Show the following statements hold:

1. Suppose that $s_1, s_2, ..., s_{m+1}$ are elements of a vector space V and that $S = \{s_1, s_2, ..., s_m\}$ is linearly independent. Then $s_{m+1} \in span(S)$ if and only if $\{s_1, s_2, ..., s_m, s_{m+1}\}$ is linearly dependent.

2. If $S = \{v_1, v_2, ..., v_m\}$ spans the vector space V then there is a basis of V which is a subset of S.

3. If V is a finite-dimensional vector space and $S = \{v_1, v_2, ..., v_k\}$ is linearly independent, then there is a basis of V which contains S.

4. Suppose that $S = \{v_1, ..., v_n\}$ spans the vector space V and $S' = \{w_1, ..., w_m\}$ is a set of m linearly independent vectors in V. Then $m \leq n$.

5. If W is a subspace of an n-dimensional vector space V then W is finite-dimensional with dimension $m \leq n$. The case $m = n$ holds if and only if $W = V$.

6. Suppose that $f : V \to W$ is a linear map, $\{w_1, ...w_r\}$ is a basis for the image of f and $\{v_1, ..., v_k\}$ is a basis for the kernel of f. For $i = 1, ..., r$, let $u_i \in V$ be such that $f(u_i) = w_i$. Then $S = \{u_1, ..., u_r, v_1, ...v_k\}$ is a basis for the vector space V.