Valuation of Convertible Bonds with Credit Risk

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Introduction

Project Objective:

- Understand and implement pricing of a convertible bond with credit risk:
    - Finite Difference Model
    - Binomial Model

Literature

Finite Difference Model:

- Ho and Pfeffer (1996) and Tsiveriotis and Fernandes (1998) modelled the derivative using geometric Brownian motion
- Ayache et al. (2003) modelled non-total default jump of the stock

Binomial Model:

- Davis and Lischka (2002) first proposed a trinomial tree
- Bardhan et al. (1994), and Hull (2011) have implemented a binomial tree
- Milanov and Kounchev (2012) implemented a binomial model that converges to the stochastic model

Outline of the Talk

Outline

Convertible Bond (1/3)

Definition. (Convertible Bond)

A convertible bond has the following components:

- *Annuity: a series of coupons with a final redemption payment*
- *Put: a put option on the bond for the bond holder*
- *Call: a call option on the bond for the bond issuer*
- *Conversion: an asset swap for stock by the bond holder*

Convertible Bond (2/3)

Definition. (Convertible Bond)

A convertible bond has the following components[a]*:*

- *Annuity: Coupons:* C_i *at* t_i*; Redemption:* R *at* T
- *Put: Strike:* K_t^p *for* $t \in \Omega^p$*; Payoff:* $K_t^p 1_{(V_t \leq K_t^p) \wedge (t \in \Omega^p)}(V_t, t)$
- *Call: Strike:* K_t^c *for* $t \in \Omega^c$*; Payoff:* $K_t^c 1_{(V_t \geq K_t^c) \wedge (t \in \Omega^c)}(V_t, t)$
- *Conversion: Stock:* $\kappa_t S_t$ *for* $t \in \Omega^v$*; Payoff:* $\kappa_t S_t 1_{(V_t \leq \kappa_t S_t) \wedge (t \in \Omega^v)}(V_t, t)$

[a] V_t denoted the intrinsic value of the derivative

Convertible Bond (3/3)

Definition. (Convertible Bond)

A convertible bond has the following components:

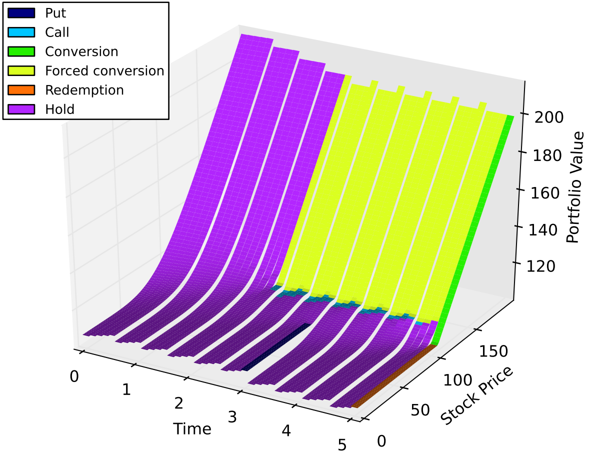
- *Annuity:* $t_i \neq T$
- *Put:* $T \notin \Omega^p$
- *Call:* $T \notin \Omega^v$; $K_t^c > K_t^p$
- *Conversion:* $\kappa > 0$; $T \in \Omega^v$; *supersedes the put and call option*

Payoff and Actions

Action	Payoff	Condition
Put	K_t^p	$(V_t \leq K_t^p) \wedge (t \in \Omega^p) \wedge [(\kappa_t S_t < V_t) \vee (t \notin \Omega^v)]$
Call	K_t^c	$(V_t \geq K_t^c) \wedge (t \in \Omega^c) \wedge [(\kappa_t S_t < K_t^c) \vee (t \notin \Omega^v)]$
Conversion	$\kappa_t S_t$	$(\kappa_t S_t \geq V_t) \wedge (t \in \Omega^v)$
Forced conversion	$\kappa_t S_t$	$(V_t > \kappa_t S_t \geq K_t^c) \wedge (t \in \Omega^v) \wedge (t \in \Omega^c)$
Redemption	R	$(t = T) \wedge [(\kappa_t S_t \leq R) \vee (t \notin \Omega^v)]$
Hold		*otherwise*

Table : Payoff for the convertible bond. V_t is the intrinsic value of the derivative

Payoff and Actions



Credit Risk

Definition. (Convertible Bond on Default)

In default the convertible bond has the following components:

- *Annuity: residual value*[a] *of* γR
- *Conversion: stock price of* $(1-\eta)\kappa_t S_t$

The value (payoff) the convertible bond is:

haircut/ reduction in the stock value in case of default

$$X_t = \max(\underbrace{\gamma R, (1-\eta)\kappa_t S_t}_{\text{recovery rate}}) \tag{1}$$

Default is considered terminal.

[a] This is assuming the bond recovery is based on the redemption value. Another possibility is to base the recovery on the bond value at time t.

Outline

Stock Process with Credit Risk (1/3)

Definition. (Geometric Brownian with Credit Risk)

A stock process with geometric Brownian motion, with a jump component to model the default event, has the following stochastic differential equation:

$$dS_t = \mu S_t dt + \sigma S_t d\tilde{W}_t - \eta S_t dq_t \tag{2}$$

with q_t being a Poisson process with intensity λ where the first event $q_t = 1$ is the default event.

Stock Process with Credit Risk (2/3)

Corollary.

The stochastic process q_t, when representing terminal default, follows an exponential distribution with:

$$q_t \sim \exp(\lambda) \tag{3}$$

$$\tilde{\mathbb{P}}(q_{t+dt} = 1 | q_t = 0) = \lambda dt \tag{4}$$

$$\tilde{\mathbb{E}}[dq_t] = \lambda dt \tag{5}$$

$$\tilde{\mathbb{V}}\mathrm{ar}[dq_t] = \lambda dt \tag{6}$$

Stock Process with Credit Risk (3/3)

Corollary.

Under risk neutral probability space the drift rate of S_t *is* $\mu = (r + \lambda\eta)$ *and the stochastic differential equation is:*

$$dS_t = (r + \lambda\eta)S_t dt + \sigma S_t d\tilde{W}_t - \eta S_t dq_t \tag{7}$$

Partial Differential Equation

Theorem. (Black-Scholes with Default)

The price of an option, on an underlying S_t, has the following partial differential equation:

$$(r+\lambda)V_t = \frac{\partial V}{\partial t} + (r+\lambda\eta)S_t\frac{\partial V}{\partial S} + \tfrac{1}{2}\sigma^2 S_t^2\frac{\partial^2 V}{\partial S^2} + \lambda X_t \tag{8}$$

Derivation

The following steps are used in the derivation for Equation 8:

1. Create a portfolio $\Pi_t = V_t - \Delta_t S_t$
2. Invest the residual of Π_t at the risk free rate: $d\Pi_t = r\Pi_t dt$
3. Equate the expected value of the derivative of Π_t with the risk free rate:

$$r\Pi_t dt = \tilde{\mathbb{E}}[d\Pi_t] = \tilde{\mathbb{P}}(q_{t+dt} = 0)\,\tilde{\mathbb{E}}[d(V_t - \Delta_t S_t)|q_{t+dt} = 0] + \tilde{\mathbb{P}}(q_{t+dt} = 1)\,\tilde{\mathbb{E}}[d(V_t - \Delta_t S_t)|q_{t+dt} = 1] \quad (9)$$

4. Note that the derivative of Π_t when going into default is:

$$\tilde{\mathbb{E}}[d(V_t - \Delta_t S_t)|q_{t+dt} = 1] = X_t - (1-\eta)\Delta_t S_t - (V_t - \Delta_t S_t) \quad (10)$$

5. Choose $\Delta_t = \frac{\partial V}{\partial S}$ to eliminate the $d\tilde{W}_t$ terms

Outline

Binomial Tree (1/2)

Definition. (Binomial Process with Default)

A binomial process with up movement u and down movement d and a terminal default event $(1-\eta)$ has the following mass distribution function:

$$f(S_{t+\delta t}) = \begin{cases} p_u & \text{if } S_{t+\delta t} = uS_t \\ p_d & \text{if } S_{t+\delta t} = dS_t \\ p_o & \text{if } S_{t+\delta t} = (1-\eta)S_t \end{cases} \tag{11}$$

Binomial Tree (2/2)

Corollary.

The expected value and variance of the Binomial Process with Default is:

$$\tilde{\mathbb{E}}\left[\frac{S_{t+\delta t}}{S_t}\right] = up_u + dp_d + (1-\eta)p_o \tag{12}$$

$$\tilde{\mathbb{V}}\mathrm{ar}\left[\frac{S_{t+\delta t}}{S_t}\right] = u^2 p_u + d^2 p_d + (1-\eta)^2 p_o - \tilde{\mathbb{E}}\left[\frac{S_{t+\delta t}}{S_t}\right]^2 \tag{13}$$

Binomial Process (1/2)

Theorem. (Binomial Process with Default)

A Binomial Process with Default has the following parameters:

$$
\begin{aligned}
u &= e^{\sigma\sqrt{\delta t}} \\
d &= e^{-\sigma\sqrt{\delta t}} \\
p_u &= \frac{e^{r\delta t} - de^{-\lambda\delta t} - (1-\eta)(1-e^{-\lambda\delta t})}{u-d} \\
p_d &= e^{-\lambda\delta t} - p_u \\
p_o &= 1 - e^{-\lambda\delta t} \\
V_t &= e^{-r\delta t}(V_t^u p_u + V_t^d p_d + X_t p_o) + c_i
\end{aligned}
$$

Binomial Process (2/2)

Theorem. (Binomial Process with Default)

A Binomial Process with Default has the following restrictions on the parameters:

$$
\begin{aligned}
0 &< \sigma \\
0 &\leq \lambda \\
0 &< \delta t \\
\delta t &\leq \frac{\sigma^2}{r^2} \\
\delta t &\leq \frac{1}{\lambda} \ln \left(\frac{u - (1 - \eta)}{e^{r\delta t} - (1 - \eta)} \right)
\end{aligned}
$$

Derivation (1/3)

The following steps are used in the derivation for the parameters of the Binomial Process with Default.

1. Note that the arrival time of a default event follows an exponential distribution with hazard rate λ:

$$p_o = 1 - e^{-\lambda\delta t} \tag{14}$$

$$p_d = e^{-\lambda\delta t} - p_u \tag{15}$$

2. Equate $\tilde{\mathbb{E}}\left[\frac{S_{t+\delta t}}{S_t}\right]$ with $\tilde{\mathbb{E}}\left[\frac{S_{t+dt}}{S_t}\right] = e^{r\delta t}$, where $dt \approxeq \delta t$
3. Equate $\tilde{\mathbb{V}}\mathtt{ar}\left[\frac{S_{t+\delta t}}{S_t}\right]$ with $\tilde{\mathbb{V}}\mathtt{ar}\left[\frac{S_{t+dt}}{S_t}\right] = (\sigma^2 + \lambda\eta^2)\delta t$, where $dt \approxeq \delta t$
   1. Assume $\delta t^2 = 0$
   2. $ud = 1$
   3. $u = e^{\sqrt{A\delta t}}$
   4. Use Taylor series expansion for all exponential terms

Derivation (2/3)

The following steps are used in the derivation for the valuation of the Binomial Process with Default.

1. Construct a portfolio $\Pi_t = V_t - \Delta_t S_t$ with:

$$\Pi_{t+\delta t} = \begin{cases} V_t^u - \Delta_t S_t u & \text{with probability } p_u \\ V_t^d - \Delta_t S_t d & \text{with probability } p_d \\ X_t - (1-\eta)\Delta_t S_t & \text{with probability } p_o \end{cases} \tag{16}$$

2. Invest the residual of Π_t at the risk free rate: $\Pi_{t+\delta t} = \Pi_t e^{r\delta t}$
3. Equate the expected value of $\Pi_{t+\delta t}$ with the risk free rate:

$$\Pi_t e^{r\delta t} = \tilde{\mathbb{E}}[\Pi_{t+\delta t}] \tag{17}$$

4. Choose $\Delta_t = \frac{V_t^u - V_t^d}{S_t(u-d)}$ to hedge against up and down movements of the stock

Derivation (3/3)

The following steps are used in the derivation for the limits of the Binomial Process with Default.

1. Note that $\{p_u, p_d, p_o\}$ is required to be a valid probability set:

$$\min(p_u, p_d, p_o) \geq 0 \tag{18}$$

$$p_u + p_d + p_o = 1 \tag{19}$$

2. Using the relationship between p_u, p_d and p_o the following inequality needs to hold:

$$0 \leq p_u \leq e^{-\lambda \delta t} \tag{20}$$

3. Impose the requirements that $\lambda \delta t \geq 0$

Outline

Parameters for the Convertible Bond

Consider the following parameters of a Convertible Bond:

Component	Parameter	Value
Annuity	Notional	100
	Coupon	8%
	Coupon frequency	Semi-annually
	Maturity	$T := 5$
	Recovery	$\gamma := 0\%$
Put	Strike	$K_t^p := 105$
	Period(s)	$\Omega^p := \{3\}$
Call	Strike	$K_t^c := 110 + C_i \frac{t \pmod{0.5}}{0.5}$
	Period(s)	$\Omega^c := [2, 5)$
Conversion	Quantity of stocks	$\kappa_t := 1$
	Period(s)	$\Omega^v := [0, 5]$

Table : Convertible Bond Parameters

Parameters for the Stock

Parameter	Value
Risk free rate	$r := 5\%$
Volatility	$\sigma := 20\%$
Hazard rate	$\lambda := 2\%$
Default (total)	$\eta := 100\%$
Default (typical[1])	$\eta := 30\%$
Default (partial)	$\eta := 0\%$

Table : Stock Parameters

[1] Beneish and Press (1995) found that stock prices typically drop 30% on announcement of default

Payoff and Action

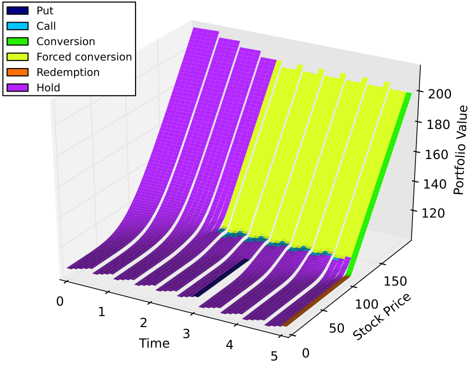


Figure : Payoff of the convertible bond with total default and colours indicating the action taken for that payoff. $\delta t = 2^{-3}$, $\delta S_t = 2^1$ and $S_t \in [0, 250]$.

Simplified Call

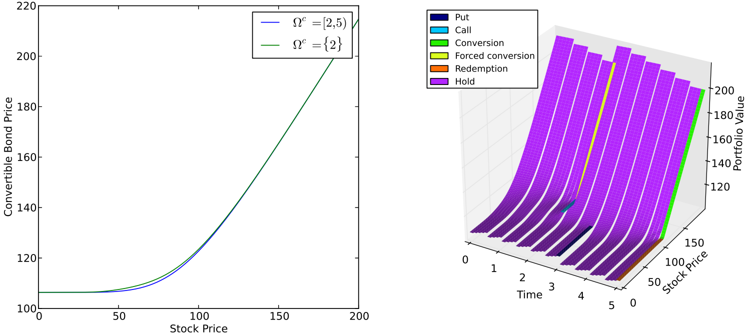


Figure : Comparison of initial value of the standard convertible bond compared to one with a singular time for the call provision. Also the payoff of the "simple call" and colours indicating the action taken for the payoff.

Varying Call Time

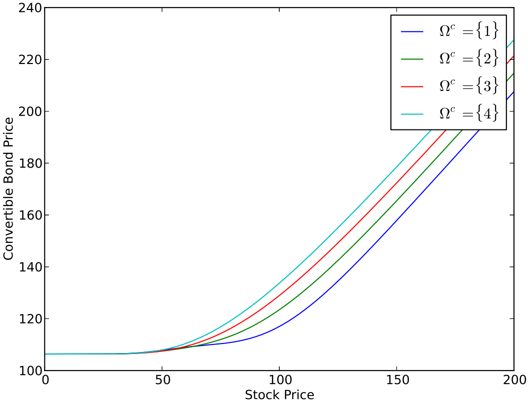


Figure : Comparison of initial value of convertible bonds with different times for the call provision.

Varying Put Time

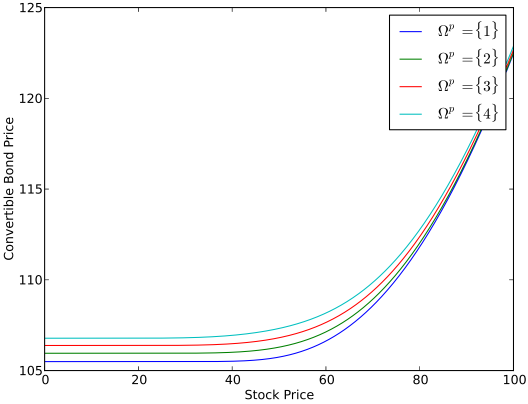
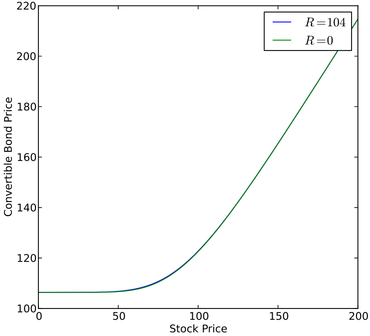


Figure : Comparison of initial value of convertible bonds with different times for the put provision.

Varying Redemption Value



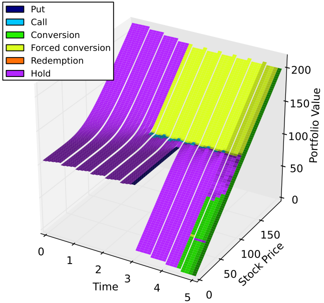


Figure : Comparison of initial value of standard convertible bonds to one with no redemption value. Also the payoff of the "no redemption" and colours indicating the action taken for the payoff.

Outline

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