ECON526: Quantitative Economics with Data Science Applications

Applications of Linear Algebra and Eigenvalues

Jesse Perla

jesse.perla@ubc.ca

University of British Columbia

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Overview

Motivation and Materials

- In this lecture, we will cover some applications of the tools we developed in the previous lecture
- The goal is to build some useful tools to sharpen your intuition on linear algebra and eigenvalues/eigenvectors, and practice some basic coding

Packages

- Some additional material and references
  - → QuantEcon Python
  - → QuantEcon DataScience
  - → A First Course in Quantitative Economics with Python

```
import numpy as np
import matplotlib.pyplot as plt
import scipy
from numpy.linalg import cond, matrix_rank, norm
from scipy.linalg import inv, solve, det, eig, lu, eigvals
from scipy.linalg import solve_triangular, eigvalsh, cholesky
```

Difference Equations

Linear Difference Equations as Iterative Maps

- Consider $\boldsymbol{A} : \mathbb{R}^N \to \mathbb{R}^N$ as the linear map for the state $\boldsymbol{x}_t \in \mathbb{R}^N$
- An example of a linear difference equation is

$$\boldsymbol{x}_{t+1} = \boldsymbol{A}\boldsymbol{x}_t$$

where

$$A \equiv \begin{bmatrix} 0.9 & 0.1 \\ 0.5 & 0.8 \end{bmatrix}$$

```
A = np.array([[0.9, 0.1], [0.5, 0.8]])
x_0 = np.array([1, 1])
x_1 = A @ x_0
print(f"x_1 = {x_1}, x_2 = {A @ x_1}")
```

```
x_1 = [1.  1.3], x_2 = [1.03 1.54]
```

Iterating with $\rho(A) > 1$

Iterate $x_{t+1} = Ax_t$ from x_0 for $t = 100$

```
1 x_0 = np.array([1, 1])
2 t = 200
3 x_t = np.linalg.matrix_power(A, t) @ x_0
4 rho_A = np.max(np.abs(eigvals(A)))
5 print(f"rho(A) = {rho_A}")
6 print(f"x_{t} = {x_t}")
```

```
rho(A) = 1.079128784747792
x_200 = [3406689.32410673 6102361.18640516]
```

- Diverges to $x_\infty = \begin{bmatrix}\infty & \infty\end{bmatrix}^T$
- $\rho = 1 + 0.079$ says in the worst case (i.e., $x_t \propto$ the eigenvector associated with $\lambda = 1.079$ eigenvalue), expands by 7.9% on each iteration

Iterating with $\rho(A) < 1$

```python
A = np.array([[0.6, 0.1], [0.5, 0.8]])
x_t = np.linalg.matrix_power(A, t) @ x_0
rho_A = np.max(np.abs(eigvals(A)))
print(f"rho(A) = {rho_A}")
print(f"x_{t} = {x_t}")
```

```
rho(A) = 0.9449489742783178
x_200 = [6.03450418e-06 2.08159603e-05]
```

- Converges to $x_\infty = \begin{bmatrix} 0 & 0 \end{bmatrix}^T$

Iterating with $\rho(A) = 1$

- To make a matrix that has $\rho(A) = 1$ reverse eigendecomposition!
- Leave previous eigenvectors in Q, change Λ to force $\rho(A)$ directly

```
Q = np.array([[-0.85065081, -0.52573111], [0.52573111, -0.85065081]])
print(f"check orthogonal: dot(x_1,x_2) approx 0: {np.dot(Q[:,0], Q[:,1])}")
Lambda = [1.0, 0.8]  # choosing eigenvalue so max_n|lambda_n| = 1
A = Q @ np.diag(Lambda) @ inv(Q)
print(f"rho(A) = {np.max(np.abs(eigvals(A)))}")
print(f"x_{t} = {np.linalg.matrix_power(A, t) @ x_0}")
```

```
check orthogonal: dot(x_1,x_2) approx 0: -1.9275984594779062e-17
rho(A) = 1.0
x_200 = [ 0.27639321 -0.17082039]
```

Unemployment Dynamics

Dynamics of Employment without Population Growth

- Consider an economy where in a given year $\alpha = 5\%$ of employed workers lose job and $\phi = 10\%$ of unemployed workers find a job
- We start with $E_0 = 900,000$ employed workers, $U_0 = 100,000$ unemployed workers, and no birth or death. Dynamics for the year:

$$E_{t+1} = (1 - \alpha)E_t + \phi U_t$$
$$U_{t+1} = \alpha E_t + (1 - \phi)U_t$$

Write as Linear System

- Use matrices and vectors to write as a linear system

$$\underbrace{\begin{bmatrix} E_{t+1} \\ U_{t+1} \end{bmatrix}}_{X_{t+1}} = \underbrace{\begin{bmatrix} 1-\alpha & \phi \\ \alpha & 1-\phi \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} E_t \\ U_t \end{bmatrix}}_{X_t}$$

Simulating

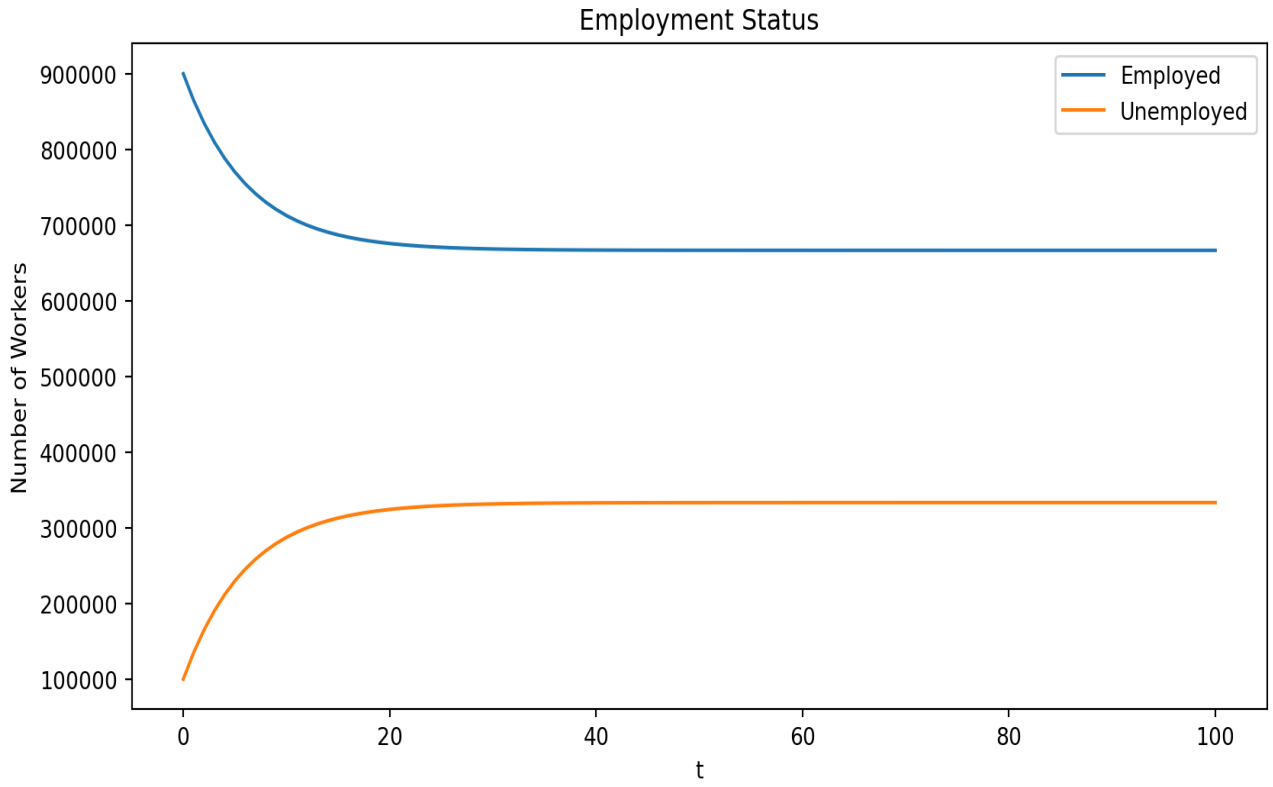
Simulate by iterating $X_{t+1} = AX_t$ from X_0 until $T = 100$

```
1  def simulate(A, X_0, T):
2      X = np.zeros((2, T+1))
3      X[:,0] = X_0
4      for t in range(T):
5          X[:,t+1] = A @ X[:,t]
6      return X
7  X_0 = np.array([900000, 100000])
8  A = np.array([[0.95, 0.1], [0.05, 0.9]])
9  T = 100
10 X = simulate(A, X_0, T)
11 print(f"X_{T} = {X[:,T]}")
```

```
X_100 = [666666.6870779  333333.31292209]
```

Dynamics of Unemployment

```
fig, ax = plt.subplots()
ax.plot(range(T+1), X.T,
  label=["Employed", "Unemployed"])
ax.set(xlabel="t",
  ylabel="Number of Workers",
  title="Employment Status")
ax.legend()
plt.show()
```



Convergence to a Longrun Distribution

- Find X_∞ by iterating $X_{t+1} = AX_t$ many times from a X_0?
  - → Check if it has converged with $X_\infty \approx AX_\infty$
  - → Is X_∞ the same from any X_0? Will discuss "ergodicity" later
- Alternatively, note that this expression is the same as

$$1 \times \bar{X} = A\bar{X}$$

  - → i.e, a $\lambda = 1$ where $\bar{X}$ is the corresponding eigenvector of A
  - → Is $\lambda = 1$ always an eigenvalue? (yes if all $\sum_{n=1}^{N} A_{ni} = 1$ for all i)
  - → Does $\bar{X} = X_\infty$? For any X_0?
  - → Multiple eigenvalues with $\lambda = 1 \implies$ multiple $\bar{X}$

Using the First Eigenvector for the Steady State

```
1 Lambda, Q = eig(A)
2 print(f"real eigenvalues = {np.real(Lambda)}")
3 print(f"eigenvectors in columns of =\n{Q}")
4 print(f"first eigenvalue = 1? \
5 {np.isclose(Lambda[0], 1.0)}")
6 X_bar = Q[:,0] / np.sum(Q[:,0]) * np.sum(X_0)
7 print(f"X_bar = {X_bar}\nX_{T} = {X[:,T]}")
```

```
real eigenvalues = [1.   0.85]
eigenvectors in columns of =
[[ 0.89442719 -0.70710678]
 [ 0.4472136   0.70710678]]
first eigenvalue = 1? True
X_bar = [666666.66666667 333333.33333333]
X_100 = [666666.6870779  333333.31292209]
```

Using the Second Eigenvalue for the Convergence Speed

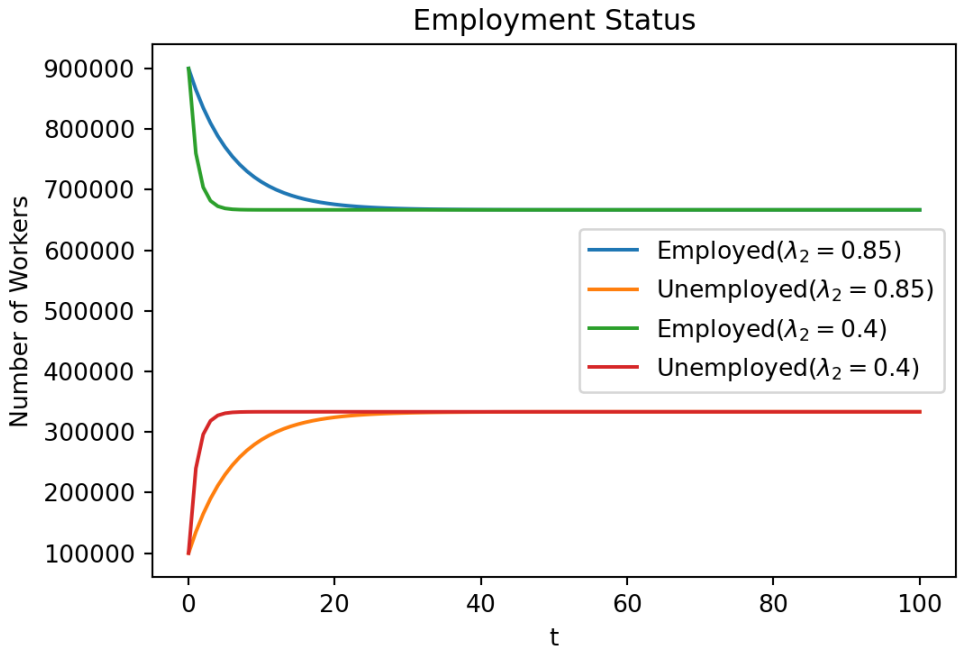
- The second largest ($\lambda_2 < 1$) provides information on the speed of convergence
  - → 0 is instantaneous convergence here
  - → 1 is no convergence here
- We will create a new matrix with the same steady state, different speed
  - → To do this, build a new matrix with the same eigenvectors (in particular the same eigenvector associated with the $\lambda = 1$ eigenvalue)
  - → But we will replace the eigenvalues $[1.0 \quad 0.85]$ with $[1.0 \quad 0.5]$
  - → Then we will reconstruct A matrix and simulate again
- Intuitively we will see the that the resulting A_{fast} implies α and ϕ which are larger by the same proportion

Simulating with Different Eigenvalues

```
Lambda_fast = np.array([1.0, 0.4])
A_fast = Q @ np.diag(Lambda_fast) @ inv(Q) # same eigenvectors
print("A_fast =\n", A_fast)
print(f"alpha_fast/alpha = {A_fast[1,0]/A[1,0]:.2g}, \
phi_fast/phi = {A_fast[0,1]/A[0,1]:.2g}")
X_fast = simulate(A_fast, X_0, T)
print(f"X_{T} = {X_fast[:,T]}")
```

```
A_fast =
 [[0.8 0.4]
 [0.2 0.6]]
alpha_fast/alpha = 4, phi_fast/phi = 4
X_100 = [666666.66666667 333333.33333333]
```

Convergence Dynamics of Unemployment



Present Discounted Values

Geometric Series

- Assume dividends follow $y_{t+1} = Gy_t$ for $t = 0, 1, \ldots$ and y_0 is given
- $G > 0$, dividends are discounted at factor $\beta > 1$ then
  $p_t = \sum_{s=0}^{\infty} \beta^s y_{t+s} = \frac{y_t}{1-\beta G}$
- More generally if $x_{t+1} = Ax_t$, $x_t \in \mathbb{R}^N$, $y_t = Gx_t$ and $A \in \mathbb{R}^{N \times N}$, then

$$
\begin{aligned}
p_t &= y_t + \beta y_{t+1} + \beta^2 y_{t+2} + \ldots = Gx_t + \beta GAx_t + \beta GAAx_t + \ldots \\
&= \sum_{s=0}^{\infty} \beta^s A^s y_t \\
&= G(I - \beta A)^{-1} x_t \quad , \text{ if } \rho(A) < 1/\beta
\end{aligned}
$$

- where $\rho(A)$ is the spectral radius

Discounting and the Spectral Radius

- Intuitively, the spectral radius of $\boldsymbol{A}$, the maximum scaling, must be less than discounting
- Intuition from univariate:
  - → If $\boldsymbol{G} \in \mathbb{R}^{1\times 1}$ then $\mathbf{eig}(\boldsymbol{G}) = \boldsymbol{G}$, so must have $\left|\beta \boldsymbol{G}\right| < \mathbf{1}$

PDV Example

Here is an example with $1 < \rho(A) < 1/\beta$. Try with different A

```
beta = 0.9
A = np.array([[0.85, 0.1], [0.2, 0.9]])
G = np.array([[1.0, 1.0]]) # row vector
x_0 = np.array([1.0, 1.0])
p_t = G @ solve(np.eye(2) - beta * A, x_0)
#p_t = G @ inv(np.eye(2) - beta * A) @ x_0 # alternative
rho_A = np.max(np.abs(np.real(eigvals(A))))
print(f"p_t = {p_t[0]:.4g}, spectral radius = {rho_A:.4g}, 1/beta = {1/beta:.4g}")
```

```
p_t = 24.43, spectral radius = 1.019, 1/beta = 1.111
```

(Optional) Matrix Conditioning and Stability

Matrix Conditioning

- Poorly conditioned matrices can lead to inaccurate or wrong solutions
- Tends to happen when matrices are close to singular or when they have very different scales - so there will be times when you need to rescale your problems

```
1 eps = 1e-7
2 A = np.array([[1, 1], [1 + eps, 1]])
3 print(f"A =\n{A}")
4 print(f"A^{-1} =\n{inv(A)}")
```

```
A =
[[1.        1.       ]
 [1.0000001 1.       ]]
A^-1 =
[[-9999999.99336215  9999999.99336215]
 [10000000.99336215 -9999999.99336215]]
```

Condition Numbers of Matrices

- $\mathbf{det}(A) \approx 0$ may say it is "almost" singular, but it is not scale-invariant
- $\mathbf{cond}(A) \equiv ||A|| \cdot ||A^{-1}||$ where $|| \cdot ||$ is the matrix norm - expensive to calculate in practice. Connected to eigenvalues $\mathbf{cond}(A) = |\frac{\lambda_{max}}{\lambda_{min}}|$
- Scale free measure of numerical issues for a variety of matrix operations
- Intuition: if $\mathbf{cond}(A) = K$, then $b \to b + \nabla b$ change in b amplifies to a $x \to x + K\nabla b$ error when solving $Ax = b$.
- See Matlab Docs on inv for example, where `inv` is a bad idea due to poor conditioning

```
print(f"condition(I) = {cond(np.eye(2))}")
print(f"condition(A) = {cond(A)}, condition(A^(-1)) = {cond(inv(A))}")
```

```
condition(I) = 1.0
condition(A) = 40000001.939191714, condition(A^(-1)) = 40000002.00307444
```

Example with Interpolation

- Consider fitting data $x \in \mathbb{R}^{N+1}$ and $y \in \mathbb{R}^{N+1}$ with an N-degree polynomial
- That is, find $c \in \mathbb{R}^{N+1}$ such that

$$
\begin{aligned}
c_0 + c_1 x_1 + c_2 x_1^2 + \ldots + c_N x_1^N &= y_1 \\
\ldots &= \ldots \\
c_0 + c_1 x_N + c_2 x_N^2 + \ldots + c_N x_N^N &= y_N
\end{aligned}
$$

- Which we can then use as $P(x) = \sum_{n=0}^{N} c_n x^n$ to interpolate between the points

Writing as a Linear System

- Define a matrix of all of the powers of the x values

$$A \equiv \begin{bmatrix} 1 & x_0 & x_0^2 & \dots & x_0^N \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & x_N & x_N^2 & \dots & x_N^N \end{bmatrix}$$

- Then solve for c as the solution (where A is invertible if x_n are unique)

$$Ac = y$$

Solving an Example

- Let's look at the numerical error here from the interpolation using the inf-norm, i.e., $||x||_\infty = \mathbf{max}_n \, |x_n|$

```
N = 5
x = np.linspace(0.0, 10.0, N + 1)
y = np.exp(x)  # example function to interpolate
A = np.array([[x_i**n for n in range(N + 1)] for x_i in x])  # or np.vander
c = solve(A, y)
c_inv = inv(A) @ y
print(f"error = {norm(A @ c - y, np.inf)}, \
error using inv(A) = {norm(A @ c_inv - y, np.inf)}")
print(f"cond(A) = {cond(A)}")
```

```
error = 2.2737367544323206e-11, error using inv(A) = 1.0986695997416973e-09
cond(A) = 564652.321404467
```

Things Getting Poorly Conditioned Quickly

```
N = 10
x = np.linspace(0.0, 10.0, N + 1)
y = np.exp(x)  # example function to interpolate
A = np.array([[x_i**n for n in range(N + 1)] for x_i in x])  # or np.vander
c = solve(A, y)
c_inv = inv(A) @ y # Solving with inv(A) instead of solve(A, y)
print(f"error = {norm(A @ c - y, np.inf)}, \
error using inv(A) = {norm(A @ c_inv - y, np.inf)}")
print(f"cond(A) = {cond(A)}")
```

```
error = 6.348273018375039e-10, error using inv(A) = 4.55108965979889e-06
cond(A) = 4462823910804.094
```

Matrix Inverses Fail Completely for $N = 20$

```
N = 20
x = np.linspace(0.0, 10.0, N + 1)
y = np.exp(x)  # example function to interpolate
A = np.array([[x_i**n for n in range(N + 1)] for x_i in x])  # or np.vander
c = solve(A, y)
c_inv = inv(A) @ y # Solving with inv(A) instead of solve(A, y)
print(f"error = {norm(A @ c - y, np.inf)}, \
error using inv(A) = {norm(A @ c_inv - y, np.inf)}")
print(f"cond(A) = {cond(A):.4g}")
```

```
error = 1.9554136088117957e-10, error using inv(A) = 21804.714723170073
cond(A) = 3.325e+24
```

Moral of this Story

- Use `solve`, which is faster and can often solve ill-conditioned problems. Rarely use `inv`, and only when you know the problem is well-conditioned
- Check conditioning of matrices when doing numerical work as an occasional diagnostic, as it is a good indicator of potential problems and collinearity
- For approximation, never use a monomial basis for polynomials

  → Prefer polynomials like Chebyshev, which are designed to be as orthogonal as possible

```
N = 40
x = np.linspace(-1, 1, N+1)  # Or any other range of x values
A = np.array([[np.polynomial.Chebyshev.basis(n)(x_i) for n in range(N+1)] for x_i in x])
A_monomial = np.array([[x_i**n for n in range(N + 1)] for x_i in x])  # or np.vander
print(f"cond(A) = {cond(A):.4g}, cond(A_monomial) = {cond(A_monomial):.4g}")
```

```
cond(A) = 3.64e+09, cond(A_monomial) = 8.903e+17
```