Senior Thesis in Mathematics

Labeling Comparability Graphs for Dualities between Non-attacking Rook Games and Graph Proper Colorings

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Abstract

This thesis covers a novel proof connecting the comparability graphs of posets and *transitive graphs*. I define transitive graphs as graphs G that have a labeling such that a non-attacking rook placement directly corresponds to the partition of G's vertices into non-adjacent cliques, i.e. a set of proper colorings of the vertices.

The proven theorem states that the class of comparability graphs is exactly the class of transitive graphs. In applications, the rook vector of a board can be directly used to find the chromatic polynomial of the board's corresponding comparability graph.

I also observe and define "bi-relevant chain labeling", which is a labeling of complementary comparability graphs such that their combined partial orders forms a coherent linear order. Further study is needed to explore how theories for rooks, graphs, chromatic polynomials and posets meet.

Chapters 1-3 provide the necessary background, and Chapters 4-5 contain the meat of this work.

Contents

Chapter 1

Introduction

Following in the footsteps of Goldman, Joichi and White [6], this thesis constructs a proof that for what I term transitive graphs, a placement of non-attacking rooks on an appropriate board maps bijectively to a partition of the graph's vertices into cliques. When each clique is assigned a color, the partition translates in the complement graph to color classes in a proper coloring. In general, the rook vector of an appropriate board may directly be used to find the complement graph's chromatic polynomial χ.

In exploring this class of transitive graphs, we define labelings of simple graphs and discover that the class of comparability graphs is exactly that of transitive graphs. That is, the formula works only for graphs that are based on a partial order. The labeling process introduces permutations on top of rook theory and graph coloring as another overlapping realm of mathematics. For further research in this direction, please consult [2], [3].

Frustrated that the initial graph is only a tool for finding χ of its complement, I explored of dimension-2 posets, whose comparability graphs and incomparability graphs both constitute transitive graphs. Thus, through the rook vector, χ can be found for each of these complementary graphs. For information on the dimension of partial orders, see [4], [7], and on partial orders and its graphs, see [1], [5], [8].

Chapters 2 and 3 introduce definitions and well-known theorems in graph coloring theory and partial order theory respectively. Chapter 4 provides a robust proof for the formula determining the chromatic polynomial of a graph solely from the rook vector of its corresponding board. Chapter 5 provides a proof that the class of transitive graphs is exactly the class of comparability graphs, and defines ways to label such graphs for this to be apparent. Implications for posets from rook theory are drawn, and examples are provided for when incomparability graphs that double as comparability graphs. Finally, Chapter 6 details the many future directions possible.

Colors used in this thesis have been tested to be colorblind-friendly.

Chapter 2

Coloring Graphs

2.1 Introduction to Graph Coloring

Suppose you want to buy gifts for every member of a 6-person mobile app start-up: the product manager (PM), business analyst (BA), designer, developer, social media specialist, and intern. The graph's edges in Figure 2.1 show which members work closely together. Each color represents a distinct gift – hand-drawn postcards with different designs, and coloring each vertex one color represents giving a certain design to that member.

We want to distribute gifts so **members who work closely together don't get the same gift**, i.e. no two adjacent vertices have the same color. In Figure 2.2, we see one way to represent each member getting a different gift, but this would require drawing 6 different designs. How many ways can we distribute 3 gifts? How about λ gifts?

What we are looking for, in mathematical terms, are different proper 3-colorings of the graph. Figure 2.3 shows two ways to do so, with the intern placed in the same color class as the developer, or the designer and Sr. developer. Assigning a color to a vertex represents giving a specific gift type to a particular person.

We can generalize this example to an unknown number of gift types, i.e. colors. It turns out each simple graph has a polynomial (the chromatic polynomial) that can express the number of proper λ-colorings for any positive integer λ of colors. In this chapter we cover the definitions and theorems regarding these concepts with examples.

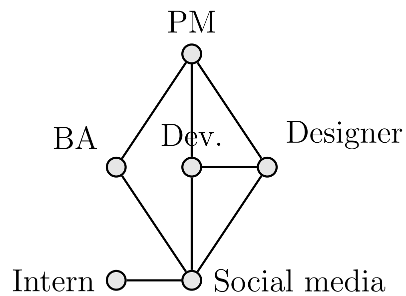


Figure 2.1: Start-up members represented by vertices, with those who work closely together connected by edges

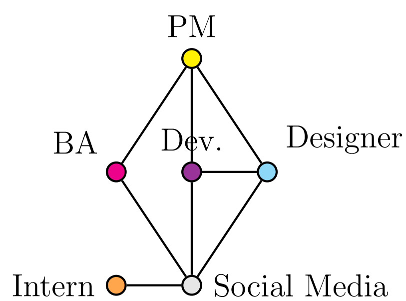


Figure 2.2: A proper coloring representing the situation where each person is given a distinct gift

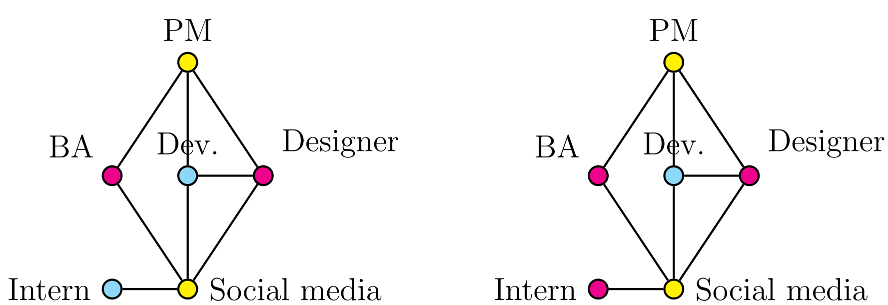


Figure 2.3: Two proper coloring representing situations where only three distinct gifts are given to the 6 people

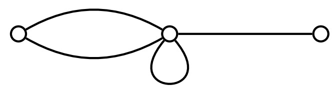


Figure 2.4: A general graph with a repeated edge and a loop

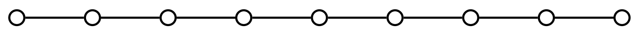


Figure 2.5: P_8, a path of length 8, has 8 edges and 9 vertices

2.2 The Basics of Graphs

Definition 1 (Graphs, Adjacency, Order)**.** A *(simple) graph* G is a pair of sets (V, E) where V is the non-empty vertex set of G, and E is the possibly empty edge set of unordered pairs of distinct elements of V.

If the edge $\{x, y\} \in E$, then x and y are *adjacent*.

The number of vertices, $|V|$, is the *order* of G. For a graph of order n, vertices are labeled 1 through n, hence the vertex set $\{1, 2, ..., n\}$ can be written as $[n]$

Definition 2 (General Graphs, Loops, Repeated Edges)**.** Let $G = (V, E)$ be a graph. If we allow repeated elements $\{x, y\}$ in E, then G has *repeated edges* between vertices x and y. If we also allow pairs of non-distinct elements $\{x, x\}$ in E, then G has *loop(s)* at vertex x. (See Figure 2.4.) If, in the definition of a graph, we allow repeated and loops, then we have a *general graph*.

General graphs will not be considered in this study since they repeated edges and loops can be removed without impacting our study focus of graph colorings.

Definition 3 (Paths)**.** Let n be a positive integer, let $V = \{1, 2, ..., n\}$ and let $E = \{\{1, 2\}, \{2, 3\}, ..., \{n-1, n\}\}$ be the set of consecutive pairs of elements of V. Then $G = (V, E)$ is the *path of length* $n - 1$, denoted as P_{n-1}. P_{n-1} has order n and $n - 1$ edges. (See Figure 2.5 for P_8.)

Definition 4 (Cycles)**.** Let $n \geq 3$ be an integer. A *cycle of length* n is a connected simple graph with n vertices and n edges such that each vertex is adjacent to exactly two other vertices. C_n, the cycle of length n, has n vertices and n edges. (See Figure 2.6.)

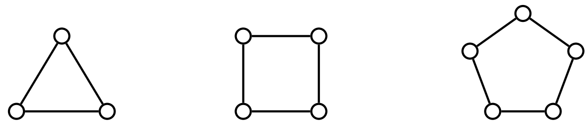


Figure 2.6: Cycles of lengths 3, 4 and 5

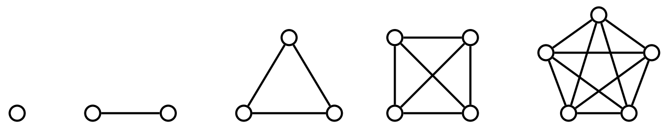


Figure 2.7: Complete graphs K_1, K_2, K_3, K_4 and K_5

Definition 5 (Complete Graphs)**.** Let n be a positive integer. The *complete graph* K_n has n vertices and all possible edges between distinct vertices, that is, no loops or repeated edges. (See Figure 2.7.)

Definition 6 (Induced Subgraph)**.** Let $G = (V, E)$ be a graph. An *induced subgraph* of G is another graph formed from a subset of V and all of the edges in E connecting pairs of vertices in that subset.

Definition 7 (Clique)**.** A *clique* is a subset of vertices of a simple graph G such that every two distinct vertices in the clique are adjacent. In other words, it is an induced subgraph of G that is complete.

Definition 8 (Complement Graphs)**.** The *complement* (or inverse of a graph G is a graph G' with the same vertices, but with edges such that two distinct vertices of G' are adjacent if and only if they are not adjacent in G. In other words, to generate the complement graph of G of order n, we would fill in all the missing edges required to form the complete graph K_n, then remove all the original edges of G.

2.3 Directed Graphs

So far, the simple graphs and multigraphs we have been talking about are also undirected graphs, i.e. their edges are not directed.

Definition 9 (Orientation, Digraph, Arc)**.** An *orientation* of an undirected graph is, for each edge $\{x, y\}$, an assignment of a direction x to y or y to x

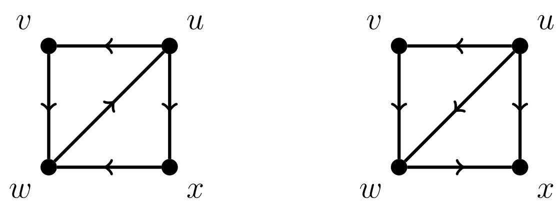


Figure 2.8: (Left) An orientation of the graph with two directed cycles – $\{u, v, w\}$ and $\{u, w, x\}$ both form C_3. (Right) An acyclic orientation that is also an *st*-orientation, with one source u and one sink x, and also a transitive orientation.

usually indicated by an arrow. This would make the graph a directed graph, or *digraph.*

More formally, for vertices x, y, an edge is an unordered pair $\{x, y\}$, while a directed edge from x to y, called an *arc*, is an ordered pair (x, y). Flipping the orientation, we get (y, x), the *reversed arc* of (x, y)

Definition 10 (*st*-orientation)**.** An *acyclic orientation* of an undirected graph assigns an orientation to each edge such that the digraph formed has no directed cycles. Hence, acyclic orientations have at least one *source* – a vertex with no incoming edges – and at least one *sink* – a vertex with no outgong edges. In particular, a *bipolar orientation* or *st-orientation* has a unique source s and a unique sink t.

Definition 11 (Transitive Orientation)**.** A graph has a *transitive orientation* if: when there is a directed edge from x to y and a directed edge from y to z, the edge $\{x, z\}$ is in the graph and it is directed from x to z. (See Figure 2.8 Right.)

2.4 Proper Colorings and the Chromatic Polynomial

Definition 12 (Proper Coloring)**.** A *proper (vertex) coloring* of some graph G is an assignment of colors to the vertices of G such that no two adjacent vertices have the same color. A proper (vertex) λ-coloring of G uses up to λ colors to properly color G.

Remark 13. Let G be a simple graph of order $[\,n\,]$ and $C = \{c_1, c_2, ..., c_\lambda\}$ be a set of λ colors. Then a coloring of G is a map $f : [\,n\,] \to C$. The function f assigns a color to each vertex, and two colorings of G are distinct if they assign distinct colors to any one of the vertices.

Remark 14. We will only consider simple graphs (instead of general graphs) since adding multiple edges between two vertices or loops do not alter properties of graph colorings.

Definition 15 (Color Class)**.** In a proper coloring of some graph G, vertices have the same color if and only if they are in the same *color class* (or *independent set*). In other words, a proper λ-coloring of G is the partition of V, the vertex set of G, into λ color classes and uniquely assigning one of λ colors to each color class.

Example 16. Considering graph G in Figure 2.10, a proper coloring of its vertex set $V = \{a, b, c, d\}$ may assign b and d to the same color class since they are non-adjacent. The possible partitions of V are $\{\{a\}, \{b\}, \{c\}, \{d\}\}$ and $\{\{a\}, \{c\}, \{b, d\}\}$.

Definition 17 (Chromatic Number)**.** The chromatic number of some graph G is the minimum number of colors needed to properly color G.

Definition 18 (Chromatic Polynomial)**.** Let G be a graph and λ be a positive integer. The *chromatic polynomial* of G is the function $\chi(G;\, \lambda)$ that denotes the number of ways G can be properly colored with *up to* λ colors. For all positive integer values of λ less than the chromatic number, we have $\chi(G;\, \lambda) = 0$.

Definition 19 (Chromatic Vector)**.** Let m be the chromatic number of some simple graph G. Then the chromatic vector of G is $c(G) = (c_0, c_1, c_2, ...)$ where c_k is the integer number of ways to partition the vertices of G into *exactly* k color classes.

If we are given $\lambda > k$ colors, for each k, those k color classes then have λ, $\lambda - 1$, $\lambda - 2, ...$ choices of colors each, because once the first class of vertices are assigned a color, other classes cannot take that color. Thus, the chromatic polynomial can be expressed as:

$$\chi(G;\, \lambda) = \sum_{k=0}^{\lambda} c_k(G) \cdot (\lambda)_k$$

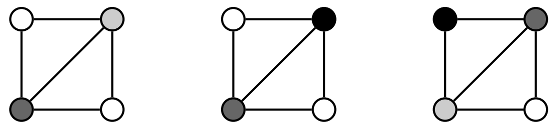


Figure 2.9: Three distinct proper colorings of G.

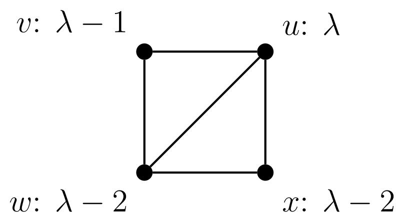


Figure 2.10: G labeled with the number of possible colors for each vertex in terms of λ.

where $(\lambda)_k$ is the falling factorial $\underbrace{\lambda(\lambda-1)(\lambda-2)\cdots(\lambda-k+1)}_{k\text{ terms}}$.

Example 20. In Figure 2.9 we see 3 proper colorings of the same graph G, the leftmost graph and middle graph are partitioned into the same 3 color classes, but are distinct because different colors are assigned to the classes. Meanwhile, the rightmost coloring uses 4 colors.

The chromatic number of these graphs is at 3; we prove this as follows: the chromatic number is at least 3 since the graphs contain triangles, and each vertex in a triangle must have a distinct color; the chromatic number is at most 3 since the example shows we can properly color the graph with 3 colors.

More generally, we can find $\chi(G;\lambda) = \lambda(\lambda-1)(\lambda-2)^2$ (Figure 2.10). One way to deduce this is by first considering vertex u, which can take any of λ colors. Then going anti-clockwise, v can take any of λ colors except that of u, and w can take neither colors of u and v. Similarly, this is the case for vertex x. Hence, multiplying these possibilities together we get the stated chromatic polynomial.

Remark 21. That the chromatic polynomial of a graph is indeed polynomial is not obvious and requires proof. In our example, we were able to calculate

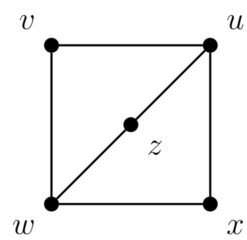


Figure 2.11: The simple graph H does not have a chromatic polynomial $\chi(H;\, \lambda)$ that is the product of some terms $\lambda - c$

$\chi(G;\, \lambda)$ as the product of the number of possible colors for each vertex, that is, the product of some terms $\lambda - c$ where c is an integer. However, this is not always possible as seen in graph H which is just G with one more vertex and edge (Figure 2.11). Through a recursive application of the deletion-contraction theorem, we can calculate $\chi(H;\, \lambda) = x(x-1)(x^3-5x^2+10x-7)$. This theorem also provides us with the scaffolding needed to prove that chromatic polynomials are polynomials.

2.5 Finding the Chromatic Polynomial

Definition 22 (Deletion, Contraction)**.** Let G be a general graph with the edge $\{x, y\}$. From G we can derive two new graphs: $G - \{e\}$ or $G\backslash e$ from the *deletion* of e, and G/e from the *contraction* of e. The graph $G - \{e\}$ has the same number of vertices as G and all edges of G except for e. The graph G/e is graph G except the vertices x, y are merged into one vertex v_{xy} and the edge between them e is discarded. Hence, all vertices adjacent to x or y in G are now adjacent to v_{xy} in G/e.

Theorem 23 (Deletion-Contraction Theorem)**.** *Let G be a simple graph and let e be an edge of G. Then*

$$\chi(G;\, \lambda) = \chi(G - \{e\};\, \lambda) - \chi(G/e;\, \lambda)$$

Proof. Let $G = (V, E)$ be a simple graph. Let $x, y \in V$ and $e = \{x, y\} \in E$. Consider the chromatic polynomial of $G - e$. In a proper coloring, the vertices x, y can either have the different colors or not. If they have different

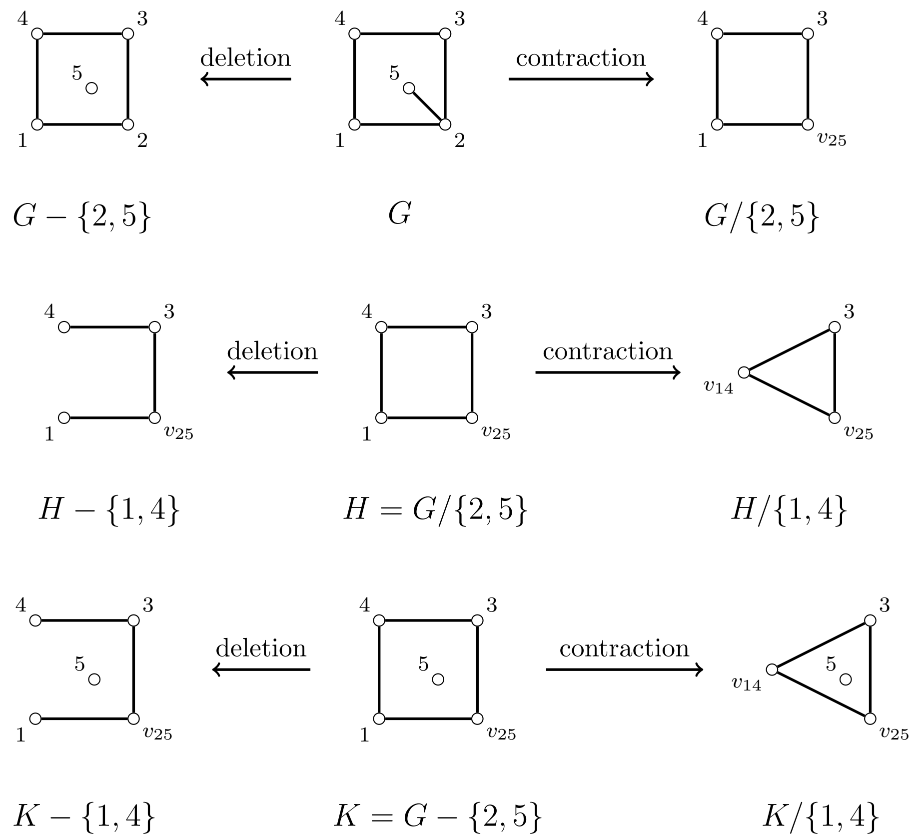


Figure 2.12: ROW 1: G is a graph of order 5 (center). Deletion of its edge $\{2,5\}$ gives a cycle of length 4 and a single disconnected vertex (left). Contraction of $\{2,5\}$ leaves just the 4-cycle (right). ROW 2: Rename $G/\{2,5\}$ as H (center). Deletion of its edge $\{2,v_{25}\}$ gives a path of length 4 (left) while contraction of it gives a 3-cycle (right). ROW 3: Similarly rename $G-\{2,5\}$ as K (center). The graphs derived from deletion and contraction of $\{2,5\}$ yield the same graphs as H did but with the extra vertex 5 remaining.

colors, the number of proper colorings of $G - e$ is equal to those of G. This is because a proper coloring of G would require the adjacent vertices x, y to have different colors. Now, if x, y have the same color, the number of proper colorings of $G - e$ is equal to those of G/e, since assigning the same color to x and y is comparable to merging them into one vertex of that color.

So the chromatic polynomial of G is equal to that of $G - e$ where x, y have different colors, to which we can add the difference between the chromatic polynomial of $G - e$ where x, y have the same color and the chromatic polynomial of G/e, which gives 0 since they are equal. Then, the sum of the chromatic polynomials of $G - e$ where x, y are the same and different simply give the chromatic polynomial of $G - e$ itself, and we are done.

$$\begin{aligned}
\chi(G;\, \lambda) &= \underbrace{\chi(G - e;\, \lambda)}_{x,y \text{ of different colors}} \\
&= \underbrace{\chi(G - e;\, \lambda)}_{x,y \text{ of different colors}} + 0 \\
&= \underbrace{\chi(G - e;\, \lambda)}_{x,y \text{ of different colors}} + \left[\underbrace{\chi(G - e;\, \lambda)}_{x,y \text{ of the same color}} - \chi(G/e;\, \lambda) \right] \\
&= \chi(G - e;\, \lambda) - \chi(G/e;\, \lambda)
\end{aligned}$$

□

Remark 24. To find the chromatic polynomial of some graph, we recursively delete and contract its edges until all the derived graphs have obvious chromatic polynomials, then apply the deletion-contraction theorem. We generally aim to derive *paths* which are strings of vertices with two ends. For example, the graph $H - \{1, 4\}$ in Figure 2.12 is a path with 4 vertices, denoted P_4. Beginning from one end, the first vertex of P_n can be assigned λ possible colors, and every remaining vertex down the path has $\lambda - 1$ possible colors. Hence, $\chi(P_n;\, \lambda) = \lambda(\lambda - 1)^{n-1}$.

Example 25. Let us find the chromatic polynomial of the graph G in Figure 2.12 using the deletion-contraction theorem. After deriving the graphs $G - \{2, 5\}$ and $G/\{2, 5\}$, we can further delete and contract edges from these until we get graphs that are paths.

Renaming the latter graph $H = G/\{2,5\}$, from a deletion we get $H - \{1,4\}$ which is P_4 and has a chromatic polynomial of $\lambda(\lambda-1)^3$, and from a contraction we get $H/\{1,4\}$ which is a 3-cycle and has a chromatic polynomial of $\lambda(\lambda-1)(\lambda-2)$.

Now, $K = G - \{2,5\}$ is the same as H but has an extra disconnected vertex. Hence it gives the same graphs as those derived H but with the extra disconnected vertex. Since that vertex can take on any of the λ colors, the chromatic polynomials of K's derived graphs are the same as those derived from H multiplied by λ.

$$\begin{aligned}
\chi(G;\,\lambda) &= \chi(G-\{2,5\};\,\lambda) - \chi(G/\{2,5\};\,\lambda)\\
&= \chi(K;\,\lambda) - \chi(H;\,\lambda)\\
&= [\chi(K-\{1,4\};\,\lambda) - \chi(K/\{1,4\};\,\lambda)] - [\chi(H-\{1,4\};\,\lambda) - \chi(H/\{1,4\};\,\lambda)]\\
&= \chi(P_4;\,\lambda)\cdot\lambda - \chi(C_3;\,\lambda)\cdot\lambda - \chi(P_4;\,\lambda) - \chi(C_3;\,\lambda)\\
&= \lambda^5 - 5\lambda^4 + 10\lambda^3 - 9\lambda^2 + 3\lambda
\end{aligned}$$

Corollary 26. *Let G be a simple graph with order n. Then $\chi(G;\,\lambda)$ is a polynomial of degree n with integer coefficients.*

Proof. We need to show that, for all G with $n > 0$ vertices, $\chi(G;\,\lambda)$ is a polynomial of degree n with integer coefficients. We will prove this by induction, inducting on the number of edges. For the graph consisting of n vertices and of no edges, $\chi(G;\,\lambda) = \lambda^n$ fulfills the base case.

Then by the inductive hypothesis, $\chi(G-e;\,\lambda)$ has degree k since $G-e$ is of order k, and $\chi(G/e;\,\lambda)$ has degree $k-1$ since G/e is of order $k-1$. By the deletion-contraction theorem, $\chi(G;\,\lambda) = \chi(G-e;\,\lambda) - \chi(G/e;\,\lambda)$ is a polynomial of degree k with integer coefficients. □

So far we have been considering natural number values of λ, but fascinatingly, $\lambda = -1$ gives a mathmatically meaningful result.

Theorem 27 (Stanley 1973)**.** *Let G be a simple graph of order n. Then $(-1)^n\chi(G;\,-1)$ is the number of possible acyclic orientations of G.*

Example 28. Applying Stanley's theorem to G in Figure 2.10, we have $\chi(G;\,-1) = (-1)^4(-1)(-1-1)(-1-2)^2 = 27$ possible acyclic orientations of G.

For more on this preliminary material, consult [8], [9].

Chapter 3

The Basics of Posets

3.1 Representations of Posets

Definition 29 (Partial Order, Poset)**.** Let P be a set, and let $\leq$ be a *partial order* on P. That is, $\leq$ is a binary relation on P that satisfies the following for all $x, y, z \in P$:

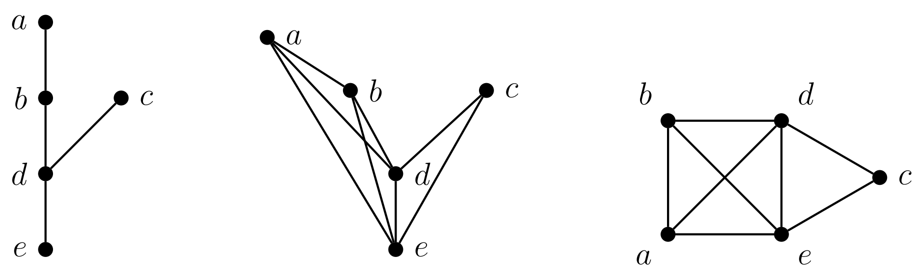
- (reflexivity) $x \leq x$
- (antisymmetry) If $x \leq y$ and $y \leq x$, then $x = y$
- (transitivity) If $x \leq y$ and $y \leq z$, then $x \leq z$

The set P together with $\leq$ form a partially ordered set, or *poset*, written as $(P, \leq)$. Since the relation $\leq$ is the only partial order referred to in this thesis, posets $(P, \leq)$ will often be referred to simply as P.

Definition 30 (Comparability relation, Covering relation)**.** Let $(P, \leq)$ be a poset, and $x, y \in P$. Then x and y are *comparable* if either $x \leq y$ or $y \leq x$. Otherwise, x and y are *incomparable*.

Write $x < y$ if $x \leq y$ and $x \neq y$. If $x < y$ and there is no z with $x < z < y$, then y *covers* x.

Definition 31 (Comparability graph, Incomparability graph)**.** Let $(P, \leq)$ be a poset with elements x, y. We can represent it through an *comparability graph* $C(P)$ by drawing all elements of P as vertices, and drawing an edge between two distinct elements x and y if and only if they are comparable, (i.e. $x \leq y$ or $y \leq x$).



$Hasse(P)$ $C(P)$ $C(P)$

Figure 3.1: (Left to Right) The Hasse diagram and two possible comparability graphs of the same poset $\{a, b, c, d, e\}$. The Hasse diagram is simpler and only has edges to represent the covering relations $a > b > d > e$ and $c > d$, where the "height" of each element matters. The middle comparability graph builds on the Hasse diagram to include the implied comparability relations $a > d$, $a > e$, $b > e$, $c > e$. The graph on the right rearranges vertices to emphasize the positions of vertices do not matter in comparability graphs.

The *incomparability graph* (or cocomparability graph) of $(P, \leq)$, notated $I(P)$, is the complement graph of its comparability graph; there is an edge between the distinct elements x and y if and only if they are incomparable (i.e. neither $x \leq y$ nor $y \leq x$).

All comparability and incomparability graphs are simple graphs, that is, there are not multiple edges between the same two distinct vertices, and there are no loops (the relation $x \leq x$ is not represented).

Definition 32 (Hasse Diagram)**.** Let $(P, \leq)$ be a poset with elements x, y. In its *Hasse Diagram* notated $Hasse(P)$, all vertices are elements of P, and an edge is drawn between the distinct elements x and y if and only if y covers x (i.e. $x < y$ with no $z \in P$ such that $x < z < y$). Additionally, if $x < y$ then x is drawn "below" y. Two elements are incomparable if one cannot be reached by only following edges downwards. For example, b and g are comparable since we can reach g from b through edges $\{b, d\}, \{d, g\}$, but b and e are incomparable since starting from e and going to g, we would then need to travel upwards to reach b.

We omit the proofs for the following two theorems.

Theorem 33 (Ghouila-Houri, 1962)**.** *A graph is a comparability graph if and* *only if it has a transitive orientation.*

Theorem 34 (Berge, 1976)**.** *A graph G is a comparability graph if and only if it has an orientation satisfying what's called the V-rule or P_3 rule: For all vertices $v, u, w \in G$ with edges $\{v, u\}, \{u, v\}$ but v not adjacent to w, the orientations are (v, u) and (w, u), or else (u, v) and (u, w). (See Figure 5.4.)*

3.2 Linear Orders

Definition 35 (Linear order, Chains, Antichains)**.** A partial order is a *linear order* if every two elements are comparable. A set with a total order is a *linearly ordered set*, or a *chain*.

Conversely, let A be a subset of a poset $(P, \leq)$ in which no elements are comparable, then A is an *antichain*. The size of the largest antichain P is the *width* of P.

Example 36. In Figure 3.2, we see the poset $P = \{a, b, c, d, e, f, g\}$ with relations $a > b > d > g, a > c > d, e > g, f > g$ represented in three ways. While the comparability graph (middle) shows all comparability relations, the Hasse diagram is clearer for identifying chains. Since e is only adjacent to g, chains containing e are only $\{e\}$ and $\{e, g\}$. Meanwhile, the longest chains in the poset can be seen from "chains" continually moving downwards; they are $\{a, b, d, g\}$ and $\{a, c, d, g\}$.

To identify chains in the comparability graph, we look for subsets that form complete graphs. Since $\{a, b, d\}$ forms K_3 and $\{a, c, d, g\}$ form K_4, they are chains. Conversely, chains in incomparability graphs are sets of elements that are all non-adjacent to each other. As such, a proper coloring of the incomparability graph of P would partition its elements into chains, where each color class forms a chain.

On the other hand, we would look for subsets of non-adjacent vertices in the Hasse diagram or comparability graph to find antichains of P. For example, $\{b, c, e, f\}$ is the largest antichain in P. An easier way to identify this antichain is by looking for complete graphs in the incomparability graph, where these four vertices form K_4.

Definition 37 (Width, Height)**.** The *height* of a poset is the maximum size of a chain in the poset. Conversely, its *width* is the maximum size of an antichain in it.

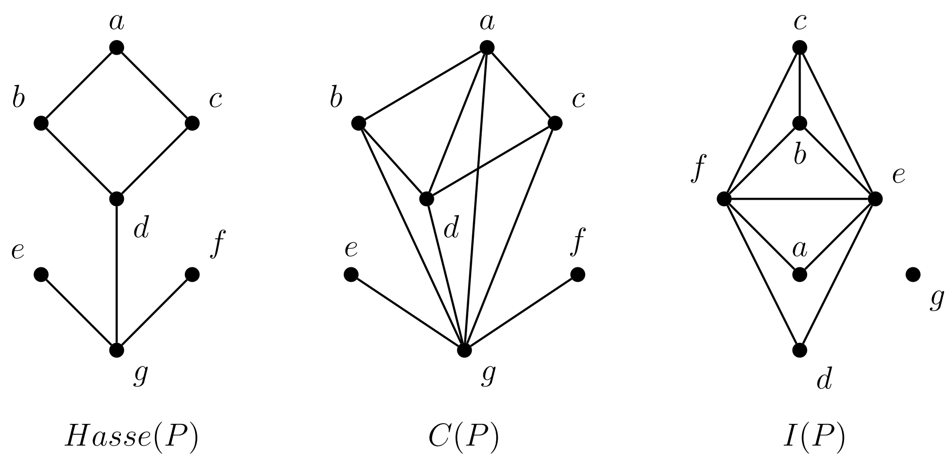


Figure 3.2: (Left to Right) The Hasse diagram, comparability and incomparability graph of the same poset $\{a, b, c, d, e, f, g\}$.

Example 38. The height of the poset in Figure 3.2 is 4, since the maximally sized chains are $\{a, b, d, g\}$ and $\{a, c, d, g\}$. The width of the poset is 4, since the maximally sized antichain is $\{b, c, e, f\}$.

We omit the proof of the following well-known theorem.

Theorem 39 (Dilworth's Theorem, Dilworth 1950)**.** *Let P be a finite poset P. Then the minimum number of chains needed to partition P is exactly equal to the width of P.*

Example 40. Returning to our poset P in Figure 3.2, the minimum number of chains needed to partition P is 4. One such partition is $\{\{b\}, \{a, c, d\}, \{e, g\}, \{f\}\}$, where each chain contains exactly 1 element from the maximally sized antichain. A

3.3 Order Dimension and Permutation Graphs

Finally, we explore the possibility of incomparability graphs being themselves a comparability graph (of another poset, in general).

Definition 41 (Linear Extension)**.** Let the linear order L and the partial order P both be defined on the same set of elements. L is a *linear extension* of P if $x < y$ in L when $x < y$ in P.

Remark 42. A linear extension of P has all the comparability relations of P, and does not have incomparable elements, i.e. for all pairs of elements x, y in P, $x < y$ or $y < x$.

Extending a partial order to a linear order is like squishing a comparability graph from the sides into a single vertical chain. Looking at the middle graph in Figure 3.2, we can squish b above or below c since they are incomparable, as long as they are between a and d. We can also elongate the edge $\{e, g\}$ and squish e above a, but we cannot place f below g since we need to retain $f > g$.

Definition 43 (Dimension, Realizer)**.** The *dimension* (or Dushnik–Miller dimension) of any poset P is the smallest number of linear orders the intersection of which is P. For a formal definition, we first define the realizer.

Let P be a poset. Let $\mathfrak{F} = (L_1, L_2, ..., L_d)$ be a family of linear extensions of P for which $x < y$ if and only if $x < y$ in every $L_i \in \mathfrak{F}$. Every poset P has such a family and $\mathfrak{F}$ is called a *realizer* of P. The dimension of P is the smallest integer d for which $\mathfrak{F}$ realizes P.

Example 44. The 3-crown graph in Figure 3.3 has dimension 3. The intersection of the following family $\mathfrak{F}$ of linear extensions is P.

$$\begin{aligned} L_1 :&\quad b < c < d < a < e < f \\ L_2 :&\quad a < c < e < b < d < f \\ L_3 :&\quad a < b < f < c < d < e \end{aligned}$$

For instance, $a < e$ is in P and in all three L_i. In terms of incomparability relations, d not comparable to e in P is reflected in the contradicting $d < e$ in L_1 and $e < d$ in L_2.

Example 45. Posets of dimension 1 do not have incomparable relations.

Definition 46 (Permutation Graph)**.** Let $\pi = (\sigma_1, \sigma_2, ..., \sigma_n)$ be a permutation of $[n]$. Then G is a *permutation graph* of order n if G has an edge $\{v_i, v_j\}$ for any two indices $i > j$ for which i appears before i in π. Another definition for a permutation graph is any graph of dimension at most two.

For the following theorems and their proofs, see [4] and [7]

Theorem 47 (Dushnik and Miller 1941)**.** *A graph G is both a comparability graph and an incomparability graph if and only if it has a dimension of at most two.*

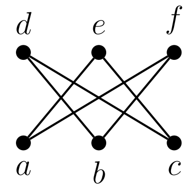


Figure 3.3: The 3-crown graph has a dimension of 3.

Theorem 48. *A graph G is a permutation graph if and only if G is both a comparability graph and an incomparability graph.*

Example 49. In Figure 3.4, the two complementary graphs are both comparability graphs since they each can be realized by a family of at least 2 linear orders. One such realizer for P is

$$\begin{aligned} L_1: &\quad c > d > b > e > a \\ L_2: &\quad b > e > c > d > a \end{aligned}$$

One such realizer for Q is

$$\begin{aligned} L_1: &\quad c > d > e > b > a \\ L_2: &\quad a > d > c > b > e \end{aligned}$$

Their respective partial orders can be represented with the permutations $\pi_P = (becda)$ and $\pi_Q = (adceb)$.

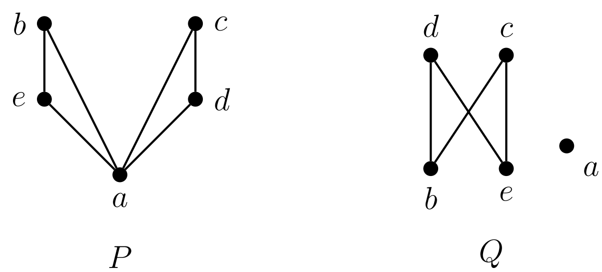


Figure 3.4: The two complementary comparability graphs both have dimension 2 as proven under Theorem 47.

Chapter 4

Rook Theory and Proper Colorings

4.1 An Introductory Problem

Let us return to the introductory proper coloring problem in Chapter 1. Unlike in Example 20, we cannot simply find $\chi(G; \lambda)$ by calculating the product of the number of possible colors for each vertex. Starting with the top vertex in Figure 4.1, it can be assigned one of λ colors. The next row down, the leftmost vertex has one less possible color since it is adjacent to the top vertex. Similarly, the middle vertex has $\lambda - 1$ possibilities, and the rightmost $\lambda - 2$ possibilities.

Now, the bottom center vertex has an ambiguous number of possible colors. It has $\lambda - 3$ possibilities if all three vertices in the row above have distinct colors, but $\lambda - 2$ possibilities if, in the row above, the leftmost vertex shares one of the other two's colors. We need another way to find the chromatic polynomial.

4.2 Non-attacking Rooks

Definition 50 (n-Board)**.** Let n be a positive integer. In the board of $n \times n$ squares, any subset of squares forms an n-board and are represented as shaded squares. That is, if B is a set of shaded squares across c columns and r rows, then B is an n-Board for any $n \geq c$ and $n \geq r$.

Note that n-boards do not have to be rectangular or square. In Figure

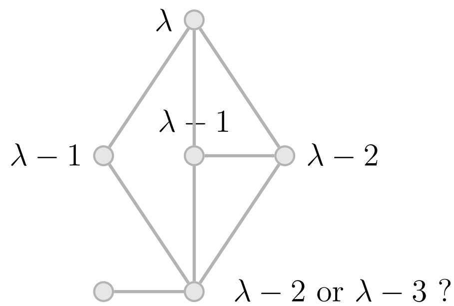


Figure 4.1: The chromatic polynomial of G cannot be calculated as a product of each vertex's color possibilities

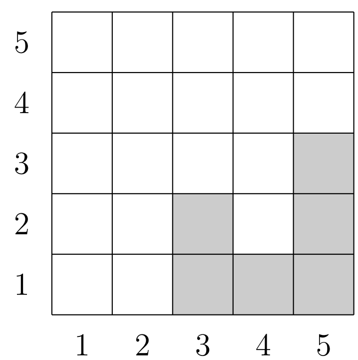


Figure 4.2: An example of a non-rectangular 5-board.

4.2, since the shaded squares have 3 columns and 3 rows, they can form a 3-board, 4-board,... After introducing proper n-boards below, we will see why we often represent n-boards in grids with extra rows and columns.

Definition 51 (Rook Vector)**.** The rook vector of an n-board B is $r(B) = (r_0, r_1, ..., r_n)$, where r_k is the number of ways to place k non-attacking rooks on B (represented as shaded squares), and $r_0 = 1$ is always true.

4.3 Connecting Graph Coloring to Rook Theory

Surprisingly, graph coloring and the non-attacking rooks are inextricably linked! We can translate our chromatic polynomial problem into one on non-attacking rooks, solve that, and translate it back to graph theory. Although we have not defined all the terms in the following theorem, we can get a general idea of this amazing connection before an in depth understanding of the underlying mechanisms. For simplicity, we sometimes denote $B(G')$ as B.

Theorem 52. *[Goldman, Joichi and White, 1975] Let G be a Γ-graph of order n, $B(G')$ be its proper n-board and the rook vector of $B(G')$ be $r(B) = (r_0, r_1, ..., r_n)$. Then the chromatic polynomial of G is*

$$\chi(G;\,\lambda) = \sum_{k=0}^{\lambda} r_k(B) \cdot (\lambda)_{n-k}$$

where $(\lambda)_{n-k}$ is the falling factorial $\underbrace{\lambda(\lambda-1)(\lambda-2)\cdots(\lambda-n+k+1)}_{n-k \text{ terms}}$.

Remark 53. Essentially, for a type of graph we term "Γ-graphs", we can derive a "proper" board from it, find its rook vector, and solely with this information find the chromatic polynomial of the original graph.

Recall that $\chi(G;\,\lambda)$ represents the number of ways to properly color G using *up to* λ colors. Then this equation would make sense if each term in the sum, $r_k \cdot (\lambda)_{n-k}$, is the number of ways to properly color G with *exactly* λ colors: if r_k is the number of ways to partition the vertices of G into $n-k$ color classes, then each of the $n-k$ terms in the falling factorial would be the number of colors available to each color class. (This is how we will prove the theorem below.)

For example, in our example graph of 6 vertices (Figure 4.1), we would have

$$\begin{aligned}\chi(G;\,\lambda) = r_0(B) &\cdot \lambda(\lambda-1)\cdots(\lambda-5)\\ + r_1(B) &\cdot \lambda(\lambda-1)\cdots(\lambda-4)\\ + r_2(B) &\cdot \lambda(\lambda-1)(\lambda-2)(\lambda-3)\\ + r_3(B) &\cdot \lambda(\lambda-1)(\lambda-2)\end{aligned}$$

Say for the term $r_2 \cdot \lambda(\lambda-1)(\lambda-2)(\lambda-3)$, we want to show that r_2 is the number of ways to partition the vertex set into 4 color classes, with the first having λ possible colors, the second cannot take on the first's color so it has $\lambda - 1$ possible colors, and so on for the next two color classes with $\lambda - 2$ and $\lambda - 3$ choices respectively.

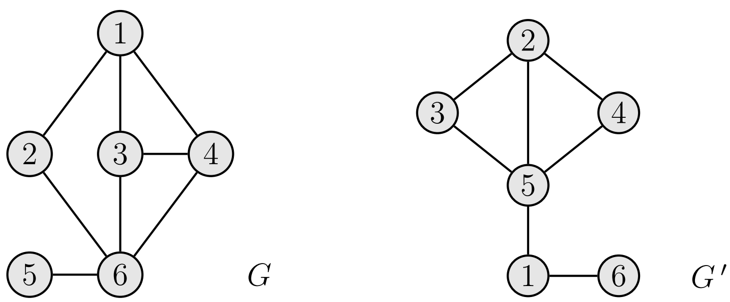


Figure 4.3: The labeled graph G of the start-up from the introductory problem in Chapter 1, and its complement G'.

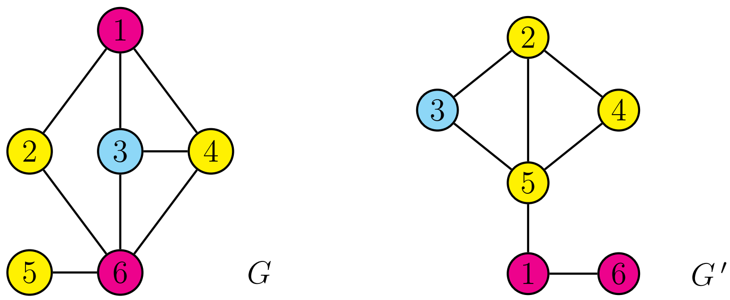


Figure 4.4: A proper 3-coloring of G and how those color classes look in G' – all vertices in the same color class are adjacent to each other.

4.4 Setting up the Correspondence

Now, we'll introduce the new terms as we follow the logic of Theorem 52. In a proper coloring, we are interested in finding color classes where, within each class, all vertices are not adjacent to one another. We can more easily identify such non-adjacent groups of vertices by considering G', the **complement** of G (Definition 8; see Figure 4.3). In the Figure 4.4, we have a partition of $[6]$, the vertices of G into three color classes $\{1,6\},\{2,4,5\},\{3\}$. On the left, vertices with the same color are non-adjacent, and on the right we see these same color classes in G' where all elements of each color class are adjacent to each other.

Now, using G', we seek to build its board $B(G')$. Starting with a graph of order n, we construct an $n \times n$ grid, with each row/column number representing a vertex. Before trying this with the graph taken from the startup example, let us consider the board of K_6. We would construct a 6×6

grid, **with the row and column labels reading like x, y coordinates** to ensure Algorithms 57 and 58 work. The square (i, j) is in $B(G')$ (is shaded) if i is adjacent to j in G'. Looking at Figure 4.5, if we draw all the edges connected to vertex 1, this is represented on $B(G')$ by shading in $(2, 1), (3, 1), (4, 1), (5, 1), (6, 1)$ (read like x, y-coordinates). Doing the same for vertex 2, we fill in most of the second row, leaving out $(1, 2)$ because that edge has already been represented in $(2, 1)$. Continuing to draw all edges to get the complete graph K_6, we see that we have shaded in a "staircase"-shape on the lower right side of $B(G')$.

This could equally have been represented by shading in the top left corner instead, i.e. swapping out each shaded (i, j) for (j, i), but for consistency we will only consider the lower right triangle. The diagonal (i, i) is never shaded since that would imply loops in G, but we are working with simple graphs since loops don't affect proper colorings. Thus, for any graph with 6 vertices, its corresponding 6-board is a subset of the "right triangle" bounded by $(2, 1), (6, 1), (6, 5)$. More generally, the n-board corresponding to a graph of order n is a subset of the "right triangle" bounded by $(2, 1), (n, 1), (n, n-1)$. Now our n-board already satisfies the first condition to make it a proper n-board.

Definition 54 (Proper n-board)**.** The n-board B is *proper* if and only if

(a) $\quad (i, j) \in B \implies i > j$

(b) $\quad (i, j), (j, k) \in B \implies (i, k) \in B$

In Figure 4.5 we see that $B(K_6)$, the 6-board of K_6, is exactly the set of squares (i, j) fulfilling $i > j$. Hence for any graph of order n, its proper n-board is a subset of $B(K_n)$.

The second condition of transitivity is not as intuitive, but is necessary for our algorithms to translate between rooks on boards and partitions of graph vertices into color classes. In Figure 4.6 we see how $B(G')$ is transitive since $(5, 3), (3, 2), (5, 2) \in B(G')$ and $(5, 4), (4, 2), (5, 2) \in B(G')$, which are the only two cases of there being $(i, j), (j, k) \in B(G')$. Remembering how G' has only two groups of 3 inter-adjacent vertices, the fact that they happened to be $\{5, 3, 2\}$ and $\{5, 4, 2\}$ is not a coincidence, as we will see below.

The definition of Γ-graphs is dependent on that of transitive graphs and proper n-boards.

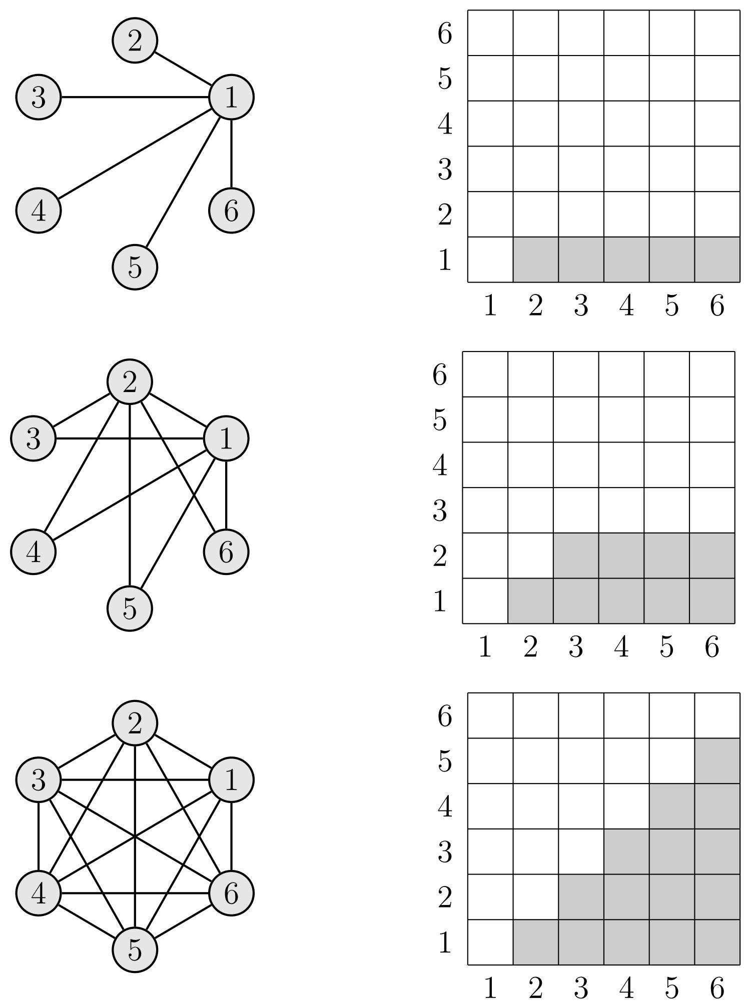


Figure 4.5: Building a proper 6-board corresponding to the complete graph K_6. The board squares are expressed like x, y-coordinates. Each edge $\{x, y\}$ in K_6, where $x > y$, is represented by the (shaded) square (x, y)

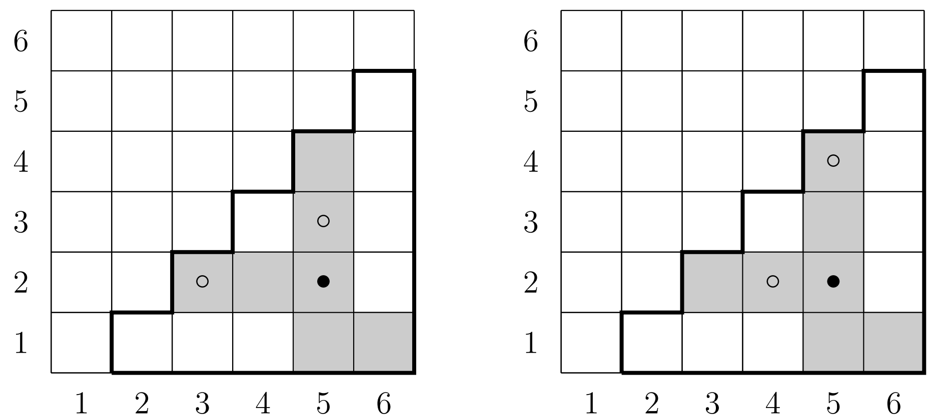


Figure 4.6: The shaded squares show $B(G')$, the 6-board corresponding to G'. It is proper since it (a) is a subset of the 6-board of K_6 (outlined in the black "staircase"), and (b) fulfills transitivity since, when white circled squares are in $B(G')$, the squares with black dots are in it too.

Definition 55 (Transitive graph)**.** Let H be a simple graph of order n. Then H is a *transitive graph* if and only if $B(H)$ is a proper n-board.

Definition 56 (Γ-graph)**.** In accordance with Goldman et al.'s definition [6], a simple graph G is a Γ-graph if $B(G')$ is a proper n-board. That is, the complement of a transitive graph is a Γ-graph, and vice versa.

To summarize, starting with G, we found its complement G' then used the latter's edges to build its proper 6-board $B(G')$. According to Theorem 52, the rook vector of $B(G')$ is directly linked to the chromatic polynomial of G. To prove this we introduce the following algorithms.

The following pseudo-code inspired by Python detail two maps between a rook placement on $B(G')$ and a partition of the vertices of G into color classes **(without colors assigned to each class yet)**. The first algorithm κ takes as an input

Algorithm 57. $[\kappa]$

Given B, a proper n-board, let r be a placement of some $k < n$ rooks on B. Define κ as a map that takes r then returns S, a partition of $[n]$ into $n-k$ color classes, as so:

Let $S = \{\}$

For each i in $\{n, n-1, ..., 1\}$:

3 If row i is empty:
4 Start a new color class $C = \{\}$.
5 Add vertex i to C.
6 While column i is not empty:
7 Name as j the row number of the rook in column i.
8 Set j as the the new i.
9 Append i to C.
10 If column i is empty:
11 Add C to S.

Algorithm 58. $[\rho]$

Let G be some Γ-graph G of order n, and $S = \{C_1, C_2, ..., C_{n-k}\}$ be the partition of $[n]$ into $n-k$ color classes. Let $B(G')$ the proper n-board of the complement of G. Let $|C|$ denote the number of elements in C. Place the vertices in each color class in *descending order*, and let $C[x]$ denote the xth element of C. Define ρ as the map that takes S and returns a placement of rooks r with the following method.

12 For each C in S:
13 If $|C| > 1$:
14 For i in $\{1, 2, ..., |C| - 1\}$:
15 Place a rook at $(C[i], C[i+1])$ on B.

4.5 The Mechanism

We can think of the function of rooks as highlighting certain non-adjacent vertex pairs for each partition into color classes. To understand the underlying mechanism, we will walk through the method here and delve into the proof later.

Example 59 (κ)**.** To find a partition of G's vertices into 4 classes, take a placement of $6 - 4 = 2$ rooks on $B(G')$, such as on $(3, 2), (5, 1)$. Then add color classes to the empty set of color classes S [line 11] by considering each row in descending order [line 2].

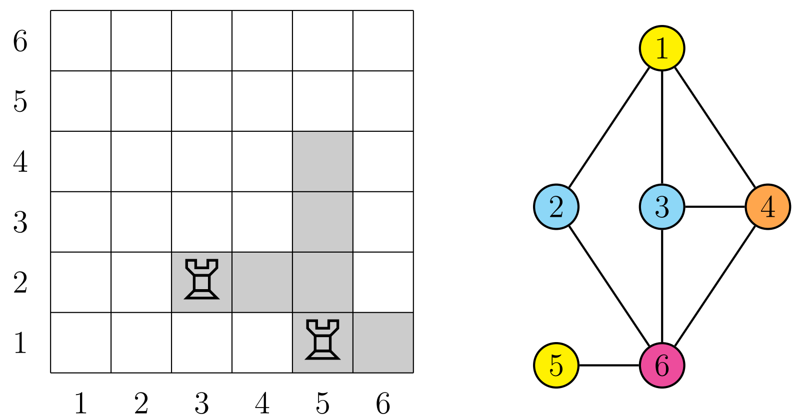


Figure 4.7: $B(G')$ with a placement of 2 rooks and the corresponding partition of G into color classes, shown as one possible proper coloring (rooks do not determine which color is assigned to which class).

- Row $i = 6$ has no rooks, so start a new color class with vertex 6, then look at column 6 which has no rooks, so we end here. Lines 6-9 are ignored. The first color class in S is the singleton $\{6\}$.
- Return to line 2 and consider row $i = 5$. It has no rooks, so begin a new class with vertex 5 [lines 3-5]. Notice on column 5 there is a rook at row 1, so add 1 to the color class [lines 6-9]. End there because column 1 is empty [lines 10-11]. Another color class is $\{5, 1\}$.
- Row 4 has no rooks; column 4 has no rooks; $\{4\}$ is another singleton.
- Row 3 has no rooks; there is a rook on $(3, 2)$; no rooks on column 2. $\{3, 2\}$ is our last color class.
- Nothing is done for $i = 2, 1$ since these rows are not empty.

Our partition of the vertex set $V = [\,6\,]$ is $S = \{\{6\}, \{5, 1\}, \{4\}, \{3, 2\}\}$.

Example 60. To find a partition of G's vertices into 3 classes, place $6-3 = 3$ rooks on B, such as on $(6, 1), (5, 4), (4, 2)$. Then find the partitions like so:

- Row $i = 6$ has no rooks, so start a new color class with vertex 6; column 6 has a rook at row 1, add 1 to the class. End there since column 1 is empty. One color class is $\{6, 1\}$.

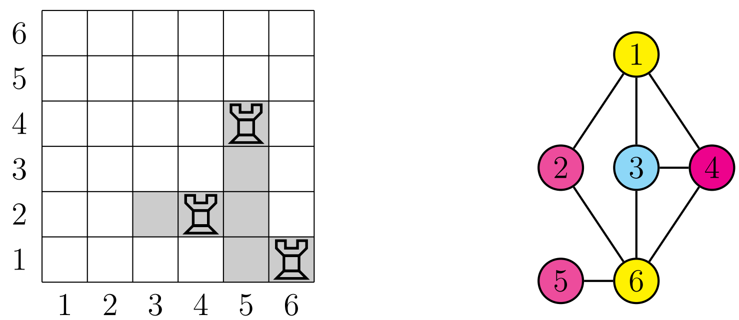


Figure 4.8: $B(G')$ with a placement of 3 rooks and the corresponding partition of G into color classes, shown as one possible proper coloring (rooks do not determine which color is assigned to which class).

- Row $i = 5$ has no rooks; begin with vertex 5. Column 5 has a rook at row 4; add 4 to the class. Column 4 has a rook at row 2; add 2 to the class. Column 2 is empty; stop here. Another color class is $\{5, 4, 2\}$.
- Do nothing for $i = 4$ since row 4 has a rook.
- Row 3 has no rooks; column 3 has no rooks; $\{3\}$ is a singleton.
- Do nothing for $i = 2, 1$ since both rows have rooks.

The reason we can "link up" rooks $(5, 4), (4, 2)$ to form the color class $\{5, 4, 2\}$ is, since these two squares are in $B(G')$, $5, 4$ and $4, 2$ are non-adjacent respectively. Now since $B(G')$ is proper, $(5, 2)$ is also in it by transitivity, so $5, 2$ are also non-adjacent. Apart from singletons, κ places all the row and column numbers $i, j, k, \ldots$ of transitive sequences of rooks $(i, j), (j, k), \ldots$ into one color class.

Remark 61. Why use non-attacking rooks? If $(i, j) \in B(G')$ means i, j are non-adjacent in G, why don't we just place these i, j pairs in the same color class?

Looking at Figure 4.9, a rook cannot be placed at $(5, 1) \in B$ for a partition of $[\,6\,]$ into 3 color classes, since $5, 1$ would be in the same color class, and three vertices $3, 4, 6$ could only choose from 2 remaining colors, since they are adjacent to one of $5, 1$. Yet $3, 4, 6$ are inter-adjacent in a triangle and must each be colored a distinct color.

Say we want to partition $[\,6\,]$ into 3 color classes C_1, C_2, C_3 and we place 5 and 1 in C_1 because $(5, 1) \in B$. Since 6 is adjacent to 5, we add 6 to C_1;

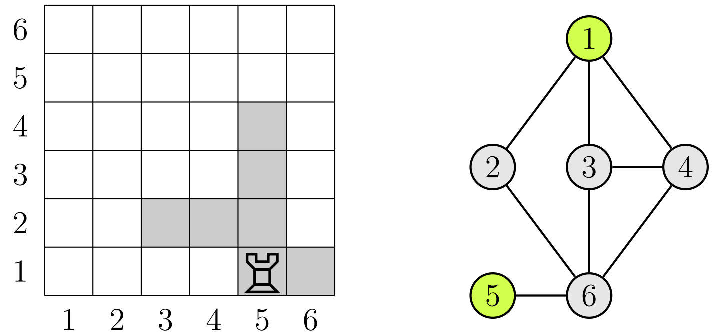


Figure 4.9: A rook cannot be placed at $(5,1) \in B$ for a partition of $[\,6\,]$ into 3 color classes, since $5,1$ would be in the same color class, and three vertices $3,4,6$ could only choose from 2 remaining colors, since they are adjacent to one of $5,1$. Yet $3,4,6$ are inter-adjacent in a triangle and must each be colored a distinct color.

and since 3 and 4 are respectively adjacent to 1, neither can be added to C_1 either. Thus the triangle 3-4-6 is left with only 2 color classes C_2, C_3 but they must each be placed in a separate class to give a proper coloring. Thus, we need rooks to help us strategize which non-adjacent vertices to place in the same color class.

If we place two rooks in the same row or column, we most likely will not get a proper coloring in G. As a counterexample, suppose we place rooks on $(4,2)$ and $(3,2)$. While $4,2$ and $3,2$ are non-adjacent vertex pairs in G, $\{4,3,2\}$ do not form a color class since 4 and 3 are adjacent. This is true for all pairs of squares on the same row/column except those related by transitivity: $(5,2),(3,2)$, $(5,3),(5,2)$, $(5,2),(4,2)$ and $(5,4),(5,2)$.

Heuristically, the existence of a long row or column i on a board $B(G')$ means that in the corresponding original graph G, vertex i is non-adjacent to many vertices. Especially when we have few colors, we want to place these "isolated" vertices in the same color class and "waste" them. Back to our example board, the long column 5 means 5 is non-adjacent to 4,3,2 and 1, while row 1 means 1 is non-adjacent to 5 and 6. It would be more strategic to have 5 in one class with some vertices it's non-adjacent to, and 1 in another class. Non-attacking rooks help to separate these rows and columns into different color classes.

4.6 Proving the Theorem

Now that we understand the general mechanism of Theorem 52, we turn to proving it. First, we prove that Algorithms 57 and 58 indeed give us valid partitions of $[n]$

Lemma 62. *Given a rook placement r, $\kappa(r)$ places all vertices of G into a color class, i.e. its output is indeed a partition of the vertex set of G.*

Proof. In κ, we only assign a vertex x to a color class through considering row x. Either we add x because row x is empty, in which case we are beginning a new color class with x [lines 3-4], or we have come across a rook in row x, and are appending x to an existing color class [lines 7-9]. Thus, if κ considers every row 1 to n, it will have placed every vertex into a color class.

Fix n, the order of G and number of rows/columns of $B(G')$. By inducting on each i in the vertex set we will prove that, given a rook placement on $B(G')$, κ can map each vertex of G to a color class.

Since all squares (a, b) of proper n-boards have column numbers a greater than row numbers b, the top row n has no squares in $B(G')$. Thus all boards have an empty row n and κ maps vertex n to a color class [lines 3-4]. κ then considers each row in descending order [line 2].

Assume that κ has considered all rows n to i and placed the corresponding vertices into color classes. For the next row $i-1$, if it is empty, it is added to a new color class [lines 3-4], if it has a rook at say $(j, i-1)$, it has already been placed in a color class by the inductive hypothesis. This is again because all column numbers are greater than row numbers in $B(G')$, and j had already been considered by κ at row $j > i - 1$. □

Lemma 63. *In the partition of vertices outputted by κ, all vertices within each color class are non-adjacent to one another in G.*

Proof. We prove this by considering outputted color classes based on their size. A one-element color class $\{a\}$ is trivially non-adjacent, and is outputted by κ when both row a and column a are empty [lines 3-5, 10-11].

Now each color class with length greater than 1 was given by at least one rook [lines 6, 9]; rooks $(i_1, i_2), (i_2, i_3), ..., (i_{m-1}, i_m)$ give the color class $\{i_1, i_2, i_3, ..., i_{m-1}, i_m\}$. Now all rooks are placed on squares in $B(G')$, and all squares $(i, j) \in B$ correspond to non-adjacent vertex pairs i, j in G by definition. Hence each vertex in a color class is non-adjacent to its sequential neighbors. Further, since $B(G')$ fulfills transitivity, $(i_1, i_2), (i_2, i_3) \in B$ means $(i_1, i_3) \in B$ and i_1, i_3 are non-adjacent. Applied to the whole color class, all pairs of vertices in the color class are non-adjacent. $\square$

Lemma 64. *κ is the right-inverse map of ρ.*

Proof. Let G be a Γ-graph with corresponding proper n-board $B(G')$. Let r be some placement of $k < n$ rooks onto $B(G')$. Fixing n, we prove by inducting on k that $(\rho \circ \kappa)(r) = r$ for all placements r.

For the case of 1 rook, let it be placed at (i_a, i_b). Since squares in $B(G')$ have column numbers are greater than row numbers, by considering each vertex in descending order, ρ will reach row i_b before row i_a [line 2]. An empty row i_b starts a color class with i_b, then column i_b has a rook at row i_a, hence i_a is appended to the color class and ends there since there are no other rooks and column i_a is empty. For all other vertices j in $[n]$, row j and column j are empty and give singletons $\{j\}$. Thus we have a partition of G's vertices into $\{i_b, i_a\}$ and $n-2$ other one-element color classes. Now, applying ρ, since only $\{i_b, i_a\}$ has length greater than one, we place a single rook at (i_a, i_b).

Assume with any placement of k rooks, the mapping κ followed by ρ can lead back to the same placement. Now, a placement of $k+1$ rooks, r_{k+1} would be equivalent to taking some r_k, a placement of k rooks from the inductive hypothesis, and adding a new rook (i_a, i_b).

Since we are able to add (i_a, i_b) to r_{k+1}, the original placement r_k did not have a rook on either column i_a or row i_b. Thus in r_k, we have i_b at the beginning of some color class $C_\flat$ [lines 3-5] and i_a is at the end of some other color class $C_\flat$ [lines 10-11]. They cannot be in the same class since color classes put their elements in descending order but $i_a > i_b$. $C_\flat$ and $C_\flat$ may be singletons.

Now consider r_{k+1} where rook (i_a, i_b) is additionally placed. When κ is adding elements to $C_\flat$ and gets to column i_a, it finds this rook and adds i_b to the class [lines 6-9]. The class ends there if $C_\flat$ is a singleton, otherwise it continues adding the other elements of $C_\flat$ just as it did in r_k. Essentially, adding rook (i_a, i_b) replaces $C_\flat$ and $C_\flat$ with some class $C_\flat \cup C_\flat = \{..., i_a, i_b, ...\}$. Hence, as required, adding 1 rook reduces the number of color classes outputted by κ by 1.

Consider r_k from the inductive hypothesis. Let $n_a = |C_\flat|$ and $n_b = |C_\flat|$. Assume for $n_a > 1$, our map ρ places $n_a - 1$ rooks at

$$(C_a[1], C_a[2]), (C_a[2], C_a[3]), ..., (C_a[n_a], C_a[n_a + 1])$$

and for $n_b > 1$, our map ρ places $n_b - 1$ rooks at

$$(C_b[1], C_b[2]), (C_b[2], C_b[3]), ..., (C_b[n_b], C_b[n_b + 1])$$

Now since $\kappa(r_{k+1})$ only affects vertices i_a, i_b and those in C_a and C_b, for $(\rho \circ \kappa)(r_{k+1})$ we only consider these vertices. The remaining vertices undergo the same mapping as in $(\rho \circ \kappa)(r_k)$. Within this scope, ρ places $n_a + n_b - 2$ rooks for r_k, and one more rook for r_{k+1}:

$$|C_a \cup C_b| - 1 = n_a + n_b - 1$$

Within this same scope, the rooks placed by $(\rho\circ\kappa)(r_{k+1})$ in order of placement are $(C_a[1], C_a[2])$,...,$(C_a[n_a], C_a[n_a+1])$ and (i_a, i_b) and $(C_b[1], C_b[2]), ..., (C_b[n_b], C_b[n_b+1])$. Hence $(\rho \circ \kappa)(r_{k+1})$ gives the same k rooks as $(\rho \circ \kappa)(r_k)$ plus (i_a, i_b). □

Lemma 65. *Let G be a Γ-graph of order n. Given a partition of $[n]$ into $n - k$ color classes, ρ returns a placement of exactly k rooks on $B(G')$.*

Proof. For color classes of size $|C| > 1$, ρ places $|C| - 1$ rooks. And if $|C| = 1$, no rooks are placed. Since $|C| \geq 1$ for all color classes, for each class, ρ places $|C| - 1$ rooks in general. Hence, given the $n - k$ color classes, the total number of rooks placed is

$$\begin{aligned}\sum_{i=1}^{n-k}(|C_i| - 1) &= \sum_{i=1}^{n-k}|C_i| - (n-k) \\ &= n - (n-k) \\ &= k\end{aligned}$$

□

Lemma 66. *The placement of rooks that ρ returns is such that all rooks are non-attacking.*

Proof. To prove he rooks are non-attacking we need to show no 2 rooks have the same row number or column number. Since the input S for ρ is a partition of the vertex set of G, each vertex only appears once in S (within some color class). Hence, rooks placed according to one color class cannot clash with those placed according to another. Within each color class $\{i_1, i_2, i_3, ...\}$, each rook placed (i_1, i_2), (i_2, i_3), ... has as its row number the column number of the previous rook. □

Lemma 67. *κ is the left-inverse map of ρ.*

Proof. Let G be a Γ-graph of order n with corresponding proper n-board $B(G')$. Let S be a partition of $[n]$ into $n-k$ color classes. We will show $(\kappa \circ \rho)(S) = S$ for all partitions S by inducting on the number of color classes $n-k$.

For the partition with 1 color class which contains all n vertices of G, ρ places $n-1$ rooks on $B(G')$. For graphs with $n = 1$, that would be 0 rooks on a proper 1-board. For graphs with $n > 1$, that would be $n-1$ rooks on a proper n-board which can only be done as a diagonal line of rooks on $(n, n-1)$, $(n-1, n-2), ..., (2, 1)$.

Now, κ takes this placement and since row n is empty, adds n to a color class [lines 3-5]. Now considering the while loop [line 6], the rook $(n, n-1)$ means κ adds $n-1$ to the color class [lines 7-9]. In the next iteration, the rook $(n-1, n-2)$ means κ adds $n-2$ to the color class, and so forth. This gives $S = \{\{n, n-1, ..., 1\}\}$ which is the partition of $[n]$ into a single color class that we started with.

Assume with any partition S into $n-k$ color classes, the mapping ρ followed by κ can lead back to the same partition. Now consider a partition S' into $n-k+1$ classes, which is the same as taking some S from the inductive hypothesis and further partitioning one of its classes C into two.

Since C must have at least 2 elements, let them be i_a and i_b with $i_a > i_b$. Assume WLOG we partition the ordered set $C = \{..., i_a, i_b, ...\}$ between these two elements into $C_a = \{..., i_a\}$ and $C_b = \{i_b, ...\}$. $\rho(S')$ would now give the same rook placement as $\rho(S)$ but without the rook (i_a, i_b), so $\rho(S')$ has an empty column i_a and empty row i_b.

Now, $(\kappa \circ \rho)(S')$ would give the same partition as $(\kappa \circ \rho)(S)$ except instead of appending i_b after i_a in C due to the rook (i_a, i_b) [lines 6-9], an empty column i_a would truncate C and leave it as $C_a = \{..., i_a\}$. Also, i_b would start a new color class since its row is empty and give $C_b = \{i_b, ...\}$. Thus $\kappa \circ \rho$ arrives back at the same partition S'. □

Theorem 68. *Algorithms 57 and 58 are bijective maps. In other words, let G be some Γ-graph of order n and $B(G')$ be its corresponding proper n-board. Then each S, a partition of $[n]$ into some $n-k$ color classes, uniquely maps to r, a placement of k rooks on $B(G')$. Conversely, each r in $B(G')$ uniquely maps to some S in G.*

Proof. First, the maps κ and ρ are well-defined since they are algorithms. By Lemmas 63 and 66, their respective outputs are indeed in the codomain.

That is, the outputted color classes and rook placements are valid – each color class's vertices are non-adjacent in G and the rooks are non-attacking in $B(G')$. By Lemmas 62 and 65, they respectively output partition n into $n-k$ color classes from a placement of k rooks and vice versa. That Algorithms 57 and 58 are bijections follow directly from Lemmas 64 and 67. $\square$

We are now ready to prove the main theorem connecting the chromatic polynomial of G and rook vector of $B(G')$.

Proof of Theorem 52. Thus, by Theorem 68 the number of ways to place exactly k rooks onto $B(G')$ is the number of ways to partition G into exactly $n-k$ color classes. That is, if G is the Γ-graph corresponding to proper n-board $B(G')$, then the kth term of the rook vector of $B(G')$ equals the $(n-k)$th term of the chromatic vector of G:

$$\begin{aligned} & r_k(B) = c_{n-k}(G) \\ \Leftrightarrow \quad & c_k(G) = r_{n-k}(B) \end{aligned}$$

Then, since the chromatic polynomial $\chi(G;\,\lambda)$ is defined as the number of ways to properly color G with *up to* λ colors, by Definition 19 we have

$$\begin{aligned} & \chi(G;\,\lambda) = \sum_{k=0}^{\lambda} c_k(G) \cdot (\lambda)_k \\ \Leftrightarrow \quad & \chi(G;\,\lambda) = \sum_{k=0}^{\lambda} r_{n-k}(B) \cdot (\lambda)_k \\ \Leftrightarrow \quad & \chi(G;\,\lambda) = \sum_{k=0}^{\lambda} r_k(B) \cdot (\lambda)_{n-k} \end{aligned}$$

$\square$

4.7 Solving the Introductory Problem

Returning to the graph G at the beginning of this chapter, we can find its chromatic polynomial by finding its complement graph G' (Figure 4.3),

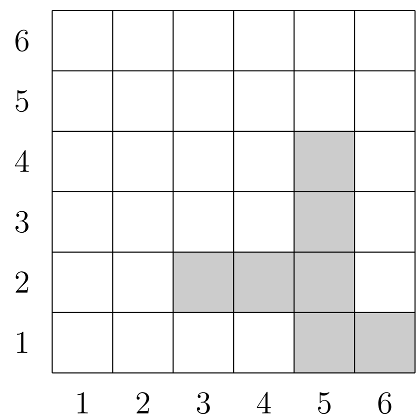


Figure 4.10: The proper 6-board of the graph G in this chapter's introductory problem.

constructing its corresponding proper 6-board $B(G')$ (Figures 4.5 and 4.6) and finding the rook vector $r(B)$.

Beginning with $k = 0$, there is exactly 1 way to put no rooks on the board, as seen in Figure 4.10. For $k = 1$, there are just as many ways to place 1 rook as there are shaded squares – 7 ways. As for $k = 2$ rooks, manual calculations will quickly find there are 11 ways, and Figure 4.11 shows there are 4 ways to place $k = 3$ rooks. Can we place more than 3 rooks? In each of the four ways we cannot place a rook on any shaded square without it "attacking" another rook on the same row or column, hence 3 is the maximum number we can place. Thus $r(B) = (1, 7, 11, 4)$. We could append zeros to the vector to represent $r_4 = 0$, $r_5 = 0$, $r_6 = 0$, but that is unnecessary for our purposes.

By Theorem 52, the chromatic polynomial of the graph can be obtained:

$$
\begin{aligned}
\chi(G;\, \lambda) &= \sum_{k=0}^{\lambda} r_k(B) \cdot (\lambda)_{n-k} \\
&= (\lambda)_6 + 7 \cdot (\lambda)_5 + 11 \cdot (\lambda)_4 + 4 \cdot (\lambda)_3 \\
&= \lambda^6 - 8\lambda^5 + 26\lambda^4 - 42\lambda^3 + 33\lambda^2 - 10\lambda
\end{aligned}
$$

And indeed, since the chromatic number of G is 3, we have $\chi(G;\, 0) = \chi(G;\, 1) = \chi(G;\, 2) = 0$. Meanwhile, $\chi(G;\, 3) = 24$ means there are 24 proper

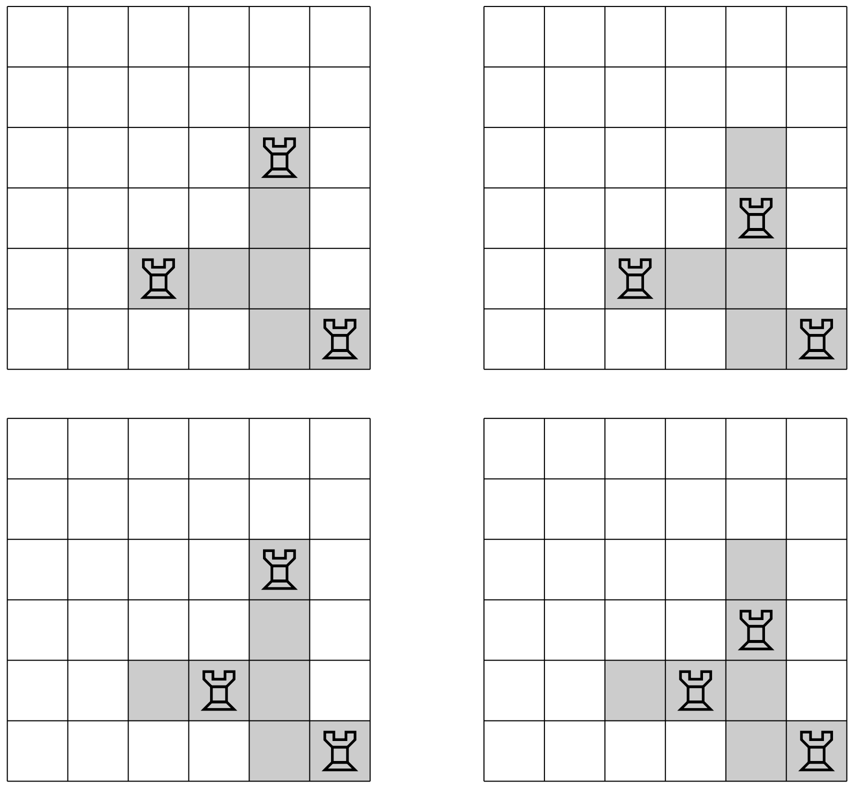


Figure 4.11: The four ways to place 3 non-attacking rooks on $B(G')$.

3-colorings of G, and the number rises quickly though at a slowing rate: $\chi(G;4) = 360$, $\chi(G;5) = 2,400$, $\chi(G;6) = 10,200$, etc.

However, this is not the end of the story! We are now able to find the chromatic polynomial of Γ-graphs, but is there a more complete heuristic of what constitutes Γ-graphs or transitive graphs? We answer this question in the next chapter by introducing posets.

Chapter 5

Comparability Graphs as Transitive Graphs

In the previous chapter we explored a fascinating link between the chromatic polynomial and rook theory. However, for which graphs G can we find $\chi(G; \lambda)$ using rooks? That is, what constitutes a Γ-graph?

In this chapter, we link transitive graphs to posets and their comparability graphs, and explore the implication that partitioning a Γ-graph into color classes is a partition of a poset into antichains.

5.1 Labeling Γ-graphs

First, we need a more concrete method to label Γ-graph, since a transitive graph with an inappropriate labeling does not correspond to a proper n-board.

Example 69. In Figure 5.1 we see a labeling of G that does not correspond to a proper n-board. The edges of its complement graph G' include $\{4, 3\}$ and $\{3, 2\}$, but not $\{4, 2\}$, thus $B(G')$ is not transitive and is not a proper 6-board.

Following the work of Goldman et al. [6], for a simple graph G of order n, if its vertices can be labeled $1, 2, ..., n$ such that $B(G')$ is a proper n-board, then G is a Γ-graph.

Definition 70 (Γ-labeling)**.** Let G be a simple graph of order n. Then the bijective map from $[n]$ to the vertex set of G is a Γ-labeling if and only if:

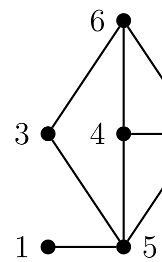


Figure 5.1: A labeling of the Γ-graph G from the previous chapter that does *not* correspond to a proper 6-board.

for all vertices u, v, w with labels $l_u, l_v, l_w \in [n]$, if v adjacent to w but u not adjacent to v and u not adjacent to w, then l_u not between l_v, l_w (in normal numerical order).

Example 71. We can check the labeling in Figure 5.2 is a Γ-labeling, or come up with it, in a systematic way. For all edges $\{v, w\} \in G$, if u is not adjacent to either v, w, we need l_u not between l_v, l_w:

- 1 not between 6,4
- 1 not between 6,3
- 1 not between 6,2
- 6 not between 5,1
- both 1 and 4 not between 3,2

We do not need to worry about edges $\{5, 4\}$, $\{5, 3\}$, $\{5, 2\}$ since they are adjacent to all other.

Remark 72. Returning to the inappropriately labeled graph in Figure 5.1, we can see it is not a Γ-labeling since 4 is adjacent to 2, and 3 is not adjacent to either 4 or 2, but numerically $2 < 3 < 4$.

Remark 73. A Γ-graph can have multiple Γ-labelings. See Figure 5.6 below for one other such labelling of the graph in Figure 5.2.

The simple proof of the following theorem illuminates the relationship between a Γ-labeling and the transitivity requirements for a proper n-board.

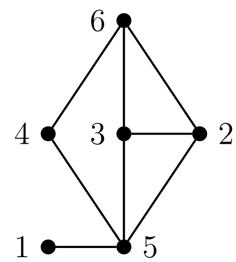


Figure 5.2: A Γ-labeling of the Γ-graph G from the previous chapter, which simply swaps the labels 3 and 4 from that in Figure 5.1.

Lemma 74. *A graph is a Γ-graph if and only if it has a Γ-labeling.*

Proof.
($\Rightarrow$) Assume G is a Γ-graph of order n. Then $B(G')$ is a proper n-board. Consider some trio of vertices $u, v, w \in G$ with labels $l_u, l_v, l_w \in [n]$, with v adjacent to w but u not adjacent to v and u not adjacent to w. We need to show that l_u not between l_v, l_w. Recall that in general, if i, j not adjacent in G with $(l_i > l_j)$, then $(l_i, l_j) \in B(G')$.

WLOG, assume $l_v > l_w$. Suppose by way of contradiction that $l_v > l_u > l_w$. Hence $(l_v, l_u), (l_u, l_w) \in B(G')$, and by the definition of a proper n-board, (l_v, l_w) must also be in $B(G')$, thus v is not adjacent to w in G which is a contradiction. We can arrive at a similar contradiction for the case $l_w > l_v$ when supposing $l_w > l_u > l_v$. Thus l_u is not between l_v and l_w.

($\Leftarrow$) We prove the contrapositive: If G does not have a Γ-labeling then G is not a Γ-graph. Assume that for all labelings of G, there exists vertices u, v, w such that v adjacent to w but u not adjacent to v and u not adjacent to w, and l_u is between l_v and l_w.

WLOG, assume $l_v > l_w$. Then $l_v > l_u > l_w$ and $(l_v, l_u), (l_u, l_w) \in B(G')$ but $(l_v, l_w) \notin B(G')$, hence $B(G')$ is not a proper n-board and G is not a Γ-graph. $\square$

5.2 Introducing Posets to Rook Theory

Incomparability graphs seem related to Γ-graphs since Buhler [2] found an equation linking the former's chromatic polynomial to drops of permutations of its vertices. Indeed, our example graphs G and G' from the previous chapter can be respectively formatted as the incomparability and comparability graphs of a poset.

Since it is well-known that a partial order can be extended to a linear order (see Definition 41), we can label the n vertices of $C(P)$ as $\{1, 2, ..., n\}$ such that the comparability relations of P align with the numerical order of the labels. This not only encourages readability of the graph, as we shall later see it also **informs us of the antichains within $I(P)$.**

Definition 75 (Chain labeling)**.** Let P be a poset with n elements and $C(P)$ be its comparability graph. Then the bijective map from $[n]$ to the elements of P is a *chain labeling* if and only if: for all elements x, y in P with labels l_x, l_y, we have $x > y$ if and only if $\{l_x, l_y\} \in C(P)$ with $l_x > l_y$.

Lemma 76. *All comparability graphs have a chain labeling.*

Proof. Let P be a poset with n elements and $C(P)$ be its comparability graph. Rearrange vertices of $C(P)$ such that for all relations $x > y \in P$, vertex x is positioned above y (no edge is a horizontal line). Then label the vertices $1, 2, ..., n$ starting from the bottom-most vertex such that for all pairs of vertices x, y, if x is positioned above y then $l_x > l_y$, even if there is no edge between x and y. Vertices with the same height can be labeled in an arbitrary order to give various chain labelings.

Then if $x > y \in P$, we have $\{x, y\} \in C(P)$ with x positioned above y, and we would have labeled $l_y < l_x$. And if $\{l_x, l_y\} \in C(P)$ with $l_x > l_y$, then that would have originated from $x > y$ in P. This algorithm is possible since the partial order can be extended to the usual linear order on integers, specifically $[n]$. □

Example 77. Starting with the comparability graph G' (taken from the previous chapter; see Figure 4.3), we want to chain label it and rearrange its vertices so that if $a > b > c$ then a is above b is above c. From an edge $\{x, y\}$, we cannot tell whether $x > y$ or $y > x$ in the poset, but we can often infer the orientation by ensuring transitivity is fulfilled across the graph.

In Figure 5.3, we notice e is not adjacent to a, hence e is incomparable to a in the poset. Similarly, e is incomparable to b, c. To avoid implying the transitive relation $b > d$ and $d > e$ hence $b > e$, we move e to a position higher than d. And since f is incomparable to d, we place f below e. The other positions do not need rearranging since $a > b > d$ implies $a > d$, and a, d are indeed adjacent. The poset G' represents with this formation has

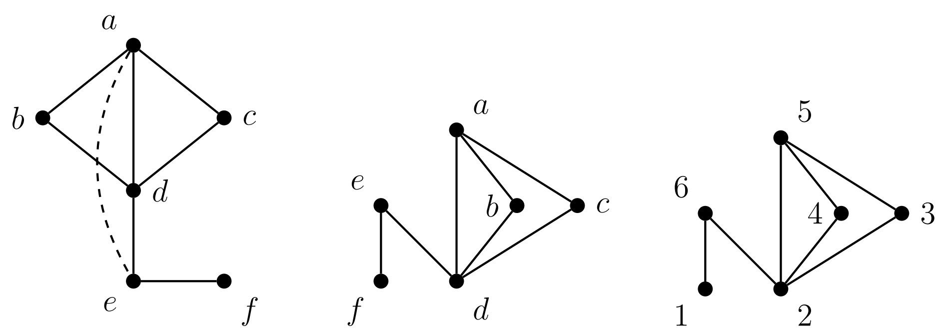


Figure 5.3: (Left to Right) Rearranging vertices of a comparability graph so that "height" reflects the order between adjacent vertices, then chain labeling it.

relations $e > f$, $e > d$, $a > b > d$ and $a > c > d$. Often a comparability graph can represent more than one poset, for example vertically flipping G' would lead to another poset with the same comparability relations except all orientations are reversed.

To chain label G', we can extend the partial order to a linear order such as $e > a > b > c > d > f$, where all the original comparability relations are preserved, and label this sequence $6, ..., 1$. We could also have labeled all the vertices $1, ..., 6$ starting from one of the bottom-most vertices and working our way upwards. For vertices with the same height, the labeling order is arbitrary, and edges can be arbitrarily shortened or lengthened in the beginning to give various chain labelings. We can check, for example, that since $e > d$ we have $6 = l_e > l_d = 2$.

5.3 Linking Transitive Graphs and Comparability Graphs

Lemma 78. *For all posets P, the comparability graph $C(P)$ is a transitive graph. Conversely, its incomparability graph $I(P)$ is a Γ-graph.*

Proof. Consider some elements $u, v, w \in P$ with labels $l_u, l_v, l_w \in [n]$ (kept constant between $C(P)$ and $I(P)$), and WLOG $l_v > l_w$. To prove $I(P)$ is a Γ-graph, we need to show it has a Γ-labeling. This is equivalent to proving in $C(P)$ that l_u is not between l_v and l_w when the adjacency conditions of $I(P)$ are reversed.

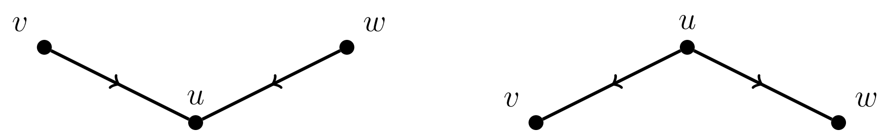


Figure 5.4: The two ways a path $\{v, u, w\}$ with v not adjacent to w can fulfill the P_3 rule: both v and w are either oriented towards or away from the middle vertex u.

Assume l_u adjacent to l_v and l_u adjacent to l_w but l_v not adjacent to l_w. Suppose by way of contradiction that l_u is between l_v and l_w, i.e. $l_v > l_u > l_w$. By Lemma 76, $C(P)$ has a chain labeling. Hence $\{l_v, l_u\}, \{l_u, l_w\} \in C(P)$ implies $v > u$ and $u > w$ in P. By poset transitivity, we must have $v > w$ which means $\{l_v, l_w\} \in C(P)$, but this is a contradiction. A similar contradiction is reached when we assume $l_w > l_v$, thus conclude $C(P)$ has the inverse of a Γ-labeling and $I(P)$ is a Γ-graph. □

Lemma 79. *For all Γ-graphs G, G is the incomparability graph of some poset P. Conversely, its complement, the transitive graph G' is the comparability graph of P.*

Proof. Let G be a Γ-graph of order n, then G' is a transitive graph. To prove that G' is the comparability graph of some poset P, we want to show that it can be transitively oriented (see Definition 11), i.e. it fulfills the P_3 or V-rule [1], [5].

By Lemma 74, G has a Γ-labeling. Thus G' has the inverse of a Γ-labeling, i.e. for some vertices $u, v, w \in G$ with labels $l_u, l_v, l_w \in [n]$, if u adjacent to v and u adjacent to w but v not adjacent to w, then either $l_u > l_v$ and $l_u > l_w$, or $l_u < l_v$ and $l_u < l_w$.

Since the labels of G' are linearly ordered, we can orient the undirected graph G' such that for all edges $\{x, y\} \in G'$, x is directed to y if $l_x > l_y$. Thus, for all P_3 paths $\{v, u, w\} \in G'$, we orient both v and w towards or away from u respectively when $l_u < l_v$ and $l_u < l_w$, or $l_u > l_v$ and $l_u > l_w$ (see Figure 5.4). □

Theorem 80. *The simple graph G is a transitive graph if and only if G is the comparability graph of some poset P. Conversely, its complement G' is a Γ-graph if and only if it is the incomparability graph $I(P)$.*

Proof. This result follows directly from Lemmas 78 and 79. □

Corollary 81. *All* Γ*-labelings are chain labelings, and all chain labelings are* Γ*-labelings.*

5.4 Posets and Rook Theory

Connecting our findings back to rook theory, each rook placement on $B(G')$ corresponds not just to cliques in G and color classes in G', but to a specific partition of G into antichains.

Applying Dilworth's theorem, we see that the width of P is not only the minimum number of antichains needed to partition P, but also the minimum number of color classes needed to properly color $I(P)$, i.e. the chromatic number of $I(P)$, and is the maximum number of non-attacking rooks on one can place on the proper n-board of $C(P)$.

However, beginning with a poset P and its comparability graph $C(P)$ we can only apply rook theory to find the chromatic polynomial of $I(P)$. The good news is, rook theory can be also be applied to the incomparability graphs of order dimension 2, since they themselves are comparability graphs of another poset! [4], [7]

As stated in Theorem 47, the incomparability graph of a poset with dimension 2 is itself the comparability graph of another poset. Thus for a dimension-2 poset P, we can chain label both $C(P)$ and $I(P)$ by Lemma 76.

Definition 82 (Bi-relevant chain labeling)**.** Let $P = (S, <)$ be a poset of dimension 2 on the set S of n elements. Let $I(P)$ be the comparability graph of another poset $Q = (S, <')$ on the same set. Let $L = ([n], <_L)$ be a linear extension of both P and Q. Define a *bi-relevant chain labeling* of $C(P)$ and $I(P)$ as a labeling such that for all $l_x <_L l_y$ in L, we have $x < y$ in one of P or Q.

Conjecture 83. Let C and C' be complementary comparability graphs of posets P and Q respectively. Then P and Q have a bi-relevant chain labeling, that is, P and Q can be extended to the same linear order L where for all relations $x < y$ in L, either $x < y$ in P or in Q.

Example 84. Take the graph G from the introductory problem of the previous chapter, which is incomparability graph of the poset P, as well as the comparability graph of another posest Q. The complement of G is $C(P)$ drawn in Figure 5.3. In Figure 5.6, both $G = I(P) = C(Q)$ and $G' = C(P)$

have been restructured so that if vertices $x < y$ in their respective poset, then x is drawn below y.

P and Q can be extended to the same linear order [6] with usual numerical order, and accordingly, a bi-relevant chain labeling is presented, combined in Figure 5.5 and separately in Figure 5.6. In fact, this bi-relevant chain labeling is unique – no other labeling gives $l_x < l_y$ if $x < y$ across both graphs.

Since the comparability graph of the linear order [6] is simply the complete graph K_n, its arcs can be partitioned into those in P (colored black) and those in Q (colored orange). Due to the linear order, the directed graph K_6 has an acyclic orientation, more specifically an st-orientation with one source 6 and one sink 1.

The "complementing" proper 6-boards of both graphs are shown in Figure 5.7. They both fulfill transitivity. $B(G)$ has a rook vector of $(1, 7, 11, 4)$, and $B(G')$ has a rook vector of $(1, 8, 14, 4)$. Thus by Theorem 52,

$$\chi(G;\, \lambda) = \lambda^6 - 8\lambda^5 + 26\lambda^4 - 42\lambda^3 + 33\lambda^2 - 10\lambda$$

and

$$\chi(G';\, \lambda) = \lambda^6 - 7\lambda^5 + 19\lambda^4 - 25\lambda^3 + 16\lambda^2 - 4\lambda$$

while

$$\chi(K_6;\, \lambda) = \lambda^6 - 15\lambda^5 + 85\lambda^4 - 225\lambda^3 + 274\lambda^2 - 120\lambda$$

As proven in Theorem 47, both G and G' have dimension 2, and by definition are permutation graphs. The labelings of G and G' respectively correspond to the permutations $\pi_P = (316452)$ and $\pi_Q = (254613)$.

Remark 85. By constructing a bijection between the number of "descents" in a comparability graph's labeling and the number of drops in the graph's corresponding permutation, Buhler note proved the chromatic polynomial of its complement, the incomparability graph, is

$$\chi(I(P);\, \lambda) = \sum_{k=0}^{n-1} \delta_P(k) \binom{\lambda + k}{n}$$

where $\delta_P(k)$ is the number of permutations of P with k drops.

Example 86. To assure that transitive graphs include comparability graphs that do not have a dimension of 1 or 2, Figure 5.8 shows an example of the

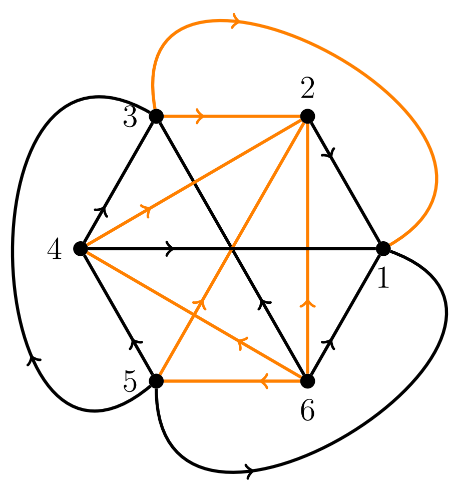


Figure 5.5: The partial orders P (in black) and Q (in orange) extended to the same linear order of [6] numerically ordered, and bi-relevantly chain labeled, give an st-orientation of K_6.

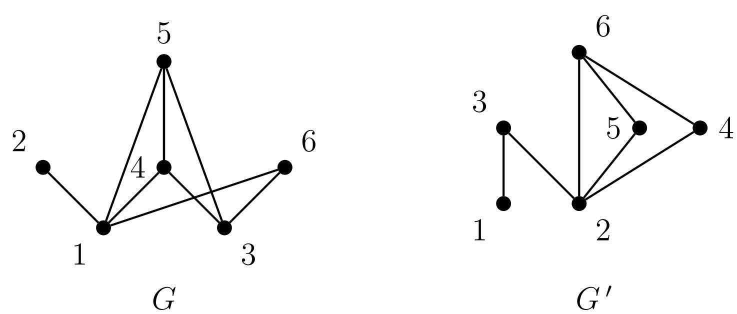


Figure 5.6: Complementary comparability graphs G and G' from Figure 4.3, restructured so that "height" reflects the order of adjacent vertices, and uniquely chain labeled such that the same labeling has $x < y$ implying $l_x < l_y$ across both graphs.

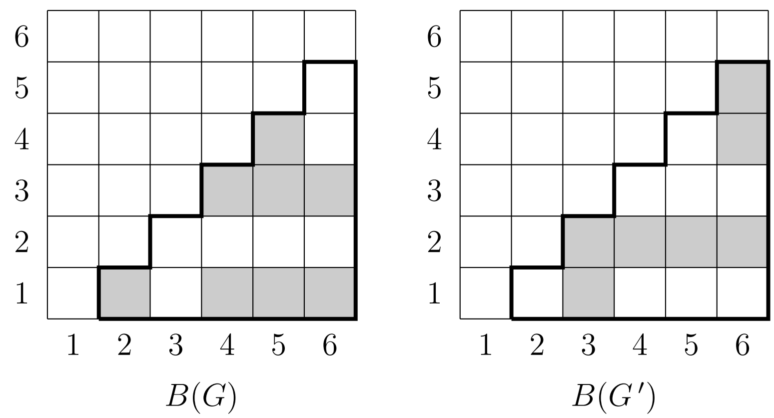


Figure 5.7: The proper 6-boards of the complementary comparability graphs G and G'.

3-crown graph C, which is the comparability graph of a 3-dimension poset. Its complement C' is not a comparability graph.

Since C is a transitive graph, C' is a Γ-graph and we can can apply Theorem 52 to find its chromatic polynomial. Figure 5.9 shows $B(C)$ which has rook vector $(1, 6, 9, 2)$. Thus

$$\chi(G;\, \lambda) = \lambda^6 - 9\lambda^5 + 34\lambda^4 - 67\lambda^3 + 67\lambda^2 - 26\lambda$$

However, C' does not have a transitive orientation (try orienting C' edge by edge by following the P_3 rule, leaving $\{e, b\}$ for last), and it cannot be chain labeled to give a proper 6-board. Hence we cannot apply Theorem 52 to find its chromatic polynomial.

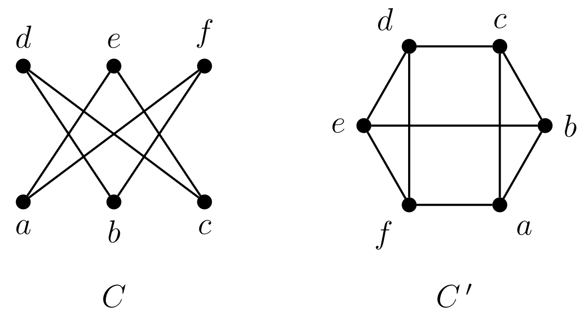


Figure 5.8: The 3-crown graph C which is the comparability graph of a 3-dimension poset, and its complement C' which is not a comparability graph but is a Γ-graph.

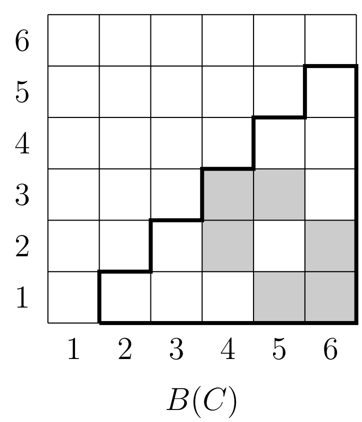


Figure 5.9: The proper 6-board of the 3-crown graph C.

Chapter 6

Conclusion

The following are possible directions for further research.

6.1 Future Directions

6.1.1 Bi-relevant Chain Labeling Conjecture

For all complementary comparability graphs, must there be a linear order that gives rise to a bi-relevant chain labeling? For which pairs of complementary comparability graphs is a bi-relevant chain labeling unique?

6.1.2 Permutations

Drops and Descents

Buhler [2] linked the chromatic polynomial to posets through drops, a property of permutations. Is there a mechanistic link between permutations and rooks? Is there a special case in Buhler's formulation for dimension-2 posets, which are all permutation graphs? How about graphs that are not permutation graphs but for which Buhler's formulation still holds?

Labelings

What is the relationship between the different posets arising from different chain labelings? Is there commonality between the proper n-boards corresponding to various Γ-labelings of Γ-graphs?

Graph Reconstruction

Can posets be reconstructed from their various comparability graphs? How about the various labelings of each comparability graph? Is this related to the open conjecture on graph reconstruction?

Since all comparability graphs correspond to a proper n-board, can these boards or their rook vectors or their chromatic polynomial illuminate anything regarding these questions?

6.1.3 Acyclic orientations

If the partial orders of complementary comparability graphs can always be extended to the same linear order, then their bi-relevant chain labeling would always give an acyclic orientation of a complete graph. Is this related to Theorem 27 and/or the number of chain labelings of each of the comparability graphs? Is $(-1)^n\chi(G; -1)$ significant with regards to Theorem 52 and the rook vector?

6.1.4 Deletion-contraction Theorem

Since a graph's chromatic polynomial can be found through Theorem 52 and the deletion-contraction theorem, are proper n-boards and rook vectors related to the latter recursive theorem?

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