ECON526: Quantitative Economics with Data Science Applications

Regression Analysis

Phil Solimine

philip.solimine@ubc.ca

University of British Columbia

Table of contents

- Overview
- Linear Regression
- Treatment Effects
- Regression with Nonlinearities
- Interactions
- Translating Causal Diagrams to Regression

Overview

Summary

- Previously we have discussed estimating treatment effects using the difference in means.
- In this lecture we will discuss how to estimate treatment effects using regression.
- I assume that you are familiar with the basics of linear regression, including the assumptions underlying OLS and the interpretation of regression coefficients.

Linear Regression

Review of Linear Regression

- Simple linear regression is a statistical method that allows us to model the relationship between two variables: a dependent variable and an independent variable.
- The goal of simple linear regression is to find the line of best fit that describes the relationship between the two variables.
- The line of best fit is defined as the line that minimizes the sum of the squared differences between the observed values and the predicted values.
- Mathematically, the linear regression equation is the *least-norm* solution to an overdetermined system of linear equations.
  - → $y = X\beta + \epsilon \longrightarrow \epsilon = y - X\beta$
  - → $\min_\beta ||\epsilon||_2^2 = \min_\beta \epsilon'\epsilon = \min_\beta (y - X\beta)'(y - X\beta)$
  - → $\hat{\beta} = \arg\min_\beta (y - X\beta)'(y - X\beta) = (X'X)^{-1}X'y$

Review of Linear Regression

- Aside: Computational complexity of OLS
- We could solve for the OLS coefficient in a number of ways. However, some ways will be substantially faster than others.

```
from scipy.optimize import minimize
import statsmodels.api as sm
import timeit

# Generate data
np.random.seed(123)
n = 10000
X = np.random.normal(size=(n, 10))
y = np.random.normal(size=n)

# Define some different OLS functions

# Most naive way - never do this!
def ols_lstsq(X, y):
    return minimize(lambda beta: np.linalg.norm(X @
```

```
Time to compute OLS coefficients numerically:
12 ms ± 259 µs per loop (mean ± std. dev. of 7 runs, 100
loops each)

Time to compute OLS coefficients with a matrix inverse:
221 µs ± 10.1 µs per loop (mean ± std. dev. of 7 runs,
1,000 loops each)

Time to compute OLS coefficients by solving a linear
system:
120 µs ± 2.09 µs per loop (mean ± std. dev. of 7 runs,
10,000 loops each)
```

Review of Linear Regression

- Aside: Computational complexity of OLS
- Solving a matrix inverse involves computing the inverse of a matrix, which is a computationally expensive operation.
  - $\rightarrow$ The most common way to find the inverse of a matrix is to solve n linear systems, where n is the number of columns in the matrix.
  - $\rightarrow$ Each linear system is of the form $(X'X)\beta = e_i$, where X is a matrix and e_i is a vector of zeros with a 1 in the i_{th} position.
  - $\rightarrow$ Solved this way, it would take $O(n^3)$ operations to find the full inverse.
  - $\rightarrow$ The fastest known way to take a matrix inverse is ~$O(n^{2.375})$.
- On the other hand, we don't need to know the full inverse of $(X'X)$. We only need to know the solution to the linear system $(X'X)\beta = X'y$.
  - $\rightarrow$ This can be solved in just one set of $O(n^2)$ operations.

Review of Linear Regression

- In this course, we will use the `statsmodels` package to estimate linear regression models.
  - → It may not be the fastest, but it gets us a whole lot more information.

```python
import statsmodels.api as sm
import statsmodels.formula.api as smf

sm_results = sm.OLS(y, X).fit()
print(sm_results.summary())
```

```
                            OLS Regression Results
=======================================================================================
Dep. Variable:                      y   R-squared (uncentered):                   0.001
Model:                            OLS   Adj. R-squared (uncentered):              0.000
Method:                 Least Squares   F-statistic:                              1.198
Date:                Sun, 29 Oct 2023   Prob (F-statistic):                       0.286
Time:                        23:27:59   Log-Likelihood:                         -14130.
No. Observations:               10000   AIC:                                  2.828e+04
Df Residuals:                    9990   BIC:                                  2.835e+04
Df Model:                          10
Covariance Type:            nonrobust
==============================================================================
                 coef    std err          t      P>|t|      [0.025      0.975]
------------------------------------------------------------------------------
```

Treatment Effects

Treatment Effects

- When estimating "treatment effects" with linear regression, we run into exactly the same problem as when we estimate using the difference in means.
  - $\rightarrow$ $Y_i = Y_{0i}(1 - T_i) + Y_{1i}T_i$
  - $\rightarrow$ If we don't know both potential outcomes, we can't estimate the treatment effect.
- However, we can estimate the treatment effect if we make some assumptions about the data generating process.
  - $\rightarrow$ The best case scenario is that we have an RCT.
  - $\rightarrow$ Otherwise, we have to make sure that we close off all other channels through which the treatment could affect the outcome.

Treatment Effects

- For starters, let's assume that we have an RCT.
- We'll compute the treatment effect both using the difference in means we used before, and also using regression.

```
df = pd.read_csv("data/online_classroom.csv")
df_no_blended = df[df["format_blended"] == 0]

# Difference in means
te = df_no_blended.groupby("format_ol")["falsexam"].mean().diff()
print(f"Estimated treatment effect: {te[1]:.3f}\n")

# Regression
sm_results = smf.ols("falsexam ~ format_ol", data=df_no_blended).fit()
print(sm_results.summary().tables[1])
```

```
Estimated treatment effect: -4.912

==============================================================================
                 coef    std err          t      P>|t|      [0.025      0.975]
------------------------------------------------------------------------------
Intercept     78.5475      1.113     70.563      0.000      76.353      80.742
format_ol     -4.9122      1.680     -2.925      0.004      -8.223      -1.601
------------------------------------------------------------------------------
```

Treatment Effects

- Because we just have a binary treatment, the difference in means and the regression coefficient are the same.
  - → $E[Y_i \mid T_i] = \beta_0 + \beta_1 T_i + \epsilon_i$
  - → $$\bar{\tau} = E[Y_i \mid T_i = 1] - E[Y_i \mid T_i = 0] = (\beta_0 + \beta_1 * 1) - (\beta_0 + \beta_1 * 0) = \beta_1$$
- With only one (binary) variable, it may not be clear why we (usually) prefer regression over the difference in means.
  - → However, regression makes it much easier to control for other variables.

Treatment Effects

- For example, suppose we want to control for the student's gender.
  - → In this example, gender is not a confounder because we have an RCT.
  - → However, we can still control for it in the regression, and it should improve the precision of our estimate.
- To "control for" a covariate using the difference in means, we have to create subsamples of the data that have no variation in that variable
  - → In other words, we look at *conditional* differences in means.
- With regression, controlling for a variable is as simple as including it on the right hand side.

Regression with Nonlinearities

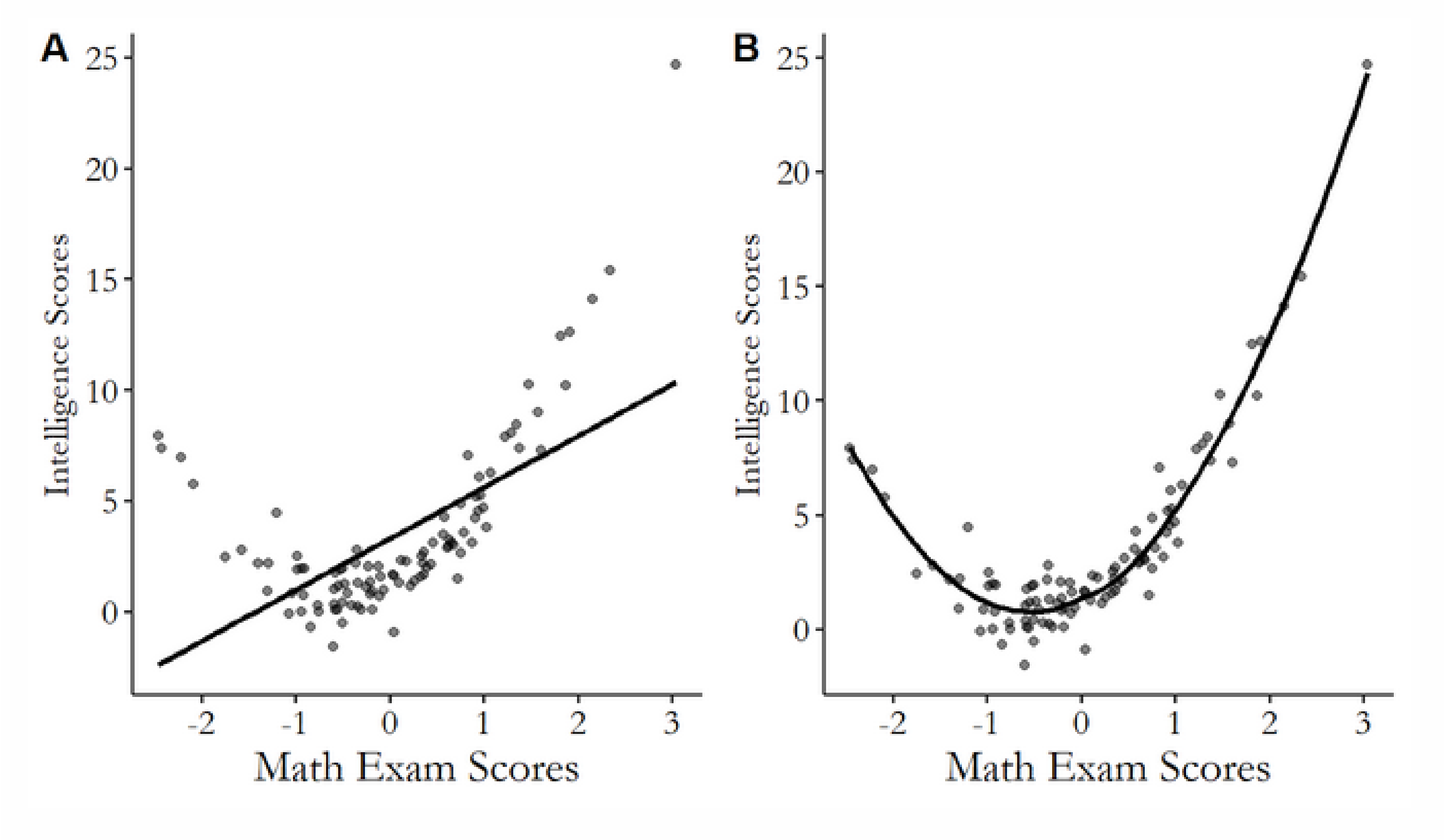
Regression with Nonlinearities

- What if we don't think that there is a linear relationship between the treatment and the outcome?
- We can still use linear regression, but we need to be careful about how we interpret the coefficients.
- For example, if we think that the treatment effect is nonlinear, we can include a quadratic term for the treatment.
  - → $Y_i = \beta_0 + \beta_1 T_i + \beta_2 T_i^2 + \epsilon_i$
  - → This works simply by creating a new column in the data that is the square of the treatment variable.

Regression with Nonlinearities

- We can also include other nonlinear transformations of the treatment variable.
  - → $Y_i = \beta_0 + \beta_1 T_i + \beta_2 \log(T_i) + \epsilon_i$
  - → $Y_i = \beta_0 + \beta_1 T_i + \beta_2 T_i^2 + \beta_3 \log(T_i) + \epsilon_i$
  - → $Y_i = \beta_0 + \beta_1 T_i + \beta_2 T_i^2 + \beta_3 \log(T_i) + \beta_4 \log(T_i)^2 + \epsilon_i$
- But we should be careful doing stuff like this. It's easy to overfit the data.
  - → In fact, if we have k data points and specify a $k-1$-th order polynomial, we can always fit the data perfectly.

Regression with Nonlinearities



Regression with Nonlinearities

- We could also transform the dependent variable, rather than the treatment variable.
- For example, let's imagine we wanted to estimate a Cobb-Douglas function.
  - → $Y = zK^{\alpha}L^{\beta}$ where z is productivity, K is capital, and L is labor.
  - → This equation is nonlinear in the parameters α and β.
  - → However, we can transform it to be linear using the log.
  - → $\ln Y = \ln z + \alpha \ln K + \beta \ln L$
- This trick works whenever the dependent variable is an invertible function of an equation that is linear in parameters.
  - → $Y = f(\beta_0 + \beta_1 X_1 + \beta_2 X_2 + \cdots + \beta_k X_k)$
  - → $f^{-1}(Y) = \beta_0 + \beta_1 X_1 + \beta_2 X_2 + \cdots + \beta_k X_k$

Limited Dependent Variables

- Sometimes the dependent variable is not continuous.
  - $\rightarrow$ For example, it might be binary (did the student pass the exam?)
  - $\rightarrow$ Or it might be a count (how many times did the student log in canvas?)
- In these cases, we shouldn't just apply OLS blindly
  - $\rightarrow$ Often we can work around this by transforming the dependent variable.
  - $\rightarrow$ For example, we can use a logistic regression to estimate the *probability* of passing the exam.
- Let $F(x) = \frac{e^x}{1+e^x}$ be the logistic function and
  - $\rightarrow$ $P(Y_i = 1) = F(\beta_0 + \beta_1 X_{1i} + \beta_2 X_{2i} + \cdots + \beta_k X_{ki})$
  - $\rightarrow$ $Y_i \sim \text{Bernoulli}(P(Y_i = 1))$

Limited Dependent Variables

```
from causaldata import restaurant_inspections
df = restaurant_inspections.load_pandas().data

df.head()
```

	business_name	inspection_score	Year	NumberofLocations	Weekend
0	MCGINLEYS PUB	94	2017	9	False
1	VILLAGE INN #1	86	2015	66	False
2	RONNIE SUSHI 2	80	2016	79	False
3	FRED MEYER - RETAIL FISH	96	2003	86	False
4	PHO GRILL	83	2017	53	False

Limited Dependent Variables

```python
# Statsmodels wants the dependent variable to be numeric
df["Weekend"] = 1*df["Weekend"]

# How is the frequency of weekend inspections changing over time?
m1 = smf.logit(formula = "Weekend ~ Year", data = df).fit()

print(m1.summary().tables[1])
```

```
Optimization terminated successfully.
         Current function value: 0.045192
         Iterations 10
==============================================================================
                 coef    std err          z      P>|z|      [0.025      0.975]
------------------------------------------------------------------------------
Intercept     44.2360     23.502      1.882      0.060      -1.828      90.300
Year          -0.0244      0.012     -2.088      0.037      -0.047      -0.002
==============================================================================
```

Limited Dependent Variables

- The coefficient on `Year` is negative, which indicates that the frequency of weekend inspections is decreasing over time.
- Notice that we have a new message: “Optimization terminated successfully”
  - → Since this GLM is nonlinear, we can’t use the formula for OLS

Limited Dependent Variables

- In order to actually interpret this coefficient, we prefer to know the **marginal effect** of `Year` rather than the coefficient value itself.

```
# Compute the marginal effect of Year
print(m1.get_margeff(at="mean").summary())
```

```
        Logit Marginal Effects
=====================================
Dep. Variable:                Weekend
Method:                          dydx
At:                              mean
==============================================================================
                dy/dx    std err          z      P>|z|      [0.025      0.975]
------------------------------------------------------------------------------
Year          -0.0002   8.79e-05     -2.110      0.035      -0.000   -1.32e-05
==============================================================================
```

Interactions

Interactions

- What if the treatment effect itself is heterogeneous?
  - → For example, maybe the treatment effect is larger for students who are more engaged in the class.
- When comparing means, we would have to create subsamples of the data and estimate the treatment effect separately for each subsample.
- In a regression equation, we can incorporate heterogeneous treatment effects using **interaction terms**.
  - → $Y_i = \beta_0 + \beta_1 T_i + \beta_2 X_i + \beta_3 T_i X_i + \epsilon_i$
  - → $E[Y_i \mid T_i, X_i] = \beta_0 + \beta_1 T_i + \beta_2 X_i + \beta_3 T_i X_i$

Interactions

```python
# Use * to include two variables independently
# plus their interaction
# (: is interaction-only, we rarely use it)
m1 = smf.ols(formula = "inspection_score ~ NumberofLocations*Weekend + Year", data = df).fit()

print(m1.summary().tables[1])
m1.t_test("NumberofLocations + NumberofLocations:Weekend = 0")
```

```
============================================================================================
                                 coef    std err          t      P>|t|      [0.025      0.975]
--------------------------------------------------------------------------------------------
Intercept                    225.1260     12.415     18.134      0.000     200.793     249.460
NumberofLocations             -0.0191      0.000    -43.759      0.000      -0.020      -0.018
Weekend                        1.7592      0.488      3.606      0.000       0.803       2.715
NumberofLocations:Weekend     -0.0098      0.008     -1.307      0.191      -0.025       0.005
Year                          -0.0648      0.006    -10.494      0.000      -0.077      -0.053
============================================================================================

<class 'statsmodels.stats.contrast.ContrastResults'>
                             Test for Constraints
==============================================================================
                 coef    std err          t      P>|t|      [0.025      0.975]
------------------------------------------------------------------------------
c0            -0.0289      0.008     -3.851      0.000      -0.044      -0.014
==============================================================================
```

Interactions

- In this example, it looks like the effect of the number of locations on the inspection score is strengthened on the weekend.
  - → (Assuming we have some causal model to justify that this is unbiased)
- Interactions are easiest to interpret when they are categorical; each group has their own treatment effect.
- However, they also work with continuous variables.
  - → In this case, the treatment effect is a function of the other variable.
  - → We could say that a certain variable "moderates" the treatment effect.

Interactions

- Why not just make everything interact with everything else?
  → We could, but it would be hard to interpret the coefficients.
  → It would also be easy to overfit the data.
  → In order to identify the interaction terms, we want lots of variation across both variables.
- Interactions are most useful when we have a *theory* about how the treatment effect varies with other variables.
  → For example, we might think that the treatment effect of online class is larger for students who are more engaged in the class, or vice versa
- Generally, it's good to include interactions if they are a primary focus of the analysis.

Translating Causal Diagrams to Regression

Translating Causal Diagrams to Regression