Kompleksni brojevi- vežbe

November 17, 2021

1. Dati su kompleksni brojevi $z_1 = 2 - 2i$ i $z_2 = -4 + 5i$. Odrediti: $\mathrm{Re}\,(z_1)$, $\mathrm{Im}\,(z_1)$, $|z_1|$, $\overline{z_1}$, $z_1 + z_2$, $z_1 - z_2$, $z_1 \cdot z_2$, $\frac{z_1}{z_2}$ i

$$\left|\frac{2z_1 + z_2 + 3 + i}{\overline{z_1}^2 - z_2 - 2} + \sqrt{3}\right|.$$

$Re(z_1) = 2$

$Im(z_1) = -2$

$|z_1| = \sqrt{2^2 + (-2)^2} = \sqrt{4+4} = \sqrt{8} = 2\sqrt{2}$

$\overline{z_1} = 2 + 2i$

$z_1 + z_2 = -2 + 3i$

$z_1 - z_2 = 6 - 7i$

$z_1 \cdot z_2 = (2-2i)(-4+5i) = -8 + 10i + 8i - 10\overset{-1}{i^2}$
$= 2 + 18i$

$\frac{z_1}{z_2} = \frac{2-2i}{-4+5i} \cdot \frac{-4-5i}{-4-5i} = \frac{-8-10i+8i+10\overset{-1}{i^2}}{16+25}$
$= \frac{-18-2i}{41} = -\frac{18}{41} - \frac{2}{41}i$

$\left|\frac{2z_1 + z_2 + 3 + i}{(\overline{z_1})^2 - z_2 - 2} + \sqrt{3}\right| =$

$= \left|\frac{4 - 4i - 4 + 5i + 3 + i}{(2+2i)^2 - (-4+5i) - 2} + \sqrt{3}\right| =$

$= \left|\frac{3+2i}{4 + 8i + 4\underset{-1}{i^2} + 4 - 5i - 2} + \sqrt{3}\right| =$

$= \left|\frac{3-2i}{2+3i} + \sqrt{3}\right| = \left|\frac{3-2i}{2+3i} \cdot \frac{2-3i}{2-3i} + \sqrt{3}\right|$

$= \left|\frac{\cancel{6} - 9i - 4i + \cancel{6i^2}}{4+9} + \sqrt{3}\right| = \left|\frac{-13i}{13} + \sqrt{3}\right|$

$= |\sqrt{3} - i| = \sqrt{(\sqrt{3})^2 + (-1)^2} = \sqrt{4} = 2$

$z = x + yi,$
$x, y \in \mathbb{R}$

2. Odrediti kompleksan broj z iz uslova da je

$$\mathrm{Re}\,(z + zi) = 2 \ \text{ i } \ \mathrm{Im}\left(\frac{1 - \bar{z}}{i}\right) = -3.$$

$\boxed{z = x + yi}, \quad x, y \in \mathbb{R}$

$x = \mathrm{Re}(z)$

$y = \mathrm{Im}(z)$

$$z + zi = x + yi + (x + yi)i = x + yi + xi - y = (x - y) + (x + y)i$$

z - није Re(Im) део јер је z комплексан број

$$\Rightarrow \mathrm{Re}\,(z + zi) = x - y \Rightarrow \boxed{x - y = 2}$$

$$\frac{1 - \bar{z}}{i} = \frac{1 - (x - yi)}{i} \cdot \frac{-i}{-i} = -i + (x - yi)i = -i + xi + y = y + (x - 1)i$$

$$\Rightarrow \mathrm{Im}\left(\frac{1 - \bar{z}}{i}\right) = x - 1 \Rightarrow \boxed{x - 1 = -3}$$

$$\left.\begin{array}{l} x = -2 \\ y = -4 \end{array}\right\} \quad z = -2 - 4i$$

ZA VEŽB URADITI KAO PRETHODNI ZADATAK.

3. Odrediti kompleksan broj z iz uslova da je

$$\mathrm{Re}\left(\underbrace{\boxed{\frac{(z-2)\,i+2\overline{z}}{i-3}}}_{\omega}\right)=-\frac{13}{10}\quad \text{i}\quad \mathrm{Im}\left(\underbrace{\boxed{\frac{(z-2)\,i+2\overline{z}}{i-3}}}_{\omega}\right)=-\frac{11}{10}.$$

$\boxed{z=x+yi}$ $x,y\in\mathbb{R}$

$x=\mathrm{Re}(z)$

$y=\mathrm{Im}(z)$

$z=\mathrm{Re}(z)+\mathrm{Im}(z)i$

$\omega=\mathrm{Re}(\omega)+\mathrm{Im}(\omega)i$

$\omega\in\mathbb{C}$

$$\omega=-\frac{13}{10}+\left(-\frac{11}{10}\right)i$$

$$\frac{(z-2)i+2\overline{z}}{i-3}=-\frac{13}{10}-\frac{11}{10}i \quad /\cdot 10(i-3)$$

$$10\left((z-2)i+2\overline{z}\right)=-13(i-3)-11i(i-3)$$

$$10(z-2)i+20\overline{z}=-13i+39+11+33i$$

$$10zi-20i+20\overline{z}=50+20i$$

$$10zi+20\overline{z}=50+40i \quad /:10$$

$$zi+2\overline{z}=5+4i$$

$$(x+yi)i+2(x-yi)=5+4i$$

$$xi-y+2x-2yi=5+4i$$

$$\underbrace{(2x-y)}+\underbrace{(x-2y)}i=\underbrace{5}+\underbrace{4}i$$

$$\left.\begin{aligned}2x-y&=5\\x-2y&=4\end{aligned}\right\}\cdot(-2)$$

$$\begin{aligned}2x-y&=5\\-3x&=-6\end{aligned}$$

$x=2$

$y=-1$

$$\Rightarrow z=2-i$$

$z = x + yi,\ x, y \in \mathbb{R}$

4. Odrediti kompleksan broj z iz uslova da je

$$\operatorname{Im}\left((3+i)\cdot \bar{z}\right) - 2i\operatorname{Re}\left(\frac{z+2}{1-i}\right) + |4-3i| = -2+i.$$

$x - 3y$ $\qquad \frac{x-y+2}{2}$ $\qquad 5$

$(3+i)\bar{z} = (3+i)(x-yi) = 3x - 3yi + xi + y = (3x+y) + (x-3y)i$

$\Rightarrow \operatorname{Im}((3+i)\bar{z}) = x - 3y$

$\frac{z+2}{1-i} = \frac{z+2}{1-i}\cdot\frac{1+i}{1+i} = \frac{z(1+i)+2+2i}{1+1} = \frac{(x+yi)(1+i)+2+2i}{2} = \frac{x+xi+yi-y+2+2i}{2}$

$= \frac{x-y+2}{2} + \frac{x+y+2}{2}i$

$\Rightarrow \operatorname{Re}\left(\frac{z+2}{1-i}\right) = \frac{x-y+2}{2}$

$|4-3i| = \sqrt{16+9} = 5$

$x - 3y - 2i\,\frac{x-y+2}{2} + 5 = -2+i$

$(x-3y+5) - (x-y+2)i = -2+i$

$x - 3y + 5 = -2$
$-(x-y+2) = 1$

$x - 3y = -7$
$-x + y = 3$

$-2y = -4$

$y = 2$
$x = -1$

$\Rightarrow z = -1 + 2i$

5. Izračunati $\sqrt{-24-10i}$ i rešenja zapisati u algebarskom obliku.

$$\sqrt{-24-10i} = x+yi \; /^2, \quad x,y\in\mathbb{R}$$

$$-24-10i = x^2+2xyi-y^2$$

$$-24-10i = (x^2-y^2)+2xyi$$

$$x^2-y^2=-24$$

$$2xy=-10 \Rightarrow \boxed{y=-\frac{5}{x}}$$

$$x^2-\left(-\frac{5}{x}\right)^2=-24$$

$$x^2-\frac{25}{x^2}=-24 \quad /\cdot x^2$$

$$x^4-25=-24x^2$$

$$x^4+24x^2-25=0$$

$$x^2=t, \quad t^2+24t-25=0$$

$$t_{1,2}=\frac{-24\pm\sqrt{576+100}}{2}=\frac{-24\pm 26}{2}$$

$$t_1=1$$

$$x^2=1$$

$$\boxed{x_1=1} \quad \wedge \quad \boxed{x_2=-1}$$

$$y_1=-5 \qquad y_2=5$$

$$1-5i \qquad -1+5i$$

$$t_2=-25$$

$$(5i)^2=-25$$

$$(-5i)^2=-25$$

$$\boxed{x^2=-25} \text{ – NE MOŽE JER JE } x\in\mathbb{R}$$

$$\sqrt{-24-10i} = \left\langle \begin{array}{l} 1-5i \\ -1+5i \end{array} \right.$$

NAPOMENA:

$$\sqrt[4]{-24-10i}=\sqrt{\sqrt{-24-10i}}=\left\langle \begin{array}{l} \sqrt{1-5i}=\langle \\ \sqrt{-1+5i}=\langle \end{array} \right.$$

6. U skupu kompleksnih brojeva rešiti jednačinu

$$((z-i)^2-2i)\cdot(1+2i)=-7-4i.$$

NAPOMENA: NIKAD NE TREBA STEPENOVATI (KVADRIRATI)!

$((z-i)^2-2i)(1+2i)=-7-4i$

$(z-i)^2-2i=\frac{-7-4i}{1+2i}\cdot\frac{1-2i}{1-2i}$

$(z-i)^2-2i=\frac{-7+14i-4i+8i^2}{1+4}$ $\quad i^2=-1$

$(z-i)^2-2i=\frac{-15+10i}{5}$

$(z-i)^2=-3+2i+2i$

$(z-i)^2=-3+4i$

$z-i=\sqrt{-3+4i}$

$\boxed{z=i+\sqrt{-3+4i}}$

$\sqrt{-3+4i}=x+yi,\quad x,y\in\mathbb{R}$

$-3+4i=x^2+2xyi-y^2$

$-3+4i=(x^2-y^2)+2xyi$

$x^2-y^2=-3$

$2xy=4 \Rightarrow \boxed{y=\frac{2}{x}}$

$x^2-\frac{4}{x^2}=-3 \quad /\cdot x^2$

$x^4+3x^2-4=0$

$x^2=t,\quad t^2+3t-4=0$

$t_{1,2}=\frac{-3\pm\sqrt{9+16}}{2}=\frac{-3\pm5}{2}$

$t_1=-4 \qquad t_2=+1$

$t_1=-4$

$\cancel{x^2=-4}$ jer $x\in\mathbb{R}$

$t_2=1$

$x^2=1$

$x_1=1 \qquad x_2=-1$

$y_1=2 \qquad y_2=-2$

$\sqrt{-3+4i}=\begin{cases}1+2i\\-1-2i\end{cases}$

$z_1=i+1+2i=1+3i$

$z_2=i+(-1-2i)=-1-i$

$2011 : 4 = 502$
$\underline{20}$
11
3

7\. Odrediti kompleksan broj z u algebarskom i eksponencijalnom obliku ako je

$$z = \frac{\left(\frac{1}{2} - \frac{\sqrt{3}}{2} i\right)^{536} - \left|\overline{3 - 7i} + \overline{3i}\right| + 6}{i^{2011}}.$$

$-\frac{1}{2} - \frac{\sqrt{3}}{2} i$ $\qquad 5 \qquad -i$

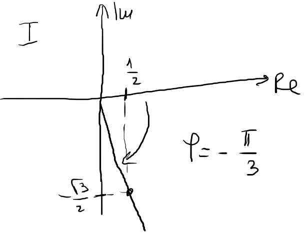
$$i^{2011} = i^{4 \cdot 502 + 3} = i^3 = i^2 \cdot i = -i$$

$$|\overline{3-7i} + \overline{3i}| = |3+7i-3i| = |3+4i| = \sqrt{9+16} = 5$$

$$\frac{1}{2} - \frac{\sqrt{3}}{2} i = 1 \cdot e^{-\frac{\pi}{3} i} = e^{-\frac{\pi}{3} i}$$

$$\left|\frac{1}{2} - \frac{\sqrt{3}}{2}\right| = \sqrt{\left(\frac{1}{2}\right)^2 + \left(-\frac{\sqrt{3}}{2}\right)^2} = \sqrt{\frac{1}{4} + \frac{3}{4}} = 1$$

I

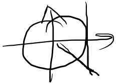


$\varphi = -\frac{\pi}{3}$

II

$$\operatorname{tg} \varphi = \frac{-\frac{\sqrt{3}}{2}}{\frac{1}{2}} = -\sqrt{3}$$

$$\varphi \in \left\{-\frac{\pi}{3}, \cancel{\frac{2\pi}{3}}\right\}$$



III

$$\left.\begin{aligned} \sin \varphi &= \frac{-\frac{\sqrt{3}}{2}}{1} = -\frac{\sqrt{3}}{2} \\ \cos \varphi &= \frac{\frac{1}{2}}{1} = \frac{1}{2} \end{aligned}\right\} \varphi = -\frac{\pi}{3}$$

$$\left(\frac{1}{2}-\frac{\sqrt{3}}{2}i\right)^{536} = \left(e^{-\frac{\pi}{3}i}\right)^{536} = e^{-\frac{\pi}{3}i\cdot 536} = e^{-\frac{536}{3}\pi i} = e^{-(178+\frac{2}{3})\pi i} =$$

$$= e^{-178\pi i - \frac{2\pi}{3}i} = e^{-178\pi i}\cdot e^{-\frac{2\pi}{3}i}$$

$$= e^{0i}\cdot e^{-\frac{2\pi}{3}i} = e^{-\frac{2\pi}{3}i}$$

$$= \cos\left(-\frac{2\pi}{3}\right) + i\sin\left(-\frac{2\pi}{3}\right)$$

$$= \cos\frac{2\pi}{3} - i\sin\frac{2\pi}{3}$$

$$= -\frac{1}{2} - \frac{\sqrt{3}}{2}i$$

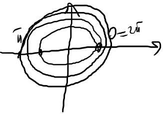
$$\frac{536}{3} \notin (-\pi,\pi)$$

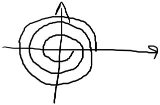
536 : 3 = 178
23
26
2

$$\frac{536}{3} = 178\frac{2}{3} = 178+\frac{2}{3}$$

$$-178\pi = 2\pi\cdot(-89)$$

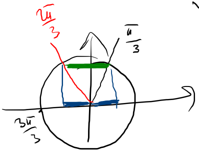
$$= 2\underbrace{(-89)}_{k}\pi$$





$$\frac{7}{2} = 3\frac{1}{2} = 3+\frac{1}{2}$$

$$2^{x+y} = 2^x\cdot 2^y$$



$$\sin\frac{2\pi}{3} = \sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$$

$$\cos\frac{2\pi}{3} = -\cos\frac{\pi}{3} = -\frac{1}{2}$$

$$z = \frac{-\frac{1}{2} - \frac{\sqrt{3}}{2}i - 5 + 6}{-i} = \frac{\frac{1}{2} - \frac{\sqrt{3}}{2}i}{-i} \cdot \frac{i}{i} = \frac{\frac{1}{2}i + \frac{\sqrt{3}}{2}}{1} = \frac{\sqrt{3}}{2} + \frac{1}{2}i$$

$(*)$ $$z = \frac{\left(\frac{\sqrt{2}}{2}+\frac{\sqrt{2}}{2}i\right)^{708} + \overline{(2-i)^3} - 2i}{\left(\frac{\sqrt{2}}{2}-\frac{\sqrt{2}}{2}i\right)^{17}}$$

1 $\quad$ $2+11i$

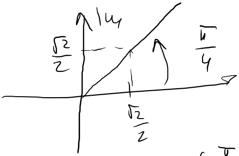
$(A-B)^3 = A^3 - 3A^2B + 3AB^2 - B^3$

$(A-B)^2(A-B)$ //

$$\overline{(2-i)^3} = \overline{8-12i-6+i} = \overline{2-11i} = 2+11i$$

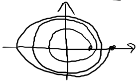
$$\frac{\sqrt{2}}{2}+\frac{\sqrt{2}}{2}i = 1e^{\frac{\pi}{4}i} = e^{\frac{\pi}{4}i}$$

$$\left|\frac{\sqrt{2}}{2}+\frac{\sqrt{2}}{2}i\right| = \sqrt{\frac{2}{4}+\frac{2}{4}} = 1$$



$$\left(\frac{\sqrt{2}}{2}+\frac{\sqrt{2}}{2}i\right)^{708} = \left(e^{\frac{\pi}{4}i}\right)^{708} = e^{\frac{\pi}{\cancel{4}}i\cdot\cancel{708}^{152}} = e^{152\pi i} = e^{0i} = 1$$

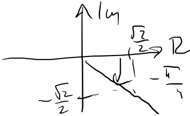
$$152\pi = 2\pi\cdot 71 = 2\cdot 71\cdot\pi = 2k\pi$$



I $$\frac{\sqrt{2}}{2}-\frac{\sqrt{2}}{2}i = \overline{\frac{\sqrt{2}}{2}+\frac{\sqrt{2}}{2}i} = \overline{e^{\frac{\pi}{4}i}} = e^{-\frac{\pi}{4}i}$$

II $$\frac{\sqrt{2}}{2}-\frac{\sqrt{2}}{2}i = e^{-\frac{\pi}{4}i}$$

$$\left|\frac{\sqrt{2}}{2}-\frac{\sqrt{2}}{2}i\right| = \sqrt{\frac{2}{4}+\frac{2}{4}} = 1$$



$$\left(\frac{\sqrt{2}}{2}-\frac{\sqrt{2}}{2}i\right)^{17} = \left(e^{-\frac{\pi}{4}i}\right)^{17} = e^{-\frac{17\pi}{4}i} = e^{-(4+\frac{1}{4})\pi i} = e^{-4\pi i}\cdot e^{-\frac{\pi}{4}i} = e^{0i}\cdot e^{-\frac{\pi}{4}i} = e^{-\frac{\pi}{4}i}$$

$$= \cos\frac{\pi}{4} - i\sin\frac{\pi}{4} = \frac{\sqrt{2}}{2}-\frac{\sqrt{2}}{2}i$$

$$\boxed{\frac{17}{4} = 4\tfrac{1}{4} = 4+\frac{1}{4}}$$

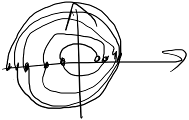
$$z = \frac{1+2+11i-2i}{\frac{\sqrt{2}}{2}-\frac{\sqrt{2}}{2}i} = \frac{3+9i}{\frac{\sqrt{2}}{2}(1-i)} \cdot \frac{1+i}{1+i}$$

$$= \frac{\cancel{2}}{\sqrt{2}} \frac{(3+9i)(1+i)}{\cancel{1+1}} = \frac{1}{\sqrt{2}}(3+3i+9i-9) =$$

$$= \frac{\sqrt{2}}{2}(-6+12i) = \frac{\sqrt{2}}{\cancel{2}} \cdot \cancel{2}(-3+6i) = \sqrt{2}(-3+6i)$$

$$\boxed{2^{x+y} = 2^x \cdot 2^y}$$

$$e^{23\pi i} = e^{\pi i} = -1$$



$$e^{\frac{23}{3}\pi i} = e^{\left(7+\frac{2}{3}\right)\pi i} = e^{7\pi i + \frac{2\pi}{3} i}$$

$$= e^{7\pi i} \cdot e^{\frac{2\pi}{3} i}$$

$$= e^{\pi i} \cdot e^{\frac{2\pi}{3} i}$$

$$= e^{\left(\pi + \frac{2\pi}{3}\right) i} = e^{\frac{5\pi}{3} i}$$

$$= e^{-\frac{\pi}{3} i}$$

23 : 3 = 7

2

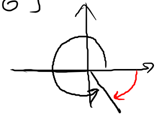
$$\frac{23}{3} = 7\frac{2}{3} = 7 + \frac{2}{3}$$

$$[-\pi, \pi]$$

$$\left[-\frac{3\pi}{3}, \frac{3\pi}{3}\right]$$

$$\left[-\frac{6\pi}{6}, \frac{6\pi}{6}\right]$$

$$\frac{5\pi}{3} \notin [-\pi, \pi]$$



$$-\frac{\pi}{3}$$

8. Jedno rešenje jednačine $\left(z-\sqrt{3}+2i\right)^6 = a$ je $z_1 = -3i$. Odrediti a i ostala rešenja ove jednačine. Rešenja zapisati u algebarskom obliku.

$$(z-\sqrt{3}+2i)^6 = a$$

$$z_1 = -3i$$

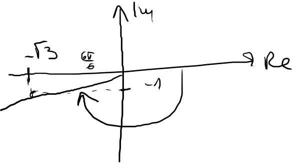
$$a = (z_1-\sqrt{3}+2i)^6 = (-3i-\sqrt{3}+2i)^6 = (-\sqrt{3}-i)^6 = \left(2\cdot e^{-\frac{5\pi}{6}i}\right)^6 =$$

$$= 2^6 e^{-\frac{5\pi}{6}i\cdot 6} = 64\cdot e^{-5\pi i}$$

$$= \boxed{64\cdot e^{\pi i}} = 64(\cos\pi + i\sin\pi)$$

$$= 64(-1+0\cdot i) = \boxed{-64}$$

$$|-\sqrt{3}-i| = \sqrt{(-\sqrt{3})^2+(-1)^2} = \sqrt{3+1} = 2$$



$$-\frac{5\pi}{6}$$

$$\boxed{(z-\sqrt{3}+2i)^6 = -64}$$

$$z-\sqrt{3}+2i = \sqrt[6]{-64}$$

$$\boxed{z = \sqrt{3}-2i+\sqrt[6]{-64}}$$

$$w = \sqrt[6]{-64} = \sqrt[6]{64e^{\pi i}} = \underbrace{\sqrt[6]{64}}_{2} \cdot e^{\frac{\pi + 2k\pi}{6} i}, \quad k = 0,1,2,3,4,5$$

$k=0 \quad w_0 = 2 \cdot e^{\frac{\pi + 2\cdot 0 \cdot \pi}{6} i} = 2e^{\frac{\pi}{6} i} = 2(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}) = 2\left(\frac{\sqrt{3}}{2} + \frac{1}{2} i\right) = \sqrt{3} + i$

$k=1 \quad w_1 = 2 \cdot e^{\frac{\pi + 2\cdot 1 \cdot \pi}{6} i} = 2e^{\frac{\pi}{2} i} = 2(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}) = 2(0 + 1\cdot i) = 2i$

$k=2 \quad w_2 = 2 \cdot e^{\frac{\pi + 2\cdot 2 \cdot \pi}{6} i} = 2e^{\frac{5\pi}{6} i} = 2(\cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6}) = 2\left(-\frac{\sqrt{3}}{2} + \frac{1}{2} i\right) = -\sqrt{3} + i$

$k=3 \quad w_3 = 2 \cdot e^{\frac{\pi + 2\cdot 3 \cdot \pi}{6} i} = 2e^{\frac{7\pi}{6} i} = 2e^{-\frac{5\pi}{6} i} = 2\left(\cos\frac{5\pi}{6} - i\sin\frac{5\pi}{6}\right) = 2\left(-\frac{\sqrt{3}}{2} - \frac{1}{2} i\right)$

$k=4 \quad w_4 = 2 \cdot e^{\frac{\pi + 2\cdot 4\pi}{6} i} = 2 \cdot e^{\frac{3\pi}{2} i} = 2e^{-\frac{\pi}{2} i} = 2\left(\cos\frac{\pi}{2} - i\sin\frac{\pi}{2}\right) = -2i$

$k=5 \quad w_5 = 2 \cdot e^{\frac{\pi + 2\cdot 5 \cdot \pi}{6} i} = 2 \cdot e^{\frac{11\pi}{6} i} = 2e^{-\frac{\pi}{6}} = 2\left(\cos\frac{\pi}{6} - i\sin\frac{\pi}{6}\right) = 2\left(\frac{\sqrt{3}}{2} - \frac{1}{2} i\right)$

$z = \sqrt{3} - 2i + \sqrt[6]{-64}$

$z_0 = \sqrt{3} - 2i + \sqrt{3} + i = 2\sqrt{3} - i$

$z_1 = \sqrt{3} - 2i + 2i = \sqrt{3}$

$z_2 = \cancel{\sqrt{3}} - 2i - \cancel{\sqrt{3}} + i = -i$

$z_3 = \cancel{\sqrt{3}} - 2i - \cancel{\sqrt{3}} - i = -3i$

$z_4 = \sqrt{3} - 2i - 2i = \sqrt{3} - 4i$

$z_5 = \sqrt{3} - 2i + \sqrt{3} - i = 2\sqrt{3} - 3i$

$$z = |z| e^{\varphi i}$$

$$\sqrt[3]{z} = \sqrt[3]{|z|} \; e^{\frac{\varphi + 2k\pi}{3} i} \quad , \; k = 0, 1, 2$$

$$\sqrt[4]{z} = \sqrt[4]{|z|} \; e^{\frac{\varphi + 2k\pi}{4} i} \quad , \; k = 0, 1, 2, 3$$

ZA VEŽBU IZ SKRIPTE
Zadatak 8.1, 8.2, 8.10, 8.11, 8.12, 8.13, 8.15, 8.17, 8.18, 8.20, 8.21, 8.23, 8.24 a, 8.25, 8.28, 8.29, 8.30, 8.31a;
Primer 8.11;