Mathematical Formulation for Optimal Delta Hedging Techniques

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Project Overview

The aim of this project is to study option pricing and hedging in and beyond the Black–Scholes framework. We start from the classical GBM model, then add numerical techniques, market frictions and more realistic dynamics to see how prices and hedging errors change.

The project consists of four parts:

1. **Black–Scholes and Monte Carlo pricing.** We introduce the Black–Scholes model, estimate the volatility from S&P 500 data and compute the analytic price of an at-the-money European call. We then simulate GBM paths and approximate the option price by Monte Carlo, comparing the estimate and its confidence interval to the closed-form value.

2. **Variance reduction with antithetic sampling.** Using the same GBM discretisation, we implement antithetic variates and compare the plain and antithetic Monte Carlo estimators. An F–test is used to check whether the variance reduction is statistically significant.

3. **Delta hedging with and without execution delays.** On simulated GBM paths we perform discrete-time delta hedging and analyse the distribution of hedging errors. We then introduce random execution delays, modelled by an $M/M/1$ queue, to study how waiting times for trades affect hedging performance.

4. **Model risk: jump–diffusion and Markov-switching volatility.** Finally, we simulate stock paths from a Lévy jump–diffusion model and from a Markov-switching volatility model, while still hedging with the Black–Scholes delta. Comparing prices and hedging errors to the GBM benchmark illustrates the impact of jumps and volatility regimes.

1 Theoretical Background

1.1 Option Pricing Based on the Black–Scholes Model

We consider a **European call option** written on a stock. An *option* is a contract whose payoff depends on the price of an underlying asset. A *call* gives the holder the right, but not the obligation, to buy one share of the stock. The adjective *European* means that this right can only be exercised at a single future time T (the *maturity*), not before. The agreed purchase price is the *strike* K. If the stock price at maturity is S_T, the payoff of one European call is

$$\text{payoff} = (S_T - K)_+ = \max(S_T - K, 0).$$

To determine a fair price for this option today (time 0), we need a model for the future evolution of the stock price S_t. In the Black–Scholes framework, S_t is described by a *stochastic process*: a random function of time that captures many small, unpredictable market shocks. A natural continuous-time candidate is *Brownian motion*. By a central-limit-type argument, the cumulative effect of many small independent shocks over short intervals is approximately Gaussian; letting the stock be driven by such a process leads to *geometric Brownian motion* (GBM), which keeps S_t positive and produces lognormally distributed prices.

In the classical Black–Scholes model we assume that the stock price $(S_t)_{t\geq 0}$ follows the stochastic differential equation

$$dS_t = \mu S_t \, dt + \sigma S_t \, dW_t, \tag{1}$$

where μ is the drift, $\sigma > 0$ is the volatility, and $(W_t)_{t\geq 0}$ is a standard Brownian motion. We do not derive the solution in this report, but use the known closed form:

$$S_t = S_0 \exp\Big((\mu - \tfrac{1}{2}\sigma^2)t + \sigma W_t \Big). \tag{2}$$

For a fixed time step $\Delta t > 0$, the log-return over $[t, t + \Delta t]$,

$$R_{t,\Delta t} = \ln \frac{S_{t+\Delta t}}{S_t},$$

is normally distributed, so the stock price itself is lognormal. In the Monte Carlo part of the project we use this discrete-time representation of GBM to generate random price paths.

Risk-neutral pricing and the Black–Scholes formula

In practice, investors are risk-averse and demand a risk premium for holding stocks. Directly modelling these preferences is complicated, so the Black–Scholes model uses a simpler *risk-neutral* viewpoint. Under this view, we imagine a "fictional" probability world in which all traded assets, once discounted at the risk-free interest rate r, have zero expected growth. In that world, the fair time-0 price of any payoff X at time T is

$$\Pi_0 = e^{-rT} \, E^Q[X] = e^{-rT} \, E^Q[(S_T - K)_+], \tag{3}$$

where E^Q denotes expectation under the risk-neutral assumption.

In theory, r is the continuously compounded return of a perfectly risk-free asset. In practice there is no truly risk-free asset, but one typically uses yields on very safe, short-maturity government bonds or money-market rates. In our assignment we fix the risk-free rate at

$$r = 0.02,$$

which can be interpreted as a stylised 2% annual yield.

Under the risk-neutral assumption, the GBM dynamics are the same as in (1) except that the drift μ is replaced by r:

$$dS_t = r S_t \, dt + \sigma S_t \, dW_t^Q. \tag{4}$$

The solution at maturity T is

$$S_T = S_0 \exp\Big((r - \tfrac{1}{2}\sigma^2)T + \sigma W_T^Q \Big), \tag{5}$$

so S_T is still lognormally distributed under this condition.

For a European call option with payoff $X = (S_T - K)_+$, inserting this lognormal distribution into (3) yields the classical Black–Scholes formula

$$C_0 = S_0 \Phi(d_1) - K e^{-rT} \Phi(d_2), \tag{6}$$

where

$$d_1 = \frac{\ln(S_0/K) + (r + \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}, \qquad d_2 = d_1 - \sigma\sqrt{T},$$

and Φ is the standard normal cumulative distribution function. These analytic prices serve as the benchmark that we compare our Monte Carlo estimates against in the numerical part of the project.

1.2 Choosing Model Parameters from Data

In our implementation we calibrate the volatility parameter σ and the initial price S_0 from recent market data, while taking the risk-free rate r as a stylised constant.

We use one year of daily closing prices of the SPDR S&P 500 ETF (ticker `SPY`) as a liquid proxy for the S&P 500 index. Let

$$S_{t_0}, S_{t_1}, \dots, S_{t_n}$$

denote the observed daily prices between 1 January 2023 and 1 January 2024, with time step $\Delta t \approx 1/252$ years. The empirical daily log-returns are

$$R_i = \ln \frac{S_{t_i}}{S_{t_{i-1}}}, \qquad i = 1, \dots, n. \tag{7}$$

We estimate the *daily* volatility by the sample standard deviation of these log-returns,

$$\hat{\sigma}_{\text{daily}} = \sqrt{\frac{1}{n-1} \sum_{i=1}^{n} (R_i - \bar{R})^2}, \qquad \bar{R} = \frac{1}{n} \sum_{i=1}^{n} R_i, \tag{8}$$

and obtain an *annual* volatility by the usual scaling with $\sqrt{252}$,

$$\hat{\sigma} = \hat{\sigma}_{\text{daily}} \sqrt{252}. \tag{9}$$

This $\hat{\sigma}$ is the volatility parameter used in all subsequent simulations.

The initial stock price S_0 in the model is taken to be the last observed price in the sample, i.e. $S_0 = S_{t_n}$. We consider an at-the-money European call, so the strike is set to

$$K = S_0,$$

and we fix the time to maturity at one year, $T = 1$.

For the risk-free rate we use the constant value

$$r = 0.02,$$

which can be interpreted as a stylised annual yield of 2% on a very safe short-term government bond. Under the risk–neutral measure, the drift of the GBM dynamics is then set equal to this r, while the volatility is given by $\sigma = \hat{\sigma}$. Note that we do not need to estimate a separate real-world drift parameter μ for our pricing and hedging experiments.

1.3 Monte Carlo Simulation of Option Prices

We approximate the risk–neutral expectation in (3) by Monte Carlo; the general idea follows the treatment in Lecture 3, slides 36–42 (Muniraj 2025c). The time interval $[0, T]$ is split into M steps of length $\Delta t = T/M$. For each of N independent paths we simulate the risk–neutral GBM dynamics. Using the closed-form solution of this SDE(2), we obtain the exact discrete-time update

$$S_{t_{k+1}}^{(i)} = S_{t_k}^{(i)} \exp\Big((r - \tfrac{1}{2}\hat{\sigma}^2)\Delta t + \hat{\sigma}\sqrt{\Delta t}\, Z_k^{(i)} \Big), \qquad k = 0, \dots, M-1, \tag{10}$$

where $t_k = k\Delta t$ and $Z_k^{(i)} \sim \mathcal{N}(0,1)$ are i.i.d.

For path i the discounted payoff of the call option is

$$Y_i = e^{-rT} \big(S_T^{(i)} - K \big)_+, \qquad S_T^{(i)} = S_{t_M}^{(i)}. \tag{11}$$

The Monte Carlo estimator of the call price is then

$$\widehat{C}_{\text{MC}} = \frac{1}{N} \sum_{i=1}^{N} Y_i. \tag{12}$$

Under the model this estimator is unbiased, with empirical variance and standard error

$$\widehat{\mathrm{Var}}(Y) = \frac{1}{N-1}\sum_{i=1}^{N}(Y_i - \widehat{C}_{\mathrm{MC}})^2, \qquad \widehat{\mathrm{SE}}(\widehat{C}_{\mathrm{MC}}) = \sqrt{\frac{\widehat{\mathrm{Var}}(Y)}{N}}. \tag{13}$$

By the central limit theorem, $\widehat{C}_{\mathrm{MC}}$ is approximately normal for large N, so we can form confidence intervals and compare $\widehat{C}_{\mathrm{MC}}$ to the analytic Black–Scholes price C_0 by one-sample T-test.

1.4 Variance Reduction with Antithetic Variates

We follow the presentation in Lecture 7, slides 24–28 (Muniraj 2025e) and use antithetic sampling to obtain more accurate estimates for a fixed N. For each pair index $j = 1, \dots, N/2$ we draw a single $Z^{(j)} \sim \mathcal{N}(0,1)$ and construct two terminal prices

$$S_T^{(j,+)} = S_0 \exp\Big((r - \tfrac{1}{2}\hat{\sigma}^2)T + \hat{\sigma}\sqrt{T}\, Z^{(j)}\Big), \tag{14}$$

$$S_T^{(j,-)} = S_0 \exp\Big((r - \tfrac{1}{2}\hat{\sigma}^2)T + \hat{\sigma}\sqrt{T}\, (-Z^{(j)})\Big), \tag{15}$$

with discounted payoffs

$$Y^{(j,+)} = e^{-rT}\big(S_T^{(j,+)} - K\big)_+, \qquad Y^{(j,-)} = e^{-rT}\big(S_T^{(j,-)} - K\big)_+. \tag{16}$$

We average the pair to obtain one antithetic observation

$$\overline{Y}^{(j)} = \frac{1}{2}\big(Y^{(j,+)} + Y^{(j,-)}\big), \qquad j = 1, \dots, N/2. \tag{17}$$

The corresponding antithetic estimator of the call price is

$$\widehat{C}_{\mathrm{anti}} = \frac{2}{N}\sum_{j=1}^{N/2}\overline{Y}^{(j)}. \tag{18}$$

This estimator remains unbiased for C_0, but we expect it to have a smaller variance than the plain Monte Carlo estimator.

F–test for variance reduction

Following the discussion in Lecture 8, slide 18 (Muniraj 2025f), we formally compare the variances of the plain and antithetic Monte Carlo estimators. Let $Y_1, \dots, Y_N$ be the discounted payoffs from the plain simulation and $\overline{Y}^{(1)}, \dots, \overline{Y}^{(N/2)}$ the antithetic payoffs defined above. We estimate their variances by the sample variances:

$$s^2_{\mathrm{plain}} = \frac{1}{N-1}\sum_{i=1}^{N}(Y_i - \widehat{C}_{\mathrm{MC}})^2, \qquad \widehat{C}_{\mathrm{MC}} = \frac{1}{N}\sum_{i=1}^{N} Y_i, \tag{19}$$

and

$$s^2_{\mathrm{anti}} = \frac{1}{N/2-1}\sum_{j=1}^{N/2}\big(\overline{Y}^{(j)} - \widehat{C}_{\mathrm{anti}}\big)^2, \qquad \widehat{C}_{\mathrm{anti}} = \frac{2}{N}\sum_{j=1}^{N/2}\overline{Y}^{(j)}. \tag{20}$$

To formally assess whether antithetic sampling reduces variance, we perform an F–test with hypotheses

$$H_0 : \sigma^2_{\mathrm{plain}} = \sigma^2_{\mathrm{anti}}, \qquad H_1 : \sigma^2_{\mathrm{plain}} > \sigma^2_{\mathrm{anti}}.$$

The test statistic is the ratio of sample variances

$$F = \frac{s^2_{\mathrm{plain}}}{s^2_{\mathrm{anti}}}. \tag{21}$$

Under H_0, F has an F–distribution with $(N-1,\, N/2-1)$ degrees of freedom. We reject H_0 in favour of H_1 if the p–value is smaller than the chosen significance level α. A significant result provides statistical evidence that the antithetic simulation achieves a lower variance than the plain Monte Carlo approach.

1.5 Delta Hedging without and with Execution Delays

1.5.1 Delta and its interpretation

Let $V(t, S)$ be the time–t value of the option when the stock price is S. The *delta* of the option is

$$\Delta(t, S) = \frac{\partial V}{\partial S}(t, S),$$

which measures how sensitive the option value is to small changes in the stock price. For a European call in the Black–Scholes model,

$$\Delta_{\text{call}}(t, S) = \Phi\big(d_1(t, S)\big),$$

where

$$d_1(t, S) = \frac{\ln(S/K) + (r + \frac{1}{2}\sigma^2)(T - t)}{\sigma\sqrt{T - t}},$$

and Φ is the standard normal cdf.

Intuitively, $\Delta(t, S)$ tells us how many shares of stock we should hold per option in order to make the portfolio almost insensitive to small price movements over a short time step. A "delta–hedged" portfolio consisting of selling one option and owning Δ shares changes very little in value when S moves slightly. In continuous time, continuously rebalancing the number of shares to match the current delta would exactly replicate the option payoff in the Black–Scholes world.

Discrete-time delta hedging (no delays)

Delta hedging aims to choose a stock position so that small moves in the stock price have almost no effect on the value of our total portfolio. If we sell one option and hold Δ shares, the portfolio value is

$$\Pi(t) = -V(t, S_t) + \Delta(t, S_t)S_t.$$

Setting $\Delta(t, S_t) = \partial V/\partial S(t, S_t)$ makes the first-order effect of a small price change dS cancel, so the loss on the option is locally offset by the gain on the stock position.

Using the simulated GBM paths $(S_{t_k}^{(i)})_{k=0}^{M}$, we rebalance at times $t_k = k\Delta t$:

1. At $t_0 = 0$ we sell one call for $V(0, S_0)$ and deposit the proceeds in the cash account.
2. For each $k = 0, \dots, M$ and each path i we compute $\Delta_k^{(i)} = \Delta_{\text{call}}(t_k, S_{t_k}^{(i)})$. The change in stock position is $q_k^{(i)} = \Delta_k^{(i)} - \Delta_{k-1}^{(i)}$ (with $\Delta_{-1}^{(i)} = 0$), traded at price $S_{t_k}^{(i)}$. The corresponding cash flow is

$$\Delta\text{cash}_k^{(i)} = -q_k^{(i)} S_{t_k}^{(i)}.$$

3. At T we sell $\Delta_M^{(i)}$ shares at $S_T^{(i)}$ and pay the payoff $(S_T^{(i)} - K)_+$.

Let $\text{Portfolio}_T^{(i)}$ be the final portfolio value along path i after all cash flows and the payoff. The *hedging error* is

$$H^{(i)} = \text{Portfolio}_T^{(i)}. \tag{22}$$

We summarize hedging performance by the empirical mean, variance and distribution of the $H^{(i)}$ across paths.

Execution delays and a queueing interpretation

In reality, trades are not executed instantaneously. Hedge orders are sent to an exchange or trading platform, where they may wait in an order book before being matched. To capture these market frictions, we introduce random *execution delays* in our delta-hedging experiment, following the modelling idea from Lecture 6, slides 71–79 (Muniraj 2025d).

When a rebalance decision $q_k^{(i)}$ is made at time t_k, the trade is executed only at a later time

$$t_k + D_k^{(i)},$$

where $D_k^{(i)} \sim \text{Exp}(\lambda)$ is an exponential delay. We approximate this by an integer number of time steps

$$m_k^{(i)} = \lfloor D_k^{(i)}/\Delta t \rfloor.$$

If $k + m_k^{(i)} \le M$, the order is eventually filled at price $S^{(i)}_{t_{k+m_k^{(i)}}}$ and the actual cash flow becomes

$$\Delta\text{cash}_k^{(i)} = -q_k^{(i)} S^{(i)}_{t_{k+m_k^{(i)}}}.$$

If $k + m_k^{(i)} > M$, the order is never executed and the intended hedge adjustment does not take place.

This mechanism is analogous to an $M/M/1$ queue: hedge orders play the role of customers, the trading system is the single server, and $D_k^{(i)}$ is the random waiting time before execution. In the implementation we choose concrete arrival and service rates that keep the queue stable. Hedge orders arrive at rate

$$\lambda = 0.5 \times 252 \approx 126$$

orders per year, corresponding to on average half an order per trading day. The trading system processes orders at rate

$$\mu = 200 \quad \text{orders per year.}$$

The traffic intensity is therefore

$$\rho = \frac{\lambda}{\mu} \approx \frac{126}{200} = 0.63 < 1,$$

so orders are, on average, executed faster than they arrive and the $M/M/1$ queue remains stable, with finite expected waiting times for hedge orders.

Running the same hedging experiment with these delays yields hedging errors $H^{(i)}_{\text{delay}}$, whose empirical distribution we compare to the no-delay case to quantify the impact of execution lags on hedging performance.

1.6 Lévy Processes and a Jump–Diffusion Extension of Black–Scholes

Empirical stock returns exhibit occasional large jumps (e.g. crashes, announcements) and heavier tails than the normal distribution. The Black–Scholes model, driven by a continuous Brownian motion with constant volatility, cannot capture these features: price paths are continuous and very large moves are extremely unlikely. In our project we address this limitation in two ways: by allowing *jumps* in a Lévy jump–diffusion model (Muniraj 2025a) , and by allowing *regime changes in volatility* in a Markov-switching model (Muniraj 2025b) (see next section).

A Lévy process $(L_t)_{t \ge 0}$ is a stochastic process with stationary, independent increments, right–continuous paths with left limits, and $L_0 = 0$ almost surely. Brownian motion and Poisson processes are basic examples. In particular, the log-price under the Black–Scholes model can be written as

$$\ln S_t = \ln S_0 + \Big(r - \tfrac{1}{2}\sigma^2\Big)t + \sigma W_t^Q,$$

so $\ln S_t$ is a Brownian motion with drift and is therefore a Lévy process.

To allow for occasional jumps in the stock price and heavier tails in the return distribution, we extend the model by adding a jump component to the log-price. A simple and widely used specification is the *Merton jump–diffusion model.* Under the risk–neutral assumption we set

$$\ln S_t = \ln S_0 + \Big(r - \tfrac{1}{2}\sigma^2 - \lambda\kappa\Big)t + \sigma W_t^Q + \sum_{k=1}^{N_t} Y_k, \tag{23}$$

where

- $(W_t^Q)_{t\geq 0}$ is a standard Brownian motion under the risk–neutral assumption;
- $(N_t)_{t\geq 0}$ is a Poisson process with intensity $\lambda > 0$, independent of W_t^Q;
- $(Y_k)_{k\geq 1}$ are i.i.d. jump sizes, independent of W^Q and N, typically assumed to be normal:

$$Y_k \sim \mathcal{N}(m_J, s_J^2).$$

The term $\sum_{k=1}^{N_t} Y_k$ represents the cumulative jump contribution to the log-price up to time t. The parameter

$$\kappa = E[e^{Y_1} - 1]$$

is the average *relative* jump size. If $Y_1 \sim \mathcal{N}(m_J, s_J^2)$, then

$$\kappa = \exp\Big(m_J + \tfrac{1}{2}s_J^2\Big) - 1. \tag{24}$$

Equation (23) defines a Lévy process for $\ln S_t$, because it is the sum of a Brownian motion with drift and a compound Poisson process with i.i.d. jumps. The corresponding stock price process $S_t = \exp(\ln S_t)$ is therefore a jump–diffusion, with multiplicative jumps of size e^{Y_k} occurring at Poisson times.

Simulation of jump–diffusion paths

To simulate this model over a time interval $[0, T]$, we discretize time into steps of length $\Delta t = T/M$ and use the following scheme. Starting from S_0, for each step $k = 0, \dots, M-1$ we generate

- a Brownian shock $Z_k \sim \mathcal{N}(0, 1)$,
- a Poisson jump count $N_k \sim \text{Poisson}(\lambda\Delta t)$,
- N_k jump sizes $Y_{k,\ell} \sim \mathcal{N}(m_J, s_J^2)$, independently.

We then update the log-price by

$$\ln S_{t_{k+1}} = \ln S_{t_k} + \Big(r - \tfrac{1}{2}\sigma^2 - \lambda\kappa\Big)\Delta t + \sigma\sqrt{\Delta t}\, Z_k + \sum_{\ell=1}^{N_k} Y_{k,\ell}, \tag{25}$$

and set $S_{t_{k+1}} = \exp(\ln S_{t_{k+1}})$. The Black–Scholes model is recovered as the special case $\lambda = 0$, i.e. no jumps.

In our project, we can use this jump–diffusion model to:

- simulate alternative stock price paths with jumps,
- compare option prices and hedging errors to the pure Brownian Black–Scholes case.

This allows us to investigate how model misspecification (ignoring jumps) affects pricing accuracy and hedging performance.

1.7 Markov-Switching Volatility

In the Black–Scholes model the volatility σ is constant over time. Empirically, financial markets exhibit *volatility clustering*: periods of low volatility are followed by low volatility, and periods of high volatility are followed by high volatility. A simple way to capture this pattern is to let volatility follow a finite-state Markov chain.

We consider a two-regime model with "low" and "high" volatility levels σ_{L} and σ_{H}. Let $(X_k)_{k=0}^M$ be a discrete-time Markov chain taking values in $\{\mathrm{L}, \mathrm{H}\}$, representing the volatility regime at time $t_k = k\Delta t$. Its transition matrix is

$$P = \begin{pmatrix} p_{\mathrm{LL}} & p_{\mathrm{LH}} \\ p_{\mathrm{HL}} & p_{\mathrm{HH}} \end{pmatrix}, \qquad p_{\mathrm{LL}} + p_{\mathrm{LH}} = 1,\ p_{\mathrm{HL}} + p_{\mathrm{HH}} = 1,$$

where, for example, p_{LH} is the probability of switching from the low-volatility regime to the high-volatility regime in one time step.

Given the regime X_k at time t_k, we set the volatility in the next time step to

$$\sigma_k = \begin{cases} \sigma_{\mathrm{L}}, & \text{if } X_k = \mathrm{L}, \\ \sigma_{\mathrm{H}}, & \text{if } X_k = \mathrm{H}. \end{cases}$$

Under the risk-neutral measure, the stock price then evolves according to a *Markov-modulated GBM*:

$$S_{t_{k+1}} = S_{t_k} \exp\Big((r - \tfrac{1}{2}\sigma_k^2)\Delta t + \sigma_k \sqrt{\Delta t}\, Z_k \Big), \qquad Z_k \sim \mathcal{N}(0,1), \tag{26}$$

where the shocks Z_k are independent standard normal variables, independent of the Markov chain (X_k). The pair (S_{t_k}, X_k) is itself a Markov chain on the joint state space "price $\times$ regime".

The classical Black–Scholes model is obtained as the special case in which the Markov chain is degenerate (e.g. $X_k \equiv \mathrm{L}$ for all k), so that $\sigma_k \equiv \sigma_{\mathrm{L}}$.

In our project we use this model to generate alternative stock price paths with time-varying volatility, while still hedging with the *constant* Black–Scholes volatility. By comparing Monte Carlo option prices and hedging errors under the Markov-switching model to those under pure Black–Scholes, we can investigate the impact of volatility misspecification and the role of Markovian volatility dynamics.

2 Conclusion

Starting from the classical Black–Scholes framework, we have examined option pricing and hedging from several angles. Under the GBM assumptions, Monte Carlo simulation reproduces the analytic Black–Scholes price well, and antithetic sampling provides a clear variance reduction confirmed by an F–test. In this idealised setting, discrete-time delta hedging leads to small and nearly symmetric hedging errors, indicating that the replication idea works well when markets behave as the model assumes.

Once we introduce more realistic features, the picture changes. Random execution delays, modelled through an $M/M/1$ queue, already widen the distribution of hedging errors even though the underlying GBM dynamics are unchanged. When we move to jump–diffusion and Markov-switching volatility models, prices become more heavy-tailed and the Black–Scholes delta hedge produces larger and more skewed hedging errors. These experiments illustrate both the usefulness and the fragility of the Black–Scholes model: it provides a clean benchmark and efficient numerical methods, but it can underestimate risk when market frictions, jumps, or regime changes are present.