Vektorski prostori

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1. U vektorskom prostoru $\boldsymbol{R}^3$ ispitati linearnu zavisnost i generatornost sledećih skupova vektora:

   1.1 $b_1 = (0,1,0)$, $b_2 = (0,0,-1)$

$\mathbb{R}^3 \Rightarrow \boxed{\dim(\mathbb{R}^3) = 3}$

$\vec{i} = (1,0,0)$

$\vec{j} = (0,1,0)$ $\quad \vec{a} = a_1\vec{i} + a_2\vec{j} + a_3\vec{k}$

$\vec{k} = (0,0,1)$

$\{\vec{i}, \vec{j}, \vec{k}\}$ - jedna baza za $\mathbb{R}^3$

- Dva vektora b_1, b_2 ne mogu da generišu prostor dimenzije 3 (jer je 3 minimalan broj generatora).
- $\boxed{\alpha b_1 + \beta b_2 = 0}$

$\alpha(0,1,0) + \beta(0,0,-1) = 0$

$(0, \alpha, -\beta) = 0$

$\alpha = 0 \wedge -\beta = 0$

$\boxed{\alpha = \beta = 0} \Rightarrow b_1, b_2$ jesu linearno nezavisni

1.2 $b_1 = (3,0,0)$, $b_2 = (0,3,0)$, $b_3 = (0,0,3)$, $b_4 = (5,7,9)$ $\boxed{\dim(\mathbb{R}^3) = 3}$

- Četiri vektora uvijek moraju biti linearno zavisni u trodimenzionalnom v.p. (jer 3 je maksimalan broj linearno nezavisnih vektora).

- Vektori b_1, b_2, b_3 su lin. nezavisni jer

$$\alpha b_1 + \beta b_2 + \gamma b_3 = 0$$
$$\alpha(3,0,0) + \beta(0,3,0) + \gamma(0,0,3) = 0$$
$$(3\alpha, 3\beta, 3\gamma) = 0$$
$$\alpha = \beta = \gamma = 0$$

pa čine bazu. Ali to znači znači da su one i generatori.

$\Rightarrow \{b_1, b_2, b_3, \underline{b_4}\}$ – je generatoran skup vektora

↳ dodavanje b_4 ne mijenja se generatornost.

1.3 $b_1 = (0, 0, 7)$

- Jedan vektor ne može da generiše trodimenzion V.P. (minimalan broj generatora je 3).
- Jedan ne-nula vektor je uvek lin. nezavisan jer

$$\alpha b_1 = 0$$

$$\alpha(0,0,7) = 0$$

$$(0,0,7\alpha) = 0$$

$$7\alpha = 0$$

$$\alpha = 0$$

1.4 $b_1 = (0,0,0)$, $b_2 = (5,3,1)$

- $\{b_1, b_2\}$ - nije generatoran za $\mathbb{R}^3$ $(\dim(\mathbb{R}^3) = 3)$ jer dva vektora ne mogu da generišu trodimenzionalni V.P.

- $$\alpha b_1 + \beta b_2 = 0$$
$$\underline{\alpha(0,0,0)} + \beta(5,3,1) = 0$$
$$(5\beta, 3\beta, \beta) = 0$$
$$\beta = 0$$
$$\underline{\underline{\alpha = ?}}$$

Svaki skup vektora koji sadrži 0 vektor je linearno zavisan.

1.5 $b_1 = (5,5,5),\ b_2 = (3,3,0), b_3 = (2,0,0)$

$\alpha b_1 + \beta b_2 + \gamma b_3 = 0$

$\alpha(5,5,5) + \beta(3,3,0) + \gamma(2,0,0) = 0$

$(5\alpha + 3\beta + 2\gamma, 5\alpha + 3\beta, 5\alpha) = 0$

$$
\begin{aligned}
5\alpha + 3\beta + 2\gamma &= 0 \\
5\alpha + 3\beta &= 0 \\
5\alpha &= 0
\end{aligned}
$$

$\alpha = \beta = \gamma = 0$

$\Rightarrow \{b_1, b_2, b_3\}$ - skup lin. nez. vektora

$\dim(\mathbb{R}^3) = 3$

$\{b_1, b_2, b_3\}$ - lin. nez. vektori

$\Downarrow$

$\{b_1, b_2, b_3\}$ - čine bazu

$\Downarrow$

$\{b_1, b_2, b_3\}$ - generatori

(1,0,0)	(1,1,1)	(1,1,1)
(0,1,0)	(1,1,0)	(0,1,1)
(0,0,1)	(1,0,0)	(0,0,1)

2. Dati su vektori a_1, a_2, a_3 i b. Odrediti realan parametar α tako da se vektor b može izraziti kao linearna kombinacija vektora a_1, a_2 i a_3.

2.1 $a_1 = (4, 4, 3)$, $a_2 = (7, 2, 1)$, $a_3 = (4, 1, 6)$ i $b = (5, 9, \underline{\alpha})$

$D_S \neq 0 \Rightarrow$ sistem određen i ima jedinstveno rješenje

$D_S = 0 \Rightarrow$ sistem je nemoguć ili neodređen

$$b = \beta a_1 + \gamma a_2 + \delta a_3$$

$$(5, 9, \alpha) = \beta(4,4,3) + \gamma(7,2,1) + \delta(4,1,6)$$

$$(5, 9, \alpha) = (4\beta + 7\gamma + 4\delta,\ 4\beta + 2\gamma + \delta,\ 3\beta + \gamma + 6\delta)$$

$$4\beta + 7\gamma + 4\delta = 5$$

$$4\beta + 2\gamma + \delta = 9$$

$$3\beta + \gamma + 6\delta = \alpha$$

$$\frac{24 \cdot 7}{168}$$

$$D_S = \left|\begin{matrix} 4 & 7 & 4 \\ 4 & 2 & 1 \\ 3 & 1 & 6 \end{matrix}\right| \begin{matrix} 4 & 7 \\ 4 & 2 \\ 3 & 1 \end{matrix} = 48 + 21 + 16 - 24 - 4 - 168 \neq 0$$

$\Rightarrow$ sistem je određen za bilo koje $\alpha \in \mathbb{R}$

$\Rightarrow$ za bilo koje $\alpha \in \mathbb{R}$ b se može izraziti preko a_1, a_2, a_3.

2.2 $a_1 = (2,1,0)$, $a_2 = (-3,2,1)$, $a_3 = (5,-1,-1)$ i $b = (8,\underline{\alpha},-2)$

$$b = \beta a_1 + \gamma a_2 + \delta a_3$$

$$(8,\alpha,-2) = \beta(2,1,0) + \gamma(-3,2,1) + \delta(5,-1,-1)$$

$$(8,\alpha,-2) = (2\beta - 3\gamma + 5\delta,\ \beta + 2\gamma - \delta,\ \gamma - \delta)$$

$$\begin{aligned} 2\beta - 3\gamma + 5\delta &= 8 \\ \beta + 2\gamma - \delta &= \alpha \\ \gamma - \delta &= -2 \end{aligned}$$

$$D_S = \begin{vmatrix} 2 & -3 & 5 \\ 1 & 2 & -1 \\ 0 & 1 & -1 \end{vmatrix} \begin{matrix} 2 & -3 \\ 1 & 2 \\ 0 & 1 \end{matrix} = -4 + 5 + 2 - 3 = 0$$

$\Rightarrow$ sistem nemoguć ili neodređen

$\rightarrow$ dalje mora Gaussov postupak

$$\left.\begin{aligned} \beta + 2\gamma - \delta &= \alpha \\ 2\beta - 3\gamma + 5\delta &= 8 \\ \gamma - \delta &= -2 \end{aligned}\right\} \cdot(-2)$$

$$\begin{aligned} \beta + 2\gamma - \delta &= \alpha \\ -7\gamma + 7\delta &= 8 - 2\alpha \\ \gamma - \delta &= -2 \end{aligned}$$

$$\begin{aligned} \beta + 2\gamma - \delta &= \alpha \\ \gamma - \delta &= -2 \\ -7\gamma + 7\delta &= 8 - 2\alpha \end{aligned} \Big\} \cdot 7$$

$$\begin{aligned} \beta + 2\gamma - \delta &= \alpha \\ \gamma - \delta &= -2 \\ 0 &= -6 - 2\alpha \end{aligned}$$

$\boxed{\alpha = -3}$ $\Rightarrow$ sistem neodređen i b se može napisati preko a_1, a_2, a_3

$\alpha \neq -3$ $\Rightarrow$ sistem je nemoguć i b se ne može napisati preko a_1, a_2, a_3

ZA VEŽBU

2.3 $a_1 = (-1, 3, -4)$, $a_2 = (1, -3, 4)$, $a_3 = (2, -6, 8)$ i $b = (0, \alpha, -1)$.

3. Ispitati linearnu zavisnost vektora:

ZA VJEŽBU

3.1 $(-4,2,-1,3)$, $(1,-3,2,4)$, $(-2,4,3,-1)$, $(-3,5,1,-2)$;

3.2 $(1,1,2,1)$, $(1,-1,1,2)$, $(-3,1,-4,-5)$, $(0,2,1,-1)$.

$$\alpha(1,1,2,1)+\beta(1,-1,1,2)+\gamma(-3,1,-4,-5)+\delta(0,2,1,-1)=0$$

$$(\alpha+\beta-3\gamma,\ \alpha-\beta+\gamma+2\delta,\ 2\alpha+\beta-4\gamma+\delta,\ \alpha+2\beta-5\gamma-\delta)=0$$

$$\begin{aligned}
\alpha+\beta-3\gamma &= 0 \quad \cdot(-1),\ \cdot(-2),\ \cdot(-1)\\
\alpha-\beta+\gamma+2\delta &= 0\\
2\alpha+\beta-4\gamma+\delta &= 0\\
\alpha+2\beta-5\gamma-\delta &= 0
\end{aligned}$$

$$\begin{aligned}
\alpha+\beta-3\gamma &= 0\\
-2\beta+4\gamma+2\delta &= 0\\
-\beta+2\gamma+\delta &= 0\\
\beta-2\gamma-\delta &= 0
\end{aligned}$$

$$\begin{aligned}
\alpha+\beta-3\gamma &= 0\\
\beta-2\gamma-\delta &= 0 \quad \cdot 2,\ \cdot(-1)\\
-2\beta+4\gamma+2\delta &= 0\\
\beta-2\gamma-\delta &= 0
\end{aligned}$$

$$\begin{aligned}
\alpha+\beta-3\gamma &= 0\\
\beta-2\gamma-\delta &= 0\\
0 &= 0\\
0 &= 0
\end{aligned}$$

$$\begin{aligned}
\alpha+\beta &= 3\gamma\\
\beta &= 2\gamma+\delta
\end{aligned}$$

$\Rightarrow$ neodređen sistem

(nije mu jedino rješenje $(0,0,0,0)$)

$\Rightarrow$ vektori su linearno zavisni

4. Dati su vektori: $a_1 = (3,1,1)$, $a_2 = (m,-1,0)$ i $a_3 = (0,1,m)$.

4.1 Za koju vrednost realnog parametra m skup $\{a_1, a_2, a_3\}$ predstavlja bazu prostora $\mathbf{R}^3$?

$\dim(\mathbb{R}^3) = 3$

Da bi tri vektora $\{a_1, a_2, a_3\}$ činili bazu dovoljno je da oni budu lin. nezavisni.

$$\boxed{\alpha a_1 + \beta a_2 + \gamma a_3 = 0}$$

$$\alpha(3,1,1) + \beta(m,-1,0) + \gamma(0,1,m) = 0$$

$$(3\alpha + m\beta,\ \alpha - \beta + \gamma,\ \alpha + m\gamma) = 0$$

$$\boxed{\begin{aligned} 3\alpha + m\beta &= 0 \\ \alpha - \beta + \gamma &= 0 \\ \alpha + m\gamma &= 0 \end{aligned}}$$

Da bi vektori bili linearno nezavisni ovaj sistem mora biti određen. (mora imati samo rešenje $\alpha = \beta = \gamma = 0$)

Sistem će biti određen ako je $D_s \neq 0$.

$$D_s = \begin{vmatrix} 3 & m & 0 \\ 1 & -1 & 1 \\ 1 & 0 & m \end{vmatrix} \begin{matrix} 3 & m \\ 1 & -1 \\ 1 & 0 \end{matrix} =$$

$$= -3m + m - m^2 = -m^2 - 2m$$

$$= -m(m+2)$$

Za $m \neq 0 \wedge m \neq -2$ sistem je određen, vektori a_1, a_2, a_3 su linearno nezavisni pa čine bazu za $\mathbb{R}^3$.

4.2 Za $m = 2$ napisati vektor $b = (4, 6, 8)$ kao linearnu kombinaciju vektora a_1, a_2 i a_3.

$$b = \alpha a_1 + \beta a_2 + \gamma a_3$$

$$(4,6,8) = \alpha(3,1,1) + \beta(2,-1,0) + \gamma(0,1,2)$$

$$(4,6,8) = (3\alpha + 2\beta,\ \alpha - \beta + \gamma,\ \alpha + 2\gamma)$$

$$\begin{aligned} 3\alpha + 2\beta \phantom{{}+2\gamma} &= 4 \\ \alpha - \beta + \gamma &= 6 \\ \alpha \phantom{{}-\beta} + 2\gamma &= 8 \end{aligned}$$

$$\begin{aligned} \alpha - \beta + \gamma &= 6 \\ 3\alpha + 2\beta \phantom{{}+2\gamma} &= 4 \quad \cdot(-3) \\ \alpha \phantom{{}-\beta} + 2\gamma &= 8 \quad \cdot(-1) \end{aligned}$$

$$\begin{aligned} \alpha - \beta + \gamma &= 6 \\ 5\beta - 3\gamma &= -14 \\ \beta + \gamma &= 2 \end{aligned}$$

$$\begin{aligned} \alpha - \beta + \gamma &= 6 \\ \beta + \gamma &= 2 \quad \cdot(-5) \\ 5\beta - 3\gamma &= -14 \end{aligned}$$

$$\begin{aligned} \alpha - \beta + \gamma &= 6 \\ \beta + \gamma &= 2 \\ -8\gamma &= -24 \end{aligned}$$

$$\gamma = 3$$
$$\beta = -1$$
$$\alpha = 2$$

$$\boxed{b = 2a_1 - a_2 + 3a_3}$$

5. Dokazati da vektori $a = (2, 0, 0, 0)$, $b = (0, -1, 2, 0)$, $c = (0, 0, -3, 0)$, $d = (-1, 0, 0, 1)$ čine bazu vektorskog prostora $\boldsymbol{R}^4$, a zatim napisati vektor $v = (1, 2, -1, 3)$ kao linearnu kombinaciju vektora a, b, c i d.

$\dim(\mathbb{R}^4) = 4$

$\{a, b, c, d\}$ - četiri vektora čine bazu V.P. dimenzije 4 ako su lin. nezavisni.

$\alpha a + \beta b + \gamma c + \delta d = 0$

$\alpha(2,0,0,0) + \beta(0,-1,2,0) + \gamma(0,0,-3,0) + \delta(-1,0,0,1) = 0$

$(2\alpha - \delta, -\beta, 2\beta - 3\gamma, \delta) = 0$

$$\begin{aligned} 2\alpha - \delta &= 0 \\ -\beta &= 0 \\ 2\beta - 3\gamma &= 0 \\ \delta &= 0 \end{aligned}$$

$\beta = 0 \quad \delta = 0$

$\gamma = 0 \quad \alpha = 0$

$\Rightarrow \{a, b, c, d\}$ je skup lin. nez. vektora pa čini bazu

$v = \alpha a + \beta b + \gamma c + \delta d$

$(1, 2, -1, 3) = (2\alpha - \delta, -\beta, 2\beta - 3\gamma, \delta)$

$$\begin{aligned} 2\alpha - \delta &= 1 \\ -\beta &= 2 \\ 2\beta - 3\gamma &= -1 \\ \delta &= 3 \end{aligned}$$

$\boxed{\beta = -2}$ $\boxed{\delta = 3}$

$\boxed{\gamma = -1}$ $\boxed{\alpha = 2}$

$v = 2a - 2b - c + 3d$

6. Skup vektora $A = \{x, y, u, v\}$ čini bazu vektorskog prostora $\boldsymbol{R}^4$. Da li je skup vektora $B = \{x + u, 2y + v, x + u - v, y - 3u\}$ baza tog prostora?

Skup B ima 4 vektora, i oni će biti baza za $\mathbb{R}^4$ ako su linearno nezavisni:

$$\boxed{\alpha(x+u)+\beta(2y+v)+\gamma(x+u-v)+\delta(y-3u)=0}$$

$$\alpha x+\alpha u+2\beta y+\beta v+\gamma x+\gamma u-\gamma v+\delta y-3\delta u=0$$

$$\boxed{(\alpha+\gamma)x+(2\beta+\delta)y+(\alpha+\gamma-3\delta)u+(\beta-\gamma)v=0}$$

Kako je $A=\{x,y,u,v\}$ baza to su vektori x,y,u,v linearno nezavisni, pa iz linearne kombinacije zaključujemo

$$\begin{aligned} \alpha+\gamma&=0\\ 2\beta+\delta&=0\\ \alpha+\gamma-3\delta&=0 \quad \cdot(-1)\\ \beta-\gamma&=0 \end{aligned}$$

$$\begin{aligned} \alpha+\gamma&=0\\ 2\beta+\delta&=0\\ -3\delta&=0\\ \beta-\gamma&=0 \end{aligned}$$

$\boxed{\beta=0}$, $\boxed{\delta=0}$, $\boxed{\gamma=0}$, $\boxed{\alpha=0}$

$\alpha=\beta=\gamma=\delta=0$

$\Rightarrow$ vektori skupa B su lin. nezavisni pa čine bazu u $\mathbb{R}^4$.

7. Vektorski prostor V generisan je vektorima $v_1 = (a, 1, 1)$, $v_2 = \left(-a, a, -a^2\right)$ i $v_3 = \left(a^3, -a, 1\right)$. Naći njegovu dimenziju i bazu u zavisnosti od realnog parametra a.

$$M = \begin{bmatrix} a & -a & a^3 \\ 1 & a & -a \\ 1 & -a^2 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & a & -a \\ 1 & -a^2 & 1 \\ a & -a & a^3 \end{bmatrix} \begin{matrix} \cdot(-1) \\ \\ \cdot(-a) \end{matrix} \sim \begin{bmatrix} 1 & a & -a \\ 0 & -a^2-a & 1+a \\ 0 & -a^2-a & a^2+a^3 \end{bmatrix} \begin{matrix} \\ \\ \cdot(-1) \end{matrix} \sim$$

$$\sim \begin{bmatrix} 1 & a & -a \\ 0 & -a^2-a & 1+a \\ 0 & 0 & a^3+a^2-a-1 \end{bmatrix} = \begin{bmatrix} 1 & a & -a \\ 0 & -a(a+1) & a+1 \\ 0 & 0 & (a+1)^2(a-1) \end{bmatrix}$$

$$a^3+a^2-a-1 = a^2(a+1)-(a+1) = (a+1)(a^2-1) = (a+1)^2(a-1)$$

$\underline{\text{I}}$ za $a \neq 0 \wedge a \neq -1 \wedge a \neq 1 \Rightarrow \text{rang}(M) = 3 \Rightarrow \dim(V) = 3 \Rightarrow \{v_1, v_2, v_3\}$ je baza (jer su v_1, v_2, v_3 3 generatorna vektora u trodimenzionalnom V.P.)

$\underline{\text{II}}$ za $a = 0 \Rightarrow M = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & -1 \end{bmatrix} \begin{matrix} \\ \\ + \end{matrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \text{rang}(M) = 2 \Rightarrow \dim(V) = 2$

$v_1 = (0,1,1)$, $v_2 = (0,0,0)$, $v_3 = (0,0,1) \Rightarrow$ baza je $\{v_1, v_3\}$ jer nula vektor ne može nikad biti u bazi pošto je svaki skup koji sadrži nula vektor linearno zavisan.

$\underline{\text{III}}$ za $a = -1 \Rightarrow M = \begin{bmatrix} 1 & -1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \text{rang}(M) = 1 \Rightarrow \dim(V) = 1$

$v_1 = (-1,1,1)$
$v_2 = (1,-1,-1)$
$v_3 = (-1,1,1)$

$\{v_1\}$-baza, $\{v_2\}$-baza, $\{v_3\}$-baza

$\underline{\overline{IV}}$ ze $a=1$ $M = \begin{bmatrix} 1 & 1 & -1 \\ 0 & -2 & 2 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow$ rang $(M) = 2 \Rightarrow$ dim $(V) = 2$

$v_1 = (1,1,1)$, $v_2 = (-1,1,-1)$, $v_3 = (1,-1,1)$

$\{v_1, v_2\}$ - baza

$\{v_1, v_3\}$ - baza

$\{v_2, v_3\}$ - nisu baze jer nisu lin. nez $v_2 = -v_3$

8. U zavisnosti od realnog parametra a odrediti bazu i dimenziju prostora S generisanog vektorima $v_1 = (a, a, a, a)$, $v_2 = (a, 2, 2, 2)$, $v_3 = (a, 2, a, a)$ i $v_4 = (a, 2, a, 3)$.

$$M = \begin{bmatrix} a & a & a & a \\ a & 2 & 2 & 2 \\ a & 2 & a & a \\ a & 2 & a & 3 \end{bmatrix} \sim \begin{bmatrix} a & a & a & a \\ 0 & 2-a & 2-a & 2-a \\ 0 & 2-a & 0 & 0 \\ 0 & 2-a & 0 & 3-a \end{bmatrix} \sim \begin{bmatrix} a & a & a & a \\ 0 & 2-a & 2-a & 2-a \\ 0 & 0 & a-2 & a-2 \\ 0 & 0 & a-2 & 1 \end{bmatrix}$$

$$\sim \begin{bmatrix} a & a & a & a \\ 0 & 2-a & 2-a & 2-a \\ 0 & 0 & a-2 & a-2 \\ 0 & 0 & 0 & 3-a \end{bmatrix}$$

I za $a \neq 0 \wedge a \neq 2 \wedge a \neq 3 \Rightarrow \operatorname{rang}(M) = 4 \Rightarrow \dim(S) = 4$

$\Rightarrow \{v_1, v_2, v_3, v_4\}$ – baza

II za $a = 0 \Rightarrow M = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 2 & 2 & 2 \\ 0 & 0 & -2 & -2 \\ 0 & 0 & 0 & 3 \end{bmatrix} \Rightarrow \operatorname{rang}(M) = 3 \Rightarrow \dim(S) = 3$

$\Rightarrow \{v_2, v_3, v_4\}$ – baza $\quad v_1 = (0, 0, 0, 0)$ pa ne može biti u bazi

II za $a=2$ $\Rightarrow M = \begin{bmatrix} 2 & 2 & 2 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \Rightarrow rang(M)=2 \Rightarrow dim(V)=2$

$v_1=(2,2,2,2)$, $v_2=(2,2,2,2)$, $v_3=(2,2,2,2)$, $v_4=(2,2,2,3)$

$\{v_4, v_1\}$, $\{v_4, v_2\}$, $\{v_4, v_3\}$ - baze

III za $a=3$

$M = \begin{bmatrix} 3 & 3 & 3 & 3 \\ 0 & -1 & -1 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow rang(M)=3 \Rightarrow dim(S)=3$

$v_1=(3,3,3,3)$, $v_2=(3,2,2,2)$, $v_3=(3,2,3,3)$, $v_4=(3,2,3,3)$

$\{v_1, v_2, v_3\}$, $\{v_1, v_2, v_4\}$ - baze

$(\mathbb{R}^3, +, \cdot, \mathbb{R})$

$(U, +, \cdot, \mathbb{R})$

$(0,0,0)$

9. Ispitati koji od sledećih podskupova $U \subseteq \mathbb{R}^3$ čine (nosače) potprostor prostora $\boldsymbol{R}^3$ i za one koji to jesu odrediti njihovu dimenziju:
   Napomena: Koristiki sledeće poznate činjenice:
   - Nula vektor vrktorskog prostora mora biti i nula vektor svakog njegovog potprostora. Kako je $0 = (0,0,0)$ nula vektor u $\boldsymbol{R}^3$ svi poskupovi koji ne sadrže vektor $0 = (0,0,0)$ ne mogu biti potprostori od $\boldsymbol{R}^3$.
   - Da bi U bio potprostor od $\boldsymbol{R}^3$ mora važiti $\forall \alpha, \beta \in \mathbb{R}, \forall u, v \in U, \alpha u + \beta v \in U$ (ili posebno $\alpha u \in U$ i $u + v \in U$).
   - Ako je skup U zadat sistemom jednačina čije su promenljive komponente elemenata skupa U, tada je $\boldsymbol{U}$ potprostor prostora $\boldsymbol{R}^3$ akko je taj sistem, sistem linearnih homogenih jednačina. U tom slučaju je je dimenzija potprostora $\boldsymbol{U}$ jednaka stepenu neodređenosti sistema jednačina.

9.1 $\mathcal{U} = \{(x, y, z) \in \mathbb{R}^3 \mid x = y = z\}$

- $(0,0,0) \in \mathcal{U}$ ✓ $\Rightarrow$ U može ali ne mora biti potprostor od $\mathbb{R}^3$
- $u, v \in \mathcal{U} \Rightarrow$ $u = (a, a, a)$ $v = (b, b, b)$

$\alpha, \beta \in \mathbb{R}$

$\alpha u = \alpha(a, a, a) = (\alpha a, \alpha a, \alpha a) \in \mathcal{U}$

$u + v = (a, a, a) + (b, b, b) = (a+b, a+b, a+b) \in \mathcal{U}$

$\Longrightarrow$ U jeste potprostor od $\mathbb{R}^3$

- $x = y = z$ je ekvivalentno sa sistemom

$x - y = 0$

$y - z = 0$

Ovo je homogen sistem pa U jeste potprostor od $\mathbb{R}^3$

9.2 $U = \{(x,y,z) \in \mathbb{R}^3 \mid \underline{x}^2 = \underline{y}^2\}$

- $(0,0,0) \in U$ ✓
- $u = (1,1,0)$, $v = (-1,1,0)$ $\Big\}$ $u+v = (1,1,0)+(-1,1,0) = (0,2,0) \notin U$
  $0^2 \neq 2^2$

$\Rightarrow U$ nije potprostor od $\mathbb{R}^3$

- $x^2 = y^2$ nije ekvivalenten so linearnim sistemom

$\Rightarrow U$ nije potprostor od $\mathbb{R}^3$

9.3 $U = \{(x, y, z) \in \mathbb{R}^3 \mid \underline{x = 1}\}$

$(\underline{1}, y, z)$

- $(0,0,0) \notin U$ pa U nije potprostor od $\mathbb{R}^3$

9.4 $U = \{(x, y, z) \in \mathbb{R}^3 \mid x + y + z = 0\}$

$x + y + z = 0$ — linearna homogena jednadžba

$\Rightarrow$ U jeste potprostor

9.5 $U = \{(x, y, z) \in \mathbb{R}^3 \mid x + y + z = 1\}$

$(0,0,0) \notin U$ jer $0+0+0 \neq 1$

$\Rightarrow$ U nije potprostor od $\mathbb{R}^3$

9.6 $U = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 0\} = \{(0, 0, 0)\}$

$\Rightarrow$ U jeste trivijalan potprostor od $\mathbb{R}^3$

9.7 $U = \{(x, y, z) \in \mathbb{R}^3 \mid x + y = 0,\ y = 0\}$

$$x + y = 0$$
$$y = 0$$

linearan homogen sustav

$\Rightarrow U$ jeste potprostor od $\mathbb{R}^3$

9.8 $U = \{(x, y, z) \in \mathbb{R}^3 \mid x + y = 0,\ xy = 0\}$

$$\begin{array}{l} x+y=0 \\ x\cdot y=0 \end{array} \quad \left\langle \begin{array}{l} x=0\ ,\ y=0 \\ y=0\ ,\ x=0 \end{array} \right.$$

$$U = \{(0,0,z) \mid z \in \mathbb{R}\}$$

- $\left.\begin{array}{l} u=(0,0,a) \\ v=(0,0,b) \end{array}\right\} \quad \begin{array}{l} u+v=(0,0,a+b) \in U \\ \alpha u = (0,0,\alpha a) \in U \end{array}$

$\rightarrow U$ jeste potprostor od $\mathbb{R}^3$

9.9 $U = \{(x, y, z) \in \mathbb{R}^3 \mid x + y + \underline{z^2} = 0\}$

$(0,0,0) \in U$ ✓

$u = (0, -1, 1) \in U \qquad 0 - 1 + 1^2 = 0$

$v = (0, -1, -1) \in U \qquad 0 - 1 + (-1)^2 = 0$

$u + v = (0, -2, 0) \notin U$

$0 - 2 + 0^2 \neq 0$

$\rightarrow$ U nije potprostor od $\mathbb{R}^3$

9.10 $U = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + z^2 = 0\}$

$$x^2 + z^2 = 0 \iff x = z = 0$$

$$U = \{(0, y, 0) \mid y \in \mathbb{R}\}$$

$$\left.\begin{aligned} u &= (0, a, 0) \in U \\ v &= (0, b, 0) \in U \end{aligned}\right\} \begin{aligned} u + v &= (0, a+b, 0) \in U \\ \alpha u &= (0, \alpha a, 0) \in U \end{aligned}$$

10. Za skup $\mathcal{R}_S$ rešenja homogenog sistema S linearnih jednačina dokazati da je potprostor vektorskog prostora $\boldsymbol{R}^4$ i odrediti jednu njegovu bazu

$$S: \begin{array}{rcrcrcrcl} x & + & 2y & + & z & - & 2u & = & 0 \\ -2x & - & 5y & + & z & + & 4u & = & 0 \\ x & - & 3y & + & 16z & - & 2u & = & 0 \end{array} \quad \cdot 2, \ \cdot(-1)$$

$$\begin{aligned} x+2y+z-2u &= 0 \\ -y+3z &= 0 \quad \cdot(-5) \\ -5y+15z &= 0 \end{aligned}$$

$$\begin{aligned} x+2y+z-2u &= 0 \\ -y+3z &= 0 \end{aligned}$$

$$\begin{aligned} x+2y &= 2u-z \\ y &= 3z \end{aligned}$$

$z=t,\ t\in\mathbb{R}$

$u=m,\ m\in\mathbb{R}$

$y=3t$

$x=2m-t-6t=2m-7t$

$$\begin{aligned} R_S &= \{(2m-7t,\ 3t,\ t,\ m) \mid m,t\in\mathbb{R}\} \\ &= \{(2m,\ 0,\ 0,\ m)+(-7t,\ 3t,\ t,\ 0) \mid m,t\in\mathbb{R}\} \\ &= \{m(2,0,0,1)+t(-7,3,1,0) \mid m,t\in\mathbb{R}\} \\ &= L((2,0,0,1),(-7,3,1,0)) \end{aligned}$$

$\leftarrow$ skup svih linearnih kombinacija ova dva vektora

Lineal je uvek potprostor odgovarajućeg V.P.

$\Rightarrow R_S$ jeste potprostor od $\mathbb{R}^4$

Kako su vektori $(2,0,0,1)$ i $(-7,3,1,0)$ linearno nezavisni oni će činiti bazu tog potprostora.

Za vežbu

11. Za skup $\mathcal{R}_S$ rešenja homogenog sistema S linearnih jednačina dokazati da je potprostor vektorskog prostora $\boldsymbol{R}^3$ i odrediti jednu njegovu bazu

$$S: \begin{array}{rcrcrcl} x & + & 2y & - & 3z & = & 0 \\ 3x & + & 4y & - & 2z & = & 0 \\ -5x & - & 2y & - & 13z & = & 0 \end{array}.$$

$R_S = \{ \quad \}$

$= L(\quad , \quad)$

ZA VEŽBU IZ SKRIPTE
Zadatak 11.2, 11.3, 11.4, 11.5, 11.6, 11.7, 11.8, 11.9, 11.10, 11.11, 11.12, 11.13, 11.14, 11.15a, 11.16a, 11.17, 11.18