7. Za sledeće funkcije ispitati, odnosno diskutovati po parametrima da li su linearne transformacije i u slučajevima kada jesu naći njihovu matricu, jezgro, sliku i rang.

7.1 $f : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$, $f(\vec{v}) = \dfrac{\vec{n} \times \vec{v}}{|\vec{n}|}$, $\vec{v} \in \mathbb{R}^3$, $\vec{n} = (-1, 1, 2)$;

$\boxed{\vec{v} = (x, y, z)}$

$$f(x,y,z) = \frac{(-1,1,2)\times(x,y,z)}{|(-1,1,2)|} = \frac{1}{\sqrt{1+1+4}} \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -1 & 1 & 2 \\ x & y & z \end{vmatrix} =$$

$$= \frac{1}{\sqrt{6}} \left(\vec{i} \begin{vmatrix} 1 & 2 \\ y & z \end{vmatrix} - \vec{j} \begin{vmatrix} -1 & 2 \\ x & z \end{vmatrix} + \vec{k} \begin{vmatrix} -1 & 1 \\ x & y \end{vmatrix} \right) = \frac{1}{\sqrt{6}} \left((z-2y)\vec{i} - (-z-2x)\vec{j} + (-y-x)\vec{k} \right)$$

$$= \frac{1}{\sqrt{6}} (z-2y,\ z+2x,\ -x-y)$$ $\leftarrow$ f jeste lin. tr. jer su sve komponente slike oblika $t_1x + t_2y + t_3z$, $t_1, t_2, t_3 \in \mathbb{R}$

$$M_f = \frac{1}{\sqrt{6}} \begin{bmatrix} 0 & -2 & 1 \\ 2 & 0 & 1 \\ -1 & -1 & 0 \end{bmatrix} \sim \begin{bmatrix} -1 & -1 & 0 \\ 2 & 0 & 1 \\ 0 & -2 & 1 \end{bmatrix} \overset{2}{\sim} \begin{bmatrix} -1 & -1 & 0 \\ 0 & -2 & 1 \\ 0 & -2 & 1 \end{bmatrix} \overset{-1}{\sim} \begin{bmatrix} -1 & -1 & 0 \\ 0 & -2 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$\Rightarrow \text{rang}(M_f) = 2 \rightarrow \boxed{\text{rang}(f) = 2}$

$f: \mathbb{R}^3 \to \mathbb{R}^3$, $\quad f(x,y,z) = \frac{1}{\sqrt{6}}(-2y+z,\ 2x+z,\ -x-y)$

JEZGRO: $\ker(f) = \{(x,y,z) \in \mathbb{R}^3 \mid f(x,y,z) = 0\}$

$f(x,y,z) = 0$

$\cancel{\frac{1}{\sqrt{6}}}(-2y+z,\ 2x+z,\ -x-y) = 0$

$$\begin{aligned} -2y+z &= 0 \\ 2x \quad +z &= 0 \\ -x-y \quad &= 0 \end{aligned}$$

$$\begin{aligned} -x-y \quad &= 0 \\ 2x \quad +z &= 0 \ \big/ \cdot 2 \\ -2y+z &= 0 \end{aligned}$$

$$\begin{aligned} -x-y \quad &= 0 \\ -2y+z &= 0 \\ \cancel{-2y+z} &= \cancel{0} \end{aligned}$$

$$\begin{aligned} -x &= y \\ z &= 2y \end{aligned}$$

1x neodređen

$y = t,\ t \in \mathbb{R}$

$z = 2t$

$x = -t$

$\ker(f) = \{(-t, t, 2t) \mid t \in \mathbb{R}\}$

$= \{t(-1,1,2) \mid t \in \mathbb{R}\}$

$\ker(f)$ je potprostor v.p. $\mathbb{R}^3$, i jedna njegova baza je

$\{(-1,1,2)\}$

SLIKA: $\operatorname{Im}(f) = \{(a,b,c) \in \mathbb{R}^3 \mid \exists (x,y,z) \in \mathbb{R}^3,\ f(x,y,z) = (a,b,c)\}$

$f(x,y,z) = (a,b,c)$

$\frac{1}{\sqrt{6}}(-2y+z,\ 2x+z,\ -x-y) = (a,b,c)$

$(-2y+z,\ 2x+z,\ -x-y) = \sqrt{6}(a,b,c)$

$$\begin{aligned} -2y+z &= \sqrt{6}a \\ 2x \quad +z &= \sqrt{6}b \\ -x-y \quad &= \sqrt{6}c \end{aligned}$$

$$\begin{aligned} -x-y \quad &= \sqrt{6}c \ \big/ \cdot 2 \\ 2x \quad +z &= \sqrt{6}b \\ -2y+z &= \sqrt{6}a \end{aligned}$$

$$\begin{aligned} -x-y \quad &= \sqrt{6}c \\ -2y+z &= \sqrt{6}b + 2\sqrt{6}c \\ -2y+z &= \sqrt{6}a \ \big/ \cdot (-1) \end{aligned}$$

$$\begin{aligned} -x-y \quad &= \sqrt{6}c \\ -2y+z &= \sqrt{6}b + 2\sqrt{6}c \\ 0 &= \sqrt{6}a - \sqrt{6}b - 2\sqrt{6}c \end{aligned}$$

$\sqrt{6}a = \sqrt{6}b + 2\sqrt{6}c \ \big/ : \sqrt{6}$

$\boxed{a = b + 2c}$

$\operatorname{Im}(f) = \{(b+2c, b, c) \mid b,c \in \mathbb{R}\}$

$= \{(1,1,0)b + (2,0,1)c \mid b,c \in \mathbb{R}\}$

$\operatorname{Im}(f)$ je potprostor v.p. $\mathbb{R}^3$, i jedna njegova baza je $\{(1,1,0), (2,0,1)\}$

$\dim(\operatorname{Im}(f)) = 2 = \operatorname{rang}(f)$

7.2 $g: \mathbb{R}^3 \longrightarrow \mathbb{R}^3$, $g(\vec{v}) = (\vec{n}\cdot\vec{v})\cdot\vec{m}$, $\vec{v}\in\mathbb{R}^3$, $\vec{n}=(1,1,q)$, $q\in\mathbb{R}$
$\vec{m}=(0,1,0)$;

$\boxed{\vec{v}=(x,y,z)}$

$$g(x,y,z) = (\underbrace{(1,1,q)\cdot(x,y,z)}_{\text{skalarni produkt}})\cdot(0,1,0) = (\overbrace{x+y+qz}^{\text{skalar, broj}})(0,1,0) = (0,\ x+y+qz,\ 0)$$

g je lin. transformacija za vsako $q\in\mathbb{R}$ ker je vsaka komponenta slike oblike $t_1x+t_2y+t_3z$, $t_1,t_2,t_3\in\mathbb{R}$

$$M_g = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 1 & q \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \text{rang}(M_g)=1 \Rightarrow \text{rang}(g)=1$$

JEDRO: $\ker(g) = \{(x,y,z)\in\mathbb{R}^3 \mid g(x,y,z)=0\}$

$g(x,y,z)=0$

$(0,\ x+y+qz,\ 0)=0$

$x+y+qz=0$

$\boxed{x=-y-qz}$

2x neodreden

$z=t$, $t\in\mathbb{R}$

$y=m$, $m\in\mathbb{R}$

$x=-m-qt$

$\ker(g) = \{(-m-qt,\ m,\ t) \mid m,t\in\mathbb{R}\}$

$= \{(-1,1,0)m + (-q,0,1)t \mid m,t\in\mathbb{R}\}$

$\ker(g)$ je podprostor v.p. $\mathbb{R}^3$, jedna baza mu je $\{(-1,1,0),(-q,0,1)\}$

SLIKA:

$\text{Im}(g) = \{(a,b,c)\in\mathbb{R}^3 \mid \exists(x,y,z)\in\mathbb{R}^3,\ g(x,y,z)=(a,b,c)\}$

$g(x,y,z)=(a,b,c)$

$(0,\ x+y+qz,\ 0) = (a,b,c)$

$\boxed{a=0}$

$b = x+y+qz$

$\boxed{c=0}$

$\text{Im}(g) = \{(0,b,0) \mid b\in\mathbb{R}\} = \{b(0,1,0) \mid b\in\mathbb{R}\}$

$\text{Im}(g)$ je podprostor v.p. $\mathbb{R}^3$, jedna baza pa je $\{(0,1,0)\}$, $\dim(\text{Im}(g)) = 1 = \text{rang}(g)$

7.3 $h : \mathbb{R}^3 \longrightarrow \mathbb{R}^2$, $h(\vec{v}) = (\vec{n} \times \vec{v}) \cdot \vec{n} \cdot \vec{m} + 2(\vec{v} \cdot \vec{n}) \cdot \vec{j}$, $\vec{v} \in \mathbb{R}^3$,
$\vec{n} = (1, 1, 2)$, $\vec{m} = (0, p)$, $\vec{j} = (0, 1)$.

$\vec{v} = (x, y, z)$

$$h(x,y,z) = \underbrace{((1,1,2)\times(x,y,z))}_{\text{vektorski proizvod}}\cdot(1,1,2)\cdot(0,p) + 2\,\overbrace{((x,y,z)\cdot(1,1,2))}^{\text{skalarni proizvod}}\cdot(0,1)$$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 1 & 2 \\ x & y & z \end{vmatrix}\cdot(1,1,2)\cdot(0,p) + \overbrace{2(x+y+2z)}^{\text{broj}}\cdot(0,1)$$

$$= \left(\vec{i}\begin{vmatrix} 1 & 2 \\ y & z \end{vmatrix} - \vec{j}\begin{vmatrix} 1 & 2 \\ x & z \end{vmatrix} + \vec{k}\begin{vmatrix} 1 & 1 \\ x & y \end{vmatrix}\right)\cdot(1,1,2)\cdot(0,p) + \overbrace{(2x+2y+4z)}^{\text{broj}}\cdot(0,1)$$

$$= \left((z-2y)\vec{i} - (z-2x)\vec{j} + (y-x)\vec{k}\right)\cdot(1,1,2)\cdot(0,p) + (0, 2x+2y+4z)$$

$$= \underbrace{(-2y+z,\ 2x-z,\ -x+y)\cdot(1,1,2)}_{\text{skalarni proizvod}}\cdot(0,p) + (0, 2x+2y+4z)$$

$$= (\cancel{-2y}+\cancel{z}+\cancel{2x}-\cancel{z}-\cancel{2x}+\cancel{2y})\cdot(0,p) + (0, 2x+2y+4z)$$

$$= 0\cdot(0,p) + (0, 2x+2y+4z) = (0, 2x+2y+4z)$$

h jeste lin. fr. jer su sve komponente oblika $t_1 x + t_2 y + t_3 z$, $t_1, t_2, t_3 \in \mathbb{R}$

$$M_h = \begin{bmatrix} 0 & 0 & 0 \\ 2 & 2 & 4 \end{bmatrix} \Rightarrow \mathrm{rang}(M_h) = 1 \Rightarrow \underline{\mathrm{rang}(h) = 1}$$

$h: \mathbb{R}^3 \to \mathbb{R}^2$, $h(x,y,z) = (0, 2x+2y+4z)$

JEZGRO $\ker(h) = \{(x,y,z) \in \mathbb{R}^3 \mid h(x,y,z) = 0\}$

$h(x,y,z) = 0$

$(0, 2x+2y+4z) = 0$

$\underline{2x+2y+4z=0}$

$x = -y-2z$

2x neodređen

$z = t, t \in \mathbb{R}$

$y = m, m \in \mathbb{R}$

$x = -m-2t$

$\ker(h) = \{(-m-2t, m, t) \mid t, m \in \mathbb{R}\}$

$= \{(-1,1,0)m + (-2,0,1)t \mid m, t \in \mathbb{R}\}$

$\ker(h)$ je potprostor V.P. $\mathbb{R}^3$ i jednu bazu mu je $\{(-1,1,0),(-2,0,1)\}$

SLIKA:

$\mathrm{Img}(h) = \{\underline{(a,b)} \in \mathbb{R}^2 \mid \exists (x,y,z) \in \mathbb{R}^3$
$\underline{f(x,y,z) = (a,b)}\}$

$f(x,y,z) = (a,b)$

$(0, 2x+2y+4z) = (a,b)$

$\boxed{a=0}$

$2x+2y+4z = b$

$\mathrm{Img}(h) = \{(0,b) \mid b \in \mathbb{R}\}$

$= \{b(0,1) \mid b \in \mathbb{R}\}$

$\mathrm{Img}(h)$ je potprostor V.P. $\mathbb{R}^2$, i jednu njemu baza je $\{(0,1)\}$

$\dim(\mathrm{Img}(h)) = 1 = \underline{\mathrm{rang}(h)}$

8. Linearna transformacija $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ data je sa $f(1,-1) = (-3,6)$ i $f(-2,1) = (2,-4)$. Odrediti $f(x,y)$ i odgovarajuću matricu M_f linearne transformacije f, a zatim naći njen rang. Da li postoji f^{-1}?

NE POSTOJI f^{-1} jer je $\det(M_f) = 0$

$f(1,-1) = (-3,6)$
$f(-2,1) = (2,-4)$

svako komponenta slike u.tr. mora biti oblika $t_1 x + t_2 y$, $t_1, t_2 \in \mathbb{R}$

$f(x,y) = (ax+by,\ cx+dy)$

$f(1,-1) = (a-b,\ c-d)$
$f(1,-1) = (-3,6)$

$(a-b, c-d) = (-3,6) \Rightarrow$ $a-b=-3$, $c-d=6$

$f(-2,1) = (-2a+b,\ -2c+d)$
$f(-2,1) = (2,-4)$

$(-2a+b, -2c+d) = (2,-4) \Rightarrow$ $-2a+b=2$, $-2c+d=-4$

sistemi

$a-b=-3$
$-2a+b=2$

$-a=-1$

$\boxed{a=1}$ $\boxed{b=4}$

$c-d=6$
$-2c+d=-4$

$-c=2$

$\boxed{c=-2}$ $\boxed{d=-8}$

$\boxed{f(x,y) = (x+4y,\ -2x-8y)}$

$M_f = \begin{bmatrix} 1 & 4 \\ -2 & -8 \end{bmatrix} \underset{2 I_1 + I_2}{\sim} \begin{bmatrix} 1 & 4 \\ 0 & 0 \end{bmatrix}$

$\operatorname{rang}(M_f) = 1 \Rightarrow \operatorname{rang}(f) = 1$

9. Linearna transformacija $f : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ data je sa $f(1,-1,1) = (0,2,1)$, $f(2,0,3) = (-2,13,3)$ i $f(-1,2,0) = (-2,5,-1)$. Odrediti $f(x,y,z)$, a zatim naći $ker(f)$ i $Img(f)$. ZA VEŽBU

PROVERITI !

$f(x,y,z) = (a_1x+b_1y+c_1z,\ a_2x+b_2y+c_2z,\ a_3x+b_3y+c_3z)$

$f(1,-1,1) = (0,2,1)$

$f(1,-1,1) = (a_1-b_1+c_1,\ a_2-b_2+c_2,\ a_3-b_3+c_3)$

$f(2,0,3) = (-2,13,3)$

$f(2,0,3) = (2a_1+3c_1,\ 2a_2+3c_2,\ 2a_3+3c_3)$

$f(-1,2,0) = (-2,5,-1)$

$f(-1,2,0) = (-a_1+2b_1,\ -a_2+2b_2,\ -a_3+2b_3)$

$a_1-b_1+c_1=0$ $\cdot(-2)$
$2a_1+3c_1=-2$
$-a_1+2b_1=-2$ $+$

$a_2-b_2+c_2=2$ $\cdot(-4)$
$2a_2+3c_2=13$
$-a_2+2b_2=5$

$a_3-b_3+c_3=1$ $\cdot(-2)$
$2a_3+3c_3=3$
$-a_3+2b_3=-1$

$a_1-b_1+c_1=0$
$2b_1+c_1=-2$ $\cdot(-2)$
$b_1+c_1=-2$

$a_1-b_1+c_1=0$
$b_1+c_1=-2$
$-c_1=2$

$\boxed{c_1=-2}$
$\boxed{b_1=0}$
$\boxed{a_1=2}$

$a_2-b_2+c_2=2$
$2b_2+c_2=9$ $\cdot(-2)$
$b_2+c_2=7$

$a_2-b_2+c_2=2$
$b_2+c_2=7$
$-c_2=-5$

$\boxed{c_2=5}$
$\boxed{b_2=2}$
$\boxed{a_2=-1}$

$a_3-b_3+c_3=1$
$2b_3+c_3=1$ $\cdot(-2)$
$b_3+c_3=0$

$a_3-b_3+c_3=1$
$b_3+c_3=0$
$-c_3=1$

$\boxed{c_3=-1}$
$\boxed{b_3=1}$
$\boxed{a_3=3}$

$f(x,y,z) = (2x-2z, \ -x+2y+5z, \ 3x+y-z)$

Rezultat za preostali dva zadatka:

$\ker(f) = \{(1,-2,1)t \mid t \in \mathbb{R}\}$

$\operatorname{Im}(f) = \{(-\frac{2}{7},1,0)b + (\frac{4}{7},0,1)c \mid b,c \in \mathbb{R}\}$

10. Linearna transformacija $f : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ data je sa $f(1,-1,0) = (1,0,1)$, $f(1,2,-4) = (0,-1,2)$ i $f(-2,0,3) = (-1,1,0)$. Odrediti $f(x,y,z)$ i odgovarajuću matricu M_f linearne transformacije f, a zatim izračunati $f(-1,3,0)$.

ZA VEŽBU

ve

$$f(x,y,z) = (a_1x + b_1y + c_1z,\ a_2x + b_2y + c_2z,\ a_3x + b_3y + c_3z)$$

Zamenite $f(1,-1,0)$, $f(1,2,-4)$, $f(-2,0,3)$, dobijete tri sistema, rešite, trebalo bi da dobijete

$$f(x,y,z) = (2x + y + z,\ x + y + z,\ 12x + 11y + 8z)$$

$$f(-1,3,0) = (-2+3+0,\ -1+3+0,\ -12+33+0) = (1,2,21)$$

ZA VEŽBU IZ SKRIPTE
Zadatak 12.1; 12.2; 12.7; 12.8 (uzeti da je $g(x, y, z) = (x, y)$)