Dot Product

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Dot Product and its Properties

Define the dot product and prove its algebraic properties.

Definition of the Dot Product

Definition 1. *Let* $\mathbf{u}$ *and* $\mathbf{v}$ *be vectors in* $\mathbb{R}^n$*. The dot product of* $\mathbf{u}$ *and* $\mathbf{v}$*, denoted by* $\mathbf{u}\cdot\mathbf{v}$*, is given by*

$$\mathbf{u}\cdot\mathbf{v}=\begin{bmatrix}u_1\\u_2\\\vdots\\u_n\end{bmatrix}\cdot\begin{bmatrix}v_1\\v_2\\\vdots\\v_n\end{bmatrix}=u_1v_1+u_2v_2+\ldots+u_nv_n.$$

Example 1. *Find* $\mathbf{u}\cdot\mathbf{v}$ *if* $\mathbf{u}=\begin{bmatrix}-2\\0\\1\end{bmatrix}$ *and* $\mathbf{v}=\begin{bmatrix}3\\2\\-4\end{bmatrix}$.

Explanation.

$$\mathbf{u}\cdot\mathbf{v}=\begin{bmatrix}-2\\0\\1\end{bmatrix}\cdot\begin{bmatrix}3\\2\\-4\end{bmatrix}=(-2)(3)+(0)(2)+(1)(-4)=-6-4=-10$$

Note that the dot product of two vectors is a **scalar**. For this reason, the dot product is sometimes called a *scalar product*.

Properties of the Dot Product

A quick examination of Example **??** will convince you that the dot product is *commutative*. In other words, $\mathbf{u}\cdot\mathbf{v}=\mathbf{v}\cdot\mathbf{u}$. This and other properties of the dot product are stated below.

Theorem 1. *The following properties hold for vectors* $\mathbf{u}$, $\mathbf{v}$ *and* $\mathbf{w}$ *in* $\mathbb{R}^n$ *and scalar* k.

(a) $\mathbf{u}\cdot\mathbf{v}=\mathbf{v}\cdot\mathbf{u}$

(b) $(\mathbf{u}+\mathbf{v})\cdot\mathbf{w}=\mathbf{u}\cdot\mathbf{w}+\mathbf{v}\cdot\mathbf{w}$

(c) $\mathbf{u}\cdot(\mathbf{v}+\mathbf{w})=\mathbf{u}\cdot\mathbf{v}+\mathbf{u}\cdot\mathbf{w}$

Learning outcomes:
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(d) $(k\mathbf{u}) \cdot \mathbf{v} = k(\mathbf{u} \cdot \mathbf{v}) = \mathbf{u} \cdot (k\mathbf{v})$

(e) $\mathbf{u} \cdot \mathbf{u} \geq 0$, *and* $\mathbf{u} \cdot \mathbf{u} = 0$ *if and only if* $\mathbf{u} = \mathbf{0}$.

(f) $||\mathbf{u}||^2 = \mathbf{u} \cdot \mathbf{u}$

We will prove Properties **??** and **??**. The remaining properties are left as exercises.

Proof of Property ??:

$$\begin{aligned}
(\mathbf{u}+\mathbf{v}) \cdot \mathbf{w} &= \left(\begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} + \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} \right) \cdot \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{bmatrix} = \begin{bmatrix} u_1+v_1 \\ u_2+v_2 \\ \vdots \\ u_n+v_n \end{bmatrix} \cdot \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{bmatrix} \\
&= (u_1+v_1)w_1 + (u_2+v_2)w_2 + \ldots + (u_n+v_n)w_n \\
&= u_1w_1 + v_1w_1 + u_2w_2 + v_2w_2 + \ldots + u_nw_n + v_nw_n \\
&= (u_1w_1 + u_2w_2 \ldots + u_nw_n) + (v_1w_1 + v_2w_2 + \ldots + v_nw_n) \\
&= \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} \cdot \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{bmatrix} + \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} \cdot \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{bmatrix} = \mathbf{u} \cdot \mathbf{w} + \mathbf{v} \cdot \mathbf{w}
\end{aligned}$$

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YouTube link: `https://www.youtube.com/watch?v=858cSuHqF-Q`

We will illustrate Property **??** with an example.

Example 2. *Let* $\mathbf{u} = \begin{bmatrix} -2 \\ 3 \end{bmatrix}$. *Use* $\mathbf{u}$ *to illustrate Property* **??** *of Theorem* **??**.

Practice Problems

Problem 1 *Find the dot product of* $\mathbf{u}$ *and* $\mathbf{v}$. $\mathbf{u} = \begin{bmatrix} -1 \\ -2 \\ 5 \\ 4 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} 2 \\ -2 \\ -3 \\ 1 \end{bmatrix}$, $\mathbf{u} \cdot \mathbf{v} = \boxed{-9}$.

Problem 2 *Find the dot product of* $\mathbf{u}$ *and* $\mathbf{v}$. $\mathbf{u} = \begin{bmatrix} 1 \\ 1 \\ \frac{}{2} \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} -2 \\ 4 \end{bmatrix}$, $\mathbf{u} \cdot \mathbf{v} = \boxed{0}$.

Problem 3 *Use the vector* $\mathbf{u} = \begin{bmatrix} 2 \\ 5 \\ -7 \end{bmatrix}$ *to illustrate Property* **??** *of Theorem* **??**.

Problem 4 *Prove Properties* **??**, **??**, **??** *and* **??** *of Theorem* **??**.

Problem 5 *From the given list of vector pairs, which pairs of vectors below lie on perpendicular lines?*

Select All Correct Answers:

(a) $\mathbf{u} = \begin{bmatrix} 1 \\ \frac{1}{2} \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} -2 \\ 4 \end{bmatrix}$ ✓

(b) $\mathbf{u} = \begin{bmatrix} -1 \\ \frac{1}{2} \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} -2 \\ 4 \end{bmatrix}$

(c) $\mathbf{u} = \begin{bmatrix} 1 \\ \frac{1}{2} \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$ ✓

(d) $\mathbf{u} = \begin{bmatrix} -1 \\ -\frac{1}{2} \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} -2 \\ 4 \end{bmatrix}$ ✓

Compute $\mathbf{u} \cdot \mathbf{v}$ *for each pair. What do you observe?*

Problem 6 *For each of the following pairs of vectors, find the value of* x *that will make the two vectors perpendicular. Check your answer by sketching the vectors.*

Problem 6.1 *If* $\mathbf{u} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} 2 \\ x \end{bmatrix}$, *then* $\mathbf{u} \cdot \mathbf{v} = 0$ *when* $x = \boxed{-1}$.

Problem 6.2 *If* $\mathbf{u} = \begin{bmatrix} 5 \\ 2 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} x \\ -4 \end{bmatrix}$, *then* $\mathbf{u} \cdot \mathbf{v} = 0$ *when* $x = \boxed{8/5}$.

Problem 6.3 *If* $\mathbf{u} = \begin{bmatrix} 4 \\ -3 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} 6 \\ x \end{bmatrix}$, *then* $\mathbf{u} \cdot \mathbf{v} = 0$ *when* $x = \boxed{8}$.

Problem 7 (a) *Vector* $\mathbf{u}$ *that lies on the line* $y = mx$, *has the form* $\mathbf{u} = k \begin{bmatrix} 1 \\ m \end{bmatrix}$. *Assuming that* $m \neq 0$, *find the general form for a vector* $\mathbf{v}$ *that lies on a line perpendicular to* $y = mx$.

Hint: *What do you know about the slopes of perpendicular lines?*

(b) *Find* $\mathbf{u} \cdot \mathbf{v}$.

(c) *Formulate a conjecture about the dot product of perpendicular vectors.*