Handling position and visibility discontinuities for physically-based differentiable rendering

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SML Weekly Talk

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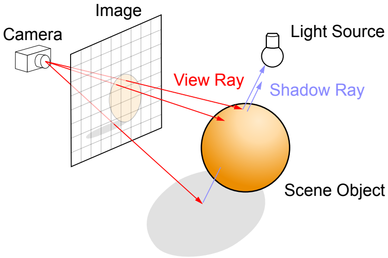
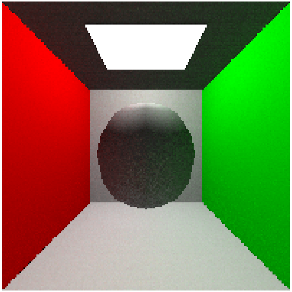
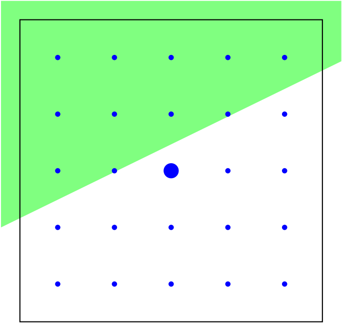
Preliminaries: The ray tracing algorithm







Preliminaries: The Rendering equation

The rendering equation is given as,

(1) $$I = \int_{\mathcal{X}} f(x, \theta)\ dx$$

where the integrand is, $f = k(x) \cdot m(x, \theta)$ and k is the filtering kernel, m is the imaging function, and θ is *all* the scene parameters. The filtering kernel function is set to *box filter*.

Use Monte-Carlo estimation,

(2) $$I \approx \frac{1}{N} \sum_{i=1}^{N} \frac{f(x_i, \theta)}{p(x_i)}$$

where p is the probability density function.

Preliminaries: Consistency and bias

- The estimation is called **consistent** if,

(3) $$\lim_{N\to\infty} \frac{1}{N} \sum_{i=1}^{N} \frac{f(x,\theta)}{p(x_i)} = I$$

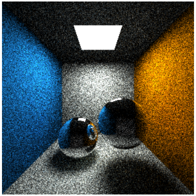
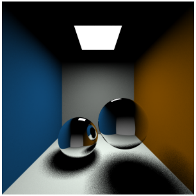
- The estimation is **unbiased** if,

(4) $$\mathbb{E}[f(x_i,\theta)] = I$$

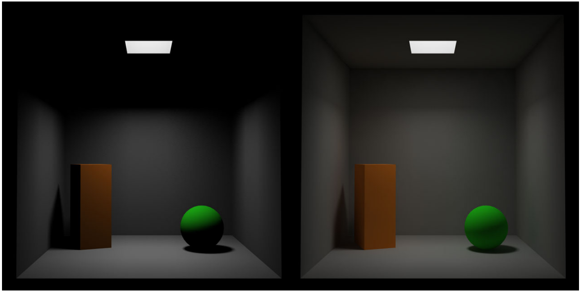
Trade-offs are made with both the above to handle the position and visibility discontinuities.

Preliminaries: Noise and direct illumination

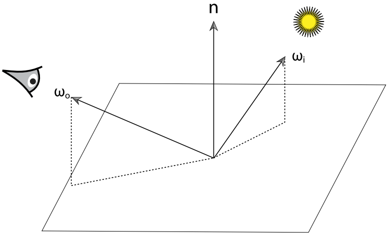
- For small and large N, the render is,

- Direct illumination vs global illumination



Preliminaries: Bi-directional reflectance distribution function



$$L_o(p,\omega_o) = L_e(p,\omega_o) + \int_{S^2} f(p,\omega_o,\omega_i) L_i(p,\omega_i) |\cos(\phi_i)| \; d\omega_i \tag{5}$$

where $f(p,\omega_o,\omega_i)$ depends on the type of surface, texture, roughness, etc.

The problem statement

For a given scene θ, the color intensity at the j^{th} pixel on the sensor is depends on the imaging function $f(x, \theta)$ given as,

$$I = \int_{\mathcal{X}} f(x, \theta)\ dx \tag{6}$$

where $\mathcal{X}$ is the domain over the camera (sensor) space co-ordinates.

Goal:

$$\boxed{\text{Find } \partial_\theta I.}$$

Initial approach

Notice:

- Fortunately, most of the scene parameters θ are differentiable.
- The ones which are not differentiable are discontinuous. There are discontinuities in both position and visibility.

But let's differentiate anyways,

$$\frac{\partial I}{\partial \theta} = \frac{\partial}{\partial \theta} \int f(x, \theta)\ dx \tag{7}$$

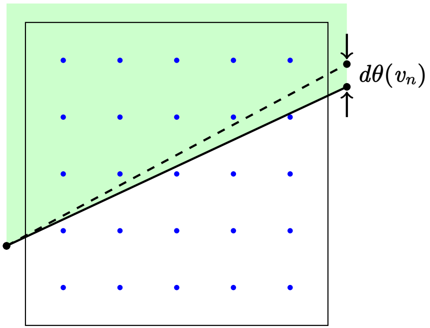
using Monte-Carlo estimator,

$$\frac{\partial I}{\partial \theta} = \frac{1}{N} \sum \frac{\partial}{\partial \theta} \frac{f(x_i, \theta)}{p(x_i, \theta)} \tag{8}$$

for unbiased sampling,

$$\frac{\partial I}{\partial \theta} = \frac{1}{N} \sum \frac{\partial f(x_i, \theta)}{\partial \theta} \tag{9}$$

Vertex position gradient



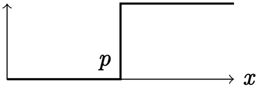
The discretization and the gradient operator do not commute for discontinuous integrand. Therefore, the local changes in θ is not detected in the final integral.

$$\frac{\partial}{\partial \theta_v} \int f(x, \theta)\, dx \neq \frac{1}{N} \sum_{i=1}^{N} \frac{\partial}{\partial \theta_v} f(x, \theta_v) = 0 \tag{10}$$

Modeling boundaries as Heaviside function

Consider one-dimensional Heaviside function,

$$\mathbb{1}_p(x) = \begin{cases} 0 & x < p \\ 1 & x \geq p \end{cases}, \tag{11}$$



The Heaviside function in higher dimension can be used to estimate the boundaries. It's derivative is,

$$\partial_x \mathbb{1}_p(x) = \delta(x - p) \tag{12}$$

where $\int_{-\infty}^{\infty} \delta(x) = 1$ called the Dirac delta distribution.

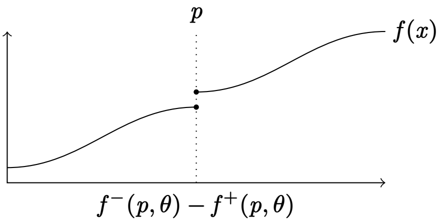
Leibintz's differentiating under the integral sign

For Dirac-delta distribution,

$$\frac{\partial}{\partial p}\int_0^1 \mathbb{1}_p \; dx = \frac{\partial}{\partial p}\int_0^p 0 \; dx + \frac{\partial}{\partial p}\int_p^1 1 \; dx = 0 + 1 = 1 \tag{13}$$

In general,

$$\frac{\partial}{\partial \theta}\int_a^b f(x,\theta) \; dx = \int_a^b f'(x,\theta) \; dx + b' f(b,\theta) - a' f(a,\theta) \tag{14}$$



Generalizing to higher dimension: Reynolds Transport Theorem

The Leibintz rule for calculating discontinuous integrals can be extended to many discontinuity points say, $\{p_0, p_1, \cdots, p_n\}$ is,

$$\frac{\partial}{\partial \theta} \int_a^b f(x,\theta)\ dx = \int_a^b f'(x,\theta)\ dx + \sum_{k=0}^{n} f^-(p_k,\theta) - f^+(p_k,\theta) \tag{15}$$

Generalizing to higher-dimension: For a, potentially discontinuous, scalar-valued function defined on some n-dimensional manifold Ω parameterized by some $\theta \in \mathbb{R}$ with surface boundary Γ, it holds that,

$$\frac{\partial}{\partial \theta} \int_\Omega f\ d\Omega = \int_\Omega f'\ d\Omega + \int_\Gamma \langle n, \partial_\theta x \rangle\ \nabla f\ d\Gamma \tag{16}$$

with the convention that $f'(x,\theta) = 0$ for discontinuous point. And,

$$\nabla f(x) = \lim_{\epsilon \to 0^+} f(x + \epsilon n) - \lim_{\epsilon \to 0^+} f(x + \epsilon n) \tag{17}$$

Monte-Carlo integrator

Again the Reynolds Transport Theorem states that,

$$\frac{\partial}{\partial\theta}\int_\Omega f\ d\Omega = \int_\Omega f'\ d\Omega + \int_\Gamma \langle n, \partial_\theta x\rangle\ \nabla f\ d\Gamma \tag{18}$$

We can now do the Monte-Carlo estimation by,

$$\int_\Omega f'\ d\Omega \approx \frac{1}{N_i}\sum_{i=1}^{N_i} f(x,\theta) \tag{19}$$

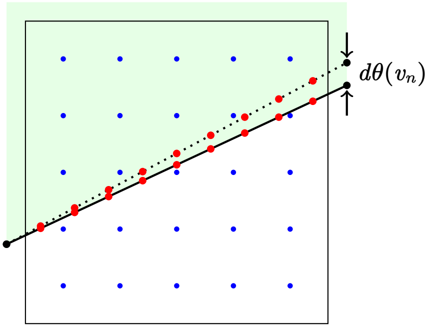
$$\int_\Gamma \langle n, \partial_\theta x\rangle\ \nabla f\ d\Gamma \approx \frac{1}{N_b}\sum_{i=1}^{N_b} \langle n, \partial_\theta x\rangle \nabla f(x,\theta) \tag{20}$$

Differentiable rendering with edge sampling

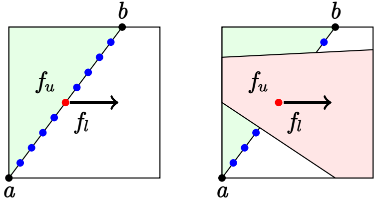
Introduce the bias back for both interior and boundary sampling.

$$\int_\Omega f' \; d\Omega \approx \frac{1}{N_i} \sum_{i=1}^{N_i} \frac{f(x,\theta)}{p_i(x)} \tag{21}$$

$$\int_\Gamma \langle n, \partial_\theta x \rangle \; \nabla f \; d\Gamma \approx \frac{1}{N_b} \sum_{i=1}^{N_b} \frac{\langle n, \partial_\theta x \rangle \nabla f(x,\theta)}{p_b(x)} \tag{22}$$



Primary visibility



A triangle edge can be modeled with Heaviside function as,

$$\mathbb{1}_{\alpha} f_u + \mathbb{1}_{-\alpha} f_l \tag{23}$$

where f_u and f_l are the two half-spaces separated by α,

$$\alpha = (a_y - b_y)x + (b_x - a_x)y + (a_x b_y - b_x a_y) \tag{24}$$

For occuluded parts the function with respect to the vertex position doesn't change, i.e, $\nabla f = 0$. No gradient backpropagates.

Computing the derivative

The scene function as a summation of Heaviside function with $\{\alpha_i\}$ edges is,

$$\iint f(x, y; \theta)\,dxdy = \sum_i \iint \mathbb{1}_{\alpha_i} f_i(x, y; \theta)\,dxdy \tag{25}$$

Then the analytical derivative of the scene function is now,

$$\nabla_\theta \sum_i \iint \mathbb{1}_{\alpha_i} f_i(x, y; \theta)\,dxdy \tag{26}$$

$$= \sum_i \iint \nabla_\theta \mathbb{1}_{\alpha_i} f_i(x, y; \theta)\,dxdy \tag{27}$$

$$+ \sum_i \iint \mathbb{1}_{\alpha_i} \nabla_\theta f_i(x, y; \theta)\,dxdy \tag{28}$$

The hard part

$$\begin{aligned}\sum_i \iint \nabla_\theta \mathbb{1}_{\alpha_i} f_i(x,y;\theta)\,dxdy &= \sum_i \iint \delta(\alpha_i)\nabla_\theta \alpha_i f_i(x,y;\theta)\,dxdy \\ &= \sum_i \int_{\alpha=0} \frac{\nabla_\theta \alpha_i}{\|\nabla_{xy}\alpha_i\|} f_i(x,y;\theta)\,d\sigma(x,y)\end{aligned}$$

The gradient of the edge equation is,

$$\|\nabla_{x,y}\alpha\| = \sqrt{(a_x - b_x)^2 + (a_y - b_y)^2} \tag{29}$$

$$\frac{\partial \alpha}{\partial a_x} = b_y - y, \quad \frac{\partial \alpha}{\partial a_y} = x - b_y \tag{30}$$

$$\frac{\partial \alpha}{\partial b_x} = y - a_y, \quad \frac{\partial \alpha}{\partial b_y} = a_x - x \tag{31}$$

$$\frac{\partial \alpha}{\partial x} = a_y - b_y, \quad \frac{\partial \alpha}{\partial y} = b_x - a_x \tag{32}$$

Bonus: Screen space gradient.

Monte carlo sampling with Dirac integral

For other scene parameters p, (like camera position, vertex color, etc), there is no discontinuities. So, their derivatives are a simple application of chain rule,

$$\frac{\partial \alpha}{\partial \Phi} = \sum_k \frac{\partial \alpha}{\partial a_k} \frac{\partial a_k}{\partial \Phi} + \frac{\partial \alpha}{\partial b_k} \frac{\partial b_k}{\partial \Phi} \tag{33}$$

And the Monte carlo estimator is,

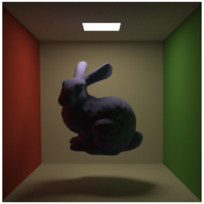
$$\frac{1}{N} \sum_{j=1}^{N} \frac{\|E\| \nabla_\Phi \alpha_i (f_u - f_l)}{\Pr[E] \|\nabla_{xy} \alpha_i\|} \tag{34}$$

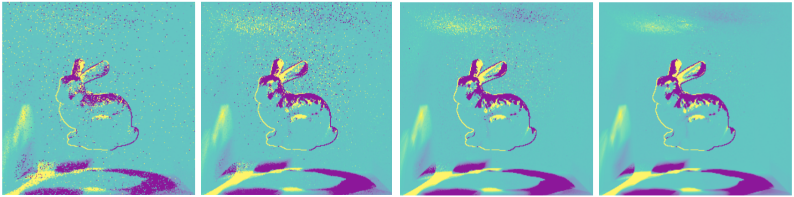
where, $\|E\|$ length of edge E and $\Pr[E]$ is the probability of selecting edge E. This only for the *silhouette* edges, where the gradient contribution is non-zero.

Sampling an edge and sampling on edge

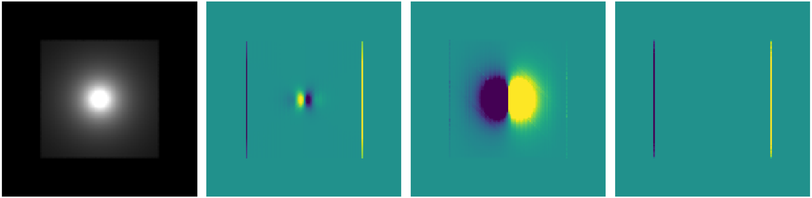
- There are millions of triangles in the scene.
- Sample an edge which contributes to the gradient, i.e, *silhouette* edge.
- Project all the edges to the screen space (in the pre-processing step), select the silhouette and visible edges and discard the other edges.
- The selected edges are proportional to the length on the screen space. (importance sampling)
- Uniformly select a point on the selected edge.

Results





(a) 1 spp (b) 16 spp (c) 128 spp (d) 1024 spp



Reparameterizing the discontinuities

Idea:

- Reparameterizing the integrand can make it free of parameter-dependent discontinuities.
- Move the derivative into the integral sign and estimate the derivative using Monte-Carlo estimation.
- Account for the distortion due to reparameterization to get unbiased gradient estimator.

The gradient to be estimated is,

$$\partial_\theta I = \partial_\theta \int_{S^2} f(\omega, \theta)\ d\omega \tag{35}$$

Reparameterization

Consider the reparameterization, $\mathcal{T} : S^2 \to S^2$ that maps the unit sphere on to itself.

Conditions:

- The reparameterization removes the discontinuities.
- The differential motion perfectly matches the motion of discontinuities on the unit sphere.

$$\partial_\theta \mathcal{T}(\omega, \theta) = \partial_\theta \omega \tag{36}$$

- The intgrand has no boundary or the integrand goes to zero as the domain approaches zero at the boundaries.

Derivative after the reparameterization

With the reparameterization conditions satisfied, the derivative can be computed as,

$$\partial_\theta I = \partial_\theta \int_{S^2} f(\omega, \theta)\ d\omega \tag{37}$$

$$= \int_{S^2} \partial_\theta \Big[f(\mathcal{T}(\omega, \theta), \theta) \| D\mathcal{T}_{\omega,\theta}(s) \times D\mathcal{T}_{\omega,\theta}(t) \| \Big]\ d\omega \tag{38}$$

where $\| D\mathcal{T}_{\omega,\theta}(s) \times D\mathcal{T}_{\omega,\theta}(t) \|$ accounts for the distortion of the integration domain due to reparameterization.

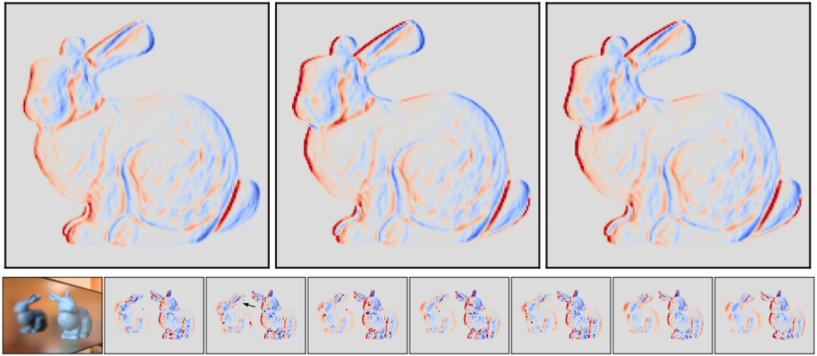
A mapping that works for *signed distance function*, $\phi(x, \theta)$ is,

$$\mathcal{T}(\omega, \theta) = t\omega - \mathcal{V}(x_t, \theta) - \mathcal{V}(x_t, \theta_0) \tag{39}$$

where $x_t = x + t\omega$ and $\mathcal{V}$ is a mapping that is the scaled multiple of the surface normal,

$$\mathcal{V} = -\frac{\partial_x \phi(x, \theta_0)}{\| \partial_x \phi(x, \theta_0) \|} \phi(x, \theta) \tag{40}$$

Results



Thank you.