# ECON526: Quantitative Economics with Data Science Applications

*Stochastic Processes, Markov Chains, and Expectations*

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# Overview

# Summary

- Here we build on the previous lecture on probability and distributions to introduce stochastic processes, Markov processes, and expectations/forecasts

We will introduce,

1. **Stochastic Processes** a sequence of events where the probability of the next event depends the past events
2. **Markov Processes** a stochastic process where the probability of the next event depends only on the current event

# Packages and Other Materials

- See the following for extra material - some of which were used in these notes
  - → QuantEcon Markov Chains
  - → Intermediate QuantEcon Markov Chains
  - → QuantEcon AR1 Processes

```python
import matplotlib.pyplot as plt
import pandas as pd
import numpy as np
import scipy.stats
import seaborn as sns
from scipy.stats import rv_discrete
from numpy.linalg import matrix_power
```

# Stochastic and Markov Processes

# Discrete-time Stochastic Process

- A **stochastic process** is a sequence of random variables $\{X_t\}_{t=0}^{\infty}$ [1]
- Events in $\Omega$ are subtle to define because they contain nested information
  - $\rightarrow$ e.g. the realized random variable $X_t$ depends on $X_{t-1}$, $X_{t-2}$, and changes the future random variables $X_{t+1}$, $X_{t+2}$, etc.
  - $\rightarrow$ Similarly, the probability of $X_{t+1}$ is effected by the realized $X_t$ and $X_{t-1}$
- Intuitively we can work with each $\{X_t\}_{t=0}^{\infty}$ and look at conditional distributions by considering independence, etc.

1. See formal definition here

# Information Sets and Forecasts

- Expectations and conditional expectations give us notation for making forecasts while carefully defining information available
  - → More general, and not specific to stochastic processes or forecasts
  - → Might to "nowcast" or "smooth" to update your previous estimates
- To formalize
  1. Define **information set** as the known random variables
  2. Provide a random variable that is **forecast** using the information set
  3. Typically, provide a function of the random variable of interest and calculate the **conditional expectation** given the information set

# Forecasts and Conditional Probability Distributions

- Take a stochastic process $\{X_t\}_{t=0}^{\infty}$
- Define the **information set** at $t$ as $\mathcal{I}_t \equiv \{X_0, X_1, \ldots, X_t\}$
- The **conditional probability** of $X_{t+1}$ given the information set $\mathcal{I}_t$ is

$$\mathbb{P}(X_{t+1} \,|\, X_t, X_{t-1}, \ldots X_0) \equiv \mathbb{P}(X_{t+1} \,|\, \mathcal{I}_t)$$

→ e.g. the probability of being unemployed, unemployed, or retired next period given the full workforce history

→ Useful in macroeconomics when you want to formalize expectations of the future, as well as econometrics when you want to update estimates given different amounts of observation

# Forecasts and Conditional Expectations

- You may instead be interested in a function, $f(\cdot)$, of the random variable (e.g., financial payoffs, utility, losses in econometrics)
- Use the conditional probability of the forecasts for **conditional expectations**

$$\mathbb{E}[f(X_{t+1}) \,|\, X_t, X_{t-1}, \ldots X_0] \equiv \mathbb{E}[f(X_{t+1}) \,|\, \mathcal{I}_t]$$

$\rightarrow$ e.g. the expected utility of being unemployed next period given the history of unemployment; or the expected dividends in a portfolio next period given the history of dividends

- Standard properties of expectations hold conditioning on information sets,
  - $\rightarrow$ $\mathbb{E}[A\,X_{t+1} + B\,Y_{t+1} \,|\, \mathcal{I}_t] = A\,\mathbb{E}[X_{t+1} \,|\, \mathcal{I}_t] + B\,\mathbb{E}[Y_{t+1} \,|\, \mathcal{I}_t]$
  - $\rightarrow$ $\mathbb{E}[X_t \,|\, \mathcal{I}_t] = X_t$, i.e., not stochastic if the information set $X_t$

# Easy Notation for Information Sets

- Information sets in stochastic processes are often just a sequence for the entire history. Hence the time, $t$, is often sufficient
- Given $\mathcal{I}_t \equiv \{X_0, X_1, \ldots, X_t\}$ for shorthand we can denote

$$
\begin{aligned}
\mathbb{E}[f(X_{t+1}) \,|\, X_t, X_{t-1}, \ldots X_0] &\equiv \mathbb{E}[f(X_{t+1}) \,|\, \mathcal{I}_t] \\
&\equiv \mathbb{E}_t[f(X_{t+1})]
\end{aligned}
$$

# Law of Iterated Expectations for Stochastic Processes

- Recall that $\mathcal{I}_t \subset \mathcal{I}_{t+1}$ since $X_{t+1}$ is known at $t+1$
- The **Law of Iterated Expectations** can be written as

$$
\begin{aligned}
\mathbb{E}\left[\mathbb{E}[X_{t+2} \mid X_{t+1}, X_t, X_{t-1}, \ldots] \mid X_t, X_{t-1}, \ldots\right] &= \mathbb{E}[X_{t+2} \mid X_t, X_{t-1}, \ldots] \\
\mathbb{E}\left[\mathbb{E}[X_{t+2} \mid \mathcal{I}_{t+1}] \mid \mathcal{I}_t\right] &= \mathbb{E}[X_{t+2} \mid \mathcal{I}_t] \\
\mathbb{E}_t[\mathbb{E}_{t+1}[X_{t+2}]] &= \mathbb{E}_t[X_{t+2}]
\end{aligned}
$$

- i.e. if I am forecasting my forecast, I can only use information available today

# Markov Processes

- **(1st-Order) Markov Process**: a stochastic process where the conditional probability of the future is independent of the past given the present

$$\mathbb{P}(X_{t+1} \mid X_t, X_{t-1}, \ldots) = \mathbb{P}(X_{t+1} \mid X_t)$$

  $\longrightarrow$ Or with information sets: $\mathbb{P}(X_{t+1} \mid \mathcal{I}_t) = \mathbb{P}(X_{t+1} \mid X_t)$

  $\longrightarrow$ i.e., the present sufficiently summarizes the past for predicting the future

- **Conditional expectations** are are then

$$\mathbb{E}[f(X_{t+1}) \mid X_t, X_{t-1}, \ldots X_0] = \mathbb{E}[f(X_{t+1}) \mid X_t]$$

# Martingales

- A stochastic process $\{X_t\}_{t=0}^{\infty}$ is a **martingale** if

$$\mathbb{E}[X_{t+1} \,|\, X_t, X_{t-1}, \ldots, X_0] = X_t$$

- Not all martingales are Markov processes, but most of the ones you will encounter are. If Markov,

$$\mathbb{E}[X_{t+1} \,|\, X_t] = X_t, \quad \text{or} \quad \mathbb{E}_t[X_{t+1}] = X_t$$

See here for a more formal definition with the complete set of requirements

# Random Walks

- Let $X_t \in \{-\infty, \ldots, -1, 0, 1, \ldots \infty\}$
- A simple two-state random walk can be written as the following transition

$$\mathbb{P}(X_{t+1} = X_t + 1 \,|\, X_t) = \mathbb{P}(X_{t+1} = X_t - 1 \,|\, X_t) = \frac{1}{2}$$

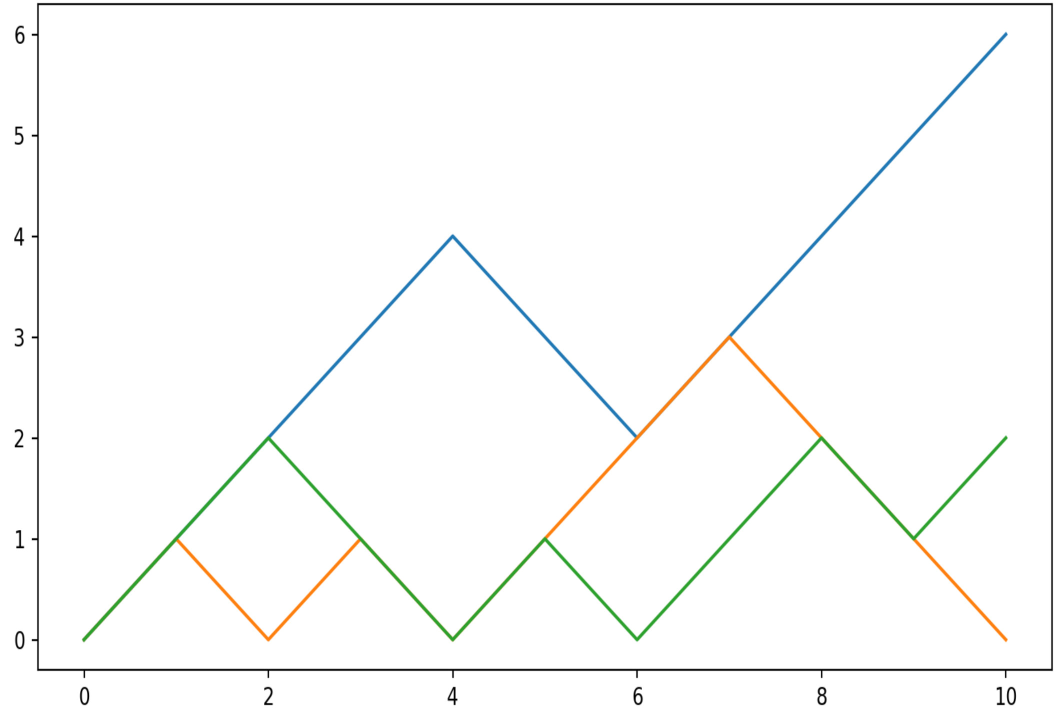
- Markov since $X_t$ summarizes the past. Martingale?

$$\begin{aligned}\mathbb{E}(X_{t+1} \,|\, X_t) &= \mathbb{P}(X_{t+1} = X_t + 1 \,|\, X_t) \times (X_t + 1) \\ &+ \mathbb{P}(X_{t+1} = X_t - 1 \,|\, X_t) \times (X_t - 1) \\ &= \frac{1}{2}(X_t + 1) + \frac{1}{2}(X_t - 1) = X_t\end{aligned}$$

# Implementation in Python

- Generic code to simulate a random walk with IID steps

```
def simulate_walk(rv, X_0, T):
  X = np.zeros((X_0.shape[0], T+1))
  X[:, 0] = X_0
  for t in range(1, T+1):
    X[:, t] = X[:, t-1] \
              +rv.rvs(size=X_0.shape[0])
  return X
steps = np.array([-1, 1])
probs = np.array([0.5, 0.5])
rv = rv_discrete(values=(steps, probs))
X_0 = np.array([0.0, 0.0, 0.0])
X = simulate_walk(rv, X_0, 10)
plt.figure()
plt.plot(X.T)
```

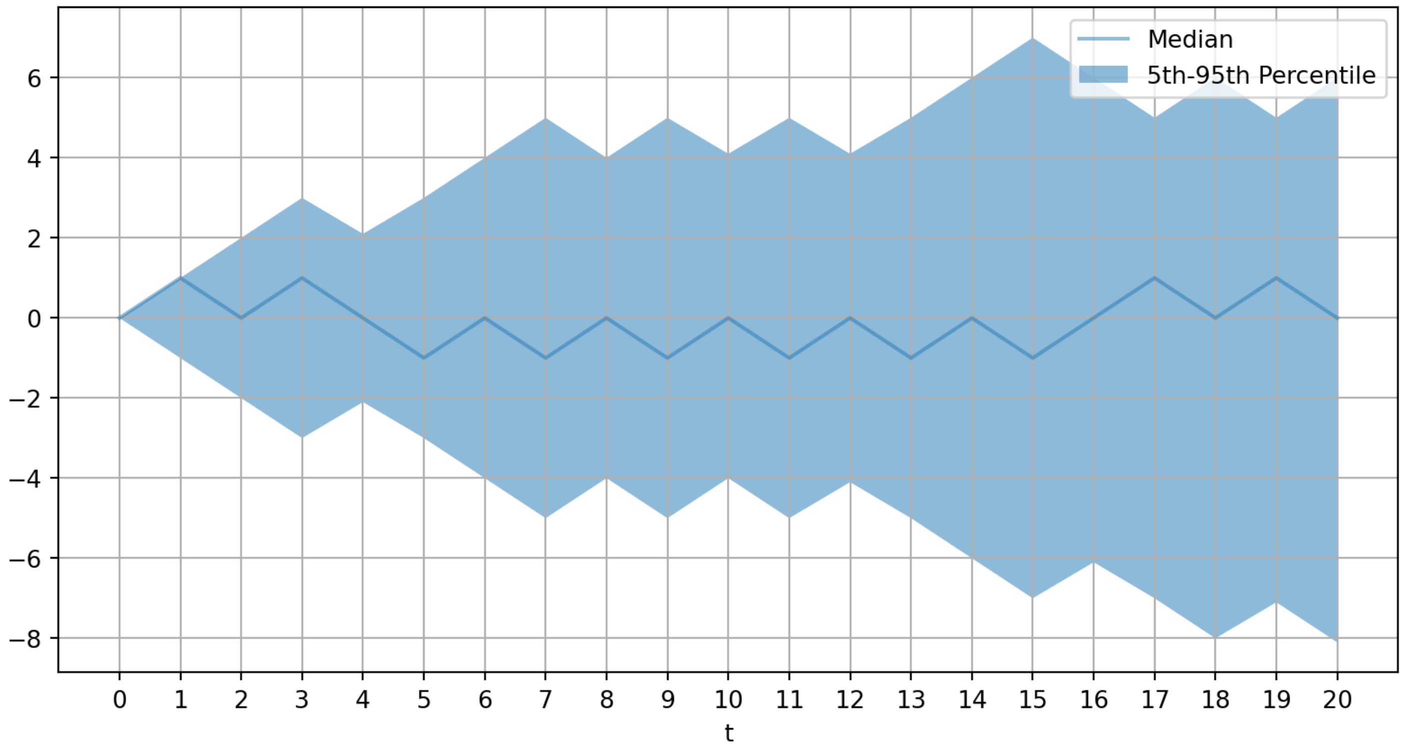


# Visualizing the Distribution of Many Trajectories

- $\mathbb{E}_0[X_t] \to 0$ for finite $t$ as $t \to \infty$
- But is there a limiting distribution of $X_t$ as $X_t \to \infty$?

```python
num_trajectories, T = 100, 20
X = simulate_walk(rv, np.zeros(num_trajectories), T)
percentiles = np.percentile(X, [50, 5, 95], axis=0)
fig, ax = plt.subplots()
plt.plot(np.arange(T+1), percentiles[0,:], alpha=0.5, label='Median')
plt.fill_between(np.arange(T+1), percentiles[1,:], percentiles[2,:],
  alpha=0.5, label='5th-95th Percentile')
plt.xlabel('t')
ax.set_xticks(np.arange(T+1))
plt.legend()
plt.grid(True)
```

# Visualizing the Distribution of Many Trajectories



# AR(1) Processes

- An **auto-regressive process** of order 1, AR(1), is the Markov process

$$X_{t+1} = \rho X_t + \sigma \epsilon_{t+1}$$

  → $\rho$ is the **persistence** of the process, $\sigma \geq 0$ is the **volatility**

  → $\epsilon_{t+1}$ is a random shock, we will assume $\mathcal{N}(0, 1)$

- Can show $X_{t+1} \,|\, X_t \sim \mathcal{N}(\rho X_t, \sigma^2)$ and hence

$$\mathbb{E}_t[X_{t+1}] = \rho X_t, \quad \mathbb{V}_t[X_{t+1}] = \sigma^2$$

For much more, see QuantEcon Lectures on AR(1)

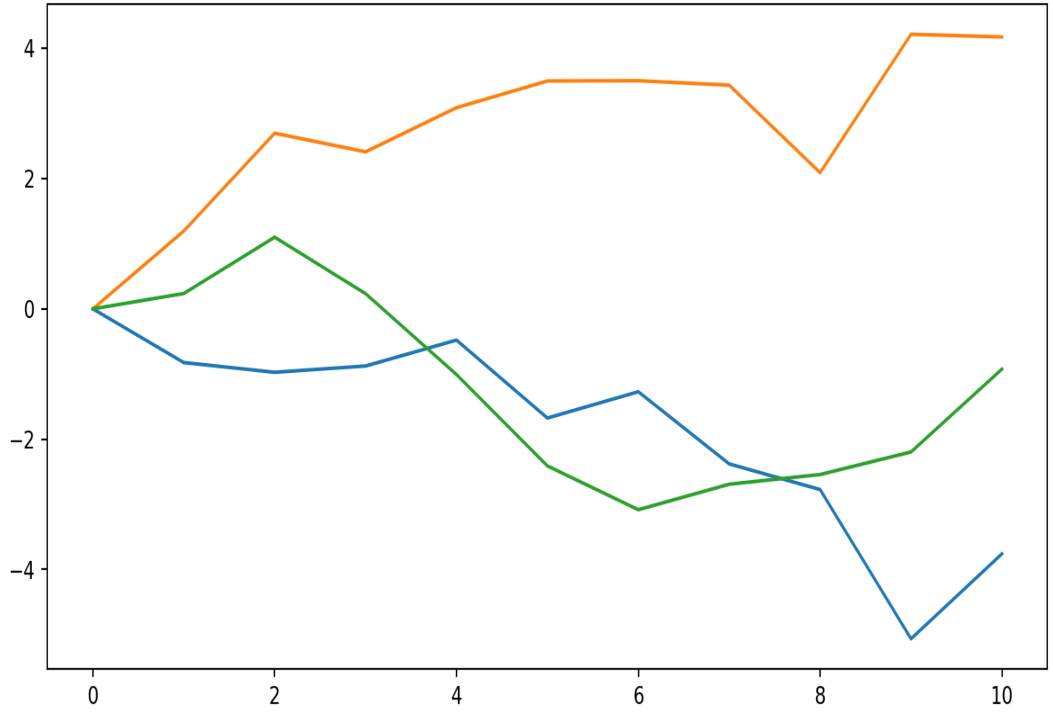
# Stationarity and Unit Roots

- **Unit roots** are a special case of AR(1) processes where $\rho = 1$
- They are important in econometrics because they tell us if processes have permanent or transitory changes
  - $\rightarrow$ The econometrics of finding whether $\rho = 1$ are subtle and important
- Note that if $\rho = 1$ then this is a **martingale** since $\mathbb{E}_t[X_{t+1}] = X_t$
- These are an important example of a **non-stationary process**.
- Intuitively: stationary if $X_t$ distribution has well-defined limit as $t \to \infty$
  - $\rightarrow$ Key requirements: $\lim_{t\to\infty} |\mathbb{E}[X_t]| < \infty$ and $\lim_{t\to\infty} \mathbb{V}(X_t) < \infty$

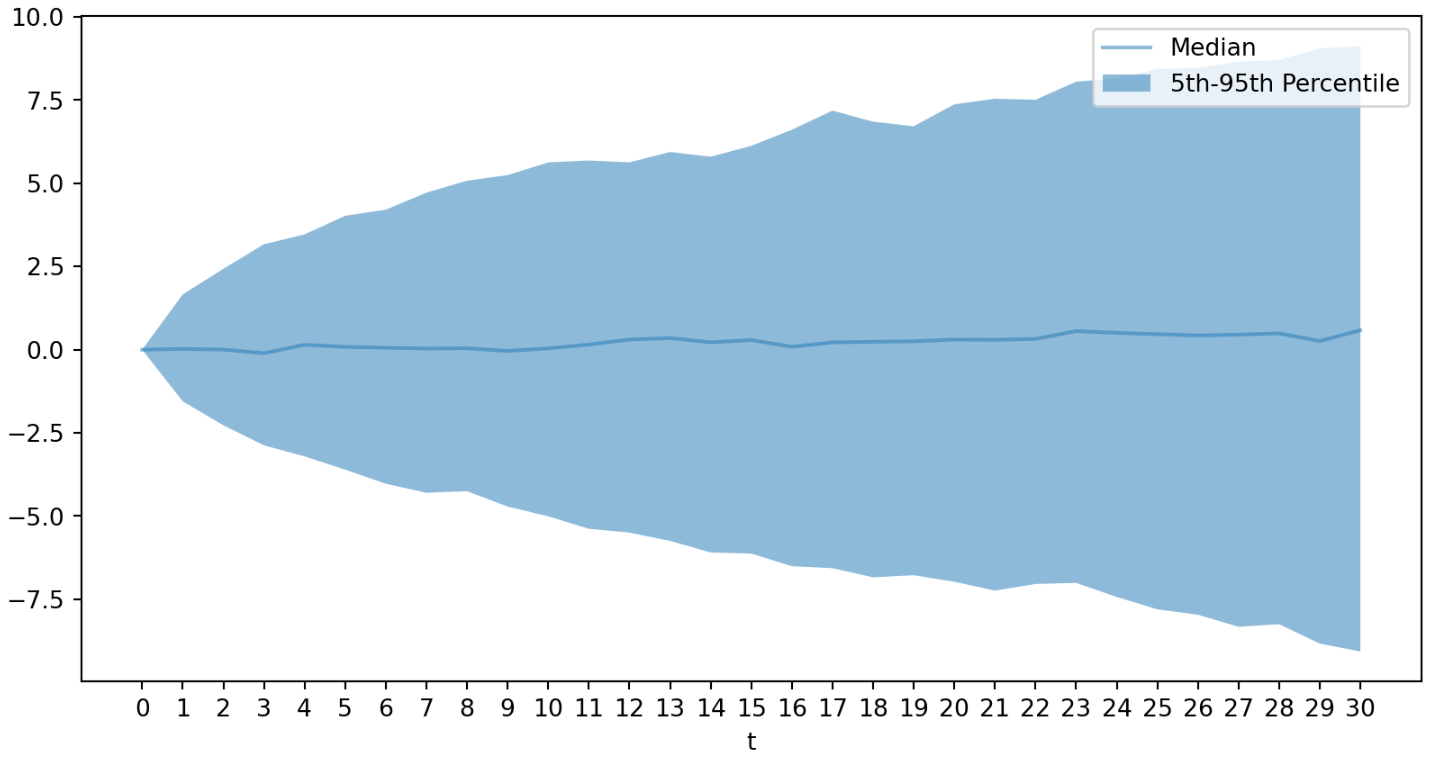
See here for a rigorous definitions and different types of stationarity and discussion of auto-covariance

# Simulating Unit Root

```
X_0 = np.array([0.0, 0.0, 0.0])
rv_epsilon = scipy.stats.norm(loc=0, scale=1)
X = simulate_walk(rv_epsilon, X_0, 10)
plt.figure()
plt.plot(X.T)
```



# Visualizing the Distribution of Many Trajectories



# Martingales and Arbitrage in Finance

- **Random Walks** are a key model in finance
  - → e.g. stock prices, exchange rates, etc.
- Central to no-arbitrage pricing, after adjusting to interest rates/risk/etc.
  - → e.g. if you could predict the future price of a stock, you could make money by buying/selling today
  - → Martingales have no systematic drift which leads to a key source of arbitrage (especially with options/derivatives)
- Does this prediction hold up in the data? Generally yes, but depends on how you handle risk/etc.
  - → If it were systematically wrong then hedge funds and traders would be far richer than they are now

# Information and Arbitrage

$$\mathbb{E}[\boldsymbol{X}_{t+1} \,|\, \mathcal{I}_t] = \boldsymbol{X}_t$$

- Given all of the information available, the best forecast of the future is the current price. Plenty of variables in $\mathcal{I}_t$ for individuals, including public prices
- Does this mean there is never arbitrage?

  $\rightarrow$ No, just that it may be short-term because prices feed back into $\mathcal{I}_t$

  $\rightarrow$ So some individuals make short term money given private information, but that information quickly becomes reflecting in other people's information sets (typically through prices)

  $\rightarrow$ How, and how quickly markets aggregate information is a key question in financial economics

# Markov Chains

# Discrete-Time Markov Chains

- A **Markov Chain** is a Markov process with a finite number of states
  - $\to$ $X_t \in \{0, \ldots, N-1\}$ be a sequence of Markov random variables
  - $\to$ In discrete time it can be represented by a **transition matrix** $P$ where

$$P_{ij} \equiv \mathbb{P}(X_{t+1} = j \,|\, X_t = i)$$

- We are counting from $0$ to $N-1$ for coding convenience in Python. Names of discrete states are arbitrary!
  - $\to$ Count from 1 in R, Julia, Matlab, Fortran, instead

A continuous-time Markov Chain instead uses a **transition rate matrix** $\Lambda$ where $\Lambda_{ij} = \lambda_{ij}$ is the rate of transitioning from state $i$ to state $j$. All rows such to $0$ rather than $1$. Many properties have analogies, for example there is an eigenvalue of $0$ rather than an eigenvalue of $1$

# Stochastic Matrices

- $P$ is a stochastic matrix if
  - $\rightarrow$ $\sum_{j=0}^{N-1} P_{ij} = 1$ for all $i$, i.e. rows are conditional distributions
- **Key Property:**
  - $\rightarrow$ One (or more) eigenvalue of $1$ with associated left-eigenvector $\pi$

$$\pi P = \pi$$

  - $\rightarrow$ Equivalently the right eigenvector with eigenvalue $= 1$

$$P^\top \pi^\top = \pi^\top$$

  - $\rightarrow$ Where we can normalize to $\sum_{n=0}^{N-1} \pi_i = 1$

# Transitions and Conditional Distributions

- The $P$ summarizes all transitions. Let $X_t$ be the state at time $t$ which in general is a probability distribution with pmf $\pi_t$
- Can show that the evolution of this distribution is given by

$$\pi_{t+1} = \pi_t \cdot P$$

- And hence given some $X_t$ we can forecast the distribution of $X_{t+j}$ with

$$X_{t+j} \mid X_t \sim \pi_t \cdot P^j$$

$\rightarrow$ i.e., using the matrix power we discussed in previous lectures

# Stationary Distribution

- Take some $X_t$ initial condition, does this converge?

$$\lim_{j \to \infty} X_{t+j} \,|\, X_t = \lim_{j \to \infty} \pi_t \cdot P^j = \pi_\infty ?$$

  $\longrightarrow$ Does it exist? Is it unique?

- How does it compare to fixed point below, i.e. does $\bar{\pi} = \pi_\infty$ for all $X_t$?

$$\bar{\pi} = \bar{\pi} \cdot P$$

  $\longrightarrow$ This is the eigenvector associated with the eigenvalue of $1$ of $P^\top$

  $\longrightarrow$ Can prove there is always at least one. If more than one, multiplicity

The conditions for stationary distributions, uniqueness, etc. are covered here

# Conditional Expectations

- Given the conditional probabilities, expectations are easy
- Now assign $X_t$ as a random variable with values $x_1, \ldots x_N$ and pmf $\pi_t$
- Define $x \equiv \begin{bmatrix} x_0 & \ldots & x_{N-1} \end{bmatrix}$
- From definition of conditional expectations

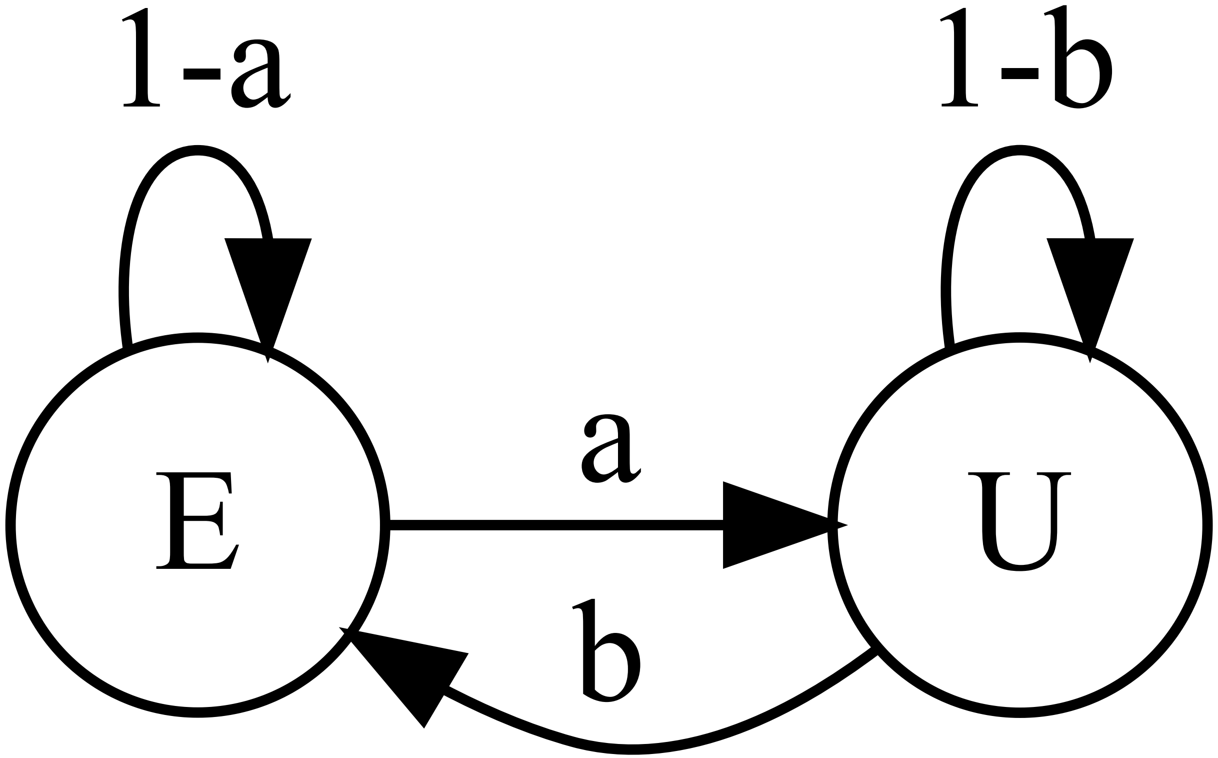
$$\mathbb{E}[X_{t+j} \,|\, X_t] = \sum_{i=0}^{N-1} x_i \pi_{t+j,i} = (\pi_t \cdot P^j) \cdot x$$

# Example of Markov Chain: Employment Status

- Employment(E) in state $\mathbf{0}$, Unemployment(U) in state $\mathbf{1}$
- $\mathbb{P}(U \mid E) = a$ and $\mathbb{P}(E \mid E) = 1 - a$
- $\mathbb{P}(E \mid U) = b$ and $\mathbb{P}(U \mid U) = 1 - b$
- Transition matrix $P \equiv$

$$
\begin{array}{rc}
 & \begin{array}{cc} \overbrace{\phantom{1-a}}^{X_{t+1}=E} & \overbrace{\phantom{1-b}}^{X_{t+1}=U} \end{array} \\
\begin{array}{r} X_t=E \,\} \\ X_t=U \,\} \end{array} & \begin{bmatrix} 1-a & a \\ b & 1-b \end{bmatrix}
\end{array}
$$

# Visualizing the Chain



# Transitions and Probabilities

- Let $\pi_0 \equiv \begin{bmatrix} 1 & 0 \end{bmatrix}^\top$, i.e. $\mathbb{P}(X_0 = E) = 1$
- The distribution of $X_1$ is then $\pi_1 = \pi_0 \cdot P$
  - $\rightarrow$ $\mathbb{P}(X_1 = E \,|\, X_0 = E) = \pi_{11}$ (first element)
  - $\rightarrow$ Can use to forecast probability of employment $j$ periods in future
- Can also use our conditional expectations to calculate expected income
  - $\rightarrow$ Define income in E state as $100,000$ and $20,000$ in the U
  - $\rightarrow$ $x \equiv \begin{bmatrix} 100,000 & 20,000 \end{bmatrix}^\top$

$$\mathbb{E}[X_{t+j} \,|\, X_t = E] = (\begin{bmatrix} 1 & 0 \end{bmatrix} \cdot P^j) \cdot x$$

# Coding Markov Chain in Python

- We can make simulation easier if turn rows into conditional distributions
- Count states from $0$ to make coding easier, i.e. $E = 0$ and $U = 1$

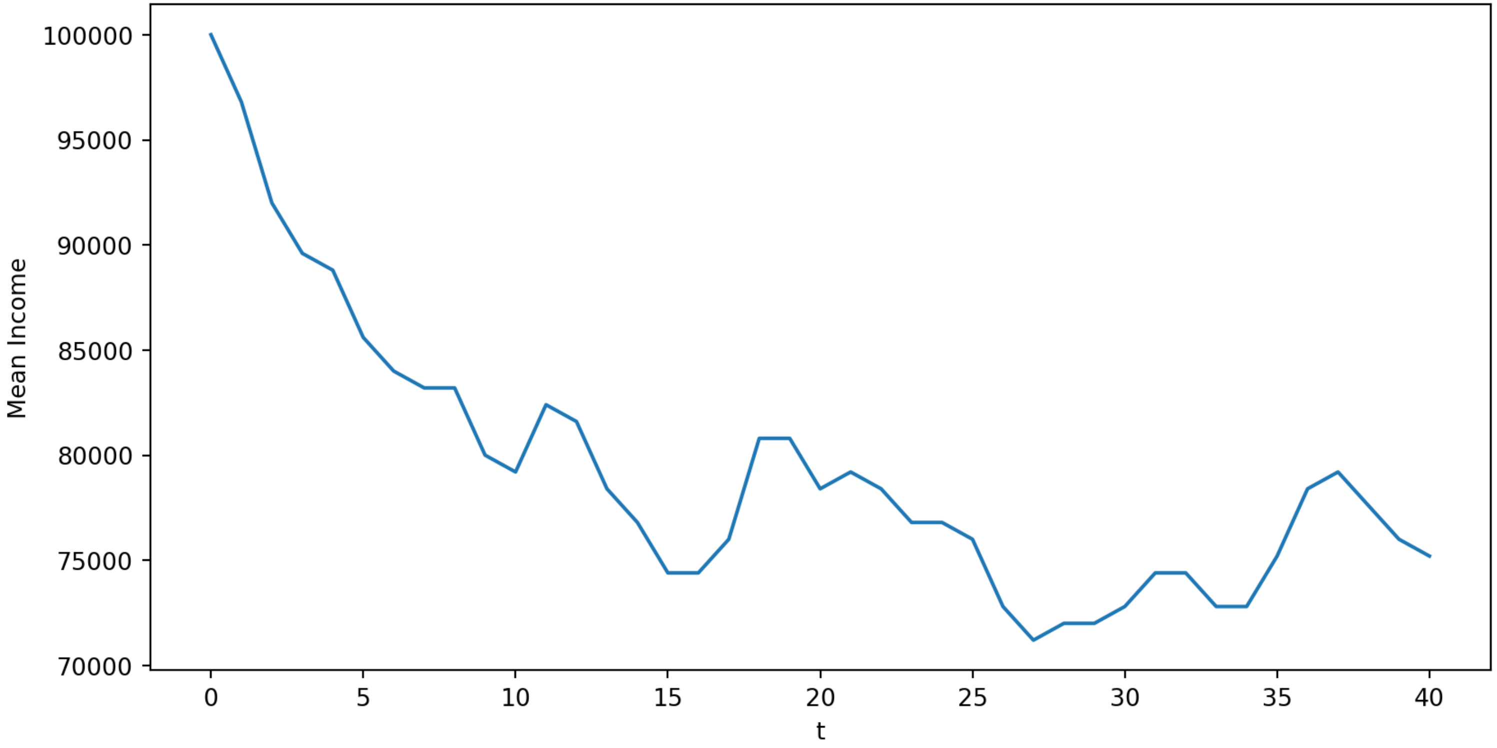
```python
a, b = 0.05, 0.1
P = np.array([[1-a, a],   # P(X | E)
              [b, 1-b]]) # P(X | U)
N = P.shape[0]
P_rv = [rv_discrete(values=(np.arange(0,N),
                    P[i,:])) for i in range(N)]
X_0 = 0 # i.e. E
X_1 = P_rv[X_0].rvs() # draw index | X_0
print(f"X_0 = {X_0}, X_1 = {X_1}")
T = 10
X = np.zeros(T+1, dtype=int)
X[0] = X_0
for t in range(T):
  X[t+1] = P_rv[X[t]].rvs() # draw given X_t
print(f"X_t indices =\n {X}")
```

```
X_0 = 0, X_1 = 0
X_t indices =
 [0 1 1 1 1 1 1 1 1 1 1]
```

# Simulating Many Trajectories

```python
def simulate_markov_chain(P, X_0, T):
  N = P.shape[0]
  num_chains = X_0.shape[0]
  P_rv = [rv_discrete(values=(np.arange(0,N),
                      P[i,:])) for i in range(N)]
  X = np.zeros((num_chains, T+1), dtype=int)
  X[:,0] = X_0
  for t in range(T):
      for n in range(num_chains):
          X[n, t+1] = P_rv[X[n, t]].rvs()
  return X
X_0 = np.zeros(100, dtype=int) # 100 people start employed
T = 40
X = simulate_markov_chain(P, X_0, T)
# Map indices to RV values
values = np.array([100000.00, 20000.00]) # map state to value
X_values = values[X] # just indexes by the X
```

# Simulating Many Trajectories



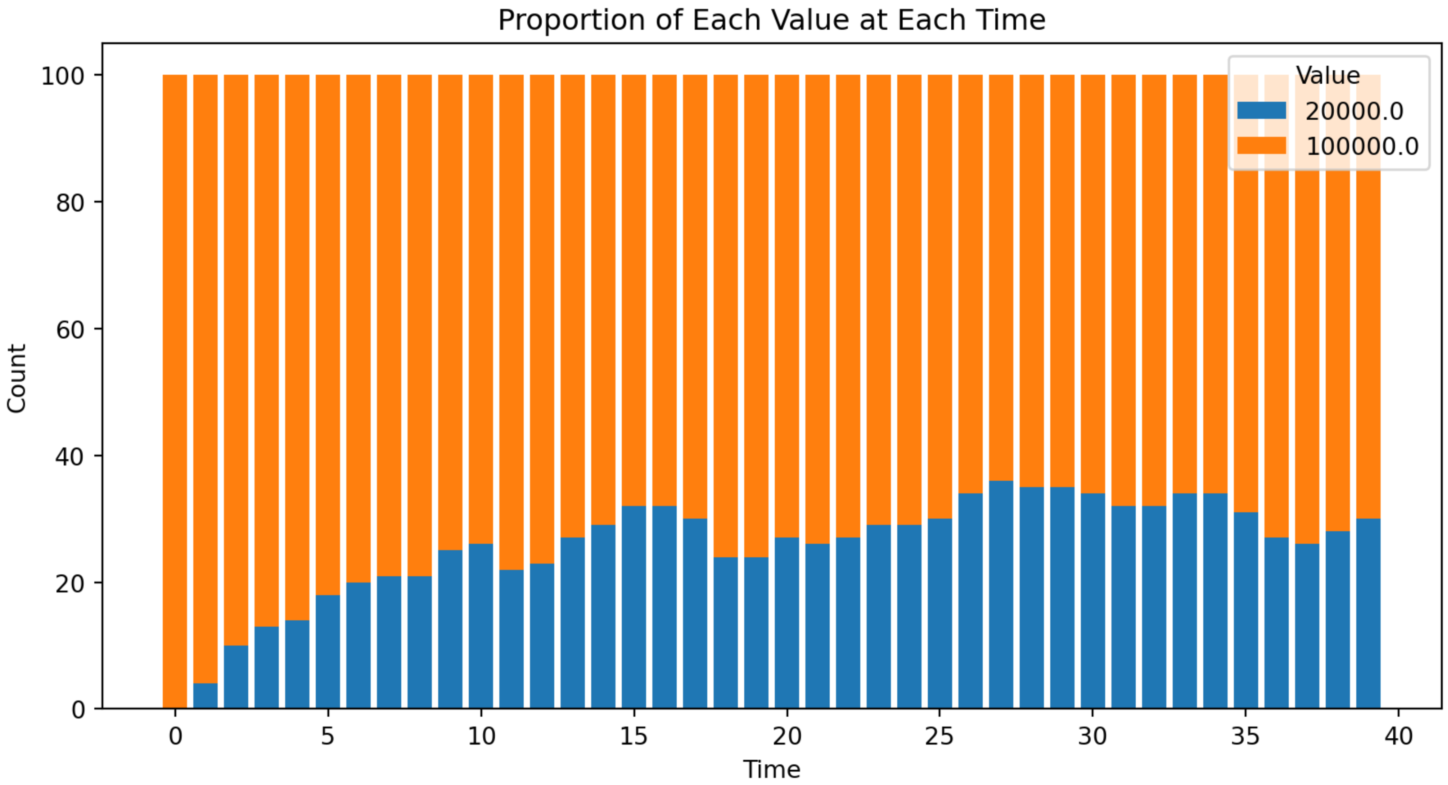
# Visualizing the Distribution of Many Trajectories

```python
# Count the occurrences of each unique value at each time step
unique_values = np.unique(X_values)
counts = np.array([[np.sum(X_values[:, t] == val) for val in unique_values] for t in range(T)])

# Create the stacked bar chart
fig, ax = plt.subplots()
bottoms = np.zeros(T)
for i, val in enumerate(unique_values):
    ax.bar(range(T), counts[:, i], bottom=bottoms, label=str(val))
    bottoms += counts[:, i]

# Labels and title
ax.set_xlabel('Time')
ax.set_ylabel('Count')
ax.set_title('Proportion of Each Value at Each Time')
ax.legend(title='Value')
plt.show()
```

# Visualizing the Distribution of Many Trajectories



# Stationary Distribution

- Recall different ways to think about steady states
  - → Left-eigenvector: $\bar{\pi} = \bar{\pi}P$
  - → Limiting distribution: $\lim_{T\to\infty} \pi_0 P^T$
- Can show that the stationary distribution is $\bar{\pi} = \begin{bmatrix} \frac{b}{a+b} & \frac{a}{a+b} \end{bmatrix}$

```python
eigvals, eigvecs = np.linalg.eig(P.T)
pi_bar = eigvecs[:, np.isclose(eigvals, 1)].flatten(
pi_bar = pi_bar / pi_bar.sum()
pi_0 = np.array([1.0, 0.0])
pi_inf = pi_0 @ matrix_power(P, 100)
print(f"pi_bar = {pi_bar}")
print(f"pi_inf = {pi_inf}")
```

```
pi_bar = [0.66666667 0.33333333]
pi_inf = [0.6666667 0.3333333]
```

# Expected Income

- Recall that $\mathbb{E}[X_{t+j} \mid X_t = E] = ([1 \quad 0] \cdot P^j) \cdot x$

```python
def forecast_distributions(P, pi_0, T):
    N = P.shape[0]
    pi = np.zeros((T+1, N))
    pi[0, :] = pi_0
    for t in range(T):
        pi[t+1, :] = pi[t, :] @ P
    return pi
x = np.array([100000.00, 20000.00])
pi_0 = np.array([1.0, 0.0])
T = 20
pi = forecast_distributions(P, pi_0, T)
E_X_t = np.dot(pi, x)
E_X_bar = pi_bar @ x
plt.plot(np.arange(0, T+1), E_X_t)
plt.axhline(E_X_bar, color='r',
  linestyle='--')
plt.show()
```