ECON526: Quantitative Economics with Data Science Applications

Practical Uses of Directed Graphical Models

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Overview

Summary

- Previously in the course, we introduced Directed Acyclic Graphs as a way to represent conditional independence relationships between variables.
- We built a DAG to represent the causal relationships between variables in a simple model of online learning.
- We used this example to discuss how we can use DAGs to build a causal model of a data generating process.
  - $\longrightarrow$ We also discussed some techniques to build concise models without losing too much information.
- Today, we will discuss how these DAGs can actually be used in practice, both to identify causal effects, and to reduce the representational complexity of a model.

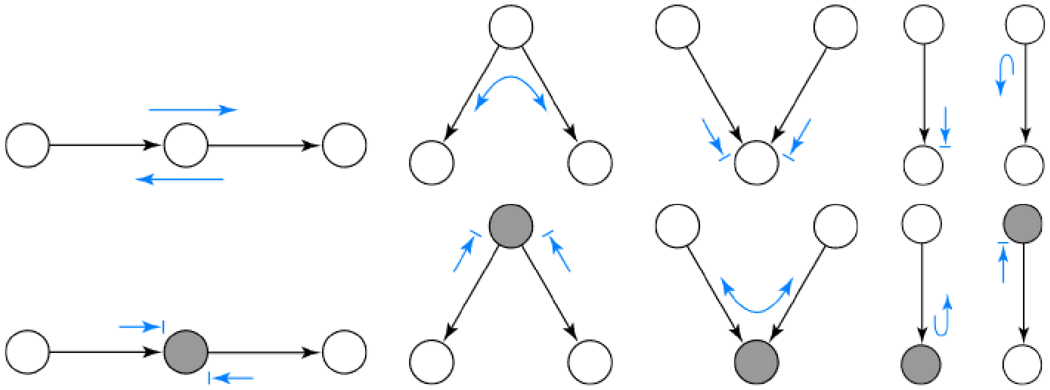
Dependence Flows

Dependence Flows

- In order to determine whether or not we can identify a treatment effect, we need to understand how dependence **flows** through a graphical model.
- We have seen that conditioning on a node can either make or break the dependence relationship between two other nodes
- To identify the treatment effect, we want the *link between the treatment and the outcome to be unblocked*.
- However, we also need to make sure that there is *no other dependence path* between the treatment and the outcome that is unblocked.

The Rules of Bayes-Ball

- We can think about the flow of dependence as a game of “Bayes-ball”
- The rules of Bayes-ball are reasonably simple. A dependence path is blocked if and only if:
    1. It contains a *non-collider* that is conditioned on
    2. It contains a *collider* that is not conditioned on, and neither are *any of its descendants*

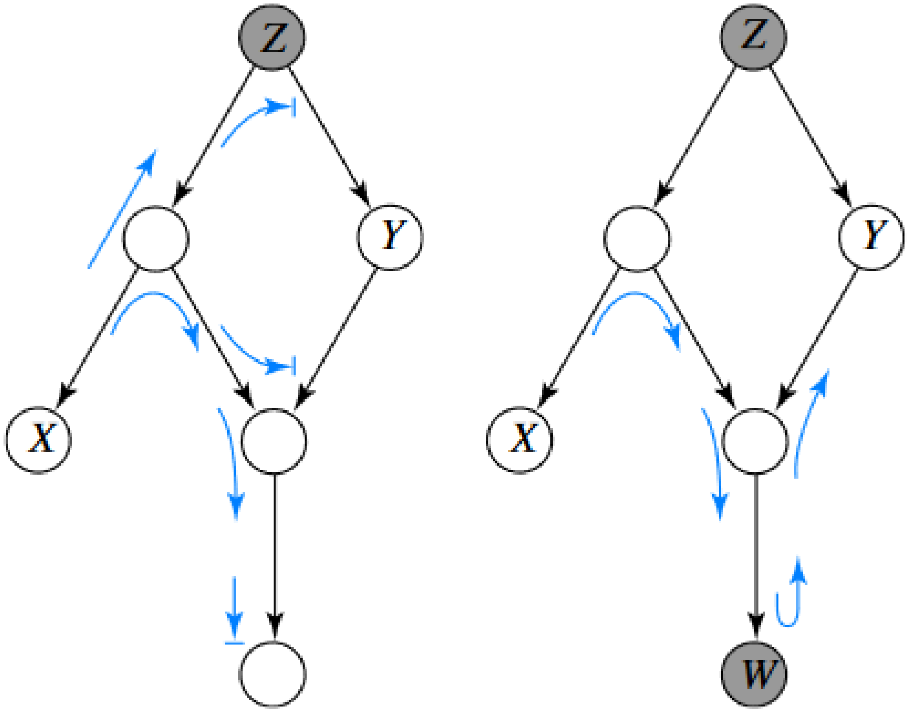


Directed Graphical Models

Turning back to our **collider** example:

- Notice that when we do not condition on C, A and B are independent.
- However, somewhat unintuitively, when we condition on X, Y and Z become dependent.
  - $\longrightarrow$ This is because conditioning on X opens the flow of dependence from Y to Z.
  - $\longrightarrow$ In any case that is *not* a collider, conditioning on a node *blocks* the flow of dependence.

Two Games of Bayes Ball



no active paths

$X \perp\!\!\!\perp Y \mid Z$

one active path

$X \not\perp\!\!\!\perp Y \mid \{W, Z\}$

Viualizing Bias

Bias and Causality

- In a causal inference framework, we can use graphical models to determine whether or not we can identify a treatment effect, and which covariates we need to condition on.
- Typically, drawing out a graphical model is not necessary, but it can be a useful exercise to help you think through the problem.
  - $\longrightarrow$ The links you draw represent the assumptions you are making about the data generating process

Bias and Causality

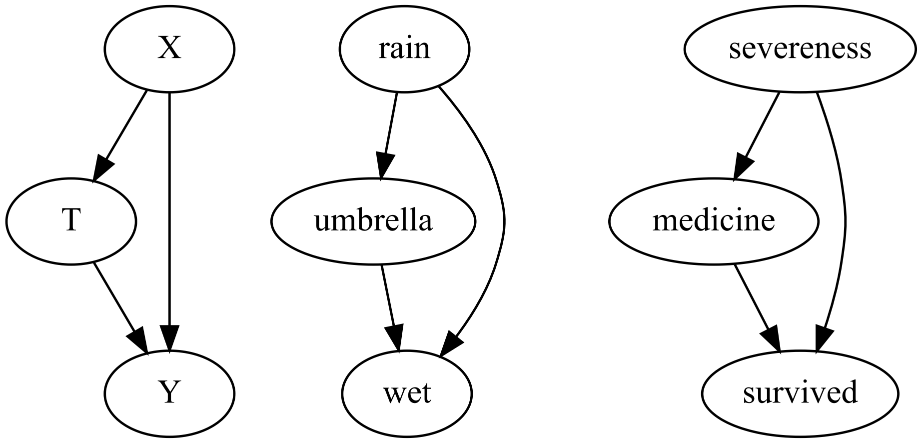
- There are two major types of bias that we need to worry about in causal inference:
  - → **Confounding**: When there is an unobserved variable that is a common cause of both the treatment and the outcome
  - → **Selection**: When there is an unobserved variable that is a common cause of both the treatment and the selection into the sample
- Both of these types of bias can be represented using a graphical model

Confounding

Confounding

- Let's look at an example of confounding:

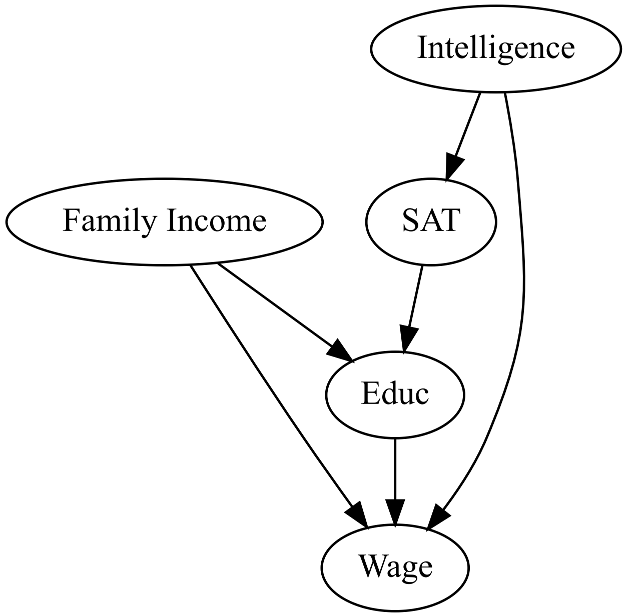
```
g = gr.Digraph()
g.edge("X", "T")
g.edge("X", "Y")
g.edge("T", "Y")

g.edge("rain", "umbrella")
g.edge("rain", "wet")
g.edge("umbrella", "wet")

g.edge("severeness", "medicine")
g.edge("severeness", "survived")
g.edge("medicine", "survived")
g
```



- To control for confounding, we need to condition on all of the common causes of the treatment and the outcome.

Confounding

```python
g = gr.Digraph()

g.node("Family Income")
g.edge("Family Income", "Educ")
g.edge("Educ", "Wage")

g.node("SAT")
g.edge("SAT", "Educ")

g.node("Family Income")
g.edge("Family Income", "Wage")

g.edge("Intelligence", "SAT")
g.edge("Intelligence", "Wage")
g
```

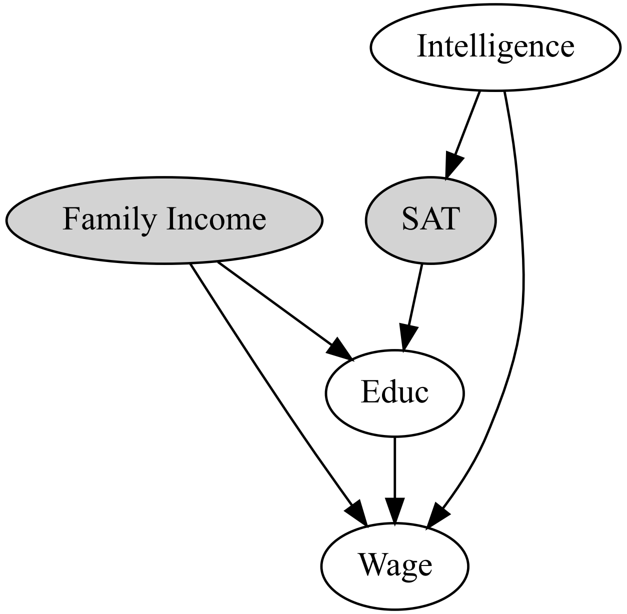


- Often, there are confounding variables that we cannot observe

  → For example, we cannot observe intelligence, but it is a common cause of both education (the treatment) and wages

Confounding

```python
g = gr.Digraph()

g.node("Family Income", style="filled")
g.edge("Family Income", "Educ")
g.edge("Educ", "Wage")

g.node("SAT", style="filled")
g.edge("SAT", "Educ")

g.node("Family Income", style="filled")
g.edge("Family Income", "Wage")

g.edge("Intelligence", "SAT")
g.edge("Intelligence", "Wage")
g
```



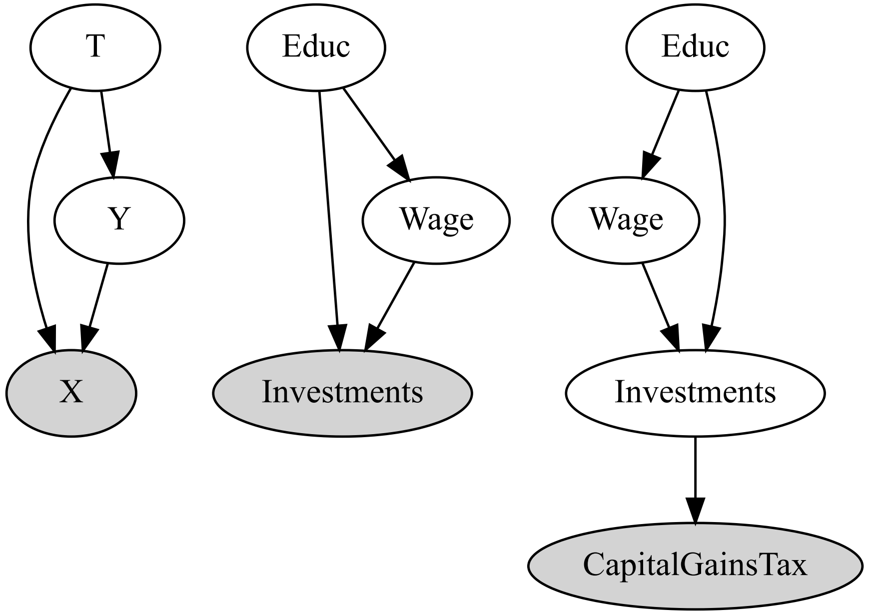
- Often, there are confounding variables that we cannot observe
  - → For example, we cannot observe intelligence, but it is a common cause of both education (the treatment) and wages
  - → But we can use SAT as a **surrogate** or **proxy** for intelligence.

Selection

Selection

- Selection bias often occurs when there is an unobserved variable that is a common cause of both the treatment and the selection into the sample, in other words, by conditioning on a variable that you shouldn't.

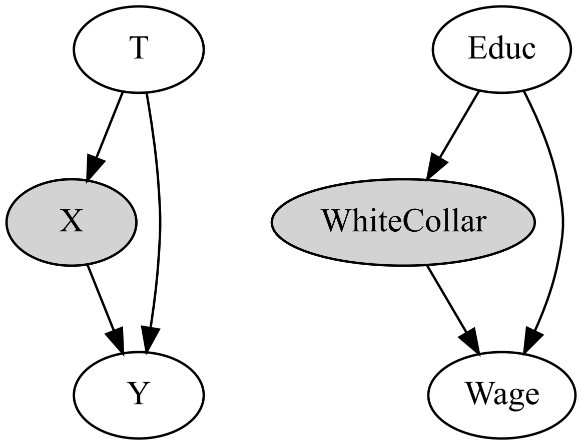
```
g = gr.Digraph()
g.node("X", style="filled")
g.edge("T", "X")
g.edge("T", "Y")
g.edge("Y", "X")
g.node("Investments", "Investments", style="filled")
g.edge("Educ", "Investments")
g.edge("Educ", "Wage")
g.edge("Wage", "Investments")
g.node("Educ2", "Educ")
g.node("Wage2", "Wage")
g.node("Investments2", "Investments")
g.edge("Educ2", "Wage2")
g.edge("Wage2", "Investments2")
g.edge("Educ2", "Investments2")
g.node("CapitalGainsTax", "CapitalGainsTax", style="
```



Selection

- Selection bias can also occur when controlling for a **mediator** between the treatment and the outcome

```
g = gr.Digraph()

g = gr.Digraph()
g.edge("T", "X")
g.edge("T", "Y")
g.edge("X", "Y")
g.node("X", "X", style="filled")

g.edge('Educ', 'WhiteCollar')
g.edge('Educ', 'Wage')
g.edge('WhiteCollar', 'Wage')
g.node('WhiteCollar', style="filled")

g
```



Choosing Covariates

Choosing Covariates

- With these types of bias in mind, we can think about how to choose which covariates to condition on.
- Notice that some controls will reduce bias, as in the case of confounding,
- But *not all controls are good!* Some controls will actually create bias, as in the case of selection.
- This means that we don't want to just "throw the kitchen sink" at the problem, and include every variable we can think of.

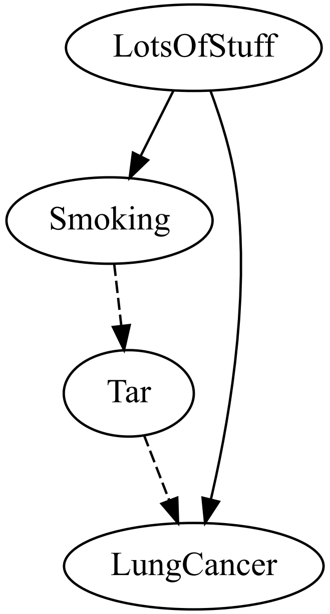
Choosing Covariates

- We want to choose covariates that will close any unblocked secondary dependence paths between the treatment and the outcome, but not open any new ones.
- To do this, we can use the “front door criterion” and the “back door criterion”

The Front Door Criterion

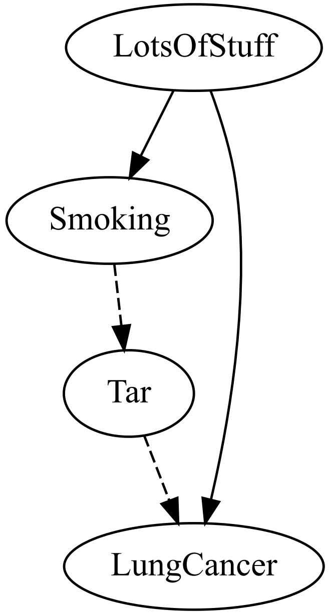
- The **front door criterion** is one way to isolate the effect of a treatment on an outcome, when there is a confounder and a mediator between the treatment and the outcome.

```python
g = gr.Digraph()
g.edge("LotsOfStuff", "Smoking")
g.edge("LotsOfStuff", "LungCancer")
g.edge("Smoking", "Tar", style="dashed")
g.edge("Tar", "LungCancer", style="dashed")

g
```



The Front Door Criterion

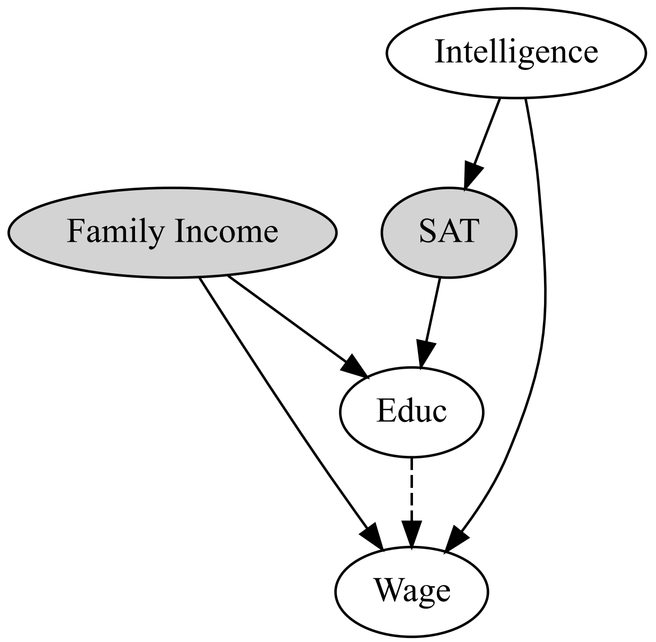
```
g = gr.Digraph()
g.edge("LotsOfStuff",
g.edge("LotsOfStuff",
g.edge("Smoking", "Tar
g.edge("Tar", "LungCar

g
```



- In this example, we want to know the effect of smoking on lung cancer, but we also know that there are lots of variables (like stress), that cause both the treatment and the outcome.
- However, stress does not cause tar, so we can condition on tar.
  - → This works by first measuring the effect of tar on lung cancer, and *then* the effect of smoking on tar.

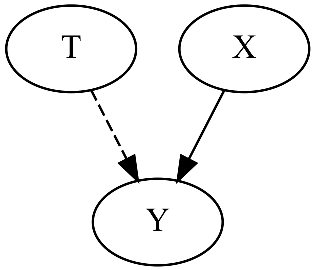
The Back Door Criterion

- It's pretty rare that we'll actually be able to use the front door criterion, because we usually don't have a mediator that we can condition on.
- Instead, we can close all of the "back door paths" from treatment to outcome. We have already seen an example of this.



Choosing Covariates: Two Examples

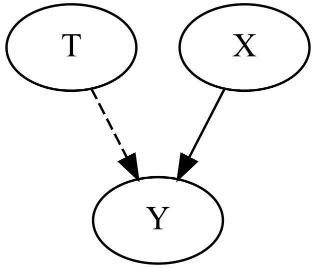
```
g = gr.Digraph()
g.edge("T", "Y", style="dashed")
g.edge("X", "Y")
g
```



- Do we need to condition on X?

Choosing Covariates: Two Examples

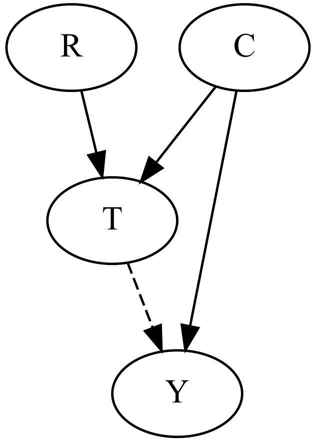
```
g = gr.Digraph()
g.edge("T", "Y", style="dashed")
g.edge("X", "Y")
g
```



- Do we need to condition on X?
  - $\longrightarrow$ No, not necessarily. There is no unblocked dependence path between T and Y that goes through X.
  - $\longrightarrow$ However, conditioning on X will reduce the variance of our estimate!
  - $\longrightarrow$ If we didn't condition on X, it would become part of our estimation error. But since $X \perp T$, it won't bias our estimate.

Choosing Covariates: Two Examples

```python
g = gr.Digraph()
g.edge("R", "T")
g.edge("C", "T")
g.edge("C", "Y")
g.edge("T", "Y", style="dashed")
g
```



- Suppose we don't have any data on C. Can we identify the treatment effect?
  - $\longrightarrow$ Yes! If we look at the effect of R on Y, we can see that it is unblocked.
  - $\longrightarrow$ Furthermore, since R only affects Y through T, the effect of R on Y is the same as the effect of T on Y.
  - $\longrightarrow$ This is called an **instrumental variable**.

Factorizing the Joint Distribution

Factorizing the Joint Distribution

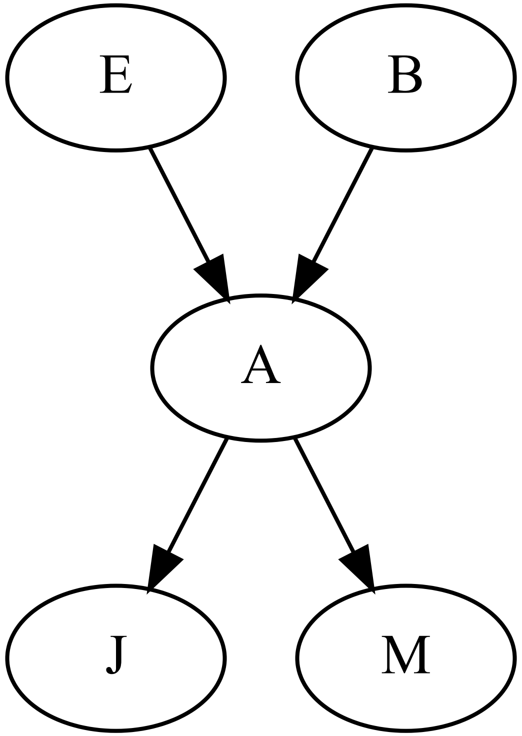
- We have seen how these graphical models can be used to determine whether or not we can identify a treatment effect.
- However, we can also use them as a computational tool, to help us find the simplest representation of the joint distribution of the data.
- This is useful because it allows us to find the simplest model that is consistent with our assumptions about conditional independence.

Factorizing the Joint Distribution

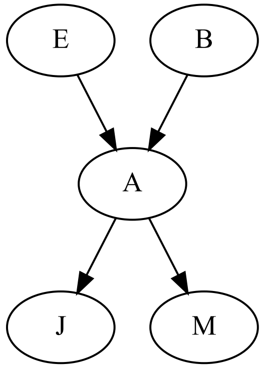
- To understand why this is useful, we'll follow this example (from Mark Paskin).
- Suppose we have a DGP with five variables:
  1. $E \in \{true, false\}$ - Has an earthquake happened? *Earthquakes are unlikely*
  2. $B \in \{true, false\}$ - Has a burglary happened? *Burglaries are unlikely, but more likely than earthquakes*
  3. $A \in \{true, false\}$ - Is the alarm going off? *The alarm is triggered by both earthquakes and burglaries*
  4. $J \in \{true, false\}$ - Is John calling? *John calls when he hears the alarm, but he often misses it*
  5. $M \in \{true, false\}$ - Is Mary calling? *Mary calls when she hears the alarm, but she also calls to chat*

Factorizing the Joint Distribution

- The goal is to compute $P(B|J = true)$ from the joint distribution $P(E, B, A, J, M)$. We'll start by drawing the graphical model, to understand the conditional independence relationships.



Factorizing the Joint Distribution

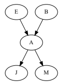


- In order to represent the full joint distribution, we could use the chain rule of probabilities:

$$P(E, B, A, J, M) = P(E)P(B|E)P(A|E, B)P(J|E, B, A)P(M|E, B, A, J)$$

- Q: How many probabilities would we need to compute to represent the joint distribution this way?

Factorizing the Joint Distribution



- In order to represent the full joint distribution, we could use the chain rule of probabilities:

$$\underbrace{P(E,B,A,J,M)}_{31} = \underbrace{P(E)}_{1}\underbrace{P(B|E)}_{2}\underbrace{P(A|E,B)}_{4}\underbrace{P(J|E,B,A)}_{8}\underbrace{P(M|E,B,A,J)}_{16}$$

- Q: How many probabilities would we need to store to represent the joint distribution this way?

  $\longrightarrow$ A: There are $\mathbf{2^5}$ possible outcomes. We need 31 probabilities. (why not 32?)

Factorizing the Joint Distribution

- However, we can use the conditional independence relationships to simplify this representation.
  - $\rightarrow$ For example, we know that $P(B|E) = P(B)$, because B and E are independent.
  - $\rightarrow$ We also know that $P(A|E,B) = P(A|B)$, because A is independent of E, given B.
- This means that we can simplify the joint distribution to:

$$\underbrace{P(E,B,A,J,M)}_{10} = \underbrace{P(E)}_{1}\underbrace{P(B)}_{1}\underbrace{P(A|E,B)}_{4}\underbrace{P(J|A)}_{2}\underbrace{P(M|A)}_{2}$$

Factorizing the Joint Distribution

- Beyond causal inference, graphical models are a useful tool for computing all kinds of conditional probabilities.
- This is useful in many inference problems, and in structural econometrics
  - → The **likelihood function** is the conditional probability of the data, given the parameter values
  - → In Bayesian inference the **posterior** is the conditional probability of the parameters, given the data (and our priors)

Variable Elimination

- In order to simplify a conditional distribution, we can use **variable elimination**.
- For example, let's say we want to compute the probability of a burglary, given that John calls.

$$p_{B|J}(b, \textbf{ true }) \propto \sum_e \sum_a \sum_m p_{EBAJM}(e, b, a, \textbf{ true }, m)$$

$$= \sum_e \sum_a \sum_m p_E(e) \cdot p_B(b) \cdot p_{A|EB}(a, e, b) \cdot p_{J|A}(\textbf{ true }, a) \cdot p_{M|A}(m, a)$$

- Then, we can reduce the computational complexity by eliminating variables one at a time, exploiting the distributive property of multiplication

  $\rightarrow$ $xy + xz = x(y + z)$.

Variable Elimination

- Variable elimination works like this:

  → Repeat the following steps:

  1. choose a variable to eliminate
  2. push in its sum as far as possible
  3. compute the sum, resulting in a new factor

Variable Elimination

$$
\begin{aligned}
&\sum_e \sum_a \sum_m p_E(e) \cdot p_B(b) \cdot p_{A|EB}(a, e, b) \cdot p_{J|A}(\textbf{ true }, a) \cdot p_{M|A}(m, a) \\
&= \sum_e \sum_a p_E(e) \cdot p_B(b) \cdot p_{A|EB}(a, e, b) \cdot p_{J|A}(\textbf{ true }, a) \cdot \sum_m p_{M|A}(m, a) \\
&= \sum_e \sum_a p_E(e) \cdot p_B(b) \cdot p_{A|EB}(a, e, b) \cdot p_{J|A}(\textbf{ true }, a) \cdot \psi_A(a) \\
&= \sum_e p_E(e) \cdot p_B(b) \cdot \sum_a p_{A|EB}(a, e, b) \cdot p_{J|A}(\textbf{ true }, a) \cdot \psi_A(a) \\
&= \sum_e p_E(e) \cdot p_B(b) \cdot \psi_{EB}(e, b) \\
&= p_B(b) \cdot \sum_e p_E(e) \cdot \psi_{EB}(e, b) \\
&= p_B(b) \cdot \psi_B(b)
\end{aligned}
$$

Variable Elimination

- That's a lot of math! But let's focus on the first step:

$$\underbrace{p_{B|J}(b, \textbf{ true }) \propto \underbrace{\sum_e \sum_a \sum_m}_{2^3=8 \text{ iterations}} \underbrace{p_E(e) \cdot p_B(b) \cdot p_{A|EB}(a, e, b) \cdot p_{J|A}(\textbf{ true }, a) \cdot p_{M|A}(m, a)}_{4 \text{ multiplications}}}_{8*4=32 \text{ multiplications } +7 \text{ additions}=39 \text{ total operations}}$$

Variable Elimination

- That's a lot of math! But let's focus on the first step:

$$p_{B|J}(b,\ \text{true}\ ) \propto$$

$$\underbrace{\underbrace{\sum_e \sum_a \sum_m}_{2^3=8 \text{ iterations}} \underbrace{p_E(e) \cdot p_B(b) \cdot p_{A|EB}(a,e,b) \cdot p_{J|A}(\ \text{true}\ ,a) \cdot p_{M|A}(m,a)}_{4 \text{ multiplications}}}_{8*4=32 \text{ multiplications } +7 \text{ additions}=39 \text{ total operations}}$$

$$= \underbrace{\underbrace{\sum_e \sum_a}_{2^2=4 \text{ iterations}} \underbrace{p_E(e) \cdot p_B(b) \cdot p_{A|EB}(a,e,b) \cdot p_{J|A}(\ \text{true}\ ,a) \cdot}_{4 \text{ multiplications}} \underbrace{\sum_m p_{M|A}(m,a)}_{1 \text{ addition}}}_{4*(4 \text{ multiplications}+1 \text{ addition})+3 \text{ additions}=23 \text{ total operations}}$$

Variable Elimination

- Variable elimination is an algorithm that exploits our conditional independence assumptions to reduce the computational complexity of conditional probabilities.
- In this case, we were able to reduce the number of operations from 39 to 23 operations in just one step, each further step will continue to reduce the complexity.
- This is a very simple example, because we only have 5 variables. In practice, we might have hundreds or thousands of variables, and the computational complexity can become very large.
- Systematic patterns of conditional independence can be exploited to reduce the complexity of inference problems in some large models.

Credits

This lecture draws heavily from Causal Inference for the Brave and True: Chapter 04 - Graphical Causal Models by Matheus Facure.

There is also material from A Short Course on Graphical Models Chapter 2: Structured Representations by Mark Paskin.

As well as The Effect: Chapter 7 - Drawing Graphical Diagrams by Nick Huntington-Klein.