Introduction.

In this project the system is a continuous bioreactor with two different inflows: one contains a highly concentrated solution of coffee ($C_{in,1} = 50mg/dl$) and the other contains water ($C_{in,2} = 0\ mg/dl$). The aim of this project is to control the concentration of coffee in the reactor by using the two inflows. So, in this case, we are talking about a MISO (Multi Input Single Output) system where the inputs are the inflows of coffee ($\mathrm{F}_{in,1} = u_1$) and water ($\mathrm{F}_{in,2} = u_2$), the output is the coffee concentration in the reactor ($C = y$) and we want to track the desired output y_r (variable over time). The inputs are subject to a constraint: in fact, the minimum and maximum inflows are respectively 0 and 66 ml/min at every instant of time (the sensing and control are not continuous but digital with a sampling time of 10 sec).
The project is divided into different tasks:

1. PID control with only 1 control input
2. Non-linear model derivation (to describe the bioreactor) and its analytical linearization.
3. MPC for both the linearized model of the reactor and 'real' (non-linear) reactor.
4. MPC with integral action (with and without input constraints).

Task 1: PID control

The PID controller was implemented in discrete-time form to compute the control input u(k) at each time step, based on the current reference signal, the measured output, and previous system values.

The proportional action is derived by multiplying the instantaneous error e(k)=r(k)−y(k) by the proportional gain Kp. The integral action is calculated by accumulating the error over time and updating it with the sampling period Ts. To prevent excessive growth of the integral term (windup), saturation limits are applied. The derivative action is approximated using the backward difference of the output, divided by the sampling time.

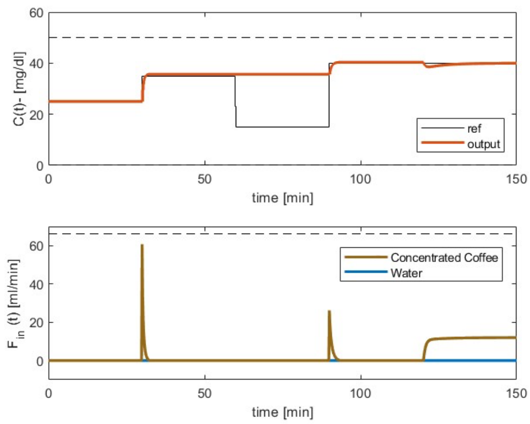
The total control signal u(k) is computed as:

$$u(t) = K_p e(t) + K_i \int_0^t e(t)dt + \frac{K_d de(t)}{dt}$$

The selected tuning parameters are:

- Proportional gain: Kp=6, providing the main corrective effort,
- Integral gain: Ki=0.1, allowing for accurate tracking of step references and effective rejection of constant disturbances,
- Derivative gain: Kd=0, excluded in this case as it was not considered beneficial for the system.

To avoid integral windup, the accumulated error is constrained within the range Ie ∈ [0, 10^6].



The results of the PID controller are shown in two plots: the top plot illustrates the concentration profile C(t), while the bottom plot displays the control inputs — the inflow rates of concentrated coffee and water.

Throughout the simulation, only the coffee inflow is regulated, while the water inflow is kept at zero. This constraint limits the system's ability to follow decreasing reference signals due to the absence of dilution.

During increasing reference steps, the system exhibits good tracking performance:

- At t=30 min, the reference increases to 35 mg/dl. The rise time is about 3 minutes, settling time around 7 minutes, with negligible overshoot and steady-state error below 0.5 mg/dl.
- At t=90 min, the reference increases again. The system responds with a 3.5-minute rise time, 6-minute settling time, and <2% overshoot.
- At t=120 min, a disturbance is introduced. The controller compensates effectively, restoring the concentration within 8 minutes.

The integral action ensures steady-state error elimination and disturbance rejection.

The control input (bottom plot) shows clear peaks corresponding to reference steps and disturbance rejection. Water inflow remains zero throughout, confirming the system's structural limitation.

Overall, the PID controller meets the design requirements:

- Settling time ≤ 10 min
- Overshoot < 5%

Further improvements will require a multi-input control strategy.

Task 2: Model Derivation, Linearization and Validation.

To carefully control the concentration of coffee exiting the system, it's crucial to understand how the provided system functions: it's a continuous bioreactor, with two input flows (pure coffee and water) and a single output flow: moreover, no chemical reaction takes place inside the reactor, and the inflow is equal to the outflow. Such considerations allow us to write the following mathematical formulation:

$$\frac{d}{dt}C_i(t) = \frac{1}{V(t)}F_{in,i}C_{in,i} - \frac{1}{V(t)}\sum_{i=1}^{N} F_{in,i}\,C, \qquad i = (1{,}2), \qquad N = 2$$

With i indicating the compound (1 coffee, 2 water), C being coffee concentration in the bioreactor in mg/dl, V being the volume of the bioreactor in ml, $F_{in,i}$ inflow of i-th compound in ml/min , $C_{in,i}$ being the inflow concentration of the i-th compound. Moreover, considering $C_1 = C$ as the state x of the system, the output will be $y = x$. This way, we can rewrite the model as:

$$\frac{d}{dt}x(t) = f(x,u) = \frac{1}{V}u_1 C_{in,1} - \frac{1}{V}(u_1 + u_2)x, \qquad y(t) = g(x,u) = x(t)$$

With u_1 being coffee inflow and u_2 being water inflow ($F_{in,1}$ and $F_{in,2}$ respectively).

From this non-linear model, the computation of the linearized model with respect to the equilibrium point (x,u) is required, with $x_{eq} = C(0) = 25\,mg/dl$ and $u_{1,eq} = u_{2,eq} = F_{in,1}(0) = F_{in,2}(0) = 0\,ml/min$. The matrices A, B, C, D can be computed thanks to the partial derivatives with respect to the state and the inputs.

$$A = \frac{\partial}{\partial x}f(x,u)\,|u_{eq}, x_{eq} = 0 - \frac{1}{V}(u\_1(t) + (u\_2(t)) = 0$$

$$B = \frac{\partial}{\partial u}f(x,u)\,|u_{eq}, x_{eq} = [\frac{1}{V}(C_{in,1} - x_{eq}), \qquad -\frac{1}{V}x_{eq}]$$

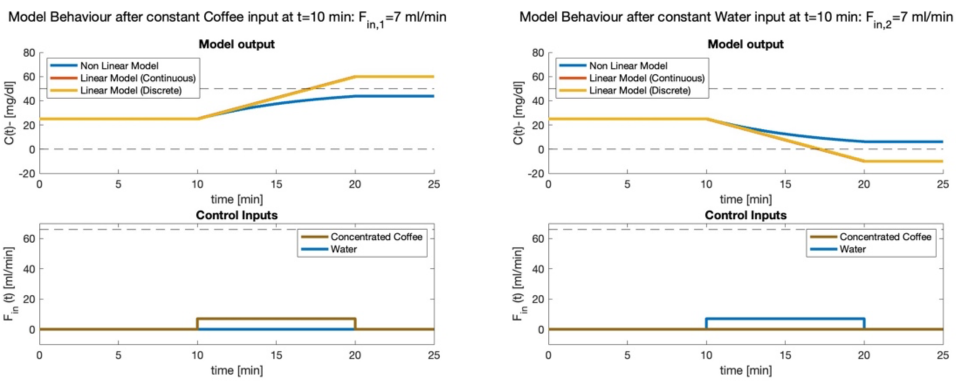
$$C = \frac{\partial}{\partial x}g(x,u)\,|u_{eq}, x_{eq} = 1, \qquad D = \frac{\partial}{\partial u}g(x,u)\,|x_{eq}, u_{eq} = [0, 0]$$

The linearized system can now be computed with respect to $\Delta u(t) = u(t) - u_{eq}$, $\Delta x(t) = x(t) - x_{eq}$, $\Delta y(t) = y(t) - y_{eq}$ (of course, $y_{eq} = x_{eq}$):

$$\frac{d}{dt}\Delta x = A\Delta x + B\Delta u, \qquad \Delta y = C\Delta x + D\Delta u = \Delta x$$

The last point of interest is the behaviour of the linearized model with respect to its non-linearized counterpart, both in discrete and continuous time conditions. Noting that the output of this new model formulation will be $\Delta y(t)$, we need to add the equilibrium y_{eq} to compare it to the original.

Two cases are highlighted: the first utilises a constant coffee inflow of $F_{in,1} = 7\,\mathrm{ml/min}$ for 10 minutes, while the second one shows the behaviour of the system after a constant water inflow of $F_{in,2} = 7\,\mathrm{ml/min}$ for 10 minutes. Simulations are performed with MATLAB, models are developed in Simulink.



From the figures, the linearized model behaves similarly to the original one when the state is close to its equilibrium, as expected from the theory. More notably, the linearized one exceeds the boundaries.

TASK 3.1: MPC for the Linearized Reactor

In this case the MPC is used to control the reactor described with the linearized state space model, introduced in the previous task.
First, we choose the prediction horizon PH and the cost function which is used to compute the optimal sequence of N control actions (until PH):

$$J = \sum_{i=1}^{N}(\hat{y}(k+i) - y_r(k+i))^T Q\,(\hat{y}(k+i) - y_r(k+i)) + \sum_{i=0}^{N-1}(u(k+i) - u_r)^T R\,(u(k+i) - u_r)$$

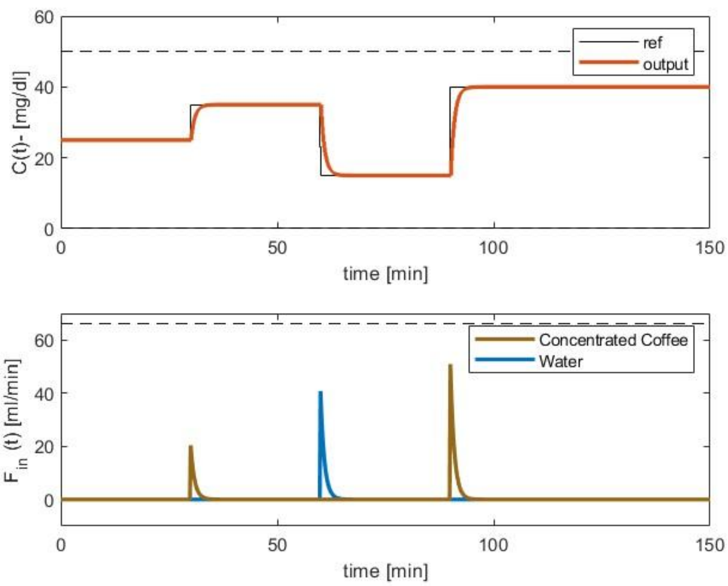
The control action is subject to saturation ($0 \leq u(k+i) \leq u_{max}$) for the generic instant of time *i*, so the linear constraints are: $Fu(k+i) \leq f$, with $F = [-1\,0; 1\,0;\ 0 - 1; 0\,1]$and $f = [0;\ u_{max}; 0;\ u_{max}]$.
Then, we introduce the matrix representation (condensed form) for all the terms:

- system evolution: $\hat{X}(k) = A_{cond}x(k) + B_{cond}U(k),\ \hat{Y}(k) = C_{cond}\hat{X}(k) = A_{call}x(k) + B_{call}U(k)$
- linear constraints: $F_{call}\ U(k) \leq f_{call}$
- cost function: $J = \left(\hat{Y}(k) - Y_r(k)\right)^T Q_{call}(\hat{Y}(k) - Y_r(k)) + \left(U(k) - U_r(k)\right)^T R_{call}\left(U(k) - U_r(k)\right)$

A_{cond} and B_{cond} are calculated such that: $\hat{x}(k+i) = A^i x(0) + \sum_{j=1}^{i} A^{i-j}Bu(k+j-1)$ for every general instant of time *i*. In this case A=0 so A_{cond}= column vector of N zeros, B_{cond}= diagonal block matrix with B as the elements of the diagonal, C = 1 so C_{cond}= identity matrix of dimension NxN.
So, we calculate the calligraphic matrices:

- A_{call}= CA_{cond} and B_{call}= CB_{cond}
- Q_{call}, R_{call}, F_{call} and f_{call}= diagonal block matrices with *Q, R, F* and *f* as the elements of the diagonal

Replacing $\hat{Y}(k)$ with the outcome prediction equation the MPC optimization problem becomes a QP (quadratic programming) problem in the form: $min(J(x))\ with\ J = \frac{1}{2}{x_{qp}}^T H_{qp}x_{qp} + {f_{qp}}^T x_{qp}$ and linear constraints. In this case: the optimization variable $x_{qp} = U(k)$, $H_{qp} = B^T{}_{call}Q_{call}B_{call} + R_{call}$ and $f_{qp} = B^T{}_{call}Q_{call}(A_{call}\hat{x}(k) - Y_r(k)) - R_{call}U_r(k)$.
In the end we can easily find U(k) with the Matlab function 'quadprog' and we can apply only the first 2 elements of the vector (which represent the current inputs), finding the current output y(k).
The matrices of weights Q and R (the tuning parameters) have a very important role because they regulate controller aggressiveness. In this case we choose Q = 5 and R = [1 0; 0 1]. The results of the MPC are shown in the following plots of the output (concentration of coffee) and the inputs (inflows of coffee and water):



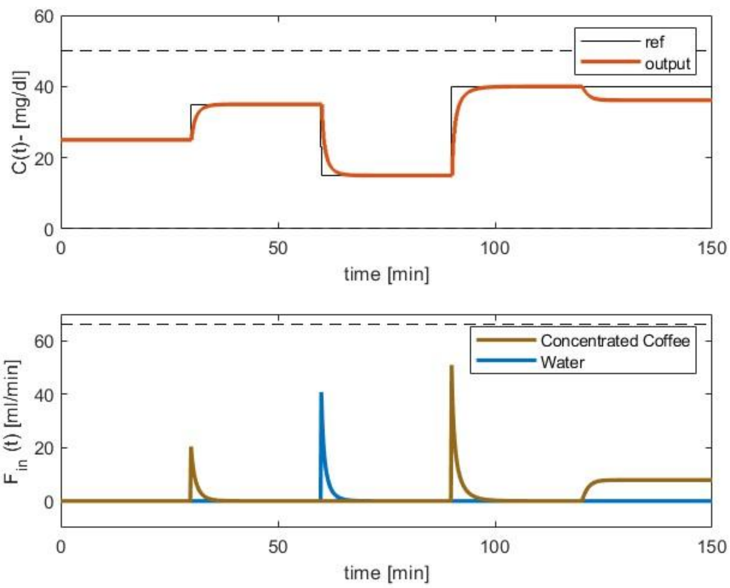
As we can see, the requests ($\pm$ settling time $\leq$ 10 min and overshoot $\leq$ 5%) are satisfied and the inputs were not saturated. There is an inflow of coffee when the reference increases and one of water when the reference decreases. Otherwise, both inflows are null, as it should be. In this case the controller is also able to reach offset free reference tracking, so the error is 0 in the steady state. In fact, we are not considering the disturbance that acts on the system: the output is not influenced by it, so the reference is tracked very well.

TASK 3.2: MPC for the nonlinear reactor

In this task we apply the MPC (implemented in the previous point) to the initial nonlinear reactor and we analyze if it works well. In this case we are considering also the constant disturbance that acts in the system after t = 120s, so the state update becomes x(k+1) = Ax(k) + Bu(k) + Md(k). We have to introduce $D(k) = [d(k);\ \ldots;\ d(k+N-1)]$ and the two block matrices M_{cond}(defined as B_{cond}, with B replaced with M) and $M_{call} = CM_{cond}$ which slightly change the optimization problem. In fact, also in this case, we can write the equivalent QP problem and the only term that changes from the previous task is

$$f_{qp} = B^T{}_{call}Q_{call}(A_{call}\hat{x}(k) + M_{call}D(k) - Y_r(k)) - R_{call}U_r(k)$$

The aim is to evaluate if the controller of the previous task works also for the real system, so the parameters Q and R are tuned in the same way. The results are shown in the following plot:



As we can see the output tracks well the reference until the steady state is reached but, after t = 120 sec, the disturbance starts to act on the system and, even if the controller tries to reach the reference by applying an inflow of coffee, the offset free reference tracking is not achieved (in steady state we have a constant difference between the output and the reference).

Task 4: MPC with Integral Action.

A clear limitation of the standard MPC application is, as expected, the inability to reject unknown disturbances: for this reason, the full-increment velocity form is considered. The basis of this new formulation is quite straightforward: instead of considering the state $x(k)$, the difference $\delta x = x(k) - x(k-1)$ is taken into account (k is the current discrete time instant). This, in turn, changes how the state is updated and how the output is obtained:

$$\delta x(k+1) = x(k+1) - x(k) = Ax(k) + Bu(k) - Ax(k-1) - Bu(k-1) = A\delta x(k) + B\delta u(k)$$

$$y(k+1) = Cx(k+1) = C\delta x(k+1) + Cx(k) = CA\delta x(k) + CB\delta u(k) + y(k)$$

This implies the need for a new state, containing both $\delta x(k)$ and $y(k)$: this is known as the "augmented state". A new objective function is therefore needed to account for this new state: starting from the original MPC formulation, $x_a(k) = [\delta x(k); y(k)]$ and $\delta u(k)$ are substituted into the objective function:

$$J = \left(\hat{y}(k) - y_{ref}(k)\right)^T Q \left(\hat{y}(k) - y_{ref}(k)\right) + u(k)^T R u(k)$$

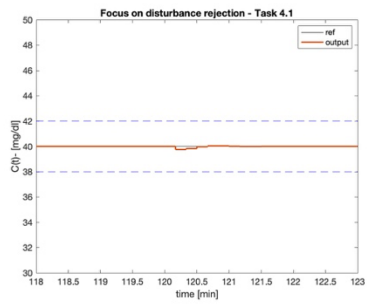
which becomes:

$$J = \left(CA\delta x(k-1) + CB\delta u(k-1) + y(k-1) - y_{ref}(k)\right)^T Q \left(CA\delta x(k-1) + CB\delta u(k-1) + y(k-1) - y_{ref}(k)\right) + \delta u(k)^T R \delta u(k)$$

Expanding the calculations, it's possible to obtain the "quadratic programming problem" formulation:

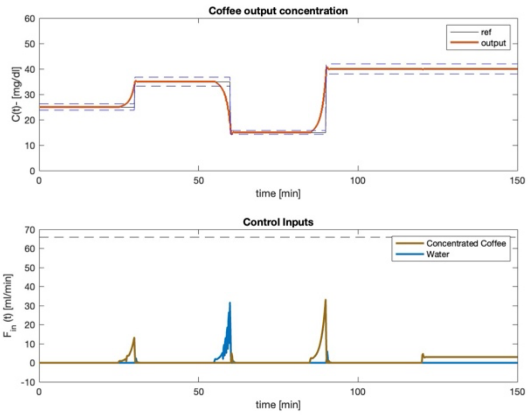
$$\delta u(k)^T (B_C^T Q B_C + R)\delta u(k) + 2\left(A_C \delta x(k-1) + y(k-1) - y_{ref}(k)\right) Q B_C \delta u(k) = \\ = \delta u(k)^T \boldsymbol{H} \delta u(k) + 2\boldsymbol{f}^{\boldsymbol{T}} \delta u(k)$$

depending on the optimization variable δu only, with $B_C = CB$, $A_C = CA$, $y(k-1)$ output of the previous iteration and $y_{ref}(k)$ current reference value. The formula as written provides a good view of a single-step application of the procedure: nonetheless, to compute the actual MPC control, we require the "calligraphic formulation". This way, the result obtained from the MPC function consists of the vector $\widehat{\delta u}(k)$ of dimension $(1\ x\ 2PH)$, of which we need the first two values only (which correspond to the coffee and water control inflows, respectively). These need to be integrated through the discrete-time filter $\frac{1}{1-z^{-1}} I$, with I being the integral gain, a parameter that can be tuned: this allows us to obtain the values $\widehat{u_1}(k)$ and $\widehat{u_2}(k)$, which are then fed to the linearized system to control the coffee concentration. Implementing the controller this way shows promising performance, rejecting the constant disturbance and allowing us to obtain the desired steady state value for the coffee concentration.

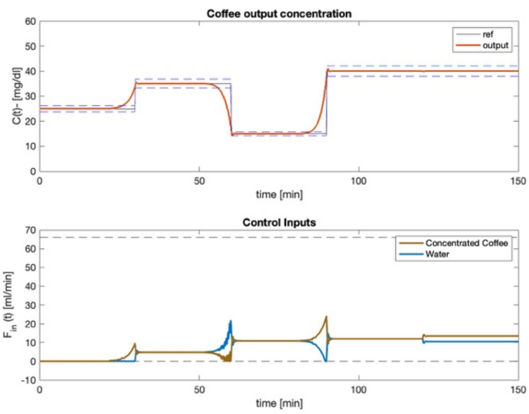


Nonetheless, some different problems arise: most notably, the large overshoots present as the reference value changes abruptly. These overshoots can't be overcome by tuning and are probably due to the programmed reference behaviour, which currently doesn't "see" ahead of the value at time instant k: it would be ideal for the controller to perform a "reference look-ahead" type of control, allowing for the output to gradually change over time, not suddenly as the current behaviour shows.

In order to do this, it's important to implement a new way of receiving reference values: this is done in the "future reference estimator" subsystem of the Simulink model, which utilizes a function receiving as input the current time instant and returns a vector containing the future reference values calculated from the given signal. This means there's no need to calculate $Yr(k)$ calligraphic in the MPC block as it is already provided by this subsystem. As we're dealing with the full increment velocity form of the controller, there's also a new part of the model that provides the state values for δx and y. Control inputs are constrained only after they've been calculated, as per assignment. The system output, along with the control inputs needed, is as follows:



Achieved thanks to tuning parameters: Q=1, r=1, PH=30 and I=12. It's clear that the model used for this process is relies more on the integral control parameter than the MPC ones, with low I associated with long settling times and high values of it associated with instability and oscillations.
Moving on to task 4.2, the assignment now requires us to include the constraint $u(k) \geq 0$ when the control inputs are calculated: this can be accomplished by considering $I * \delta u(k) = u(k) - u(k-1)$, implying $u(k) = I * \delta u(k) + u(k-1) \geq 0$, and therefore $-\delta u(k) \leq \frac{u(k-1)}{I}$: this formulation is required as the controller estimates $\delta u(k)$ rather than $\mathrm{u}(k)$ itself, so any constraints need to be developed for the former quantity. This implies a constraint that changes with each step of the process, meaning the MPC now requires, as inputs, the previous-step values of the control inputs to obtain A_qp and b_qp and the integral control gain I. This leads to us obtaining the following behaviour:

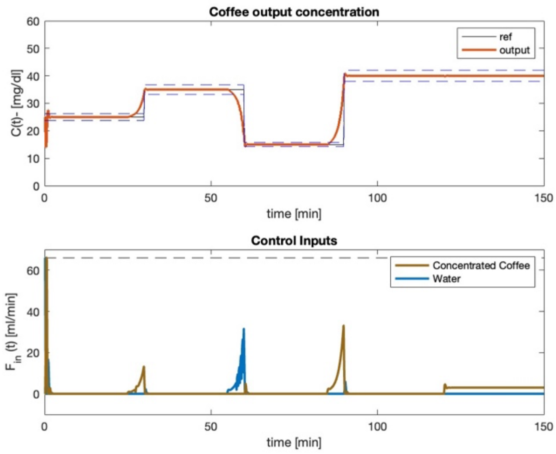


Thanks to tuning parameters: Q=1, r=1, PH=50, I=5. Again, the effect of the integral gain value seems more important than the ones of the MPC controller. In both tasks, the overshoot value remains less than 5% for any reference value (indicated by the blue dotted line) and the settling time, considered when the signal starts deviating from the reference, is kept below 10 minutes at every change.

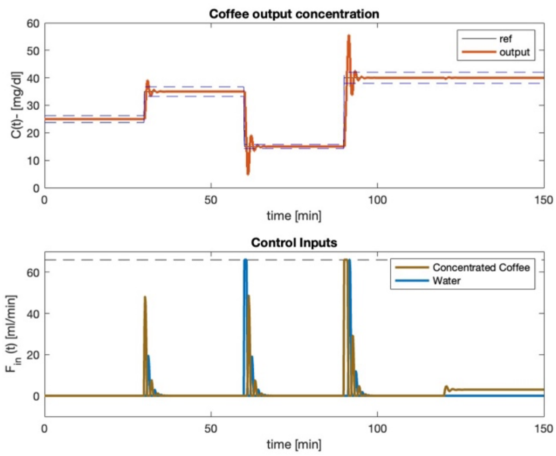
Extra space for the description of problems limitation, or special solutions implemented.

Tasks 2 and 3:
Regarding model linearization, the assignment suggested to formulate the equations in such a way to include the differences between the current values and the equilibrium point, as it is standard for the linearization process. While this has been taken into account for the calculations, the Simulink implementation is slightly different: making use of the “Discrete State Space” block, we set the starting conditions for the state to 25 mg/dl, therefore implicitly considering the output as the sum between the starting value x_{eq} and the output of the linearized model, $\Delta x(k) = x(k) - x_{eq}$. This was still the case for the control of the linearized model in Task 3. This is quite useful as $u_{eq} = 0\ ml/min$, therefore there’s no need to modify the control input in any way.

Task 4:
An important consideration is the insertion of a block that brings the first output value of the system to 0: this block is titled “Correction for delta_x(0)” in the Simulink model, and allows us to bring the initial value of δx to 0 mg/dl for the first time instant k=0, as the linearized model already starts at $x = 25\ mg/dl$ as explained above. This is done since a starting value of $\delta x(0) = 25\ mg/dl$ would cause the controller to try and lead this value to 0, making it so the control input would already be active as of the first time instant.



These results were obtained using the same configuration as the “Task 4.1” section of this report, but without the correction for $\delta x(0)$.
The most glaring limitation when dealing with task 4 was implementing the MPC with no reference look ahead, which made it so the output overshot way beyond its intended values, as pictured below:



This output was obtained with r=1, Q=1, PH=50, I=5 for task 4.1 without considering reference look ahead. As results did not improve when tuning the parameters differently, we considered the overshootings as being related to the immediate change of the reference, therefore implemented the reference look ahead technique as seen in the section for task 4, which proved to be effective in reducing overshoots.

Conclusions.

Control of coffee concentration in the proposed bioreactor is reasonably achieved utilizing proper inflows, carefully managing either the coffee or water inputs in order to achieve the desired steady state values based on the reference. PID control can be easily applied and, although during the first task only the coffee input was considered, disturbance rejection was easily achieved. Analysis was further improved thanks to the linearization of the model itself, simplifying the model and opening the door to an MPC approach, which proved to be quite successful in leading the output to the desired steady state values. Disturbance rejection was then introduced thanks to the full increment velocity form of the controller, which together with the "reference look ahead" method allowed us to achieved the desired levels of coffee concentration inside the bioreactor itself, both in a bounded and unbounded environment, while also providing reasonably contained output values, with limited or no overshoot present. Tuning proved to be quite an important part of the process, especially with the integral control gain being of interest to negate the effect of any disturbance which might act on the system. Overall, control of coffee concentration was achieved and can be applied to the physical plant.