Linearne transformacije

January 16, 2021

2. Za sledeće funkcije diskutovari po realnim parametrima kada su linearne transformacije i u slučaju kada jesu naći njihove matrice i odrediti rang.

2.1 $f : \mathbb{R}^3 \longrightarrow \mathbb{R}^2$, $f(x, y, z) = (\underbrace{ax + y^b}, \underbrace{bx - z})$

$\underline{t_1 x + t_2 y + t_3 z}$, $t_1, t_2, t_3 \in \mathbb{R}$

$ax + y^b$ $\qquad\qquad$ $bx - z$ ✓

$\boxed{b = 1}$, $\boxed{a \in \mathbb{R}}$ $\qquad\qquad$ $\boxed{b \in \mathbb{R}}$

$ax + y$ ✓ $\qquad\qquad$ $x - z$

$f(x, y, z) = (ax + y, \; x - z)$, $b = 1$, $a \in \mathbb{R}$

$$M_f = \begin{bmatrix} a & 1 & 0 \\ 1 & 0 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & a \\ 0 & -1 & 1 \end{bmatrix} \Rightarrow \text{rang}(M_f) = 2$$

$\Rightarrow \text{rang}(f) = 2$

2.2 $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$, $f(x, y) = \underbrace{ax + bxy + cy}$

$t_1 x + t_2 y$, $t_1, t_2 \in \mathbb{R}$

$ax + b\boxed{xy} + cy$

$\boxed{b=0}$, $a, c \in \mathbb{R}$

$ax + cy$ ✓

$f(x,y) = ax + cy$

$M_f = \begin{bmatrix} a & c \end{bmatrix}$, $a, c \in \mathbb{R}$, $b = 0$

$a = c = 0 \implies \operatorname{rang}(M_f) = 0$ per $M_f = [0 \ \ 0] \implies \operatorname{rang}(f) = 0$

$a \neq 0 \ \vee \ c \neq 0 \rightarrow \operatorname{rang}(M_f) = 1 \implies \operatorname{rang}(f) = 1$

2.3 $f : \mathbb{R}^3 \longrightarrow \mathbb{R}^2$, $f(x,y,z) = (\underbrace{2^{ay}x + yz^b}, \underbrace{ax + by + cz})$

$t_1x + t_2y + t_3z$, $t_1, t_2, t_3 \in R$

$2^{\boxed{ay}}x + \underline{yz}^b$; $ax + by + cz$ ✓

$\boxed{a=0}$, $\boxed{b=0}$ cz

$x + y$ ✓

$\boxed{2^{3y}} x + \ldots$

$\uparrow$

функција y

$\ln x$

$f(x,y,z) = (x+y, cz)$, $c \in R$, $a=0$, $b=0$

$$M_f = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & c \end{bmatrix}$$

$c = 0 \Rightarrow \text{rang}(M_f) = 1 \Rightarrow \text{rang}(f) = 1$ $\begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$c \neq 0 \Rightarrow \text{rang}(M_f) = 2 \Rightarrow \text{rang}(f) = 2$

2.4 $f:\mathbb{R}^2 \longrightarrow \mathbb{R}^2$, $f(x,y) = ((ax-b)y, x+ab)$

$t_1x + t_2y$, $\quad t_1, t_2 \in \mathbb{R}$

$(ax-b)y$ $\qquad\qquad\qquad$ $x+ab$

$axy - by$ $\qquad\qquad\qquad$ x ✓

$\boxed{a=0}$, $b \in \mathbb{R}$

$-by$ ✓

$f(x,y) = (-by, x)$

$$M_f = \begin{bmatrix} 0 & -b \\ 1 & 0 \end{bmatrix}$$

$b=0 \Rightarrow \text{rang}(M_f)=1 \Rightarrow \text{rang}(f)=1$ $\quad \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$

$b\neq 0 \Rightarrow \text{rang}(M_f)=2 \Rightarrow \text{rang}(f)=2$

2.5 $f:\mathbb{R}^3 \longrightarrow \mathbb{R}^2$, $f(x,y,z) = \left(\underbrace{\frac{ax+b}{bx+a} + y}, \underbrace{\sin(bx) + az}\right)$

$t_1x + t_2y + t_3z, \quad t_1, t_2, t_3 \in \mathbb{R}$

$\frac{ax+b}{bx+a} + y \quad ; \quad \longleftarrow \quad \sin(b\boxed{x}) + az$

$\boxed{b=0}, \quad a \in \mathbb{R}$

$\frac{ax+0}{0x+a} + y \qquad az \ \checkmark$

$\frac{\cancel{a}x}{\cancel{a}} + y, \quad \boxed{a \neq 0}$

$x + y \ \checkmark$

$f(x,y,z) = (x+y, az), \quad a \in \mathbb{R}\setminus\{0\}$

$$M_f = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & a \end{bmatrix}$$

Kako je $a \in \mathbb{R}\setminus\{0\}$

$\Rightarrow$ rang $(M_f) = 2$

$\Rightarrow$ rang $(f) = 2$

2.6 $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$, $f(x,y) = \left(\underbrace{x \cdot 5^{(a-1)y+b}}, \underbrace{(\ln b)\, y^2}, \underbrace{ax + cy}\right)$

$\ln A = B \Leftrightarrow A = e^B$

$t_1 x + t_2 y$, $t_1, t_2 \in \mathbb{R}$

$x \cdot 5^{(a-1)y+b}$, $(\ln b)\, \boxed{y^2}$, $ax + cy$ ✓

$a - 1 = 0$

$\boxed{a = 1}$

$\ln b = 0$

$b = e^0$

$\boxed{b = 1}$

$0 y^2$

0 ✓

$x + cy$

$x \cdot 5^1$

$5x$ ✓

$f(x,y) = (5x, 0, x + cy)$, $c \in \mathbb{R}$, $a = b = 1$

$$M_f = \begin{bmatrix} 5 & 0 \\ 0 & 0 \\ 1 & c \end{bmatrix} \sim \begin{bmatrix} 5 & 0 \\ 1 & c \\ 0 & 0 \end{bmatrix}$$

$c \neq 0 \Rightarrow \operatorname{rang}(M_f) = 2$

$\operatorname{rang}(f) = 2$

$c = 0$ $\quad \begin{bmatrix} 5 & 0 \\ 1 & 0 \\ 0 & 0 \end{bmatrix} \overset{\cdot(-5)}{\sim} \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \end{bmatrix}$

$\Rightarrow \operatorname{rang}(M_f) = 1$

$\Rightarrow \operatorname{rang}(f) = 1$

$$M_f = \begin{bmatrix} 5 & 0 \\ 0 & 0 \\ 1 & c \end{bmatrix} \underset{\cdot\left(-\frac{1}{5}\right)}{\sim} \begin{bmatrix} 5 & 0 \\ 0 & 0 \\ 0 & c \end{bmatrix} \sim \begin{bmatrix} 5 & 0 \\ 0 & c \\ 0 & 0 \end{bmatrix}$$

3. Za date funkcije ispitati da li su linearne transformacije i za one koje jesu naći jezgro, sliku, rang i matricu.

3.1 $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$, $f(x,y) = (x-3y, -2x+6y, 3x-9y)$

$\mathbb{R}^2 \xrightarrow{f} \mathbb{R}^3$

f jeste linearna transformacija jer su sve komponente slike oblika $t_1x + t_2y$, $t_1, t_2 \in \mathbb{R}$.

$$M_f = \begin{bmatrix} 1 & -3 \\ -2 & 6 \\ 3 & -9 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \Rightarrow rang(M_f) = 1 \Rightarrow rang(f) = 1$$

JEZGRO: $ker(f) = \{(x,y) \in \mathbb{R}^2 \mid f(x,y) = 0\}$

$f(x,y) = 0$

$(x-3y, -2x+6y, 3x-9y) = 0$

$$\begin{aligned} x-3y &= 0 \\ -2x+6y &= 0 \quad /\cdot 2 \\ 3x-9y &= 0 \quad /\cdot(-3) \end{aligned}$$

$$\begin{aligned} x-3y &= 0 \\ 0 &= 0 \\ 0 &= 0 \end{aligned}$$

$x = 3y$

1x neodređen

$y = t$, $t \in \mathbb{R}$

$x = 3t$

$ker(f) = \{(3t, t) \mid t \in \mathbb{R}\}$

$= \{(3,1)t \mid t \in \mathbb{R}\}$

SA PREDAVANJA: znamo da je $ker(f)$ potprostor v.p. $\mathbb{R}^2$, i njegova jedna baza je $\{(3,1)\}$.

SLIKA: $Im(f) = \{(a,b,c) \in \mathbb{R}^3 \mid \exists (x,y) \in \mathbb{R}^2, f(x,y) = (a,b,c)\}$

$f(x,y) = (a,b,c)$

$(x-3y, -2x+6y, 3x-9y) = (a,b,c)$

$$\begin{aligned} x-3y &= a \\ -2x+6y &= b \quad /\cdot 2 \\ 3x-9y &= c \quad /\cdot(-3) \end{aligned}$$

$$\begin{aligned} x-3y &= a \\ 0 &= 2a+b \\ 0 &= -3a+c \end{aligned}$$

$$\boxed{\begin{aligned} c &= 3a \\ b &= -2a \end{aligned}}$$

$Im(f) = \{(a, -2a, 3a) \mid a \in \mathbb{R}\}$

$= \{(1,-2,3)a \mid a \in \mathbb{R}\}$

SA PREDAVANJA: znamo da je $Im(f)$ potprostor v.p. $\mathbb{R}^3$, jedna njegova baza je $\{(1,-2,3)\} \Rightarrow dim(Im(f)) = 1$

$\Rightarrow rang(f) = dim(Im(f)) = 1$

što smo već bili odredili preko $rang(M_f)$

3.2 $f : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ $f(x,y,z) = (x+y+z, 2x-3y+5z, -2x-7y+z)$

f jeste lin. tr. jer su sve komponente slike oblika $t_1x+t_2y+t_3z$, $t_1,t_2,t_3 \in \mathbb{R}$

$$M_f = \begin{bmatrix} 1 & 1 & 1 \\ 2 & -3 & 5 \\ -2 & -7 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & -5 & 3 \\ 0 & -5 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & -5 & 3 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \text{rang}(M_f) = 2$$

$\Rightarrow \text{rang}(f) = 2$

JEZGRO: $\ker(f) = \{(x,y,z) \in \mathbb{R}^3 \mid \underline{f(x,y,z)=0}\}$

$f(x,y,z) = 0$

$(x+y+z, 2x-3y+5z, -2x-7y+z) = 0$

$$\begin{aligned} x+y+z&=0 \\ 2x-3y+5z&=0 \\ -2x-7y+z&=0 \end{aligned}$$

$$\begin{aligned} x+y+z&=0 \\ -5y+3z&=0 \\ \cancel{-5y+3z=0} \end{aligned}$$

$$\begin{aligned} x+y&=-z \\ -5y&=-3z \end{aligned}$$

1x neodređen

$z=t$, $t \in \mathbb{R}$

$y = \frac{3}{5}t$, $x = -\frac{8}{5}t$

$\ker(f) = \{(-\frac{8}{5}t, \frac{3}{5}t, t) \mid t \in \mathbb{R}\}$

$= \{(-\frac{8}{5}, \frac{3}{5}, 1)t \mid t \in \mathbb{R}\}$

$\ker(f)$ je potprostor v.p. $\mathbb{R}^3$ i jedna njegova baza je

$\{(-\frac{8}{5}, \frac{3}{5}, 1)\}$

SLIKA: $\text{Im}(f) = \{(a,b,c) \in \mathbb{R}^3 \mid \exists (x,y,z) \in \mathbb{R}^3,$
$\underline{f(x,y,z) = (a,b,c)}\}$

$f(x,y,z) = (a,b,c)$

$(x+y+z, 2x-3y+5z, -2x-7y+z) = (a,b,c)$

$$\begin{aligned} x+y+z&=a \\ 2x-3y+5z&=b \\ -2x-7y+z&=c \end{aligned}$$

$$\begin{aligned} x+y+z&=a \\ -5y+3z&=-2a+b \\ -5y+3z&=2a+c \end{aligned}$$

$$\begin{aligned} x+y+z&=a \\ -5y+3z&=-2a+b \\ \boxed{0} &\boxed{= 4a-b+c} \end{aligned}$$

$\boxed{b = 4a+c}$

$\text{Im}(f) = \{(a, 4a+c, c) \mid a,c \in \mathbb{R}\}$

$= \{(1,4,0)a + (0,1,1)c \mid a,c \in \mathbb{R}\}$

$\text{Im}(f)$ je potprostor v.p. $\mathbb{R}^3$, i jedna njegova baza je

$\{(1,4,0), (0,1,1)\}$

$\dim(\text{Im}(f)) = 2 = \text{rang}(f)$

4. Neka su linearne transformacije $f, g : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ definisane sa $f(x, y) = (2x - y, x + 3y)$ i $g(x, y) = (-x + y, 3x - 2y)$.

4.1 Odrediti kopoziciju $f \circ g$.

4.2 Napisati matrice M_f i M_g linearnih transformacija f i g.

4.3 Naći linearnu transformaciju h koja odgovara matrici $M_f \cdot M_g$ i uporediti je sa $f \circ g$.

4.4 Odrediti f^{-1} i g^{-1} ako postoje.

4.1. $f\circ g(x,y) = f(g(x,y)) = f(-x+y, 3x-2y) = (2\cdot(-x+y) - (3x-2y), -x+y+3(3x-2y))$

$= (-2x+2y-3x+2y, -x+y+9x-6y) = (-5x+4y, 8x-5y)$

4.2. $M_f = \begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix}$, $M_g = \begin{bmatrix} -1 & 1 \\ 3 & -2 \end{bmatrix}$

4.3. $M_f \cdot M_g = \underset{2\times 2}{\begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix}} \underset{2\times 2}{\begin{bmatrix} -1 & 1 \\ 3 & -2 \end{bmatrix}} = \begin{bmatrix} -5 & 4 \\ 8 & -5 \end{bmatrix}$

$\Rightarrow h(x,y) = (-5x+4y, 8x-5y)$

ЗАКЉУЧУЈЕМО ДА ЈЕ $h = f\circ g$!

4.4. $M_f^{-1} = ?$

$\det(M_f) = \begin{vmatrix} 2 & -1 \\ 1 & 3 \end{vmatrix} = 6+1 = 7 \neq 0$

$\operatorname{adj}(M_f) = \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}$

$M_f^{-1} = \frac{1}{\det(M_f)} \cdot \operatorname{adj}(M_f)$

$= \frac{1}{7}\begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} \frac{3}{7} & \frac{1}{7} \\ -\frac{1}{7} & \frac{2}{7} \end{bmatrix}$

$\Rightarrow f^{-1}(x,y) = \left(\frac{3}{7}x + \frac{1}{7}y, -\frac{1}{7}x + \frac{2}{7}y\right)$

$M_g^{-1} = ?$

$\det(M_g) = \begin{vmatrix} -1 & 1 \\ 3 & -2 \end{vmatrix} = 2-3 = -1 \neq 0$

$\operatorname{adj}(M_g) = \begin{bmatrix} -2 & -1 \\ -3 & -1 \end{bmatrix}$

$M_g^{-1} = \frac{1}{\det(M_g)} \operatorname{adj}(M_g) = \begin{bmatrix} 2 & 1 \\ 3 & 1 \end{bmatrix}$

$\Rightarrow g^{-1}(x,y) = (2x+y, 3x+y)$

5. Neka su linearne transformacije $f, g : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ definisane sa
$f(x, y, z) = (x - 2y, y + z, -2x + y - z)$ i
$g(x, y, z) = (x + 2y + 3z, x - z, 2y + 4z)$.

5.1 Napisati matrice M_f i M_g linearnih transformacija f i g i odrediti njihov rang.

5.2 Odrediti $f \circ g.$, f^{-1}, $ker(g)$ i $Img(g)$.

5.1. $M_f = \begin{bmatrix} 1 & -2 & 0 \\ 0 & 1 & 1 \\ -2 & 1 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 0 \\ 0 & 1 & 1 \\ 0 & -3 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{bmatrix}$ $\Rightarrow rang(M_f) = 3$
$\Rightarrow rang(f) = 3$

$M_g = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 0 & -1 \\ 0 & 2 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & -2 & -4 \\ 0 & 2 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & -2 & -4 \\ 0 & 0 & 0 \end{bmatrix}$ $\Rightarrow rang(M_g) = 2$
$\Rightarrow rang(g) = 2$

5.2.

$f \circ g(x,y,z) = f(g(x,y,z)) = f((x+2y+3z), (x-z), (2y+4z)) = \ldots$

$$M_f \cdot M_g = \underset{3\times3}{\begin{bmatrix} 1 & -2 & 0 \\ 0 & 1 & 1 \\ -2 & 1 & -1 \end{bmatrix}} \cdot \underset{3\times3}{\begin{bmatrix} 1 & 2 & 3 \\ 1 & 0 & -1 \\ 0 & 2 & 4 \end{bmatrix}} = \begin{bmatrix} -1 & 2 & 5 \\ 1 & 2 & 3 \\ -1 & -6 & -11 \end{bmatrix}$$

$\Rightarrow f \circ g(x,y,z) = (-x + 2y + 5z, x + 2y + 3z, -x - 6y - 11z)$

$$M_f = \begin{bmatrix} 1 & -2 & 0 \\ 0 & 1 & 1 \\ -2 & 1 & -1 \end{bmatrix}$$

$$\det(M_f) = \begin{vmatrix} 1 & -2 & 0 \\ 0 & 1 & 1 \\ -2 & 1 & -1 \end{vmatrix} \begin{matrix} 1 & -2 \\ 0 & 1 \\ -2 & 1 \end{matrix}$$

$$= -1 + 4 - 1 = 2 \neq 0$$

$$\operatorname{adj}(M_f) = \begin{bmatrix} +\begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} & -\begin{vmatrix} 0 & 1 \\ -2 & -1 \end{vmatrix} & +\begin{vmatrix} 0 & 1 \\ -2 & 1 \end{vmatrix} \\ -\begin{vmatrix} -2 & 0 \\ 1 & -1 \end{vmatrix} & +\begin{vmatrix} 1 & 0 \\ -2 & -1 \end{vmatrix} & -\begin{vmatrix} 1 & -2 \\ -2 & 1 \end{vmatrix} \\ +\begin{vmatrix} -2 & 0 \\ 1 & 1 \end{vmatrix} & -\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} & +\begin{vmatrix} 1 & -2 \\ 0 & 1 \end{vmatrix} \end{bmatrix}^T = \begin{bmatrix} -2 & -2 & 2 \\ -2 & -1 & 3 \\ -2 & -1 & 1 \end{bmatrix}^T = \begin{bmatrix} -2 & -2 & -2 \\ -2 & -1 & -1 \\ 2 & 3 & 1 \end{bmatrix}$$

$$\Rightarrow M_f^{-1} = \frac{1}{\det(M_f)} \operatorname{adj}(M_f) = \begin{bmatrix} -1 & -1 & -1 \\ -1 & -\frac{1}{2} & -\frac{1}{2} \\ 1 & \frac{3}{2} & \frac{1}{2} \end{bmatrix} \Rightarrow f^{-1}(x,y,z) = \left(-x-y-z,\ -x-\tfrac{1}{2}y-\tfrac{1}{2}z,\ x+\tfrac{3}{2}y+\tfrac{1}{2}z\right)$$

$g: \mathbb{R}^3 \to \mathbb{R}^3, \quad g(x,y,z) = (x+2y+3z,\ x-z,\ 2y+4z)$

JEZGRO $\ker(g) = \{(x,y,z) \in \mathbb{R}^3 \mid g(x,y,z) = 0\}$

$g(x,y,z) = 0$

$(x+2y+3z,\ x-z,\ 2y+4z) = 0$

$$\begin{aligned} x+2y+3z &= 0 \\ x \quad\ -z &= 0 \quad /\cdot(-1) \\ 2y+4z &= 0 \end{aligned}$$

$$\begin{aligned} x+2y+3z &= 0 \\ -2y-4z &= 0 \\ 2y+4z &= 0 \quad /+ \end{aligned}$$

$$\begin{aligned} x+2y+3z &= 0 \\ -2y-4z &= 0 \\ 0 &= 0 \end{aligned}$$

$$\begin{aligned} x+2y &= -3z \\ -2y &= 4z \end{aligned}$$

1x neodređen

$z = t,\ t \in \mathbb{R}$

$y = -2t$

$x = t$

$\ker(g) = \{(t,-2t,t) \mid t \in \mathbb{R}\}$

$= \{t(1,-2,1) \mid t \in \mathbb{R}\}$

$\ker(g)$ je potprostor v.p. $\mathbb{R}^3$ i jedna baza mu je

$\{(1,-2,1)\}$

SLIKA $\operatorname{Im}(g) = \{(a,b,c) \in \mathbb{R}^3 \mid \exists (x,y,z) \in \mathbb{R}^3,\ f(x,y,z) = (a,b,c)\}$

$f(x,y,z) = (a,b,c)$

$(x+2y+3z,\ x-z,\ 2y+4z) = (a,b,c)$

$$\begin{aligned} x+2y+3z &= a \\ x \quad\ -z &= b \quad /\cdot(-1) \\ 2y+4z &= c \end{aligned}$$

$$\begin{aligned} x+2y+3z &= a \\ -2y-4z &= -a+b \\ 2y+4z &= c \quad /+ \end{aligned}$$

$$\begin{aligned} x+2y+3z &= a \\ -2y-4z &= b-a \\ \boxed{0 = -a+b+c} \end{aligned}$$

$\boxed{a = b+c}$

$\operatorname{Im}(g) = \{(b+c,b,c) \mid b,c \in \mathbb{R}\}$

$= \{(1,1,0)b + (1,0,1)c \mid b,c \in \mathbb{R}^3\}$

$\operatorname{Im}(g)$ je potprostor v.p. $\mathbb{R}^3$, jedna baza mu je $\{(1,1,0),(1,0,1)\}$

$\dim(\operatorname{Im}(g)) = 2 = \operatorname{rang}(g)$

6. Dati su vektori $\vec{a} = (1, 2, -1)$, $\vec{b} = (3, -1, 1)$ i $\vec{v} = (x, y, z)$. Neka su f, g i h funkcije date sa:
$f : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$, $\boxed{f(x, y, z) = \vec{a} \times \vec{v} + (\vec{a} \cdot \vec{v}) \cdot \vec{b};}$
$g : \mathbb{R}^3 \longrightarrow \mathbb{R}^2$, $g(x, y, z) = (y, z)$;
$h : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$, $h(x, y) = (x - 2y, 2x + y, -y)$.
Dokazati da je funkcija $F = h \circ g \circ f$ linearna transformacija i odrediti njenu matricu.

skalarni proizvod

$$f(x,y,z) = \vec{a}\times\vec{v} + (\vec{a}\cdot\vec{v})\cdot\vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 2 & -1 \\ x & y & z \end{vmatrix} + \left(\overbrace{(1,2,-1)\cdot(x,y,z)}\right)\cdot(3,-1,1)$$

$$= \vec{i}\begin{vmatrix} 2 & -1 \\ y & z \end{vmatrix} - \vec{j}\begin{vmatrix} 1 & -1 \\ x & z \end{vmatrix} + \vec{k}\begin{vmatrix} 1 & 2 \\ x & y \end{vmatrix} + (\overbrace{x+2y-z}^{\text{broj}})\cdot(3,-1,1)$$

proizvod skalara i vektora

$$= (2z+y)\vec{i} - (z+x)\vec{j} + (y-2x)\vec{k} + (3\cdot(x+2y-z), -(x+2y-z), x+2y-z)$$

$$= (2z+y, -z-x, y-2x) + (3x+6y-3z, -x-2y+z, x+2y-z)$$

$$= (3x+7y-z, -2x-2y, -x+3y-z)$$

$$M_f = \begin{bmatrix} 3 & 7 & -1 \\ -2 & -2 & 0 \\ -1 & 3 & -1 \end{bmatrix} \qquad F = \underline{\underline{h \circ g \circ f}}$$

$$M_g = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$M_h = \begin{bmatrix} 1 & -2 \\ 2 & 1 \\ 0 & -1 \end{bmatrix}$$

$$M_F = M_h \cdot M_g \cdot M_f = \begin{bmatrix} 1 & -2 \\ 2 & 1 \\ 0 & -1 \end{bmatrix} \left(\underset{2\times 3}{\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}} \underset{3\times 3}{\begin{bmatrix} 3 & 7 & -1 \\ -2 & -2 & 0 \\ -1 & 3 & -1 \end{bmatrix}} \right) =$$

$$= \underset{3\times 2}{\begin{bmatrix} 1 & -2 \\ 2 & 1 \\ 0 & -1 \end{bmatrix}} \underset{2\times 3}{\begin{bmatrix} -2 & -2 & 0 \\ -1 & 3 & -1 \end{bmatrix}} = \begin{bmatrix} 0 & -8 & 2 \\ -5 & -1 & -1 \\ 1 & -3 & 1 \end{bmatrix}$$

$$F(x,y,z) = (-8y+2z,\ -5x-y-z,\ x-3y+z)$$

ZA VEŽBU IZ SKRIPTE

Zadatak 12.1; 12.2; 12.7; 12.8 (uzeti da je $g(x, y, z) = (x, y)$)