1. Show that a matrix is unitarily diagonalizable if and only if it is normal.

   First, we show that if a matrix A is unitarily diagonalizable, then it is normal. We know that $A = UDU^*$ if A is unitarily diagonalizable for some unitary matrix U and diagonal matrix D. Then, $A^* = (UDU^*)^* = (U^*)^*D^*U^* = UD^*U^*$, so $AA^* = (UDU^*)(UD^*U^*) = U(DD^*)U^*$. Similarly, $A^*A = U(D^*D)U^*$. Since D is diagonal, $D^* = \overline{D}$ and $DD^* = D^*D$, so A is normal.

   Second, we show that if A is normal, it is unitarily diagonalizable. Supposing A is normal, we have $U^*AU = T$ for some unitary matrix U and upper-triangular matrix T. Since $AA^* = A^*A$ and $T^* = (U^*AU)^* = U^*A^*(A^*)^* = U^*A^*U$, we can conclude that $TT^* = (U^*AU)(U^*A^*U) = U^*A(UU^*)A^*U = U^*(AA^*)U$. Rearranging we have, $U^*(AA^*)U = U^*(A^*A)U = U^*A^*UU^*AU = T^*T$. So, T is normal.

2. Show that a real matrix is orthogonally diagonalizable if and only if it is symmetric.

   First, we show if A is a real orthogonally diagonalizable matrix, then it is symmetric. We know that $A = PDP^t$ if A is orthogonally diagonalizable for some orthogonal matrix P and diagonal matrix D. Then, $A^t = (PDP^t)^t = (P^t)^tD^tP^t = PDP^t = A$, so A is symmetric.

   Second, we show if A is real symmetric, then it is orthogonally diagonalizable. Since A is symmetric, the eigenvalues of A are all real, so we know it is orthogonally similar to an upper-triangular matrix, $P^tAP = T$ for some orthogonal matrix P and upper-triangular T. So $T^t = P^tA^tP = P^tAP = T$, so T is symmetric. A symmetric upper-triangular matrix is clearly diagonal.

3. Show that if Q is an invertible $n \times n$ complex matrix, then $Q = UT$ for some unitary matrix U and upper-triangular matrix T.

   Let $v_1, v_2, ..., v_n$ be the columns of Q. If A is invertible, then $rank(Q) = n$, so the columns span C^n, hence form a basis. Now apply the Gram-Schmidt process to $S = \{v_1, v_2, ..., v_3\}$ to get an orthonormal basis $S' = \{u_1, u_2, ..., u_n\}$. Then the matrix whose columns are $u_1, u_2, ..., u_n$ is unitary. Then $T = U^{-1}Q$ is the transition matrix from S to S'. The jth column of T is given by the coordinates of v_j with respect to S'. Since v_j is in the span of $\{u_1, u_2, ..., u_j\}$, its i^{th} coordinate is 0 for $i > j$, which means that T is upper triangular.

4. Show that if a matrix is Hermitian then its eigenvalues are real.

   If $\lambda \in \mathbb{C}$ is an eigenvalue of a matrix A, then $Av = \lambda v$ for some $v \neq 0$. Then, $\langle Av, v\rangle = \langle \lambda v, v\rangle = \lambda\langle v, v\rangle$. We have that $A = A^*$ so this implies $\langle Av, v\rangle = \langle v, A^*v\rangle = \langle v, Av\rangle = \langle v, \lambda v\rangle = \overline{\lambda}\langle v, v\rangle$. Since $\lambda\langle v, v\rangle = \overline{\lambda}\langle v, v\rangle$ and $\langle v, v\rangle \neq 0$, it follows that $\lambda = \overline{\lambda}$, so $\lambda \in \mathbb{R}$.

5. Show that if a matrix is Hermitian then its eigenvectors with distinct eigenvalues are orthogonal.

   To start, we state that $Av = \lambda v$ and $Aw = \mu w$ for some $v, w \neq 0 \in \mathbb{C}^k$, and some $\lambda \neq \mu \in \mathbb{C}$. Then we have $\langle Av, w\rangle = \langle \lambda v, w\rangle = \lambda\langle v, w\rangle$, and $\langle Av, w\rangle = \langle v, Aw\rangle = \langle v, \mu w\rangle = \overline{\mu}\langle v, w\rangle = \mu\langle v, w\rangle$ since $\mu \in \mathbb{R}$. Since $\lambda\langle v, w\rangle = \mu\langle v, w\rangle$ and $\lambda \neq \mu$, we have that $\langle v, w\rangle = 0$.

6. Show that if matrix A is Hermitian with n distinct eigenvalues, then A is unitarily diagonalizable. Let $\{v_1, v_2, ..., v_n\}$ be a basis consisting of eigenvectors for the distinct eigenvalues $\lambda_1, \lambda_2, ..., \lambda_n$. Let

   $u_i = \frac{v_i}{||v_i||}$, for $i = 1, ..., n$. Then each u_i is an eigenvector with eigenvalue λ_i. Since a Hermitian matrix has real eigenvalues and orthogonal eigenvectors (those of distinct eigenvalues), we know $u_i, u_j = 0$ for $i \neq j$. Since $||u_i|| = 1$ for $i =, 1..., n$, we have $\{u_1, u_2, ..., u_n\}$ is an orthonormal basis eigenvectors for A, so A is unitarily diagonalizable.

The End.