Convertible Bond Pricing

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Convertible bond investments may have a special role under Solvency II and the Swiss Solvency Test as they are favoured over equities due to their convex payoff profile. Especially under Solvency II, replacing equities by an optimized convertible bond portfolio can reduce the solvency capital requirement (SCR) significantly while keeping risk/return characteristics virtually unchanged. Therefore, we believe that for small and mid size insurance companies where constraints on investment volume are less relevant, Solvency II optimized convertible bond strategies will be an attractive alternative to equities.

Before formulating such investment strategies convertible bond pricing functions have to be correctly implemented and tested. As convertibles are sensitive to a variety of market risk factors defined under Solvency II's standard model (i.e. interest rate, spread, equity, currency) and exhibit various optional features, a thorough understanding of different convertible pricing functions is needed. The aim of this paper is to consider the practical and theoretical aspects of convertible bond pricing, and create a pricing model using relevant convertible bond features and risk factors. This paper is relevant as one step in the process of structuring a Solvency II optimized portfolio (which will be treated in separate publications) as investors and regulators are likely to require a properly tested asset pricing model.

Convertible bonds are not at all a new investment innovation; the first convertible bonds were issued by US railroad companies in the 19th century. Nowadays global convertible bond markets offer a broad country and sector diversification with roughly 50% US, 30% Europe and 20% Asia (and a few but quite interesting issues from other regions) with all major sectors being represented. In the exceptionally low interest rate environment this asset class's popularity has increased also among fixed income investors searching for return opportunities. Apart from the equity-like return potential that global CB markets have demonstrated, convertibles have the advantage of typically performing positively if interest rates rise and therefore often work well for hedging duration risk. But also compared to equities, convertibles can be attractive: their downside risk is lower due to the bond floor's protective nature. While historically convertible bonds have even outperformed equities and this at a volatility which is two third of equities, they are of course not a free lunch. As convertibles are typically less liquid than equities with a liquid CB market volume at roughly 250bn USD, they are likely to be an investment opportunity to small to mid size institutional investors rather than their larger peers.

From the technical perspective, convertible bonds (CBs) are corporate bonds with an embedded option to be converted into a predefined number of shares. Consequently, convertible bonds have equity-like and bond-like characteristics. They are priced similarly as corporate bonds if the equity price is low. If the equity's price is high, a convertible bond is likely to

be converted and the price behaviour is similar to the underlying share. Due to the dynamic change of their risk profile (between debt and equity) CBs are often called *hybrids* and sometimes referred to as a separate asset class.

One of the major challenges of convertible bonds is to find an efficient pricing method that is consistent with observed market prices. This paper evaluates various pricing models for convertible bonds and shows the implementation of three different modifications of a binomial tree-based pricing model using real world CB features.

There are several questions that need to be addressed when specifying a convertible bond pricing model. *First*, most often observed convertible bond features need to be categorized as they have to be considered as input to the pricing function. Plain-vanilla convertible bonds can be converted into a predetermined number of shares if certain conditions are met. However, there are many other possible features which are used by many issuers, such as callability, putability, contingent conversion or soft callability and have to be treated explicitly by pricing functions.

Second, pricing models need to include all relevant risk factors of convertible bonds. There are several considerations due to their hybrid characteristics: the price of a convertible bond depends on the underlying share's risk factors and also on the risk factors of the bond part. There are several options discussed in the literature such as constant or stochastic volatility and constant, deterministic or stochastic interest rates.

Third, after the selection of the risk factors, one needs to select the appropriate model type. There are several models in the literature including binomial tree models, trinomial tree models, Monte Carlo methods and analytical solutions. There are several questions about the implementation too, for example computational speed which may become crucial for certain applications.

Chapter 1 discusses the most important characteristics and definitions regarding convertible bonds and aims to provide the relevant foundations of this asset class. Chapter 2 introduces the relevant literature of the pricing models and describes a binomial-tree based pricing model and its extension using different implementations of credit risk. The first version of the model uses constant credit spread, the second version discusses a so-called credit-adjusted discount rate (Goldman Sachs (1994)) and the last extension of the model uses a default intensity-based credit model. Finally, Chapter 3 investigates the convergence of the model and conducts sensitivity tests.

Contents

1 Introduction to the Pricing of Convertible Bonds

This chapter introduces the key characteristics of convertible bonds (CBs). While a variety of notations can be found in the literature, the one introduced in the following chapter will be pursued and used throughout this paper.

1.1 Definition of Convertible Bonds

Convertible bonds are typically referred to as "hybrid securities" as they combine the characteristics of debt and equity.

Hull (2000) defines a convertible bond as a corporate bond that can be converted into a predetermined amount of the company's share at certain times during its life. This definition suggests that convertible bonds are derivative securities whose value is derived from the value of the underlying equity and the underlying bond.

According to Goldman Sachs (1994) European-style convertible bonds can be decomposed in the following two ways:

1. Fixed income point of view:
   - a straight bond plus a call option that allows the investor to exchange the bond for equity.

2. Equity point of view:
   - an equity plus a put to exchange the equity for a straight bond; and
   - a swap to maturity that gives the investor the bond's coupons in exchange for the equity's dividends.

These two ways are useful to understand the basic structure of a European-style convertible bond from an equity and a bond investor's point of view. However, in real life most convertible bonds are American-style that can be exercised at any time during the life of the product and they often exhibit additional features such as contingent conversion, putability and callability, which need to be included in the pricing model. These features clearly complicate the decomposition into simpler components.

1.2 Basic Terminology of Convertible Bonds

This section introduces the most important concepts of convertible bonds with the content and notation based on Spiegeleer and Schoutens (2011).

Without the conversion option a convertible bond has the same specifications as a corporate bond, therefore it has a maturity date, a face value, redemption and a coupon which are defined as:

Maturity Date (T): Date on which the convertible bond expires.

Face Value (N): The notional amount of a convertible bond.

Redemption (R): The amount paid out to the investor at the maturity date if the CB has not been converted prior to maturity. It is often expressed as a percentage of the face value and is

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typically 100%.

Coupon Payment (c): The annual interest payment of the bond expressed as a percentage of the face value. The frequency (f_c) of the coupon is also important, which determines the number of the payment periods per annum.

These characteristics are determined by the CB's prospectus.

The relation between the bond and the equity part of the convertible bond is defined by the conversion ratio.

Conversion Ratio (C_r): It shows the number of shares the bond holder gets when converting the convertible bond into shares.

Figure 1 looks at convertible bonds from two different angles. On the one hand it shows the CBs value decomposition into an equity and a bond part, on the other hand it divides CBs depending on their moneyness in 4 categories: distressed, bond-like, balanced and equity-like.
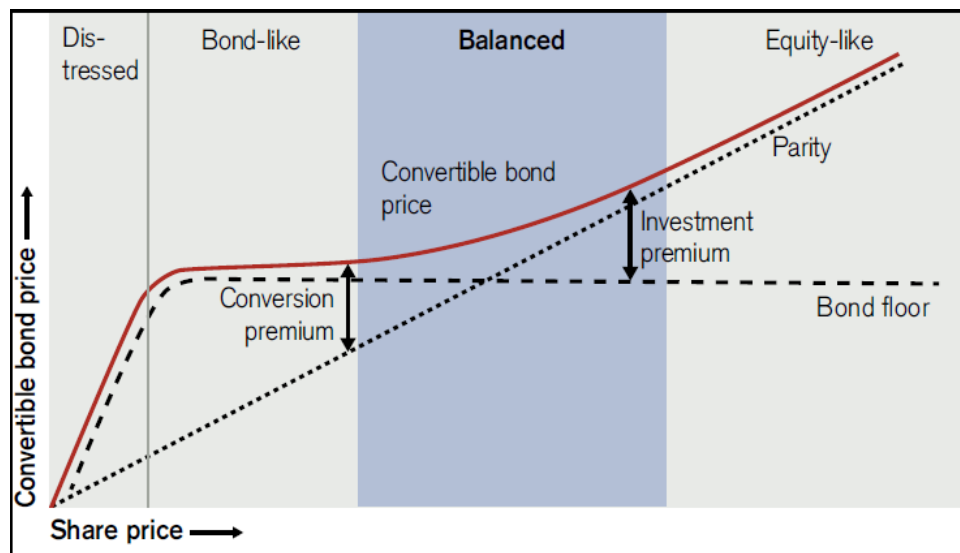

Figure 1: Convertible bond price, parity and bond floor - Source: Credit Suisse (2014)

Figure 1 shows that the price of the CB can be decomposed into the bond floor plus the investment premium or the parity plus the conversion premium. The definitions of these are the followings:

Bond Floor (B_F): This is the present value of the cash flows of the convertible bond while neglecting any possible conversion. It excludes any income from the equity part of the convertible bond.

Investment Premium (IP): It shows how much an investor is willing to pay for the option embedded in the CB. It can be calculated as

$$IP = P - B_F \tag{1}$$

where P is the price of the CB. It can also be calculated as a percentage of the bond floor:

$$\widetilde{IP} = \frac{P - B_F}{B_F} \tag{2}$$

Parity (P_a): This is the value of the shares one would hold if the CB was immediately converted. It can be calculated as

$$P_a = S \cdot C_r \tag{3}$$

where S is the value of the underlying stock.

Conversion Premium (C_p): It shows how much a CB investor is willing to pay to own the CB as opposed to the underlying shares.[1] It is defined by

$$C_p = P - P_a \tag{4}$$

If defined as a percentage of the parity, it can be written as:

$$\widetilde{C_p} = \frac{P - P_a}{P_a} \tag{5}$$

A further definition relevant in the context of Figure 1 is the conversion price. This is the value of the share price where parity equals to the face value of the CB. It is given by

$$S^* = \frac{N}{C_r} \tag{6}$$

Apart from the CBs value decomposition Figure 1 shows how CBs are categorised in 4 types depending on moneyness:

- **Equity-like (In The Money (ITM)):** The share price trades above the conversion price, conversion takes place with high probability and the conversion premium is low. The price of the CB is near to the parity and it is more sensitive to the changes in the price of the underlying equity than to interest rate and credit spread shifts.

- **Balanced (At The Money (ATM)):** The share price trades close to the conversion price and the CB is sensitive to the changes in the price of the equity as well as to interest rate and credit spread shifts.

- **Bond-like (Out of The Money (OTM)):** The share price trades below the conversion price, the conversion is unlikely and the investment premium is low. The price of the CB is near to the bond floor and the sensitivity to changes in the price of the underlying equity is negligible.

- **Distressed:** In this case the share price is sufficiently low and the default risk of the CB is high, the value of the CB converges to the value of the parity or the recovery value.

These four categories have a particular importance when choosing an adequate pricing model for CBs. Empirical studies have shown that the pricing error of the models could be significantly different in case of an ITM and an OTM convertible bond.

The next section introduces four commonly used CB features which is important in the construction of the pricing model as well.

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[1]Some authors define the conversion premium as the difference between the price of the convertible bond and the greater of the parity or the bond floor. Using the definitions above it means that they define the conversion premium as the maximum of the investment premium and the conversion premium.

1.3 Convertible Bond Features

The previous section introduced the basic properties of convertible bonds. The focus was on the standard case where investors have only one choice: either they convert into a predetermined number of shares or they do not. However, the market for convertible bonds is innovative and therefore a variety of specifications on the rights and obligations of issuers and investors can be found. This section introduces some commonly used features of CBs while relying on the definitions and notations in Spiegeleer and Schoutens (2011).

This section introduces the following four features: callability, putability, contingent conversion (CoCo) and soft callability.

Callability: In the call periods ($t \in \Omega_{Call}$) the issuer can enforce the call of the CB by paying the early redemption amount (K_c: call price). As the bond holder has the right to convert the CB into shares and will do so if he earns more than K_c, such a conversion is called forced conversion.

Putability: In the put periods ($t \in \Omega_{Put}$) the bond holder can give the convertible bond back to the issuer at a predetermined amount called the put price (K_p).

Contingent Conversion: In the contingent conversion periods ($t \in \Omega_{CoCo}$) there is a restriction on the conversion, the investor cannot convert the bond into shares unless the equity price is higher than a contingent conversion threshold (K_{CoCo}). [2]

Soft Callability: In the periods when soft callability applies ($t \in \Omega_{SoftCall}$) the issuer is restricted from calling the bond unless the underlying equity trades above a certain threshold ($K_{SoftCall}$).

These features can be active in discrete set of dates (discrete type) or during a specific time interval (continuous type). Callability, soft callability and contingent conversion are typically continuous features while putability is typically discrete.

The above section shows how a CB's life can end before maturity through conversion, a call or put. These features must be clearly be reflected in a pricing model. The following section reviews the different ways for a CB to reach maturity or a premature ending.

1.4 Exit Scenarios of Convertible Bonds

According to Spiegeleer and Schoutens (2011) CBs have seven 'exit' scenarios:

Default: In case of default, the bond holders will get an amount based on the ranking of the bond in the company's capital structure (recovery value). Time until settlement is typically considerable.

Put: The bondholder can put the bond back to the issuer at the put price in the predetermined put periods.

Forced Conversion: The issuer can call the bond back from the investor at the call price in the predetermined call periods. Technically the issuer gives a call notice to the investor about the

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[2]It is important to notice that this contingent conversion is different from the product CoCo, which is an other financial instrument. Here the contingent conversion term is used to a feature of convertible bonds. For further reference to the definition of CoCo see Spiegeleer and Schoutens (2011 p. 135).

call action, and the investor can decide whether to convert after the call or not. If the investor converts the bond, it is called forced conversion.

Optional Conversion: The investor decides to convert the CB into shares (without a call notice).

Call: The issuer calls the bond back from the bond holder and the bond holder is not converting the bond.

Redemption at Maturity: The bond has not been called, put or converted before maturity and at maturity the conversion value is below the payout of the redemption amount plus the final coupon payment.

Conversion at Maturity: The bond has not been called, put or converted before maturity and at maturity the investor chooses to convert the CB into shares.

As most of these exit scenarios are path-dependent the pricing of convertible bonds is more complicated than the decomposition method showed in Section 1.1. Clearly, pricing models need to capture this path-dependency.

2 The Pricing Model

This chapter introduces the relevant literature of convertible bond pricing models and shows the elements of the pricing model described later in this paper.

2.1 Review of Literature

There is a variety of ways modelling convertible bonds including closed-form and numerical solutions. Closed-form solutions can only be used with a restricted set of assumptions, therefore numerical solutions are frequently used in practice. Numerical solutions such as lattice methods, finite difference methods or Monte Carlo simulations can include path-dependent payoff structure allowing more realistic implementation of convertible bond features. The primary focus of this paper is on a lattice-based pricing model which is introduced in this section. Due to the high degree of generalization and flexibility to include exotic features lattice-based models are the most appropriate to our applications.

While there were some general principles to convertible bond pricing in the early 20th century, the seminal work of Black and Scholes (1973) has introduced an academic rigour into this field.

Ingersoll (1977) and Brennan and Schwartz (1977) were the first to use the contingent claims approach to price convertible bonds based on Merton's (1974) work using a closed-form solution. Cox, Ross and Rubinstein (CRR model) (1979) specified their well-known binomial option pricing method which allows for implementing path-dependent CB features. Firstly, Cheung and Nelken (1994) applied a tree method for pricing CBs, they used a quadro tree which combines a binomial stock price tree and a binomial interest rate tree. Goldman Sachs (1994) also uses the CRR binomial model to construct the stock price tree, but they assumed interest rates to be deterministic. These methods are capable of including putability and callability into the pricing model due to the flexibility of the lattice method.

The previously mentioned models are structural models based on the principles of Merton's work (1974). Hung and Wang (2002) use a reduced-form model based on the Jarrow and Turnbull (1995) framework. In their model they use stochastic stock price tree and combine the risk-free and the risky rate into one tree. Carayannopoulos and Kalimipalli (2003) discuss a reduced-form trinomial tree method following the Duffie and Singleton (1999) framework. Their model uses a stochastic stock price and the hazard rate depends on the movements of the stock price.

Chambers and Lu (2007) expand the Hung and Wang (2002) model using correlation between the stock price tree and the interest rate tree. Xu (2011) propose a trinomial tree method incorporating market risk and counterparty credit risk in the CB pricing model. The model includes a mean-reverting interest rate process and a constant elasticity of variance (CEV) model for the equity price process. Huang, Liu and Rao (2013) propose a model with constant and time-varying volatility for the interest rate tree and a CRR tree for the equity price process.

2.2 The Building Blocks of the Pricing Model

In this paper we build a pricing model based on a binomial tree using the research note of Goldman Sachs (1994) as a starting point, but improving the model with other exotic features on the market and introducing different approaches to include credit risk into the model. The model uses a stochastic stock price process, a deterministic interest rate term structure and constant volatility. In case of credit risk it introduces three different approaches from a constant

==credit spread to a default intensity-based credit risk model.== The next sections introduces the theoretical details of the model.

2.2.1 Stock Price Process

The stock price is stochastic in the model and follows the ==CRR binomial tree model.== The derivation of the model is based on Cox et al. (1979) and Hull (2000).

Assume that the price of the underlying equity under the risk-neutral measure (Q) follows a Geometric Brownian Motion (GBM):

$$dS(t) = rS(t)dt + \sigma S(t)dW(t)$$

where $S(t)$ is the price of the stock at time t, r is the risk-free interest rate, σ is the volatility of the equity and W is a Wiener process. The discretization of this GBM process can be done using the CRR binomial tree (see below).

For the purpose of pricing a CB the model needs a stock price for every period, where it calculates the value of the CB, from the analysis date to the maturity date. Assume that the analysis date is 0 and the maturity date is T and divide the $[0, T]$ interval into n equal time interval of length Δt.

Let today's stock price be defined as S, and assume it can move only in two ways, to Su or Sd, where $u > 1 + r > d > 0$.[3] The movements Su and Sd are usually called "up" and "down" movements, respectively. The probability of an up movement is denoted by q, and the probability of a down movement is denoted by $1 - q$.

The pricing model uses this discrete model for the discretization of the GBM process. The distribution of the stock price at time t is lognormal using the GBM process and one can create a binomial tree matching the first two moments of the lognormal distribution. The model has three parameters (u, d, q) and two conditions (equating expected value, variance), therefore it has one degree of freedom and it uses the condition used by Cox et al. (1979) as the third equation.

The expected value of the stock price at the end of a time interval of length Δt – where the stock price is S at the beginning – is $Se^{r\Delta t}$. The following equation must be satisfied:

$$Se^{r\Delta t} = qSu + (1-q)Sd$$

Dividing by S one obtains

$$e^{r\Delta t} = qu + (1-q)d \tag{7}$$

The variance of the stock process in a time interval Δt is $\sigma^2 \Delta t$, therefore the following equation must be satisfied:

$$\sigma^2 \Delta t = qu^2 + (1-q)d^2 - [qu + (1-q)d]^2 \tag{8}$$

There are 3 unknown variables in Equation (7) and (8), a third condition is required to obtain a unique solution. Cox et al. (1979) use the condition

$$u = \frac{1}{d} \tag{9}$$

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[3]This assumption of $u > 1 + r > d > 0$ ensures the no-arbitrage condition of the model is met.

With these three conditions there is a unique solution for Equations (7)-(9):

$$q = \frac{e^{r\Delta t} - d}{u - d} \tag{10}$$

$$u = e^{\sigma\sqrt{\Delta t}} \tag{11}$$

$$d = e^{-\sigma\sqrt{\Delta t}} \tag{12}$$

Here q is called as the risk-neutral probability. It is important to notice that q is defined under the Q measure (in the risk-neutral world) and different from the real world probabilities.

Figure 2 shows one branch of the binomial tree calculated according to the method showed previously.
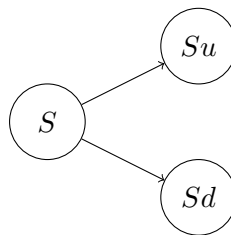

Figure 2: One branch of the binomial tree

Figure 3 shows a two-period recombining binomial tree which was built from three branches. Using the CRR tree construction method the tree is recombining: an up step followed by a down step leads to the same node as a down step followed by an up step. In case of a non-recombining tree the second level in Figure 3 would consist of four nodes instead of three.
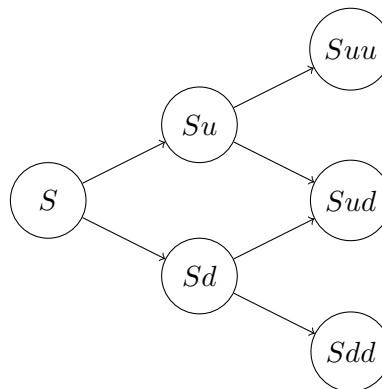

Figure 3: A two-period stock price tree

This model relies on several assumptions about the behaviour of the market. The model assumes constant risk-free interest rate, constant volatility and no dividend yield of the underlying equity. The model can be extended using a deterministic risk-free interest rate structure and a non-zero dividend yield of the underlying stock.

Assume that the stock price is $S(t)$ at a node on the tree at time t. The model assumes that the risk-free interest rate between t and $t + \Delta t$ is $f(t)$, where $f(t)$ denotes the forward interest rate between t and $t + \Delta t$ today (on the pricing date).

It means that in every level of the tree it needs to construct a new risk-neutral probability, denoted by $q(t)$, which is the risk-neutral probability between t and $t + \Delta t$. The calculation of the risk-neutral probability is based on Equation (10).

$$q(t) = \frac{e^{f(t)\Delta t} - d}{u - d} \qquad (13)$$

The calculation of u and d remains unchanged as it depends only on the volatility and the time step and not on the risk-free interest rate.

The second extension of the model is the inclusion of a non-zero dividend yield of the underlying stock. Assume that δ is the annualized continuously compounded dividend yield of the underlying stock. In case of the dividend yield it uses a constant dividend yield of the underlying equity during the life of the convertible bond. In this case Equation (13) needs another modification including the dividend yield into the pricing model. The equation of the risk-neutral probability is the following:

$$q(t) = \frac{e^{(f(t)-\delta)\Delta t} - d}{u - d} \qquad (14)$$

Using Equations (11), (12) and (14) the stock price process can be built using a deterministic interest rate term structure and a non-zero dividend yield. The next section introduces a convertible bond pricing model, which uses this stock price tree as the stochastic process of the underlying equity of the convertible bond.

2.2.2 Convertible Bond Valuation Tree

This section discusses the construction of the convertible bond valuation tree. Pricing convertible bonds with a binomial tree is quite similar to American option pricing. The first step is to compute the value of an American option at maturity using the payoff function. The second step is to move backwards in the tree by one time step and calculate the continuation value of the nodes, which is the discounted value of the expected payoff under the risk-neutral measure. The third step is to take the maximum of the continuation value and the exercise value of each nodes. Finally, the iterative process continues by moving back one time step and repeating the procedure until the first node of the tree is reached. This section shows a similar algorithm for CBs.

Assume that the analysis date is 0 and the maturity date is T as it was shown in case of the stock price tree. Divide the $[0, T]$ interval into n equal time intervals of length Δt. Let V be the value of the convertible bond at time 0 and the value of the CB is denoted by V_u if the underlying stock moves up and V_d if the underlying stock moves down. The risk-neutral probabilities are the same as above, so the probability of an up movement on the tree from time t to time $t + \Delta t$ is $q(t)$ and the probability of a down movement is $1 - q(t)$.

Figure 4 shows a two-period convertible bond valuation tree which was built from three branches. For simplicity the description of the pricing model does not use the t time index to denote the time parameter (the current level of the tree). V denotes the value of the CB at the current node, V_u and V_d denote the "up" and "down" value of the CB at the next node and S is the value of the underlying stock price at the current node.

The value of the CB at each node of the last level can be calculated as follows:

$$V = max\left(N \cdot R + N \cdot \frac{c}{f_c}, S \cdot C_r\right) \qquad (15)$$
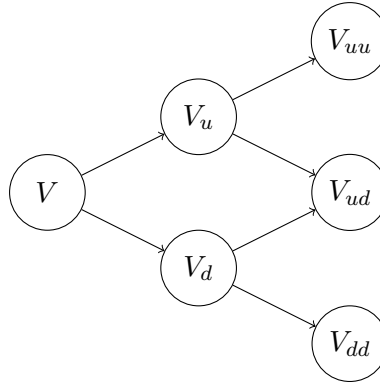
Figure 4: A two-period convertible bond valuation tree

where N is the face value, R is the redemption in percent of the face value, c is the annual coupon payment expressed as a percent of the face value, f_c is the coupon frequency and C_r is the conversion ratio. For simplicity the notation c_{USD} is used for $N \cdot \frac{c}{f_c}$ as it is the dollar amount of the coupon at time t.[4] The model further assumes that c_{USD} is a vector defined for every t, and the value of $c_{USD}(t)$ is the following:

$$c_{USD}(t) = \left\{ \begin{array}{ll} N \cdot \frac{c}{f_c} & \text{if there is a coupon payment at time t} \\ 0 & \text{otherwise} \end{array} \right.$$

In the following we use c_{USD} (without the parameter t) to denote the value of the c_{USD} vector at time t. Equation (15) is the following using the new notation:

$$V = max(N \cdot R + c_{USD}, S \cdot C_r) \tag{16}$$

After the calculation of each node on the last level of the tree, the second step is to move backwards in the tree by one time step and calculate the continuation value (H) of the nodes. The continuation value is the present value of the expected values of V_u and V_d under the risk-neutral measure. The formula for the continuation value at time t is given by

$$H = \exp(-f(t)\Delta t)(q(t)V_u + (1 - q(t))V_d) + c_{USD} \tag{17}$$

where $f(t)$ and $q(t)$ were introduced in Section 2.2.1.

At each node before the maturity the model calculates the value of the convertible bond as

$$V = max(H, S \cdot C_r) \tag{18}$$

The calculation using Equation (17) and (18) continues by moving back one time step and re-peating the procedure until the first node of the tree is reached. At the first node one obtains the dollar denominated price of the CB. As CB prices are quoted typically as a percentage of the face value the model divides V by the face value at time 0.

This pricing model implicitly assumes that the investor will not get a coupon payment in case of conversion. This assumption is used by Goldman Sachs (1994), but for example Spiegeleer et al. (2011) used $S \cdot C_r + c_{USD}$ instead of $S \cdot C_r$. It seems more consequent to use $S \cdot C_r$ as it separates the bond part and the equity part better in the pricing model.

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[4]The c_{USD} notation simply means that the coupon is expressed in a currency, it can be different from USD (e.g. CHF), the model only uses the "USD" notation for simplicity.

This simple pricing model assumes no credit risk; instead the model uses the risk-free rate for discounting (more precisely the forward rates calculated from the risk-free discount curve). Convertible bonds pay coupons and return the redemption amount at maturity, therefore both are subject to default risk. To implement this risk the first form of the model applies a constant credit spread (CS).

Let r_b be the discount rate used in the model to discount the values of the convertible bond. This changes only the calculation of the continuation value

$$H = \exp(-r_b(t)\Delta t)(q(t)V_u + (1-q(t))V_d) + c_{USD} \tag{19}$$

where $r_b(t)$ is the value of the discount rate equal to $f(t) + CS$ at time t.

The basics of the pricing model were introduced in Section 2.2.1 and 2.2.2. This simple model can only be used to price a convertible bond with no exotic features and therefore the practical application of the model is restricted to these simple cases. The next section extends the model to include exotic features like callability, putability, soft callability and contingent conversion, which were introduced previously in Chapter 1.

2.2.3 Extension of the Model Using Exotic Features

This section adds exotic features to the pricing model introduced above. First, it considers the case of callability and putability and then introduces other restrictions such as soft callability and contingent conversion.

It is important to notice that the pricing model assumes that both the issuer and the investor act rationally and there is no asymmetry of information between them. In reality there is always an asymmetry of information, the issuer has a better knowledge about the company which has an effect on call events. However, this goes beyond the goals of this study, for further information see e.g. Bechmann et al. (2014) or Dutordoir et al. (2014).

The pricing model also assumes that there is no call and put event at maturity (this is usual on the market too), therefore the calculation of the value of the CB at maturity is the same as previously (see Equation (18)). The calculation of the continuation value remains the same as in Equation (19).

In case of callability the call price is usually given as a percentage amount of the face value, denoted by K_c. The model needs the nominal amount (USD) of the call price, which is $K_c \cdot N$ and denoted by K_c^{USD}. For $t \in \Omega_{Call}$ the value of the convertible bond at time t is given by modifying Equation (18) to

$$V = max(min(H, K_c^{USD}), S \cdot C_r) \tag{20}$$

According to Equation (20), if the continuation value of the CB is higher than the call price, the bond gets called. In this case the investor gets K_c^{USD} or converts the bond into shares. The model assumes that a rational investor convert the bond into shares if the parity is higher than the call price, hence it takes the maximum of the two values.

In case of putability the put price is usually given as a percentage amount of the face value, denoted by K_p. The model needs the nominal amount (USD) of the put price too, therefore the K_p^{USD} variable was introduced as $K_p^{USD} = K_p \cdot N$. For $t \in \Omega_{Put}$ the value of the CB at time t is given by modifying Equation (18) to

$$V = max(H, K_p^{USD}, S \cdot C_r) \tag{21}$$

The formula shows if the continuation value and the parity is below the put price, a rational investor decides to put the CB.

Considering that a CB behaves as a corporate bond without the conversion option, one needs to include accrued interest (AI) into the calculation of the bond. According to Fabozzi (2012) accrued interest is given by:

$$AI = \frac{\text{Number of days from last coupon payment to settlement date}}{\text{Number of days in coupon period}} \cdot c$$

The pricing model rather defines accrued interest as a dollar denominated amount:

$$AI_{USD} = \frac{\text{Number of days from last coupon payment to settlement date}}{\text{Number of days in coupon period}} \cdot \frac{c}{f_c} \cdot N$$

Using the accrued interest the generalization of Equation (18) is the following for $t < T$:

$$V(t) = \begin{cases} max(H, S \cdot C_r) & \text{if } t \notin \Omega_{Call} \text{ and } t \notin \Omega_{Put} \\ max(min(H, K_c^{USD} + AI_{USD}), S \cdot C_r) & \text{if } t \in \Omega_{Call} \text{ and } t \notin \Omega_{Put} \\ max(H, K_p^{USD} + AI_{USD}, S \cdot C_r) & \text{if } t \notin \Omega_{Call} \text{ and } t \in \Omega_{Put} \\ max(min(H, K_c^{USD} + AI_{USD}), K_p^{USD} + AI_{USD}, S \cdot C_r) & \text{if } t \in \Omega_{Call} \text{ and } t \in \Omega_{Put} \end{cases}$$

When the node falls on the point where a coupon c_{USD} is paid and the AI_{USD} is zero, the value of the convertible bond at $t < T$ is given by

$$V(t) = \begin{cases} max(H, S \cdot C_r) & \text{if } t \notin \Omega_{Call} \text{ and } t \notin \Omega_{Put} \\ max(min(H, K_c^{USD} + c_{USD}), S \cdot C_r) & \text{if } t \in \Omega_{Call} \text{ and } t \notin \Omega_{Put} \\ max(H, K_p^{USD} + c_{USD}, S \cdot C_r) & \text{if } t \notin \Omega_{Call} \text{ and } t \in \Omega_{Put} \\ max(min(H, K_c^{USD} + c_{USD}), K_p^{USD} + c_{USD}, S \cdot C_r) & \text{if } t \in \Omega_{Call} \text{ and } t \in \Omega_{Put} \end{cases}$$

It is important to notice that the continuation value also includes a coupon payment if the node falls on a coupon date (see Equation (19)).

After the implementation of callability and putability we turn to soft callability and contingent conversion.

Soft callability introduces an extra condition to the pricing model, the issuer can only call the bond back if the underlying equity trades above a certain threshold ($K_{SoftCall}$). Usually this threshold is expressed as a percentage of the conversion price, so the model needs to calculate the nominal (USD) amount of the threshold, which can be compared to the stock price. This can be calculated as $K_{SoftCall} \cdot S^*$, where S^* is the conversion price defined in Section 1.2.

If $K_{SoftCall} \cdot S^* > S$, the issuer cannot call the bond back, so one does not have to take into account the $K_c^{USD} + c_{USD}$ part in the pricing formula, even if $t \in \Omega_{Call}$. This condition could increase the price of the convertible bond, as the model does not use the minimum function in the pricing formula if the equity price is below the soft callability threshold.

The other extension of the model is the implementation of contingent conversion. Contingent conversion is a restriction on the conversion of the bond, the investor cannot convert the bond into shares unless the equity price is higher than the contingent conversion threshold K_{CoCo}. Here the nominal (USD) amount of the threshold can also be calculated using the conversion price: $K_{CoCo} \cdot S^*$.

If $K_{CoCo} \cdot S^* > S$ and $t \in \Omega_{CoCo}$ the investor cannot convert the bond into shares, so the pricing model does not take into consideration the $S \cdot C_r$ part.

There is only one case where contingent conversion does not apply if the $K_{CoCo} \cdot S^* > S$ condition is met: this is called forced conversion. If the issuer calls the bond back, the investor can decide whether to convert the bond or not, even if the contingent conversion threshold is above the equity price. In the pricing model it can happen if $t \in \Omega_{Call}$ and $t \in \Omega_{CoCo}$ and $K_c^{USD} + AI_{USD} < S \cdot C_r < H$. If the CB is putable at time t, another condition is required: $K_P^{USD} + AI_{USD} < S \cdot C_r$.

If these conditions meet the issuer call the bond back, because $K_c^{USD} + AI_{USD} < H$, therefore it is advantageous to call the bond back rather than let the investor hold the bond until maturity. In case of the call event the investor can decide to convert the bond or put the bond if $t \in \Omega_{Put}$. The investor will convert the bond, because the parity is higher than the amount that he gets in case of the call or the put event.

This section introduced four exotic path-dependent features of CBs, which makes the pricing model more realistic and usable in a real market situation. The next section extends the model using more complex implementations of credit risk.

2.2.4 Extension of the Model Using Credit-Adjusted Discount Rate

In the literature credit risk models are classified into two groups, reduced form and structural models. Reduced form models assume a given form of default intensity (as an exogenous factor), while structural models based on the variables of the firm, e.g. the first-passage of assets of the firm to a default boundary (Duffie et al. (2003)).

Spiegeleer and Schoutens (2011) classify credit risk models of convertible bonds into structural, credit spread and reduced form models. The definition of structural and reduced form models are the same as above while credit spread models are modifications of structural models where the equity price is used to model firm value.

The previous section introduced a model where a constant credit spread was added to the risk-free rate. In order to relax the assumption of constant credit spreads two models will be discussed in the following: a modified version of the constant credit spread model based on Goldman Sachs (1994) and a reduced form model using an exogenous Poisson process to model the default intensity.

Before introducing variable credit spreads it is worthwhile to recall the calculation of the continuation value in Equation (19):

$$H = \exp(-r_b(t)\Delta t)(q(t)V_u + (1 - q(t))V_d) + c_{USD}$$

where $r_b(t)$ is the value of the discount rate equal to $f(t) + CS$ at time t. This equation assumes that the credit spread is constant during the life of the CB and the same in case of deep ITM or distressed CBs.

Assume that the CB is deep ITM and the probability of the CB being converted is virtually 1. Here the appropriate discount rate is the risk-free rate as the investor is certain to obtain the underlying shares, thus, there is no default risk.[5] However, in case of an OTM CB, the CB will most likely not be exercised: the CB behaves as a corporate bond and therefore it is subject to default risk. The appropriate discount rate in this case is $r_b(t)$, which is the risk-free rate plus the credit spread.

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[5]This is similar to stock option pricing, where the pricing model also uses the risk-free discount rate.

Goldman Sachs (1994) provides a solution with a discount rate which lies between the risk-free rate and a credit-adjusted discount rate ($r_b(t) = f(t) + CS$). Let p_{conv} be the probability that the CB will be converted into stocks in the future and $1 - p_{conv}$ the probability that it will behave like a corporate bond. Moreover, let $y(t)$ be the credit-adjusted discount rate at time t given by[6]

$$y(t) = p_{conv}f(t) + (1 - p_{conv})r_b(t) \qquad (22)$$

Equation (22) shows that $y(t)$ is the weighted average of the riskless forward rate between t and $t + \Delta t$ and the risky rate ($r_b(t)$) using the conversion probabilities (p_{conv}). This introduces two new binomial trees in the pricing model: the first tree models the conversion probabilities while the second tree models credit-adjusted discount rates. The calculation of the tree of the conversion probabilities is the following:

1. The value of p_{conv} at every node is equal to 1 if the CB is converted into stocks at the node (even if it is a result of forced conversion) and the value of p_{conv} at every node is equal to 0 if the CB is redeemed or put at the node.

2. If the CB is not put, redeemed or converted at the current node, the calculation of p_{conv} is the following:

$$p_{conv} = q(t)p_{conv}^{up} + (1 - q(t))p_{conv}^{down}$$

where p_{conv}^{up} and p_{conv}^{down} are the conversion probabilities and $q(t)$ is the risk-neutral probability at time t.

The interpretation of the algorithm is straightforward. At the last level of the tree one can decide whether the CB is converted into shares or redeemed, therefore at the last level p_{conv} is 0 or 1. At the previous levels if one can decide whether the CB is converted, put or called p_{conv} is also 0 or 1. If one cannot decide then p_{conv} is the weighted average of the conversion probabilities of the up and down nodes of the next level using the transition probabilities ($q(t)$) of the stock price process. Using this implementation the model can include a stock price dependent discount rate and a credit-adjusted discount rate.

2.2.5 Extension of the Model Using a Reduced Form Credit Risk Model

This section integrates a reduced form credit risk model into the pricing model. First it describes the basics of the credit risk model based on the default intensity (Duffie et al. (2003)) and then shows the necessary modifications for including this theory into the pricing model.

Let λ be the so-called default intensity and let us define the default as the first arrival time (τ) of a Poisson process with a constant λ parameter.[7] If $N(t)$ is a Poisson process with a constant default intensity λ, the probability of k default events to occur in a time interval $[t, T]$ is given by

$$Q[N(T) - N(t) = k] = e^{-\lambda(T-t)}\frac{(\lambda(T - t))^k}{k!}, \qquad k = 0, 1, 2, \ldots$$

where Q is the probability of an event under the risk-neutral measure. Using the above formula one can calculate the so-called survival probability $q_s(t)$ which is the probability that the CB survives up to a certain time t:

$$q_s(t) = Q[N(t) - N(0) = 0] = Q[N(t) = 0] = e^{-\lambda t}$$

As default and survival are complementary events, the probability of default up to a certain time t can be written as
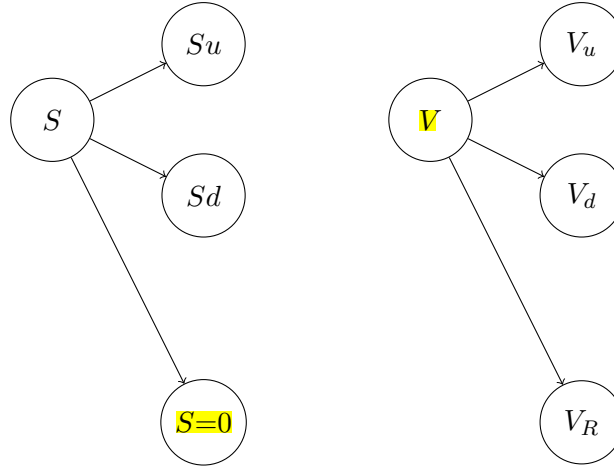
$$1 - q_s(t) = 1 - e^{-\lambda t}$$

Figure 5: One branch of the stock price tree and CB valuation tree using default risk

The pricing model assumes that in case of default the stock price will drop to zero. The left hand side of Figure 5 shows one branch of the stock price tree used in this modification of the model. Comparing this with Figure 2 the difference is that in Figure 5 there is a positive probability of a default event of the company where the stock price drops to zero.

Using that the price of the underlying share (S) today is the present value of the expected payoff under the risk-neutral measure one can use the following formula:

$$S = e^{-r\Delta t}[q_s(qSu + (1-q)Sd) + (1-q_s)S_{Default}] \tag{23}$$

where q_s is the survival probability up to a certain time Δt, q is the probability of the up movement to Su on the binomial tree and $S_{Default}$ is the value of S in case of default, which is zero in the model. Using that $q_s = e^{-\lambda \Delta t}$ and $S_{Default} = 0$ one can rewrite Equation (23) as

$$S = e^{-r\Delta t}[e^{-\lambda \Delta t}(qSu + (1-q)Sd)] \tag{24}$$

After rearranging the equation and dividing by S one obtains

$$e^{(r+\lambda)\Delta t} = qu + (1-q)d$$

This equation is quite similar to Equation (7) and one can calculate the risk-neutral probability with the same conditions as before in Equations (8) and (9) in Section 2.2.1:

$$q = \frac{e^{(r+\lambda)\Delta t} - d}{u - d}$$

The calculation of u and d are the same as in Section 2.2.1 and as seen previously, one can extend the tree using a deterministic interest rate term structure and a non-zero dividend yield. The risk-neutral probability at time t on the tree (similar to Equation (14)) is given by

$$q(t) = \frac{e^{(f(t)+\lambda-\delta)\Delta t} - d}{u - d}$$

The right hand side of Figure 5 shows the CB valuation tree: there is a positive probability of default in the tree with the value of the CB dropping to the recovery value (denoted by V_R).

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[6]For simplicity the dependence of p_{conv} and $y(t)$ on the position of the current node at the current level of the tree is omitted in the notation.

[7]This is a simplification of the credit risk models based on default intensity. The default intensity could be also a deterministic function of time or even a stochastic process.

The recovery value is the cash amount that a bond holder gets in case of the CB's default. If expressed as a percentage of the face value, this is typically referred to as recovery rate (RR). The model uses a more advanced method to calculate the recovery value given by

$$V_R = RR \cdot (N exp(-r(T-t)) \tag{25}$$

According to Equation (25) at time t (when the default event occurs) the investor gets the percentage of the present value of the face value, where this percentage amount is expressed by the recovery rate.

Modelling the recovery rate is not trivial as one cannot observe its value on the market. Thus, the model needs some assumptions on it. In this paper we discuss a constant recovery rate during the life of the CB: the value of the recovery rate is determined using the seniority of the convertible bond based on the historical observations of Moody's in the period 1982-2008. Table 1 shows the data used by Moody's as recovery rates for the convertible bonds with different seniority.

Bond Seniority	Recovery Rate
Senior secured	52.30%
Senior unsecured	36.40%
Senior subordinated	31.70%
Subordinated	31.00%
Junior subordinated	24.00%

Table 1: Recovery rates for bonds as a percentage of face value in the period 1982-2008. (Source: www.moodys.com)

Applying the recovery rate and the default intensity introduced above one can calculate the values of the convertible bond valuation tree. The values at the last level of the tree are the same as above and can be obtained by Equation (16) described in Section 2.2.2.

However, the calculation of the continuation tree will change. Let us recall Equation (17), the continuation value at time t in a default-free world, where $\lambda = 0$

$$H = \exp(-f(t)\Delta t)(q(t)V_u + (1-q(t))V_d) + c_{USD}$$

In case of a non-zero default probability, the value of H at time t is equal to

$$H = \underbrace{\exp(-(f(t)+\lambda)\Delta t)(q(t)V_u + (1-q(t))V_d)}_{a)} + \\ + \underbrace{\exp(-\lambda\Delta t)c_{USD}}_{b)} + \underbrace{\exp(-f(t)\Delta t)(1-\exp(-\lambda\Delta t))V_R}_{c)} \tag{26}$$

For interpretation purposes, Equation (26) can be decomposed into three parts a), b) and c). Part a) and b) represent the tree in case of survival (hence the $\exp(-\lambda\Delta t)$ part), while part c) represents the default event (hence the $1-\exp(-\lambda\Delta t)$ part). Part a) is the present value of the expected value of the up and down values (V_u and V_d) of the CB on the next level of the tree in case of survival, while part b) is the present value of expected coupons. Part c) represents the present value of the recovery value in case of default.

The calculation algorithm of the CB valuation tree is almost the same as introduced previously. The only difference is that the default intensity model applies a different formula for the calculation of the risk-neutral probabilities $q(t)$ of the CRR stock tree and a different formula for the calculation of the continuation value H. The continuation value can also include a constant dividend yield, which is a modification of Equation (26).

The next chapter is based on the basic model and its extensions introduced in this chapter and it analyzes the convergence of the model and shows the behaviour of the model for different values of the underlying risk factors.

3 Convergence and Sensitivity Analysis of the Pricing Model

This chapter shows the behaviour of the pricing model introduced in Chapter 2 using a sample convertible bond. First it shows the convergence of the model, then a sensitivity analysis for changing underlying parameters.

3.1 Convergence of the Binomial Model

Chapter 2 introduced a binomial tree-based pricing model used for pricing convertible bonds. The basic idea behind this model is the discretization of a continuous model. One can achieve more accuracy with the discrete model if one uses smaller time step (Δt). The discrete version of the model is the same as the continuous model if $\Delta t \to 0$, but a Δt close to zero means a higher computational effort.

Derman et al. (1995) defined two sources of inaccuracy in modelling options on a lattice.[8] The first source is the *quantization error* which is the unavoidable existence of the lattice itself. It means that the continuous stock price process is replaced by a discrete set of points. If one wants to use a tree model that matches the continuous behaviour of the stock price, one needs to use an infinitesimally small time step (Δt). In practice a well-designed tree (that can match the lognormal distribution of the stock price process) can achieve an acceptable convergence using a larger time step (Δt) too.

The second source of inaccuracy is the *specification error* which means that the lattice is not able to represent the terms of the convertible bond. It can happen if some of the characteristics of the CB do not fall on a particular node of the tree.
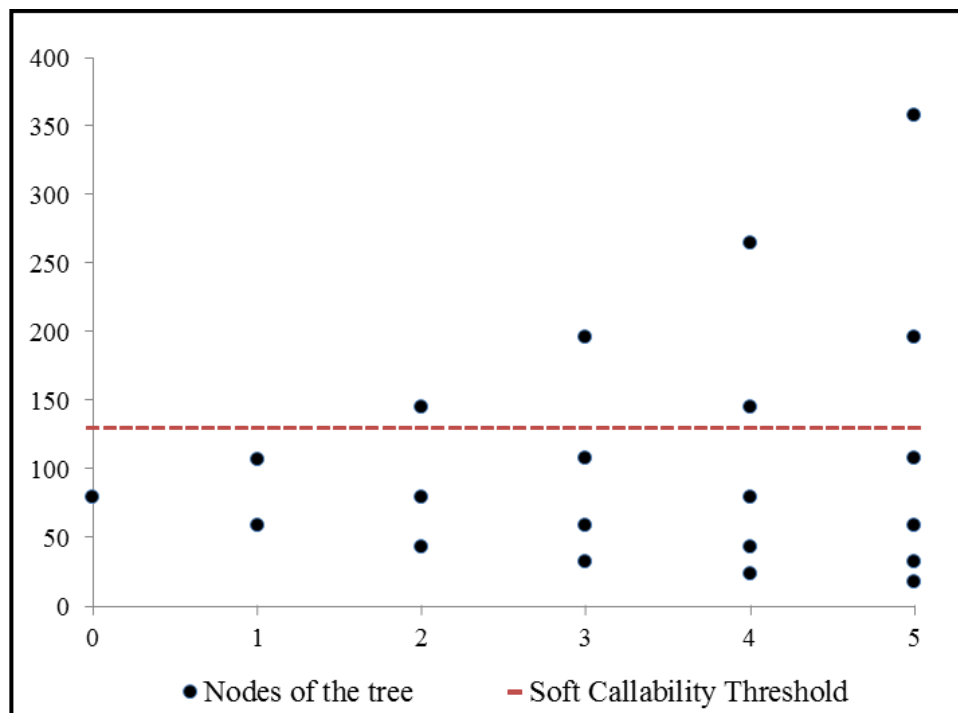

Figure 6: Specification error of the binomial tree using a soft callable CB

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[8]While they investigated barrier options their conclusions are also valid for CBs with soft callability and contingent conversion features.

Figure 6 shows an example of the specification error of a CB, where the soft callability threshold does not fall on the nodes of the tree. By using a smaller time step (Δt) the specification error can be reduced.

Table 2 shows a sample convertible bond that is used later to show the convergence of the pricing model introduced in Chapter 2. The sample bond is a convertible bond without exotic features (e.g. soft callability) and has a maturity of 5 years. It has a constant interest rate, dividend yield and credit spread during the life of the CB and it pays an annual coupon of 5 percent.[9]

Description	Value	Description	Value
Maturity	5 Years	Interest Rate	3.00%
Stock Price	20	Dividend Yield	2.00%
Face Value	100	Credit Spread	100 bps
Volatility	30.00%	Coupon	5.00%
Conversion Ratio	4	Coupon Frequency	Annual

Table 2: Summary of the sample convertible bond

Figure 7 shows the convergence of the sample CB. The vertical axis shows the value of the CB and the horizontal axis shows the number of time periods used for the calculation of the CB price. Price fluctuations are mainly caused by quantization error.
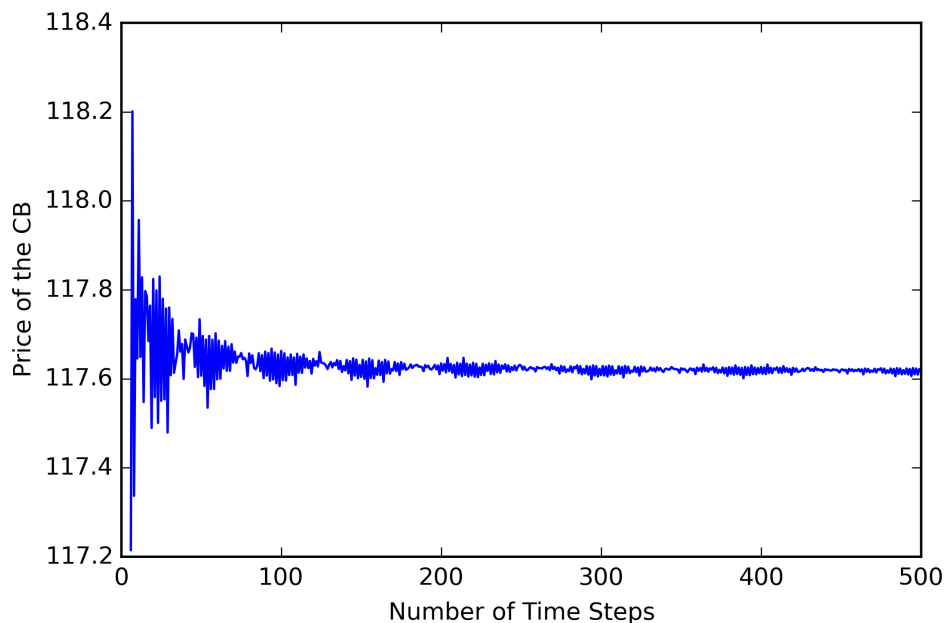

Figure 7: Convergence of the model in case of the sample CB

Suppose that the sample CB is also callable at $K_c = 100$ and there is a soft call level, at $K_{SoftCall} = 130$. In this case there is a specification error too due to the soft callability trigger. Figure 8 shows the price fluctuation of the sample bond with soft callability using different number of time steps in the pricing model.

There are several ways to reduce quantization and specification errors discussed in the literature. The first idea is to increase the branching order of the tree and implement trinomial,

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[9]This simple specification should be sufficient to show the key characteristics of the pricing function.
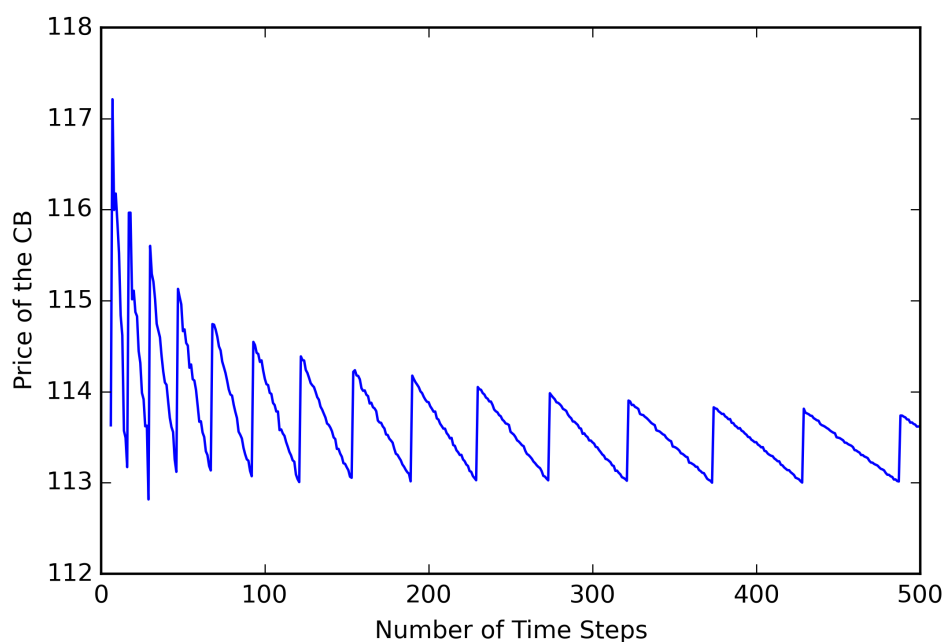
Figure 8: Convergence of the model in case of the sample CB with soft callability

heptanomial and other multinomial tree models. Using these models one can match higher moments of the distribution of the underlying variable (skewness, kurtosis etc.) and the convergence is faster to the theoretical distribution. It can decrease the quantization error of the lattice method.

There are optimization methods to decrease the specification error of the tree, for example smoothing, the adaptive mesh method and the BDKE correction. Smoothing techniques are important to make the payoff function of the CB continuous – Heston and Zhou (2000) show that the convergence of a lattice model depends on the smoothness of the payoff function. The adaptive mesh method introduced by Ahn, Figlewski and Gao (1999) uses a higher resolution tree near to the barrier and a smaller resolution tree elsewhere and it reduces the pricing error of the tree. The BDKE correction used in Derman et al. (1995) applies an interpolation technique between the effective trigger and the specified trigger and reduces effectively the specification error.[10]

3.2 Sensitivity Analysis of the Pricing Model

This section investigates the behaviour of the pricing model using the relevant risk factors. It uses the sample convertible bond introduced in Section 3.1 and its modifications for the analysis.

First, we analyse the sensitivity of the CB price to the price of the underlying equity, the volatility of the underlying equity, the risk-free interest rate and the credit risk. Figure 9 shows the sensitivities of the model to these four risk factors. The analysis uses the parameters of the sample convertible bond in Table 2 with different maturities of 3, 5, and 10 years.

The first subfigure of Figure 9 shows the sensitivity of the model to the parity of the convertible bond, which basically means the sensitivity to the underlying equity. According to the figure the

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[10]The specified trigger means the trigger set by the CB's prospectus while the effective trigger is the collection of the lowest nodes of the tree above the specified trigger. For further details see Derman et al. (1995).

price of the convertible bond is an increasing function of the price of the underlying equity and the maturity as well due to the embedded conversion option. This figure is similar to Figure 1 introduced in Chapter 1: the CB price converges to parity in case of ITM CBs, and converges to the bond floor in case of OTM CBs. The difference compared to Figure 1 is the behaviour of the CB in the distressed period. Using a stock price dependent credit spread the model would be similar to Figure 1 (this example used a constant credit spread with changing equity price ceteris paribus).

The sensitivity of the CB to the volatility of the underlying equity is shown on the second subfigure of Figure 9. If the volatility increases, the value of the conversion option and therefore the price of the CB increases. This case also assumes that the other parameters remain unchanged. In real life the increase of volatility could increase the risk of the underlying equity and consequently also the credit spread which has an offsetting effect.
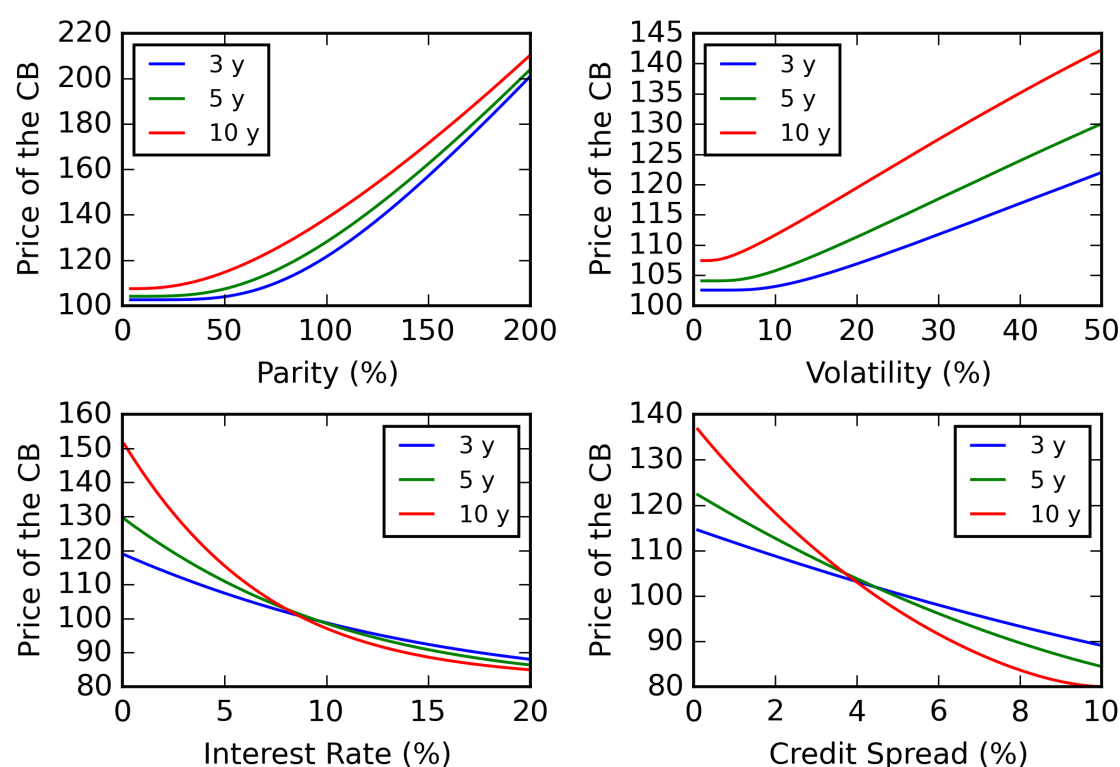

Figure 9: Sensitivities of the model to the equity price, volatility, interest rate and credit spread

The third and fourth subfigures of Figure 9 show the sensitivity of the model to interest rate and credit spread changes, respectively. The effects are similar: the CB price decreases when lowering interest rate or credit spread. It is important to notice that the price of the 10-year CB is lower than the price of the 3-year CB in case of high interest rate or credit spread. This is not too surprising as a CB is basically a corporate bond plus the conversion option. The conversion option is higher for the 10-year CB due to the longer maturity, therefore using low interest rate and credit spread the value of the 10-year CB is higher compared to the other two. However, using a higher interest rate or credit spread the value of the bond part will dominate causing CBs with longer maturities to decrease faster when moving rightwards in the figure. The exact shape of the curves depends of course on the coupons, too (which is 5% per annum in this example).

Figure 10 shows the sensitivities of callable and putable convertible bonds to the changes
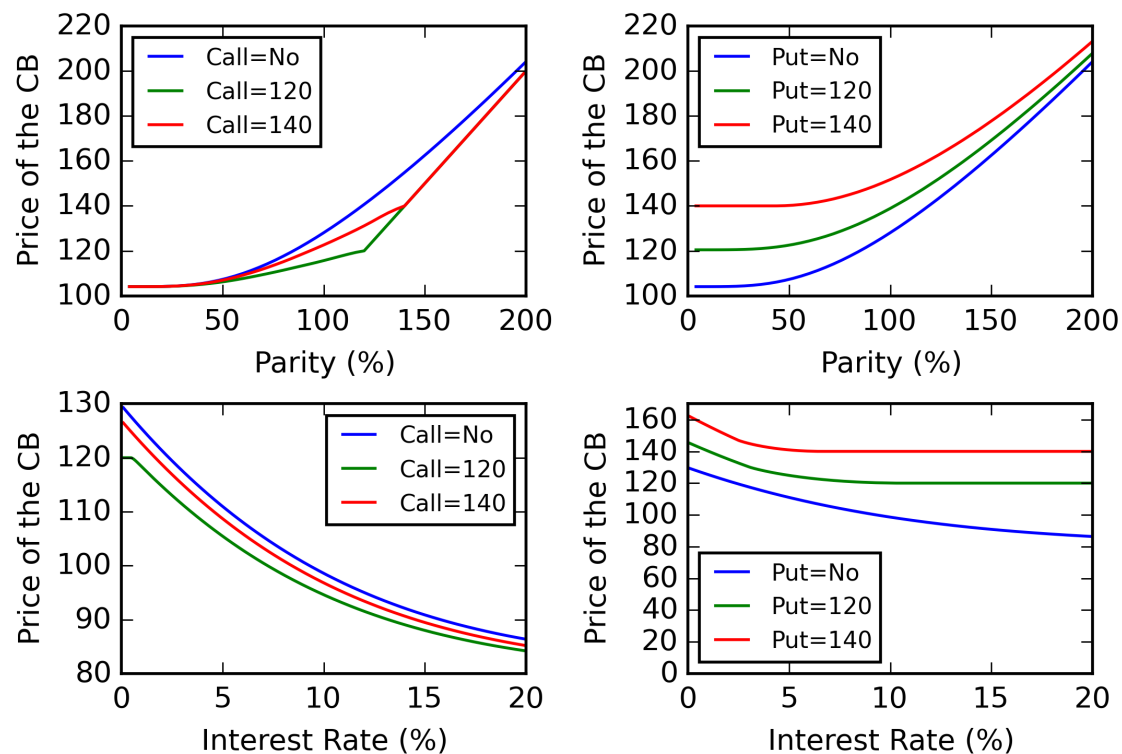
Figure 10: Sensitivities of the model using callability and putability to the equity price and interest rate

of the parity and the risk-free interest rate. Each case uses the sample convertible bond introduced above as a benchmark and compare it with two CBs with continuous callability at $K_c = 120$ and $K_c = 140$ and two CBs with a continuous putability at $K_p = 120$ and $K_p = 140$.

The sensitivity of the callable CBs to the underlying equity is shown on the top left corner of Figure 10. Callability is advantageous to the issuer, therefore a lower callability threshold causes a lower CB price as shown in the figure. In case of callable CBs there is a point in the curve where the curve is not differentiable due to the break of the curve. This is caused by the CB's forced conversion when the issuer calls the bond back and the investor is forced to convert into shares. Due to the forced conversion the convergence to the parity is faster in case of a callable CB than in case of a non-callable CB.

The top right corner of Figure 10 shows the sensitivity of the putable CBs to the underlying equity. As putability is advantageous for the investor, because it provides an opportunity for the investor to put back the CB to the issuer at a predetermined price, the price of a putable CB is higher in case of higher putability threshold. The price of the putable CBs converge to parity if the equity price is high and converge to the putability threshold if the equity price is low.

The third subfigure of Figure 10 shows the sensitivity of the callable CBs to the risk-free interest rate. It is not surprising that the price of the non-callable CB is the highest and the price of the callable CB at $K_c = 120$ is the lowest. As seen before the values of the CBs are a decreasing function of the risk-free interest rate assuming that the other parameters remain unchanged. The price of the callable CBs converge into the callability threshold if the risk-free interest rate is sufficiently low, which is the case in the left side of the curve of the callable CB with $K_c = 120$.

The last subfigure of Figure 10 shows the sensitivity of the putable CBs to the risk-free interest rate. The price of the putable CBs is also a decreasing function of the interest rate and the CBs converge into the putability threshold if the risk-free interest rate is sufficiently high.
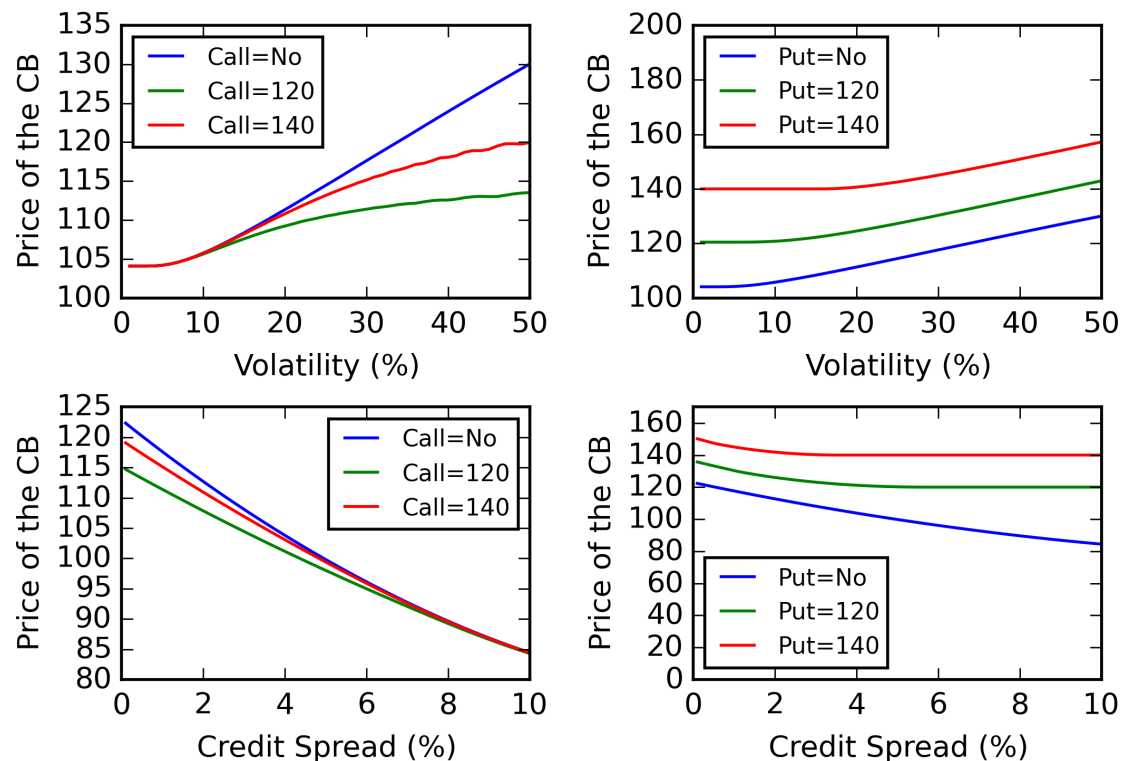

Figure 11: Sensitivities of the model using callability and putability to volatility and credit spread

Figure 11 shows the sensitivity of callable and putable convertible bonds to the changes of the volatility and credit spread for the same sample CB.

The top left corner of Figure 11 shows the sensitivity of the callable CBs to the volatility. The price of the CBs are the same in case of low volatility, but this is specific to this sample bond, as in case of low volatility the price of the underlying equity reaches the callability threshold with lower probability. In case of higher volatility the price of the CBs are different and as usual the price of the CBs with lower callability threshold are lower.

The sensitivity of the putable CBs to the volatility is shown on the top right corner of Figure 11. Because the price of the sample CB is below the putability thresholds in case of small volatility the prices of the putable CBs converge to the putability threshold.

The third subfigure of Figure 11 shows the sensitivity of the callable CBs to the credit spread. The effect is similar to the effect of the risk-free interest rate: the price of the CB is a decreasing function of the credit spread. The prices of the CBs converge to each other in case of high credit spread, because the probability of the call event is low in case of higher credit spread, therefore the callability option is worthless to the issuer.

The sensitivity of the putable CBs to the credit spread is shown on the last subfigure of Figure 11. The effect is similar to the effect of the risk-free rate: the price of the CB decreases as the

credit spread increases, therefore the investor put the CB back to the issuer and the price of the CBs converge to the putability threshold.

This section investigated the sensitivities of the pricing model to the most important risk factors of convertible bonds, such as the price of the underlying equity, risk-free interest rate, volatility and credit spread. The sensitivity analysis showed that the behaviour of the model is quite complex even if one parameter is changed in each case while supposing the other parameters remain unchanged.

Conclusion

Pricing convertible bonds is a challenging area due to the complicated payoff structure, the exotic features and the possible links between the risk factors of the equity part and the bond part of the convertible bond. There are a wide range of pricing models available in the literature including numerical and analytical solutions.

This paper introduced the basic properties and features of convertible bonds and showed the implementation of a pricing model based on a binomial tree covering callable, putable, contingent convertible and soft callable CBs. The basic pricing model was extended using more sophisticated credit risk models and the convergence and sensitivity of the model were investigated.

Our further research on convertible bond pricing will consider some modifications of the pricing model. Using this binomial tree framework there are at least three general ways of changing the model. First, one can improve the model including other features used in the market for example step coupon CBs, resets or perpetual CBs. Second, one can reconsider the behaviour of the risk factors such as equity price, interest rate and credit spread (e.g. modelling stochastic interest rates). Finally, one could improve the convergence of the model and reduce the quantization and specification errors with the methods introduced in Section 3.1.

Glossary of Notations

$\mathbf{AI}$:   Accrued interest as a percent of the face value

$\mathbf{AI_{USD}}$:   Accrued interest denominated in USD

$\mathbf{B_F}$:   Bond floor of the convertible bond

$\mathbf{c}$:   Annual coupon payment of the convertible bond (%)

$\mathbf{c_{USD}}$:   Coupon of the convertible bond at time t denominated in USD

$\mathbf{CB}$:   Convertible bond

$\mathbf{CS}$:   Credit spread

$\mathbf{C_p}$:   Conversion premium of the convertible bond

$\mathbf{C_r}$:   Conversion ratio of the convertible bond

$\mathbf{d}$:   Down movement of the binomial equity tree

$\mathbf{f(t)}$:   Forward rate between t and $t + \Delta t$

$\mathbf{f_c}$:   Coupon frequency of the convertible bond

$\mathbf{H}$:   Continuation value of the convertible bond

$\mathbf{IP}$:   Investment premium of the convertible bond

$\mathbf{K_c}$:   Call price of the convertible bond expressed as a percentage of the face value

$\mathbf{K_c^{USD}}$:   Callability threshold denominated in USD

$\mathbf{K_{CoCo}}$:   Contingent conversion threshold of the convertible bond expressed as a percentage of the conversion price

$\mathbf{K_p}$:   Put price of the convertible bond expressed as a percentage of the face value

$\mathbf{K_p^{USD}}$:   Putability threshold denominated in USD

$\mathbf{K_{SoftCall}}$:   Soft callability threshold of the convertible bond expressed as a percentage of the conversion price

$\mathbf{n}$:   Number of periods used in the pricing model

$\mathbf{N}$:   Face value of the convertible bond

$\mathbf{N(t)}$:   The value of a Poisson process at time t

$\mathbf{p_{conv}}$:   The probability of conversion at the current node of the valuation tree

$\mathbf{p_{conv}^{down}}$:   The probability of conversion at the down node of the consecutive level of the valuation tree

$\mathbf{p_{conv}^{up}}$:   The probability of conversion at the up node of the consecutive level of the valuation tree

$\mathbf{P}$:   Price of the convertible bond

$\mathbf{P_a}$:   Parity of the convertible bond

$\mathbf{q}$:   Constant risk-neutral probability

$\mathbf{q(t)}$:   Risk-neutral probability between t and $t + \Delta t$

$\mathbf{q_s}$:   Constant survival probability

$\mathbf{q_s(t)}$:   Survival probability between t and $t + \Delta t$

$\mathbf{Q}$:   Risk-neutral measure

$\mathbf{r}$:   Continuously compounded risk-free interest rate

$\mathbf{r_b}$:   Continuously compounded risky interest rate

$\mathbf{R}$:   Redemption of the convertible bond

$\mathbf{RR}$:   Recovery rate

$\mathbf{S}$:   Price of the underlying stock

$\mathbf{S_{Default}}$:   Price of the underlying stock in case of default

S^* :	Conversion price of the convertible bond
t :	Time parameter
T :	Maturity date of the convertible bond
u :	Up movement of the binomial equity tree
V :	Value of the convertible bond at the current node of the valuation tree
V_d :	Value of the convertible bond at the down node of the consecutive level of the valuation tree
V_R :	Recovery value of the convertible bond in case of default
V_u :	Value of the convertible bond at the up node of the consecutive level of the valuation tree
W :	Wiener process
$y(t)$:	Credit-adjusted discount rate
δ :	Continuously compounded dividend yield of the underlying equity
Δt :	One time step of the binomial model
λ:	Default intensity
Ω_{Call} :	Periods when the convertible bond is callable
Ω_{CoCo} :	Periods when the convertible bond is contingent convertible
Ω_{Put} :	Periods when the convertible bond is putable
$\Omega_{SoftCall}$:	Periods when the convertible bond is soft callable
σ:	Volatility of the underlying stock
τ:	First arrival time of a Poisson process

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