Polinomi

November 30, 2021

1. Podeliti polinome:

1.1 $p(x) = 3x^4 + 2x^3 + 5x^2 - x + 1$ sa $q(x) = x^2 - 2x + 2$;

$7 : 2 = 3$
1
$7 = 2 \cdot 3 + 1$

$$(3x^4 + 2x^3 + 5x^2 - x + 1) : (x^2 - 2x + 2) = \boxed{3x^2 + 8x + 15}$$

KOLIČNIK

$$\begin{array}{l}
-(3x^4 - 6x^3 + 6x^2) \\
\hline
\quad 8x^3 - x^2 - x + 1 \\
\quad -(8x^3 - 16x^2 + 16x) \\
\hline
\qquad 15x^2 - 17x + 1 \\
\qquad -(15x^2 - 30x + 30) \\
\hline
\qquad\qquad \boxed{13x - 29}
\end{array}$$

OSTATAK

$$\boxed{p(x) = q(x)(3x^2 + 8x + 15) + 13x - 29}$$

1.2 $p(x) = x^3 + 2x^2 + 7$ sa $q(x) = 3x^2 + 2x - 5$.

$+0x$

$$(x^3 + 2x^2 + 7) : (3x^2 + 2x - 5) = \boxed{\frac{1}{3}x + \frac{4}{9}}$$

KOLICNIK

$$-\left(x^3 + \frac{2}{3}x^2 - \frac{5}{3}x\right)$$

$$\frac{4}{3}x^2 + \frac{5}{3}x + 7$$

$$-\left(\frac{4}{3}x^2 + \frac{8}{9}x - \frac{20}{9}\right)$$

$$\boxed{\frac{7}{9}x + \frac{83}{9}}$$

OSTATAK

$$\boxed{p(x) = q(x)\left(\frac{1}{3}x + \frac{4}{9}\right) + \frac{7}{9}x + \frac{83}{9}}$$

2. Naći najveći zajednički delilac (NZD) polinoma
$p(x) = x^5 - x^4 - x^3 + x^2 - 2x + 2$ i $q(x) = x^4 + x^3 - x^2 + x - 2$.

$$(x^5 - x^4 - x^3 + x^2 - 2x + 2) : \boxed{(x^4 + x^3 - x^2 + x - 2)} = \underbrace{x - 2}_{\text{KOLIČNIK}}$$
$$-(x^5 + x^4 - x^3 + x^2 - 2x)$$
$$-2x^4 \qquad + 2$$
$$-(-2x^4 - 2x^3 + 2x^2 - 2x + 4)$$
$$\boxed{2x^3 - 2x^2 + 2x - 2} \neq 0$$

OSTATAK

$$= 2\boxed{(x^3 - x^2 + x - 1)}$$

$$(x^4 + x^3 - x^2 + x - 2) : \boxed{2x^3 - 2x^2 + 2x - 2} = \frac{1}{2}x + 1$$
$$-(x^4 - x^3 + x^2 - x)$$
$$2x^3 - 2x^2 + 2x - 2$$
$$-(2x^3 - 2x^2 + 2x - 2)$$
$$\boxed{0}$$

OSTATAK

POSLEDNJI OSTATAK RAZLIČIT OD NULE JE

$$\boxed{NZD(p(x), q(x)) = 2x^3 - 2x^2 + 2x - 2}$$

$$NZD(p(x), q(x)) = 2(x^3 - x^2 + x - 1)$$
$$= 2\left(x^2(x-1) + (x-1)\right)$$
$$= 2(x-1)(x^2+1)$$

$$NZD(p(x), q(x)) = d$$
$$\Rightarrow NZD(p(x), q(x)) = \alpha d, \quad \alpha \in \mathbb{R}$$

DAKLE, MOŽEMO REĆI DA JE

$$NZD(p(x), q(x)) = x^3 - x^2 + x - 1$$
$$NZD(p(x), q(x)) = -7x^3 + 7x^2 - 7x + 7$$

3. Nad poljem realnih brojeva dati su polinomi $p(x) = x^4 + x^3 - x - 1$ i $q(x) = 3x^3 - 2x^2 - 1$. Odrediti njihove zajedničke korene.

ODREDITI ZAJEDNIČKE KORENE ZNAČI NAĆI NULE (KORENE) NJIHOVOG NZD.

$$(x^4 + x^3 - x - 1) : (3x^3 - 2x^2 - 1) = \frac{1}{3}x + \frac{5}{9}$$
$$-\left(x^4 - \frac{2}{3}x^3 - \frac{1}{3}x\right)$$
$$\frac{5}{3}x^3 - \frac{2}{3}x - 1$$
$$-\left(\frac{5}{3}x^3 - \frac{10}{9}x^2 - \frac{5}{9}\right)$$
$$\boxed{\frac{10}{9}x^2 - \frac{2}{3}x - \frac{4}{9}} = \frac{2}{9}\left(5x^2 - 3x - 2\right) \neq 0$$

NE UTIČE NA NZD

$$(3x^3 - 2x^2 - 1) : (5x^2 - 3x - 2) = \frac{3}{5}x - \frac{1}{25}$$
$$-\left(3x^3 - \frac{9}{5}x^2 - \frac{6}{5}x\right)$$
$$-\frac{1}{5}x^2 + \frac{6}{5}x - 1$$
$$-\left(-\frac{1}{5}x^2 + \frac{3}{25}x + \frac{2}{25}\right)$$
$$\boxed{\frac{27}{25}x - \frac{27}{25}} = \frac{27}{25}(x-1) \neq 0$$

$$(5x^2 - 3x - 2) : (x-1) = 5x + 2$$
$$-(5x^2 - 5x)$$
$$2x - 2$$
$$-(2x - 2)$$
$$\boxed{0}$$

$\Rightarrow NZD(p(x), q(x)) = x - 1$

KOREN (NULA) POLINOMA

$x_0 \quad p(x_0) = 0$

$\Leftrightarrow$

$x_0 \quad (x - x_0) \mid p(x)$

PRIMER:

$p(x) = x^2 - 1 = (x-1)(x+1)$

NULE ZA $p(x)$: ± 1

$q(x) = x^3 - 1 = (x-1)(x^2 + x + 1)$

NULE ZA $q(x)$: 1, $\frac{-1 \pm \sqrt{3}i}{2}$

ZAJEDNIČKA NULA ZA $p(x)$ i $q(x)$ JE 1

$NZD(p(x), q(x)) = x - 1$

$\Rightarrow (x-1) \mid p(x)$
$(x-1) \mid q(x)$

1 JEDINA ZAJEDNIČKA NULA ZA $p(x)$ i $q(x)$

4. Nad poljem realnih brojeva dati su polinomi
$p(x) = x^4 - 3x^3 + 3x^2 - 3x - 4$ i $q(x) = x^3 - x^2 - 2x + 8$.
Dokazati da su polinomi p i q uzajamno prosti nad poljem $\mathbb{R}$.

$$(x^4-3x^3+3x^2-3x-4):(x^3-x^2-2x+8)=x-2$$
$$-(x^4-x^3-2x^2+8x)$$
$$-2x^3+5x^2-11x-4$$
$$-(-2x^3+2x^2+4x-16)$$
$$\boxed{3x^2-15x+12} = \underline{3}\,(x^2-5x+4) \neq 0$$

$$(x^3-x^2-2x+8):(x^2-5x+4)=x+4$$
$$-(x^3-5x^2+4x)$$
$$4x^2-6x+8$$
$$-(4x^2-20x+16)$$
$$\boxed{14x-8} = \underline{2}\,(7x-4) \neq 0$$

$$(x^2-5x+4):(7x-4)=\frac{1}{7}x-\frac{31}{49}$$
$$-\left(x^2-\frac{4}{7}x\right)$$
$$-\frac{31}{7}x+4$$
$$-\left(-\frac{31}{7}x+\frac{124}{49}\right)$$
$$\boxed{\frac{72}{49}} = \underline{\frac{72}{49}}\cdot\boxed{1} \neq 0$$

$$(7x-4):\boxed{1}=7x-4$$
$$-(7x-4)$$
$$\boxed{0}$$

$\mathrm{NZD}(p(x),q(x))=1$

$\Rightarrow$ $p(x)$ i $q(x)$ su UZAJAMNO PROSTI

5. Za koje realne vrednosti parametra a je polinom $p(x) = x^3 + ax^2 + 3x - 5$ deljiv polinomom $x + 1$?

$\dfrac{p(x)}{x+1}$

PO USLOVU ZADATKA OSTATAK JE NULA

$\dfrac{p(x)}{x+1}$

PO BEZUOVOM STAVU OSTATAK JE $p(-1)$

$$p(-1) = 0$$

$$\begin{aligned} p(-1) &= (-1)^3 + a(-1)^2 + 3\cdot(-1) - 5 \\ &= -1 + a - 3 - 5 \\ &= a - 9 \end{aligned}$$

$$a - 9 = 0$$

$$\boxed{a = 9}$$

BEZUOV STAV:
AKO POLINOM $p(x)$ DELIM POLINOMOM $x-a$ OSTATAK JE $p(a)$, T.J. VREDNOST POLINOMA U a

$\dfrac{p(x)}{x-3}$ OSTATAK JE $p(3)$

$\dfrac{p(x)}{x+13}$ OSTATAK JE $p(-13)$

$\dfrac{p(x)}{x+5}$ OSTATAK JE $p(-5)$

$\dfrac{p(x)}{x-6}$ $p(6)$

POLINOM JE DELJIV NEKIM DRUGIM POLINOMOM AKO JE OSTATAK PRI NJIHOVOM DELJENJU JEDNAK 0

6. Odrediti koeficijente a, b i c polinoma $p(x) = x^3 + ax^2 + bx + c$ tako da bude deljiv polinomima $x-1$ i $x+2$, a da pri deljenju sa polinomom $x-4$ daje ostatak 18.

$p(x) = x^3 + ax^2 + bx + C$

$\frac{p(x)}{x-1}$

- OSTATAK PO USLOVU ZADATKA JE 0
- OSTATAK PO BEZUOVOM STAVU JE $p(1)$

$\Rightarrow \boxed{p(1) = 0}$

$\frac{p(x)}{x+2}$

- OSTATAK PO USLOVU ZADATKA JE 0
- OSTATAK PO BEZUOVOM STAVU JE $p(-2)$

$\Rightarrow \boxed{p(-2) = 0}$

$\frac{p(x)}{x-4}$

- OSTATAK PO USLOVU ZADATKA JE 18
- OSTATAK PO BEZUOVOM STAVU JE $p(4)$

$\Rightarrow \boxed{p(4) = 18}$

$$p(1) = 0 \Rightarrow p(1) = 1 + a + b + C \Rightarrow 1 + a + b + C = 0 \Rightarrow \underline{a + b + C = -1}$$

$$p(-2) = 0 \Rightarrow p(-2) = (-2)^3 + a(-2)^2 + b(-2) + C \Rightarrow -8 + 4a - 2b + C = 0 \Rightarrow \underline{4a - 2b + C = 8}$$

$$p(4) = 18 \Rightarrow p(4) = 4^3 + a \cdot 4^2 + b \cdot 4 + C \Rightarrow 64 + 16a + 4b + C = 18 \Rightarrow \underline{16a + 4b + C = -46}$$

$$p(x) = x^3 - 2x^2 - 5x + 6$$

$$a + b + c = -1 \qquad -4$$
$$\boxed{4a} - 2b + c = 8 \qquad -16$$
$$\boxed{16a} + 4b + c = -46$$

$$a + b + c = 1$$
$$-6b - 3c = 12 \qquad -2$$
$$\boxed{-12b} - 15c = -30$$

$$a + b + c = 1$$
$$-6b - 3c = 12$$
$$-9c = -54$$

$$\uparrow$$

$$c = \frac{-54}{-9} = 6$$

$$-6b - 18 = 12$$
$$-6b = 30$$
$$b = -5$$

$$a - 5 + 6 = 1$$
$$a = -2$$

7. Ostatak pri deljenju polinoma $p(x)$ sa $x-1$ je -5, a sa $x-2$ je -25. Koliki je ostatak pri deljenju polinoma $p(x)$ sa $(x-1)(x-2)$?

$7:2=3 \Rightarrow 7=2\cdot 3+1$

1

$\frac{p(x)}{x-1}$

PO BEZUOVOM OSTATAK JE $p(1)$

PO USLOVU ZAD. -5

$\Rightarrow \boxed{p(1)=-5}$

$\frac{p(x)}{x-2}$

PO BEZUOVOM OSTATAK JE $p(2)$

PO USLOVU ZAD. -25

$\Rightarrow \boxed{p(2)=-25}$

$\frac{p(x)}{(x-1)(x-2)}$

OSTATAK ?

NIJE $(-5)\cdot(-25)$!

$p(x):(x-1)(x-2)=q(x)$

$r(x)$

$\deg(r(x)) < \deg((x-1)(x-2))=2$

$\Rightarrow r(x)$ je polinom najviše prvog stepena

$\Rightarrow r(x)=ax+b$

$p(x)=(x-1)(x-2)q(x)+r(x)$

$\boxed{p(x)=(x-1)(x-2)q(x)+ax+b}$

$p(1)=-5 \Rightarrow p(1)=a+b \Rightarrow a+b=-5$

$p(2)=-25 \Rightarrow p(2)=2a+b \Rightarrow 2a+b=-25 \quad \big/ \cdot(-2)$

$a+b=-5$

$-b=-15$

$b=15 \quad a=-20$

$\boxed{r(x)=-20x+15}$

8. Ostatak pri deljenju polinoma $p(x)$ sa $x+1$ je 2, sa $x-1$ je 3, a sa $x-2$ je -1. Koliki je ostatak pri deljenju polinoma $p(x)$ sa $(x+1)(x-1)(x-2)$?

$p(x) : x+1$

$\boxed{p(-1)=2}$

$p(x) : x-1$

$\boxed{p(1)=3}$

$p(x) : x-2$

$\boxed{p(2)=-1}$

$p(x) : (x+1)(x-1)(x-2) = q(x)$

$r(x)$

$\deg(r(x)) < \deg((x+1)(x-1)(x-2)) = 3$

$\Rightarrow r(x)$ je polinom drugog stepena

$\Rightarrow r(x) = ax^2+bx+c$

$p(x) = (x+1)(x-1)(x-2) \cdot q(x) + r(x)$

$$\boxed{p(x) = (x+1)(x-1)(x-2) q(x) + ax^2+bx+c}$$

$p(-1)=2 \Rightarrow p(-1) = a-b+c \Rightarrow a-b+c=2$

$p(1)=3 \Rightarrow p(1) = a+b+c \Rightarrow a+b+c=3$

$p(2)=-1 \Rightarrow p(2) = 4a+2b+c \Rightarrow 4a+2b+c=-1$

$a-b+c=2 \quad \cdot(-1),\ \cdot(-4)$

$a+b+c=3$

$4a+2b+c=-1$

$a-b+c=2$

$2b = 1 \quad \cdot(-3)$

$6b-3c=-9$

$a-b+c=2$

$2b=1$

$-3c=-12$

$c=4$

$b=\frac{1}{2}$

$a-\frac{1}{2}+4=2$

$a=-\frac{3}{2}$

$$\boxed{r(x) = -\frac{3}{2}x^2+\frac{1}{2}x+4}$$

9. Koristeći Hornerovu šemu podeliti polinome
$p(x) = 3x^5 + 11x^4 + 4x^3 - x^2 + 8x - 22$ i $q(x) = x + 3$. $= x - (-3)$ $\qquad x - \alpha$

	3	11	4	−1	8	−22
−3	3	2	−2	5	−7	$\boxed{-1}$

KOEFICIJENTI POLINOMA KOLIČNIKA

OSTATAK

$r(x) = -1$ — OSTATAK

$S(x) = 3x^4 + 2x^3 - 2x^2 + 5x - 7$ — OSTATAK

$$p(x) = (x+3)(3x^4 + 2x^3 - 2x^2 + 5x - 7) - 1$$

$p(x) = 17x^3 + 4x^2 + 1$

$x = 1$

	17	4	0	1
1	17	21	21	$\boxed{22}$ → OSTATAK
		$1\cdot 17+4$	$1\cdot 21+0$	$1\cdot 21+1$

$7 : 2 = 3$

1

$7 = \underbrace{2\cdot 3}_{\text{KOLIČNIK}} + \underbrace{1}_{\text{OSTATAK}}$

$$p(x) = \underbrace{(x-1)}_{\text{DJELIOC}} \cdot \underbrace{(17x^2 + 21x + 21)}_{\text{KOLIČNIK}} + \underbrace{22}_{\text{OSTATAK}}$$

10. Napisati polinom $p(x) = -3x^5 - 8x^2 + 8x - 13$ po stepenima od $x+1$.

$$p(x) = -3x^5 - 8x^2 + 8x - 13 = -3\,(x+1)^5 + 15\,(x+1)^4 + (-30)\,(x+1)^3 + 22\,(x+1)^2 + 9\,(x+1) + (-26)$$

	-3	0	0	-8	8	-13
-1	-3	3	-3	-5	13	$\boxed{-26}$
-1	-3	6	-9	4	$\boxed{9}$	
-1	-3	9	-18	$\boxed{22}$		
-1	-3	12	$\boxed{-30}$			
-1	-3	$\boxed{15}$				
-1	$\boxed{-3}$					

$$p(x) = -3(x+1)^5 + 15(x+1)^4 - 30(x+1)^3 + 22(x+1)^2 + 9(x+1) - 26$$

OVO NE TREBA DA RADITE:

$$p(x) = (x+1)(-3x^4+3x^3-3x^2-5x+13) - 26$$
$$= (x+1)\left[(x+1)(-3x^3+6x^2-9x+4)+9\right] - 26$$
$$= (x+1)^2(-3x^3+6x^2-9x+4) + 9(x+1) - 26$$
$$= (x+1)^2\left[(x+1)(-3x^2+9x-18)+22\right] + 9(x+1) - 26$$
$$= (x+1)^3(-3x^2+9x-18) + 22(x+1)^2 + 9(x+1) - 26$$
$$= (x+1)^3\left[(x+1)(-3x+12) - 30\right] + 22(x+1)^2 + 9(x+1) - 26$$
$$= (x+1)^4(-3x+12) - 30(x+1)^3 + 22(x+1)^2 + 9(x+1) - 26$$
$$= (x+1)^4\left[(x+1)\cdot(-3)+15\right] - 30(x+1)^3 + 22(x+1)^2 + 9(x+1) - 26$$
$$= -3(x+1)^5 + 15(x+1)^4 - 30(x+1)^3 + 22(x+1)^2 + 9(x+1) - 26$$

$$p(x) = x^6 - 2x^4 + 3x^3 + 2x - 1$$

Rozpisati po stepenima od $x-2$.

	1	0	−2	3	0	2	−1
2	1	2	2	7	14	30	$\boxed{59}$
2	1	4	10	27	68	$\boxed{166}$	
2	1	6	22	71	$\boxed{210}$		
2	1	8	38	$\boxed{147}$			
2	1	10	$\boxed{58}$				
2	1	$\boxed{12}$					
2	$\boxed{1}$						

$$p(x) = 1\cdot(x-2)^6 + 12\cdot(x-2)^5 + 58(x-2)^4 + 147(x-2)^3 + 210(x-2)^2 + 166(x-2) + 59$$

11. Naći sve nule polinoma $p(x)$:

11.1 $p(x) = 9x^4 - 12x^3 - 17x^2 + 8x + 4$;

18 BROJEVA

$p \mid 4 \Rightarrow p \in \{\pm 1, \pm 2, \pm 4\}$

$q \mid 9 \Rightarrow q \in \{1, 3, 9\}$

$$\frac{p}{q} \in \underbrace{\left\{\pm 1, \pm 2, \pm 4, \pm \frac{1}{3}, \pm \frac{2}{3}, \pm \frac{4}{3}, \pm \frac{1}{9}, \pm \frac{2}{9}, \pm \frac{4}{9}\right\}}_{18 \text{ BROJEVA}}$$

	9	−12	−17	8	4
~~1~~	~~9~~	~~−3~~	~~−20~~	~~−12~~	~~−8~~
−1	9	−21	4	4	$\boxed{0}$
~~−1~~	~~9~~	~~−30~~	~~34~~	~~−30~~	
2	9	−3	−2	$\boxed{0}$	

$\Rightarrow$ $\boxed{-1}$ jeste nula polinoma $p(x)$

$\Rightarrow p(x) = (x+1)(9x^3 - 21x^2 + 4x + 4)$

$\Rightarrow$ $\boxed{2}$ jeste nula polinoma $p(x)$

$\Rightarrow p(x) = (x+1)(x-2)(9x^2 - 3x - 2)$

$9x^2 - 3x - 2 = 0$

$$x_{1,2} = \frac{3 \pm \sqrt{9+72}}{18} = \frac{3 \pm 9}{18} = \begin{cases} \frac{2}{3} \\ -\frac{1}{3} \end{cases}$$

$\Rightarrow$ $\boxed{\frac{2}{3}}$, $\boxed{-\frac{1}{3}}$ jesu nule polinoma $p(x)$

$\Rightarrow p(x) = 9(x+1)(x-2)\left(x - \frac{2}{3}\right)\left(x + \frac{1}{3}\right)$