

we obtain

$$\lambda(i) = 0.49999999999852$$

and

$$j(i) = 1728.0000000000004542.$$

This gives us isomorphism class of the curve $y^2 = x^3 - x$.

The generalization: Genus-g curves

Does it make sense?

We consider a genus-g curve C with a distinguished point P.

Does it make sense?

We consider a genus-g curve C with a distinguished point P.
We have

$$\dim H^0(\omega_C) = g, \quad \dim H_1(C, \mathbb{Z}) = 2g,$$

so we define

$$\begin{aligned} \alpha \colon C(\mathbb{C}) &\to \mathbb{C}^g / \Lambda \\ Q &\mapsto \left(\int_P^Q \omega_1, \ldots, \int_P^Q \omega_g \right) \end{aligned}$$

with

$$\Lambda = \left\{ \left(\int_\gamma \omega_1, \ldots, \int_\gamma \omega_g \right) : \gamma \in H_1(C, \mathbb{Z}) \right\}.$$

Does it make sense?

With the right basis we obtain

$$\Lambda \simeq \mathbb{Z}^g + \tau \mathbb{Z}^g$$

for $\tau \in \mathbb{H}_g = \{M \in \mathbb{C}^{g \times g} : M = M^T, \mathrm{Im}(M) > 0\}$ a *period matrix* of C.
We define $J(C) = \mathbb{C}^g / \Lambda$ the *Jacobian of the curve*

CONSTRUCTION OF CM CURVES

$$\begin{tikzcd}[column sep=huge]
\{C\}/\cong \arrow[r, "J(C)\cong\mathbb{C}^g/(\mathbb{Z}^g+\tau\mathbb{Z}^g)"'] \arrow[d, dashed, leftrightarrow] & \tau\in\mathbb{C}^{g\times g}/\mathrm{Sp}(2g,\mathbb{Z})^{*} \arrow[l, bend right, squiggly, "\text{explicit \& numerical?}"'] \\
\text{Invariant theory} & \text{CM-fields of degree } 2g \arrow[u, dashed, "\text{CM method}"']
\end{tikzcd}$$

State of the art

$g = 2$ All curves are hyperelliptic. Done.

$g = 3$ Hyperelliptic curves: Done.

Picard curves $y^3 = x(x-1)(x-\lambda)(x-\mu)$:
We will see this next.

Plane quartics: Done.

$g = 6$ Superelliptic curves: $y^5 = x(x-1)(x-\lambda)(x-\mu)$:
Follows the same principle as Picard curves.

Definition

The Riemann theta function is the function $\theta : \mathbb{C}^g \times \mathbb{H}_g \to \mathbb{C}$ given by

$$\theta(z, \Omega) = \sum_{n \in \mathbb{Z}^g} \exp(\pi i n^t \Omega n + 2\pi i n^t z).$$

Theorem (Riemann's Vanishing theorem)

Let C be a curve over $\mathbb{C}$ of genus g, let $J(C)$ be the Jacobian of C with period matrix $\Omega \in \mathbb{H}_g$ and let α be an Abel-Jacobi map.

There is an element $\Delta \in J(C)$, called a Riemann constant with respect to α, such that the function $\theta(\,\cdot\,, \Omega)$ vanishes at $z \in \mathbb{C}^g$ if and only if there exist $Q_1, \dots, Q_{g-1} \in C$ that satisfy

$$z \equiv \alpha(Q_1 + \cdots + Q_{g-1}) - \Delta \mod \Omega\mathbb{Z}^g + \mathbb{Z}^g.$$

The formula

Theorem (Siegel)

Let C be a curve of genus g over $\mathbb{C}$ with distinguished point $P \in C$ and let ω be a basis of $H^0(\omega_C)$ that gives the period matrix $\Omega \in \mathbb{H}_g$. Let ϕ be a function on C with $\operatorname{div}(\phi) = A_1 + \cdots + A_m - B_1 - \cdots - B_m$.
Let Δ be the Riemann constant with respect to P, and choose paths from the base point P to A_i and B_i that satisfy

$$\sum_{i=1}^{m} \int_P^{A_i} \omega = \sum_{i=1}^{m} \int_P^{B_i} \omega.$$

Then, given an effective non-special divisor $D = Q_1 + \cdots + Q_g$ that satisfies $Q_j \notin \{A_i, B_i \colon 1 \leq i \leq m\}$, one has

$$\phi(P_1) \ldots \phi(P_g) = E \prod_{i=1}^{m} \frac{\theta(\sum_{j=1}^{g} \int_P^{Q_j} \omega - \int_P^{A_i} \omega - \Delta, \Omega)}{\theta(\sum_{j=1}^{g} \int_P^{Q_j} \omega - \int_P^{B_i} \omega - \Delta, \Omega)},$$

where $E \in \mathbb{C}^\times$ is independent of D, and the integrals from P to Q_j take the same paths both in the numerator and the denominator.

Picard curves

A Picard curve C over $\mathbb{C}$ is given by

$$y^3 = x(x-1)(x-\lambda)(x-\mu)$$

Picard curves

A Picard curve C over $\mathbb{C}$ is given by

$$y^3 = x(x-1)(x-\lambda)(x-\mu)$$

Branch points: $P_t = (t, 0)$ for $t \in \mathbb{C}$ with $f(t) = 0$.

Picard curves

A Picard curve C over $\mathbb{C}$ is given by

$$y^3 = x(x-1)(x-\lambda)(x-\mu)$$

Branch points: $P_t = (t, 0)$ for $t \in \mathbb{C}$ with $f(t) = 0$.
Non-trivial automorphism: $\rho(x, y) = (x, \zeta_3 y)$.

Picard curves

A Picard curve C over $\mathbb{C}$ is given by

$$y^3 = x(x-1)(x-\lambda)(x-\mu)$$

Branch points: $P_t = (t, 0)$ for $t \in \mathbb{C}$ with $f(t) = 0$.
Non-trivial automorphism: $\rho(x, y) = (x, \zeta_3 y)$.

Picard curves

A Picard curve C over $\mathbb{C}$ is given by

$$y^3 = x(x-1)(x-\lambda)(x-\mu)$$

Branch points: $P_t = (t, 0)$ for $t \in \mathbb{C}$ with $f(t) = 0$.
Non-trivial automorphism: $\rho(x, y) = (x, \zeta_3 y)$.

Then $\rho(P_t) = P_t$.

THE FORMULA FOR PICARD CURVES

Theorem

Let C be a Picard curve and let ω be a basis of $H^0(\omega_C)$ that gives the period matrix $\Omega \in \mathbb{H}_g$. Let $\phi = x$ be a function on C with $\operatorname{div}(\phi) = 3P_0 - 3P_\infty$.
Let Δ be the Riemann constant with respect to P_∞, and choose paths a_i, b_i from the base point P_∞ to P_0 and P_∞ that satisfy

$$\sum_{i=1}^{3} \int_{a_i} \omega = \sum_{i=1}^{3} \int_{b_i} \omega.$$

Then, given an effective non-special divisor $D = Q_1 + Q_2 + Q_3$ that satisfies $Q_j \notin \{P_0, P_\infty\}$, one has

$$x(Q_1)x(Q_2)x(Q_3) = E \prod_{i=1}^{3} \frac{\theta\left(\sum_{j=1}^{3} \int_{P_\infty}^{Q_j} \omega - \int_{a_i} \omega - \Delta, \Omega\right)}{\theta\left(\sum_{j=1}^{3} \int_{P_\infty}^{Q_j} \omega - \int_{b_i} \omega - \Delta, \Omega\right)}, \tag{*}$$

where $E \in \mathbb{C}^\times$ is independent of D, and the integrals from P_∞ to Q_j take the same paths both in the numerator and the denominator.

The formula for Picard curves

Theorem

Let C be a Picard curve and let ω be a basis of $H^0(\omega_C)$ that gives the period matrix $\Omega \in \mathbb{H}_g$. Let $\phi = x$ be a function on C with $\mathrm{div}(\phi) = 3P_0 - 3P_\infty$.
Let Δ be the Riemann constant with respect to P_∞, and choose paths a_i, b_i from the base point P_∞ to P_0 and P_∞ that satisfy

$$\sum_{i=1}^{3} \int_{a_i} \omega = \sum_{i=1}^{3} \int_{b_i} \omega.$$

Then, given an effective non-special divisor $D = Q_1 + Q_2 + Q_3$ that satisfies $Q_j \notin \{P_0, P_\infty\}$, one has

$$x(Q_1)x(Q_2)x(Q_3) = E \prod_{i=1}^{3} \frac{\theta\left(\sum_{j=1}^{3} \int_{P_\infty}^{Q_j} \omega - \int_{a_i} \omega - \Delta, \Omega\right)}{\theta\left(\sum_{j=1}^{3} \int_{P_\infty}^{Q_j} \omega - \int_{b_i} \omega - \Delta, \Omega\right)}, \qquad (*)$$

where $E \in \mathbb{C}^\times$ is independent of D, and the integrals from P_∞ to Q_j take the same paths both in the numerator and the denominator.

We take the quotient of the expression (*) for $D_1 = P_1 + 2P_\lambda, D_2 = 2P_1 + P_\lambda$.

Identifying the branch points

Properties

!! The branch points are the points appearing at the formula.

Identifying the branch points

Properties

!! The branch points are the points appearing at the formula.

We have the following results:

Identifying the branch points
Properties

!! The branch points are the points appearing at the formula.

We have the following results:

- $\rho(P_t) = P_t \rightarrow \alpha(P_t) \in J(C)[1-\rho_*] \subset J(C)[3]$.

!! The branch points are the points appearing at the formula.

We have the following results:

- $\rho(P_t) = P_t \rightarrow \alpha(P_t) \in J(C)[1-\rho_*] \subset J(C)[3]$.
- $\langle \alpha(P_t) \rangle = J(C)[1-\rho_*]$.

Identifying the branch points
Properties

!! The branch points are the points appearing at the formula.

We have the following results:

- $\rho(P_t) = P_t \rightarrow \alpha(P_t) \in J(C)[1-\rho_*] \subset J(C)[3]$.
- $\langle \alpha(P_t) \rangle = J(C)[1-\rho_*]$.
- $\mathrm{div}(y) = P_0 + P_1 + P_\lambda + P_\mu - 4P_\infty \rightarrow$
  $\alpha(P_0) + \alpha(P_1) + \alpha(P_\lambda) + \alpha(P_\mu) = 0.$

Identifying the branch points
Properties

!! The branch points are the points appearing at the formula.

We have the following results:

- $\rho(P_t) = P_t \rightarrow \alpha(P_t) \in J(C)[1-\rho_*] \subset J(C)[3]$.
- $\langle \alpha(P_t) \rangle = J(C)[1-\rho_*]$.
- $\mathrm{div}(y) = P_0 + P_1 + P_\lambda + P_\mu - 4P_\infty \rightarrow$
$$\alpha(P_0) + \alpha(P_1) + \alpha(P_\lambda) + \alpha(P_\mu) = 0.$$

Reminder: Riemann Vanishing theorem

$$\theta(x, \Omega) = 0 \Leftrightarrow \exists P_1, \ldots, P_{g-1} \in C : x \equiv \alpha(P_1) + \cdots + \alpha(P_{g-1}) - \Delta$$

The algorithm
Main idea

The bijection $\underline{\cdot} : J(C) \to \mathbb{R}^{2g}/\mathbb{Z}^{2g}$ given by

$$\Omega x_1 + x_2 \mapsto (x_1, x_2)$$

maps the m-torsion of $J(C)$ to $\frac{1}{m}\mathbb{Z}^{2g}/\mathbb{Z}^{2g}$.

The bijection $\underline{\cdot} : J(C) \to \mathbb{R}^{2g}/\mathbb{Z}^{2g}$ given by

$$\Omega x_1 + x_2 \mapsto (x_1, x_2)$$

maps the m-torsion of $J(C)$ to $\frac{1}{m}\mathbb{Z}^{2g}/\mathbb{Z}^{2g}$.

We define

$$\Theta_3 := \left\{ x \in \frac{1}{3}\mathbb{Z}^6/\mathbb{Z}^6 : \theta[x + \underline{\Delta}](\Omega) = 0 \right\}.$$

Then $\alpha(\mathcal{B})$ and $-\alpha(\mathcal{B})$ are the only subsets $\mathcal{T} \subset J(C)$ of four elements such that:

(i) the sum $\sum_{x \in \mathcal{T}} x$ is zero,

(ii) $\mathcal{T}$ is a set of generators of $J(C)[1 - \rho_*]$, and

(iii) the set $\mathcal{O}(\mathcal{T}) := \{\sum_{x \in \mathcal{T}} a_x x : a \in \mathbb{Z}^4_{\geq 0}, \sum_{x \in \mathcal{T}} a_x \leq 2\}$ satisfies

$$\mathcal{O}(\mathcal{T}) = \Theta_3.$$

Final theorem

Theorem

Let C be a Picard curve over $\mathbb{C}$ given by

$$y^3 = x(x-1)(x-\lambda)(x-\mu),$$

let $\Omega \in \mathbb{H}_3$ be a period matrix of the Jacobian $J(C)$, let α be the Abel-Jacobi map with base point $(0:1:0)$, and let Δ be the Riemann constant with respect to α. Let $P_t = (t, 0)$ for $t \in \{0, 1, \lambda, \mu\}$ and let $\eta \in \{\lambda, \mu\}$. Then we have

$$\eta = \varepsilon_\eta \left(\frac{\theta[\tilde{P}_1 + 2\tilde{P}_\eta - \tilde{P}_0 - \tilde{\Delta}](\Omega)}{\theta[2\tilde{P}_1 + \tilde{P}_\eta - \tilde{P}_0 - \tilde{\Delta}](\Omega)} \right)^3, \tag{1}$$

with $\varepsilon_\eta = \exp(6\pi i((\tilde{P}_\eta - \tilde{P}_1)_1(\tilde{P}_0)_2 + (\tilde{P}_1 + 2\tilde{P}_\eta - \tilde{\Delta})_1(2\tilde{\Delta} - 3(\tilde{P}_1 + \tilde{P}_\eta))_2))$.

Construction of CM Picard curves

$$
\begin{tikzcd}
\{C : y^3 = x^4 + ax^2 + bx + c\}/\cong \arrow[r, "J(C)\cong\mathbb{C}^3/(\mathbb{Z}^3+\tau\mathbb{Z}^3)"'] \arrow[d, leftrightarrow] & (\tau, \rho_*) \in \mathbb{H} \times \mathbb{Z}^{6\times 6} \arrow[l, bend right, squiggly, "\text{explicit \& numerical}"'] \\
\left\{ j_1 = \frac{b^2}{a^3},\ j_2 = \frac{c}{a^2} \right\} & \begin{array}{c}\text{CM-fields of degree 6}\\ \text{containing } \mathbb{Q}(\zeta_3)\end{array} \arrow[u, dashed, "\text{CM method}"']
\end{tikzcd}
$$

Related problems

Riemann-Schottky problem

Describe the image of $J : M_g \to A_g$. (More on this on Friday)

Related problems

Riemann-Schottky problem

Describe the image of $J : M_g \to A_g$. (More on this on Friday)

List all CM-fields whose ring of integers occurs as the endomorphism ring over $\mathbb{C}$ of the Jacobian of a curve defined over $\mathbb{Q}$.

Related problems

Riemann-Schottky problem

Describe the image of $J : M_g \to A_g$. (More on this on Friday)

List all CM-fields whose ring of integers occurs as the endomorphism ring over $\mathbb{C}$ of the Jacobian of a curve defined over $\mathbb{Q}$.

How to prove that a curve has CM by a certain field?

Examples

- For $K = \mathbb{Q}(\zeta_3, v)$ with $v^3 - 21v - 28 = 0$,

$$y^3 = x^4 - 2 \cdot 3^2 \cdot 5^2 \cdot 7^2\, x^2 + 2^9 \cdot 7^2 \cdot 71\, x - 3^2 \cdot 5 \cdot 7^3 \cdot 2621,$$

Examples

- For $K = \mathbb{Q}(\zeta_3, v)$ with $v^3 - 21v - 28 = 0$,
$$y^3 = x^4 - 2 \cdot 3^2 \cdot 5^2 \cdot 7^2\, x^2 + 2^9 \cdot 7^2 \cdot 71\, x - 3^2 \cdot 5 \cdot 7^3 \cdot 2621,$$
- for $K = \mathbb{Q}(\zeta_5, v)$ with $v^3 - v^2 - 2v + 1 = 0$,
$$y^5 = x^4 - 7x^2 + 7x.$$

CIMPA COURSE @ IMBM

THE INVERSE JACOBIAN PROBLEM

Anna Somoza
Université de Rennes 1

25th August 2021