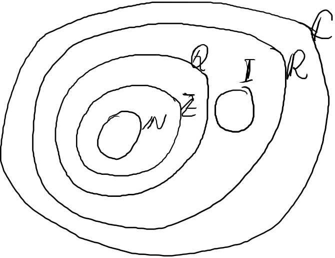


$$\mathbb{Q} = \left\{ \frac{p}{q} \;\middle|\; p \in \mathbb{Z}, q \in \mathbb{N} \right\}$$

Kompleksni brojevi

November 8, 2021

Kako jednačina $x^2 + 1 = 0$ (kao i mnoge druge jednačine) nema rešenje u skupu realnih brojeva (jer je $x^2 \geq 0$ za svako $x \in \mathbb{R}$), potrebno je skup realnih brojeva proširiti tako da u novom skupu data jednačina ima rešenje. Zbog toga, se uvodi pojam **imaginarne jedinice** i koja se definiše kao

$$\boxed{i^2 = -1.}$$

i
$i^2 = -1$
$i^3 = i^2 \cdot i = -i$
$i^4 = i^2 \cdot i^2 = 1$
$i^5 = i^4 \cdot i = i$
$i^6 = -1$
$i^7 = -i$
$i^8 = 1$

Iz ove definicije sledi da za svaki ceo broj k, važi:

$$\boxed{i^{4k} = 1,} \qquad i^{4k+1} = i, \qquad i^{4k+2} = -1, \qquad i^{4k+3} = -i \; .$$

Primer: $i^{17} = i^{4\cdot 4+1} = i, \quad i^{203} = i^{4\cdot 50+3} = -i, \quad i^{2020} = i^{4\cdot 505} = 1.$

$i^{23} = i^{4\cdot 5+3} = i^3 = i^2 \cdot i = -i$

$i^{254} = i^{4\cdot 63+2} = i^2 = -1$

$254 : 4 = 63$
-24
14
-12
2

ALGEBARSKI OBLIK KOMPLEKSNOG BROJA

$\boxed{z = x + yi,\ x, y \in \mathbb{R}}$

$x = Re(z)$

$y = Im(z)$

Kompleksan broj z zapisan **u algebarskom obliku** je broj $z = x + yi$, gde $x, y \in \mathbb{R}$, a i je imaginarna jedinica. Skup svih kompleksnih brojeva označava se sa $\mathbb{C}$, tj.

$$\mathbb{C} = \left\{ x + yi \,|\, x \in \mathbb{R} \wedge y \in \mathbb{R} \wedge i^2 = -1 \right\}.$$

Realni deo kompleksnog broja $z = x + yi$ je $\mathrm{Re}\,(z) = x$, a **imaginarni deo** je $\mathrm{Im}\,(z) = y$.

Primer:

$$\begin{array}{lll}
\mathrm{Re}\,(2+3i) = 2, & \mathrm{Re}\,(-3+7i) = -3, & \mathrm{Re}\,(2-i) = 2, \\
\mathrm{Im}\,(2+3i) = 3, & \mathrm{Im}\,(-3+7i) = 7, & \mathrm{Im}\,(2-i) = -1, \\
\mathrm{Re}\,(5i) = 0, & \mathrm{Re}\,(-3) = -3, & \mathrm{Re}\,(0) = 0, \\
\mathrm{Im}\,(5i) = 5, & \mathrm{Im}\,(-3) = 0, & \mathrm{Im}\,(0) = 0.
\end{array}$$

$-3 \in \mathbb{C}$ ✓✓

Dva kompleksna broja data u algebarskom obliku su **jednaka** akko su im jednaki realni i imaginarni delovi, tj.

$$z_1 = z_2 \iff \mathrm{Re}\,(z_1) = \mathrm{Re}\,(z_2) \land \mathrm{Im}\,(z_1) = \mathrm{Im}\,(z_2)\,.$$

$z = 3+5i$
$\overline{z} = 3-5i$

$z = i+6$
$\overline{z} = -i+6$

Konjugovano kompleksan broj broja $z = x + yi$ je $\overline{z} = x - yi$.

Primer:

$z = 2 + 3i \Longrightarrow \overline{z} = 2 - 3i,$ $\qquad$ $z = -3 + 7i \Longrightarrow \overline{z} = -3 - 7i,$

$z = 2 - i \Longrightarrow \overline{z} = 2 + i,$ $\qquad$ $z = 5i \Longrightarrow \overline{z} = -5i,$

$z = -3 \Longrightarrow \overline{z} = -3,$ $\qquad$ $z = 0 \Longrightarrow \overline{z} = 0.$

Operacije sa kompleksnim brojevima u algebarskom obliku

Neka je $z_1 = x_1 + y_1 i$, a $z_2 = x_2 + y_2 i$. Tada je:

$$z_1 \pm z_2 = (x_1 + y_1 i) \pm (x_2 + y_2 i) = (x_1 \pm x_2) + (y_1 \pm y_2)\, i,$$

$$z_1 \cdot z_2 = (x_1 + y_1 i) \cdot (x_2 + y_2 i) = (x_1 x_2 - y_1 y_2) + (x_1 y_2 + x_2 y_1)\, i,$$

$$\frac{z_1}{z_2} = \frac{x_1 + y_1 i}{x_2 + y_2 i} = \frac{x_1 + y_1 i}{x_2 + y_2 i} \cdot \frac{x_2 - y_2 i}{x_2 - y_2 i} = \frac{x_1 x_2 + y_1 y_2}{x_2^2 + y_2^2} + \frac{x_2 y_1 - x_1 y_2}{x_2^2 + y_2^2}\, i.$$

$(2+5i)(2-5i) =$

$= 2^2 - (5i)^2$

$= 4 - 25 i^2$ $\quad -1$

$= 4 + 25 = 29$

Primer: Neka je $z_1 = 3 + 5i$ i $z_2 = -1 - 3i$. Tada je:

$z_1 + z_2 = 2 + 2i,$

$z_1 - z_2 = 4 + 8i,$

$z_1 \cdot z_2 = (3 + 5i) \cdot (-1 - 3i) = -3 - 9i - 5i - 15i^2 = 12 - 14i,$

$$\frac{z_1}{z_2} = \frac{3 + 5i}{-1 - 3i} \cdot \frac{-1 + 3i}{-1 + 3i} = \frac{-3 + 9i - 5i + 15i^2}{1 + 9} = -\frac{9}{5} + \frac{2}{5} i.$$

Za proizvoljne kompleksane brojeve z i ω važi da je

$$z + \overline{z} = 2\mathrm{Re}\,(z)\,, \qquad z - \overline{z} = 2i\mathrm{Im}\,(z)\,, \qquad \overline{\overline{z}} = z,$$

$$\overline{z \pm \omega} = \overline{z} \pm \overline{\omega}, \qquad \overline{z \cdot \omega} = \overline{z} \cdot \overline{\omega}, \qquad \overline{\left(\frac{z}{\omega}\right)} = \frac{\overline{z}}{\overline{\omega}}.$$

$z = x + yi$

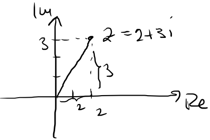
$z + \overline{z} = x + \cancel{yi} + x - \cancel{yi} = 2x = 2\mathrm{Re}(z)$

$z - \overline{z} = x + yi - (x - yi) = 2yi = 2i\,\mathrm{Im}(z)$

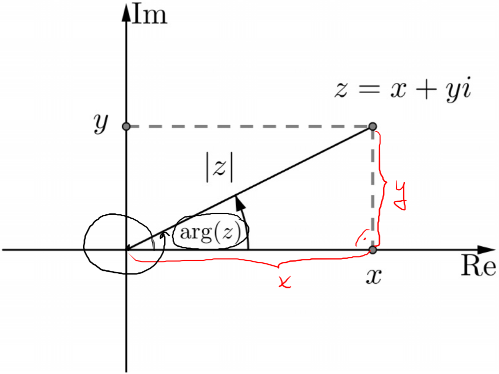
$\overline{\overline{z}} = \overline{x - yi} = x + yi = z$

Kompleksna (Gausova) ravan

$z = 2 + 3i$

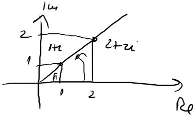


$|z| = \sqrt{2^2+3^2}$
$= \sqrt{4+9}$
$= \sqrt{13}$



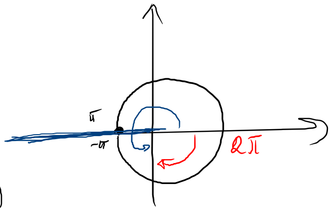
MODUO

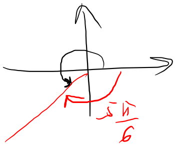
$|z| = \sqrt{x^2+y^2}$



Svaki kompleksan broj se može jednoznačno predstaviti kao tačka u ravni koja se naziva **kompleksna** ili **Gausova ravan**.
Kompleksna ravan je određena **realnom** (Re) i **imaginarnom** (Im) osom koje dele ravan na četiri kvadranta.

$\arg(z) \in (-\pi, \pi]$





$\arg(z) = \cancel{\frac{3\pi}{2}} \quad -\frac{\pi}{2}$

$\frac{7\pi}{6} \quad -\frac{5\pi}{6}$

$\frac{4\pi}{3} \quad -\frac{2\pi}{3}$

$-\frac{8\pi}{5} \quad +\frac{2\pi}{5}$

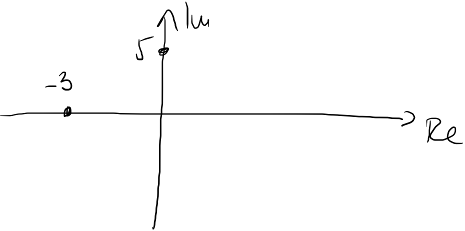
Moduo kompleksnog broja $z = x + yi$ je rastojanje tačke koja odgovara kompleksnom broju z od koordinatnog početka. Obeležava se sa $|z|$ i izračunava kao $|z| = \sqrt{x^2 + y^2}$.

Primer:

$$|2+3i| = \sqrt{2^2+3^2} = \sqrt{13}, \qquad |-3+7i| = \sqrt{(-3)^2+7^2} = \sqrt{58},$$

$$|2-i| = \sqrt{2^2+(-1)^2} = \sqrt{5}, \qquad |5i| = 5, \qquad |-3| = 3.$$

Za proizvoljan kompleksan broj z važi da je $\underline{|z|^2 = z \cdot \overline{z}}$.



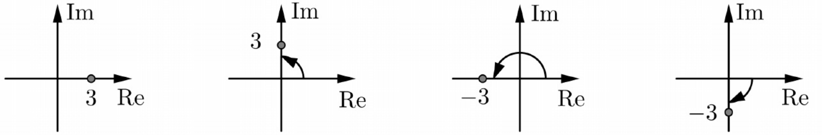
$$\begin{aligned} z \cdot \overline{z} &= (x+yi)(x-yi) \\ &= x^2 - (yi)^2 \\ &= x^2 + y^2 \\ &= |z|^2 \end{aligned}$$

(glavna vrednost argumenta)

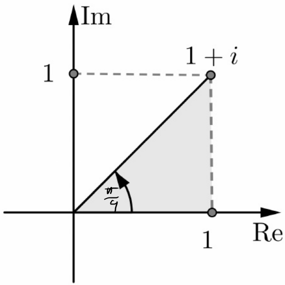
Argument kompleksnog broja z, u oznaci $\arg(z)$, je merni broj orijentisanog ugla čiji je prvi krak pozitivna realna osa, a drugi krak je poluprava $0z$. Argument kompleksnog broja je uvek u intervalu $(-\pi, \pi]$. Argument kompleksnog broja nula se ne definiše.

$\arg z \in (-\pi, \pi]$

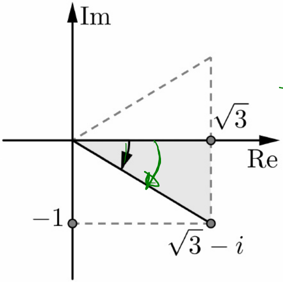
Primer:



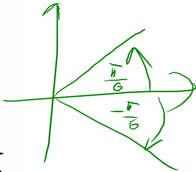
$\arg(3) = 0$ $\quad \arg(3i) = \dfrac{\pi}{2}$ $\quad \arg(-3) = \pi$ $\quad \arg(-3i) = -\dfrac{\pi}{2}$

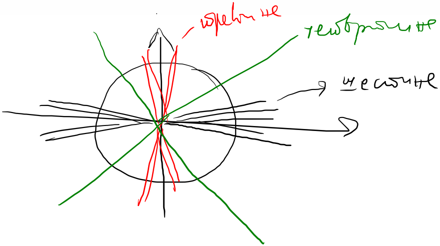


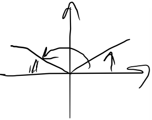
$$\arg(1+i) = \frac{\pi}{4}$$

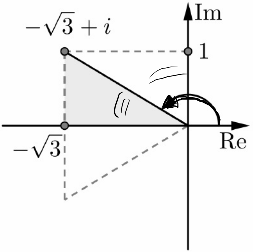


$$\arg(\sqrt{3}-i) = -\frac{\pi}{6}$$

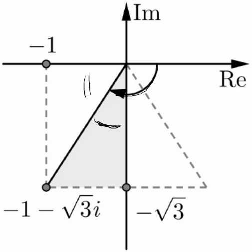




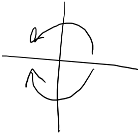




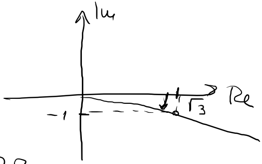
$$\arg(-\sqrt{3}+i) = \frac{5\pi}{6}$$



$$\arg(-1-\sqrt{3}i) = -\frac{2\pi}{3}$$



$$\arg(\sqrt{3}-i) = -\frac{\pi}{6}$$

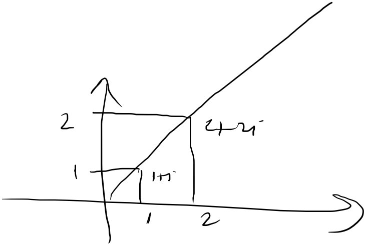


$$\arg(-1+\sqrt{3}i) = \frac{2\pi}{3}$$

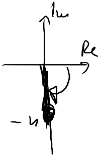
Kompleksni brojevi koji se nalaze na istoj polupravoj čija je početna tačka koordinatni početak imaju iste argumente.

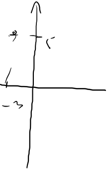
Primer:

$$\arg(5+5i) = \arg\left(\frac{1}{2}+\frac{1}{2}i\right) = \arg(13+13i) = \arg(1+i) = \frac{\pi}{4},$$
$$\arg\left(4\sqrt{3}-4i\right) = \arg\left(\frac{1}{5}\sqrt{3}-\frac{1}{5}i\right) = \arg\left(\sqrt{3}-i\right) = -\frac{\pi}{6},$$
$$\arg\left(-11\sqrt{3}+11i\right) = \arg\left(-\frac{3}{4}\sqrt{3}+\frac{3}{4}i\right) = \arg\left(-\sqrt{3}+i\right) = \frac{5\pi}{6},$$
$$\arg\left(-21-21\sqrt{3}i\right) = \arg\left(-5-5\sqrt{3}i\right) = \arg\left(-1-\sqrt{3}i\right) = -\frac{2\pi}{3}.$$



Primer: Neka je $z_1 = -4i$ i $z_2 = -3 + 5i$. Odrediti: $\mathrm{Re}\,(z_1)$, $\mathrm{Re}\,(z_2)$, $\mathrm{Im}\,(z_1)$, $\mathrm{Im}\,(z_2)$, $\overline{z_1}$, $\overline{z_2}$, $|z_1|$, $|z_2|$, $\arg(z_1)$, $z_1 + z_2$, $z_1 - z_2$, $z_1 \cdot z_2$ i $\dfrac{z_1}{z_2}$.





$z_1 = -4i$

$z_2 = -3+5i$

$\mathrm{Re}(z_1) = 0$

$\mathrm{Re}(z_2) = -3$

$\mathrm{Im}(z_1) = -4$

$\mathrm{Im}(z_2) = 5$

$\overline{z_1} = 4i$

$\overline{z_2} = -3-5i$

$|z_1| = 4 \quad \left(= \sqrt{0+(-4)^2} = \sqrt{16} = 4\right)$

$|z_2| = \sqrt{(-3)^2+5^2} = \sqrt{34}$

$\arg(z_1) = -\frac{\pi}{2}$

$z_1 + z_2 = (-4i) + (-3+5i) = -3 + i$

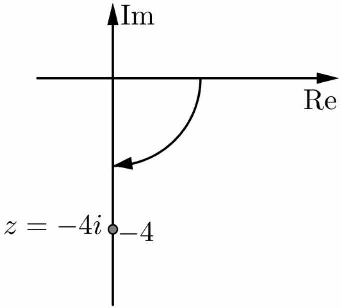
$z_1 - z_2 = -4i - (-3+5i) = -4i + 3 - 5i = 3 - 9i$

$z_1 \cdot z_2 = -4i \cdot (-3+5i) = 12i - 20 \overset{-1}{i^2} = 12i + 20$

$$\frac{z_1}{z_2} = \frac{-4i}{-3+5i} \cdot \frac{-3-5i}{-3-5i} = \frac{12i+20i^2}{9+25} =$$

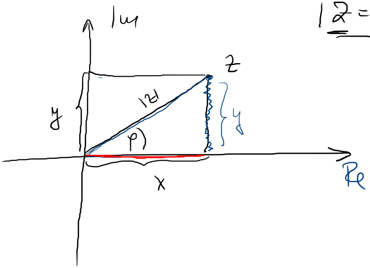
$$= \frac{12i-20}{34} = -\frac{20}{34} + \frac{12}{34}i$$

Rešenje: Iz $z_1 = -4i$ sledi da je
$\mathrm{Re}\,(z_1) = 0$, $\mathrm{Im}\,(z_1) = -4$,
a $\overline{z_1} = 4i$.
Sa slike se vidi da je $|z_1| = 4$ i
$\arg(z_1) = -\dfrac{\pi}{2}$.
Iz $z_2 = -3 + 5i$ sledi da je
$\mathrm{Re}\,(z_2) = -3$, $\mathrm{Im}\,(z_2) = 5$,
$\overline{z_2} = -3 - 5i$,
a $|z_2| = \sqrt{(-3)^2 + 5^2} = \sqrt{34}$.



Dalje je,
$z_1 + z_2 = -4i + (-3 + 5i) = -3 + i$,
$z_1 - z_2 = -4i - (-3 + 5i) = 3 - 9i$,
$z_1 \cdot z_2 = -4i \cdot (-3 + 5i) = 12i - 20i^2 = 20 + 12i$,
$\dfrac{z_1}{z_2} = \dfrac{-4i}{-3 + 5i} \cdot \dfrac{-3 - 5i}{-3 - 5i} = \dfrac{12i - 20}{9 + 25} = -\dfrac{10}{17} + \dfrac{6}{17}i$.

NAUČITI ZA USMENI!



$\boxed{z = x + yi}$

$\varphi = \arg(z)$

$\sin\varphi = \frac{y}{|z|} \Rightarrow y = |z|\sin\varphi$

$\cos\varphi = \frac{x}{|z|} \Rightarrow x = |z|\cos\varphi$

$$\operatorname{tg}\varphi = \frac{\sin\varphi}{\cos\varphi} = \frac{\frac{y}{|z|}}{\frac{x}{|z|}} = \frac{y}{x}$$

$$\Rightarrow \boxed{\operatorname{tg}\varphi = \frac{y}{x}}$$

$$z = x + yi = |z|\cos\varphi + |z|\cdot\sin\varphi\cdot i$$

$$\boxed{z = |z|(\cos\varphi + i\sin\varphi)}$$

TRIGONOMETRIJSKI OBLIK K.B.

$$\boxed{\cos\varphi + i\sin\varphi \overset{\text{DEF}}{=} e^{\varphi i}}$$

$$\boxed{z = |z|\, e^{\varphi i}}$$

EKSPONENCIJALNI OBLIK K.B.

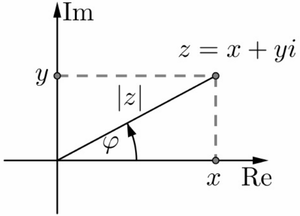
$2 + 2i$

$\operatorname{tg}\varphi = \frac{2}{2} = 1$

$\varphi \in \{\frac{\pi}{4}, -\frac{\pi}{4}\}$

TRIGONOMETRIJSKI I EKSPONENCIJALNI OBLIK KOMPLEKSNOG BROJA

Kada se kompleksan broj $z = x + yi$, $z \neq 0$, predstavi u kompleksnoj ravni, iz pravouglog trougla sa slike vidi se da je:



$$\sin\varphi = \frac{y}{|z|} \quad \Longrightarrow \quad y = |z|\sin\varphi,$$

$$\cos\varphi = \frac{x}{|z|} \quad \Longrightarrow \quad x = |z|\cos\varphi,$$

$$\operatorname{tg}\varphi = \frac{y}{x}, \quad x \neq 0,$$

gde je $|z|$ moduo kompleksnog broja z, a $\varphi \in (-\pi, \pi]$ njegov argument.

Zamenom ovih jednakosti u algebarski oblik kompleksnog broja dobija se **trigonometrijski oblik kompleksnog broja**

$$z = |z|(\cos\varphi + i\sin\varphi).$$

Napomena: Kako su sinus i kosinus periodične funkcije sa periodom $2k\pi$, $k \in \mathbb{Z}$ za ugao φ se može uzeti $\varphi = \arg(z) + 2k\pi$, $k \in \mathbb{Z}$. Uobičajeno je da se uzima glavna vrednost koja je iz intervala $(-\pi, \pi]$, te će u slučaju da vrednost ugla izlazi iz tog intervala biti potrebno transformisati je u njega.

Na osnovu Ojlerove formule $\cos\varphi + i\sin\varphi \stackrel{\text{def}}{=} e^{\varphi i}$, iz trigonometrijskog oblika dobija se **eksponencijalni oblik kompleksnog broja**

$$z = |z|e^{\varphi i}.$$

$$z = 3e^{\frac{5\pi}{4}i} = 3e^{-\frac{3\pi}{4}i}$$