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Pseudo-viscous modelling of self-gravitating discs and the formation of low mass ratio binaries

C. J. Clarke*
Institute of Astronomy, Madingley Rd, Cambridge CB3 0HA

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ABSTRACT

We present analytic models for the local structure of self-regulated self-gravitating accretion discs that are subject to realistic cooling. Such an approach can be used to predict the secular evolution of self-gravitating discs (which can usefully be compared with future radiation hydrodynamical simulations) and to define various physical regimes as a function of radius and equivalent steady state accretion rate. We show that fragmentation is inevitable, given realistic rates of infall into the disc, once the disc extends to radii >70 au (in the case of a solar mass central object). Owing to the outward redistribution of disc material by gravitational torques, we also predict fragmentation at >70 au even in the case of low angular momentum cores which initially collapse to a much smaller radius. We point out that 70 au is close to the median binary separation and propose that such delayed fragmentation, at the point that the disc expands to >70 au, ensures the creation of low mass ratio companions that can avoid substantial further growth and consequent evolution towards unit mass ratio. We thus propose this as a promising mechanism for producing low mass ratio binaries, which, while abundant observationally, are severely underproduced in hydrodynamical models.

Key words: accretion, accretion discs – circumstellar matter.

1 INTRODUCTION

Since the early attempts of Paczynski (1978) and Lin & Pringle (1987), there has been considerable interest in describing the action of self-gravity in accretion discs through adaption of the standard formulation of viscous disc theory. In such an approach, the net effect of self-gravitating modes in discs is to provide a phenomenological viscosity, in much the same way that the action of the magneto-rotational instability (MRI) is often discussed as a source of a pseudo-viscosity. Indeed, self-gravity is a prime candidate 'viscosity mechanism' in the early stages of evolution of protostellar discs, both because the MRI is unlikely to be efficacious in dense regions of cool, weakly ionized discs (Gammie 1996) and also because it is hard to avoid the conclusion – in a scenario where the star forms from material channelled by the disc – that at early times the disc's *self* –gravity should be important.

There are obvious advantages of being able to model the effect of disc self-gravity (or indeed, the MRI) using a pseudo-viscous prescription, most notably the fact that one can use many elements of the apparatus of 'α' accretion disc theory (Shakura & Sunyaev 1973) to compute the structure and evolution of self-gravitating discs. For example, one may compute the structure of a steady state disc (as a function of accretion rate: see Rafikov 2009) or else, given a disc's surface density and temperature distribution, can evaluate the disc's evolution using the viscous diffusion equation.

The exploration of this approach has been encouraged by some obvious superficial similarities between the action of viscous torques and those produced by self-gravitating features in the disc. Non-axisymmetric modes produce torques that redistribute angular momentum within the disc and the work done by these torques is dissipated in the disc, providing a net conversion of mechanical energy into heat which permits the driving of an accretion flow. In these respects, therefore, the action of self-gravity is analogous to that of viscosity and it is tempting to describe it as such. In this case, the only difference between a self-gravitating disc and a conventional viscous disc is that the magnitude of the gravitational pseudo-viscosity is not known a priori (cf. Bertin 1997), but instead self-adjusts in order to keep the disc in a state of marginal gravitational stability. Such a picture is again consistent with a range of simulations of self-gravitating gas discs, which demonstrate such self-regulation (e.g. Lodato & Rice 2004; Boley et al. 2006; Mayer et al. 2007; Stamatellos et al. 2007; Cossins, Lodato & Clarke 2009): the amplitude of spiral features self-adjusts so as to heat the disc to a point of marginal gravitational stability (specifically such that the Toomre Q parameter – see Section 2.1 – is of order unity).

However, this broad similarity between some of the effects of viscosity, and of self-gravitating modes, is not in itself sufficient for one to adopt a pseudo-viscous description: in addition, one also

*E-mail: cclarke@ast.cam.ac.uk

needs a more precise requirement to be satisfied, namely that, as in the viscous situation, the rate of work done on the flow locally is entirely specified by the local torque. As pointed out by Balbus & Papaloizou (1999), there is an alternative situation where the energy extracted from the flow is not necessarily dissipated *locally* but is instead transported in a propagating wave, to be dissipated at some other radial location. Whereas this does not alter the global energy balance (i.e. the rate of mechanical energy lost by the accretion flow is still equal to the total rate of energy dissipation, integrated over the disc), it obviously prevents a simple relationship between local torques and the local thermodynamic state of the disc (see Lodato & Bertin 2001 for a discussion of how such transport could affect the spectrum generated by self-gravitating discs).

Such global energy transport is important in the case that waves, which are launched at corotation resonances, are able to propagate over significant radial distances. Thus, a measure of the importance of non-local effects is provided by how far waves persist away from corotation. Balbus & Papaloizou were thus able to compute the importance of global effects via an 'anomalous flux' which depends on $(\Omega - \Omega_{\rm p})/\Omega$, the fractional deviation between the local flow speed and the mode's pattern speed. However, one has to rely on numerical simulations in order to discover the spectrum of modes (and their pattern speeds) that are excited in the disc and thus cannot evaluate the importance of this effect a priori.

Numerical simulations however also allow one to test the pseudo-viscous hypothesis directly, by measuring both the torques in the disc and the local energy dissipation rate, and then comparing this relationship with that expected in the case of local dissipation. (Historically, this was first undertaken by Gammie 2001, although, as stressed by Balbus & Papaloizou, the form of the boundary conditions in shearing box simulations ensures locality in any case.) More notably, a variety of global simulations, both smoothed particle hydrodynamics (SPH)and grid-based calculations, show that the relationship between torques and energy dissipation rate is indeed similar to what would apply in the viscous case (Lodato & Rice 2004, 2005; Boley et al. 2006). This essentially local behaviour appears to be set by the requirement that the flow speed of the gas into spiral arms is marginally supersonic, which prevents waves from propagating far from corotation (Cossins et al. 2009). Such an argument implies that in more massive discs, which are hotter in a state of marginal gravitational stability, the higher sound speed should allow waves to propagate further from corotation, and that non-local effects are expected to be important. Simulations corroborate this tendency, in that deviations from local behaviour are more pronounced for more massive discs (Lodato & Rice 2005; Cossins et al. 2009). Local energy dissipation however appears to be an adequate approximation in the case of discs with masses up to several tenths of the central object mass (Lodato & Rice 2005; Cossins et al. 2009).

The above discussion suggests that in the case of self-gravitating protostellar discs, with $M_{\rm d}/M_* < 0.5$, it is adequate to model the effects of self-gravity in pseudo-viscous terms, and thus reap all the benefits of being able to derive both steady state disc structures as well as the secular evolution of the disc. This latter is not currently feasible in the case of hydrodynamic simulations: such simulations are typically run for hundreds of disc outer orbital time-scales, but the torques measured in these simulations imply secular evolution time-scales that exceed this by at least an order of magnitude. Likewise, it is only possible to model self-gravitating discs over a rather limited range of parameter space: if the spiral modes are too strong, the disc fragments (Gammie 2001; Rice, Lodato & Armitage 2005), whereas if they are too weak, the angular momentum transfer in the disc is dominated by numerical viscosity. In practice, this means that it is possible to perform hydrodynamic modelling over a range of mode strengths which, when parametrized in terms of an equivalent viscous α value (Shakura & Sunyaev 1973), span only about a factor of 3 in α. As we will see below, the α values that are expected in discs with realistic cooling regimes and mass input rates instead span many orders of magnitude.

The structure of the paper is as follows. In Section 2, we set out an analytic description of self-regulated, self-gravitating discs in various cooling regimes, discussing also the range of parameter space for which this description applies. In Section 3, we discuss the resultant solutions for disc properties as a function of steady state accretion rate and radius. In Section 4, we set out how such solutions can be used also to calculate the secular evolution of discs formed from collapsing cores, which can be usefully compared with hydrodynamic solutions. We also apply our results to binary star formation and point out that the use of realistic disc cooling offers the prospect of being able to create binaries with low mass ratio, an outcome that has an outcome that has been under-represented in previous hydrodynamical models that do not incorporate such effects. Section 5 recapitulates our main conclusions.

2 SELF-REGULATED SELF-GRAVITATING DISCS

2.1 General description

The importance of self-gravity in discs is measured by the Toomre Q parameter, which, in the case of discs which are considerably less massive than their central object and where the rotation curve is thus approximately Keplerian, takes the form:

$$Q = \frac{c_{\rm s}\Omega}{\pi G \Sigma}. \qquad (1)$$

Here, Σ is the disc surface density, Ω is the angular frequency and $c_{\rm s}$ is the sound speed. Formally, the condition $Q = 1$ marks the stability boundary of discs to *axisymmetric* instability (Toomre 1964). More generally, however, it is found numerically that self-gravitating discs that are subject to cooling evolve to a marginally unstable self-regulated state with Q in the range 1–2. This result may be readily explained in heuristic terms: gravitational collapse is opposed on small scales by pressure and on large scales by shear. If $Q > 1$, there is an overlap between the regimes of spatial scale that are stabilized by each of these effects, and thus collapse cannot occur on any scale. Self-gravity plays a minor role in discs with $Q \gg 1$, whereas discs with $Q \sim 1$, which will play an important role in our subsequent discussions, are in a state of marginal stability.

Since the sound speed appears in the numerator of the definition of Q, the fate of a disc set up with a given surface density profile is largely set by its temperature, and hence on the thermal equilibrium that it attains between radiative cooling and the dissipation of mechanical energy (or any other source of heating such as external irradiation). Internal dissipation is associated with the redistribution of mass and angular momentum in the disc resulting from internal torques, which may be either viscous (e.g. stresses associated with the magnetorotational instability) or gravitational in origin. In the latter case, therefore, the development of self-gravitating modes can provide an effective heat source which acts so as to stabilize the disc. If this heating however increases Q significantly above unity, then the modes themselves shut off, the disc cools and the modes begin to develop again. In this way, the disc achieves a state of *self-regulation* where Q is maintained close to unity.[1]

The above discussion therefore implies that the setting up of a disc in the gravitationally unstable regime does not *necessarily* lead to fragmentation, since an alternative route is the establishment of a self-regulated state of marginal gravitational stability. The important work of Gammie established the criteria that determine which route a disc takes in practice: through local (shearing box) simulations of self-gravitating discs subject to cooling on a time-scale $t_{\rm cool}$, he demonstrated that the boundary between fragmentation and self-regulation is set by the requirement $t_{\rm cool} = 3\Omega^{-1}$. Similar delineation of the fragmentation boundary in terms of the ratio of cooling to dynamical time-scale has been found in a range of other (global) simulations (e.g. Rice et al. 2005; Clarke, Harper-Clark & Lodato 2007), though the exact location of this boundary depends on the dimensionality of the simulations. A fragmentation boundary of this type may be qualitatively understood inasmuch as fragmentation requires that the $p\mathrm{d}V$ work done on overdense regions is radiated away on the (roughly dynamical) time-scale on which perturbations grow.

A critical value of the cooling time-scale can also be understood, in the framework of the pseudo-viscous description of self-gravitating discs, in terms of a critical value of the stress in the disc. As noted in Section 1, a range of simulations have shown that the relationship between the R, Φ component of the stress tensor and the local dissipation rate is roughly the same as that expected for a viscous process; in thermal equilibrium, this then implies (in the case of a Keplerian disc) the relationship:

$$t_{\rm cool} \sim \frac{4\Omega^{-1}}{9\gamma(\gamma-1)\alpha}, \qquad (2)$$

where γ is the ratio of specific heats and α is the usual parametrization of the R, Φ component of the stress tensor, $[W(R, \Phi)]$, as a fraction of the thermal pressure (Shakura & Sunyaev 1973). Thus, a given ratio of $t_{\rm cool}/\Omega^{-1}$ also reflects a given value of α (at fixed γ). Rice et al. furthermore delineated the fragmentation boundary as a function of the ratio of specific heats, γ, and established that the fragmentation boundary actually corresponds to a fixed maximum α value ($\sim$0.06).

2.2 Calculation of disc structure

We set out below a self-gravitating disc model which assumes (a) self-regulation of the Toomre Q parameter to a value $\sim$ unity and (b) local thermal and hydrostatic equilibrium of the disc under the assumption that the dissipation of energy associated with gravitational modes can be modelled as a pseudo-viscous process. We therefore parametrize the effect of self-gravity as an agent for angular momentum transport in terms of the conventional viscous α parameter, though stress that, since this is no more than a convenient parametrization, we make no assumption that this α is spatially constant (cf. Bertin & Lodato 1999). We also recognize that since self-regulated discs are typified by transient but regenerative spiral features, the local α values derived from simulations undergo considerable fluctuations, and thus that the α values that we derive here should correspond to time averages over several orbital periods.[2]

We will find that there are several regions of parameter space to which the model described above does not apply because some other agent is providing energy input (and possibly angular momentum transport). Thus, for example, the disc may be non-self-gravitating (with angular momentum transport provided by the MRI) or else (although self-gravitating and subject to associated angular momentum transport) may have its thermal properties mainly set by external irradiation. We therefore clarify the self-consistency checks that we run before assigning a solution to the ‘standard self-gravitating’ category.

2.3 Self-consistency checks

We will find that over much of the relevant parameter space (in terms of radius and steady state accretion rate), the disc exists in a self-gravitating state with thermal equilibrium between heating associated with dissipation of energy contained in gravitational modes and radiative cooling. We however also allow other possibilities by running three checks on the resulting solutions: (i) we check that the effective value of α for the solution does not exceed $\alpha_{\rm max} \sim 0.06$ and, if it does, designate this a fragmenting solution (see Section 2.1), (ii) we check that the equilibrium temperature does not fall below a minimum value $T_{\rm min}$ which, following Hartmann et al. 1998, we set to 10 K as representing the minimum temperature that gas can attain when subject to ambient heat sources in molecular clouds. We designate such solutions as isothermal and replace the thermal equilibrium condition by imposing $T = T_{\rm min}$, (iii) we check the effective value of α for the solution in order to see whether it is less than the value of α which we would expect from the operation of the MRI in this region of the disc. This check therefore involves both an assumption about the region of the disc that is potentially subject to the MRI and about the typical value of α ($\alpha_{\rm MRI}$) delivered by the MRI. Both these quantities are rather uncertain. In the former regard, we need the ionization fraction to exceed 10^{-13} (Gammie 1996), either as a result of thermal ionization, in warm inner disc regions, or through ionization (by cosmic rays or X-rays, Glassgold, Najita & Igea 1997) in the outer (low column density) regions of the disc . We thus assume that the MRI may be activated either if the mid-plane temperature exceeds 1000 K or at a radius $> R_{\rm dead}$. In reality, $R_{\rm dead}$ should itself be a function of accretion rate; since a full investigation of this effect is beyond the scope of this paper, we instead choose a fixed value $R_{\rm dead} \sim 100$ au. This is towards the upper range of values quoted in the literature (Sano et al. 2000; Fromang, Terquem & Balbus 2002; Matsumura & Pudritz 2003), which we justify by the fact that we are mainly focusing on rather massive discs where $R_{\rm dead}$ is likely to be higher. The value of α delivered by the MRI is a fluctuating quantity, whose time average has been reported over a wide range, mainly in the domain <0.01 (see discussion in King, Pringle & Livio 2007); here we adopt $\alpha_{\rm MRI} = 0.01$. We note that our analytical solutions can be readily generalized to other choices of $R_{\rm dead}$, $\alpha_{\rm MRI}$ and $T_{\rm min}$ (see Rafikov 2009).

If $T < 1000$ K and $R < R_{\rm dead}$ then we assume that there is no possibility of the MRI being activated (in other words, we do not here consider the possibility of layered accretion or else of activation of the dead disc mid-plane by interaction with the surface active

[1] A similar mechanism sustains stellar discs that are subject to ‘cooling’ in the marginally stable state, except here gravitational instability feeds energy into non-circular motions of the stars rather than the internal energy of the fluid and ‘cooling’ consists of re-supply of stars on circular orbits Sellwood & Carlberg (1984).

[2] The secular evolution time-scales that correspond to the α values that we derive are many orders of magnitude greater than the orbital time-scale, so that such averaging is appropriate.

layer; Fleming & Stone 2003). Since we are only considering two sources of angular momentum redistribution (i.e. the MRI and self-gravity) then this implies that the disc *must* be self-gravitating in this regime if it is also accreting. In other regions of the disc, which are MRI active in principle, we then have to check whether the resulting MRI solution (at given accretion rate and radial location) is self-consistent (in the sense of being non-self gravitating). If instead the Toomre Q parameter corresponding to this MRI solution is <1, we infer that the angular momentum transport is dominated by self-gravity and revert to the standard self-gravitating solution. We note that the interface between the MRI and self-gravitating regimes corresponds to $\alpha = 0.01$, $Q = 1$ and that therefore once the disc becomes self-gravitating (in regions where the MRI should be active in principle) then the corresponding $\alpha > 0.01$. Since we deem that the disc fragments once $\alpha > \alpha_{\rm frag} = 0.06$, this implies that even in parts of the disc that are potentially MRI active, there is a wedge of parameter space (for which α is in the range 0.01–0.06) where the disc is self-gravitating and self-regulated. We note that the existence of such a region is a consequence of the fact that $\alpha_{\rm MRI}$ is somewhat below $\alpha_{\rm frag}$. Finally, we note that there is a wedge of parameter space for which we cannot find self-consistent steady state solutions using either the MRI or self-gravity as angular momentum transfer agents – in this case, the self-gravitating solution has $T > 1000$ K and so the MRI should be active in this region, delivering a value of α that is higher than the corresponding self-gravitating value. However, if one instead seeks an MRI solution at this location, the higher α value implies a cooler disc (at given accretion rate) so that the temperature is now < 1000 K and the MRI should not be activated. We denote regions of parameter space for which this is the case by the ??? in Figs 1–4 and note that in practice this will cause time-dependent behaviour, as envisaged by Gammie (1999) and Armitage, Livio & Pringle (2001).

2.4 The standard self-gravitating regime

Given a location in the disc, radius R, and local surface density, Σ, the self-regulated condition $Q = 1$ (equation 1) immediately fixes the local temperature, T. Given T and Σ, hydrostatic equilibrium yields the disc scaleheight through the standard thin disc relation $H = c_s/\Omega$ and the mid-plane density $\rho = \Sigma/2H$.[3]

We then use the power-law fits to the Rosseland mean opacity $\kappa(\rho, T)$ contained in Bell & Lin (1994) in order to compute the optical depth ($\tau = \kappa\ \Sigma$) and hence to determine whether the disc is locally optically thick or optically thin. As it turns out that the disc

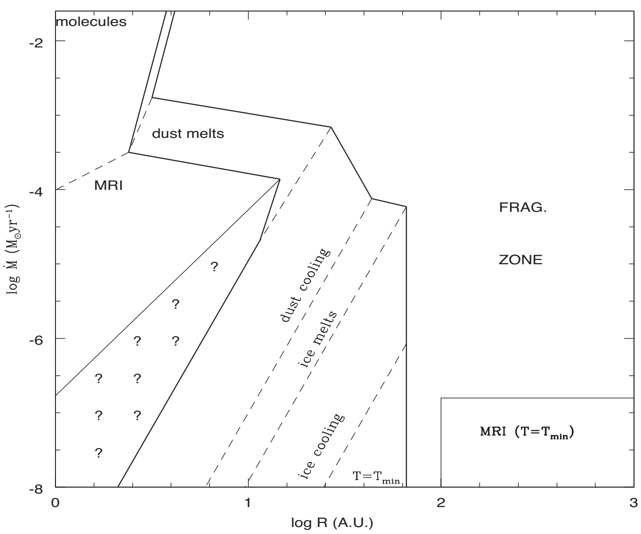

Figure 1. Regimes in the plane of steady state accretion rate versus radius in the case of a disc surrounding an object of mass 1 $M_\odot$. The non-fragmenting, self-regulated self-gravitating regime lies between the bold lines. See text for details.

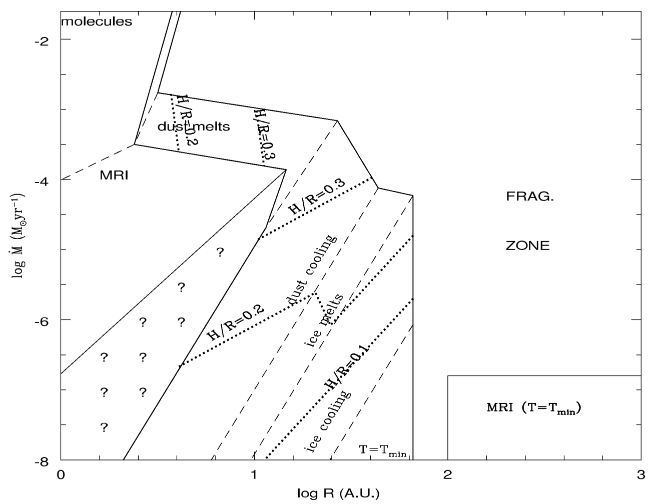

Figure 2. Contours of disc equal axis ratio (H/R) in the case of a central object of mass 1 $M_\odot$.

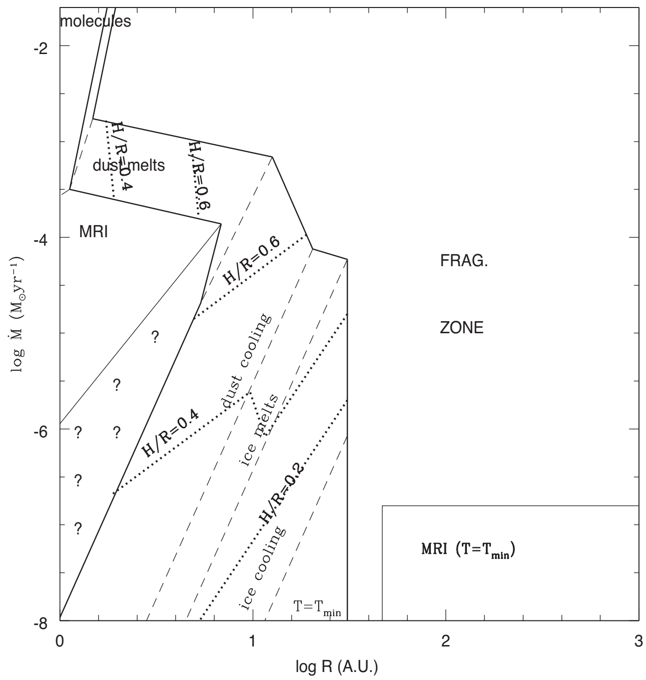

Figure 3. Contours of disc equal axis ratio (H/R) in the case of a central object of mass 0.1 $M_\odot$.

[3] This expression for H is appropriate to the case that the main vertical component of the gravitational force is provided by the central star (or disc at much smaller radius), rather than the disc's gravity locally. Actually, the two effects are competitive with each other for a disc with $Q \sim 1$, and we have therefore over-estimated H (and underestimated ρ) only slightly. See Bertin & Lodato (1999) for expressions approximating H in the self-gravitating regime.

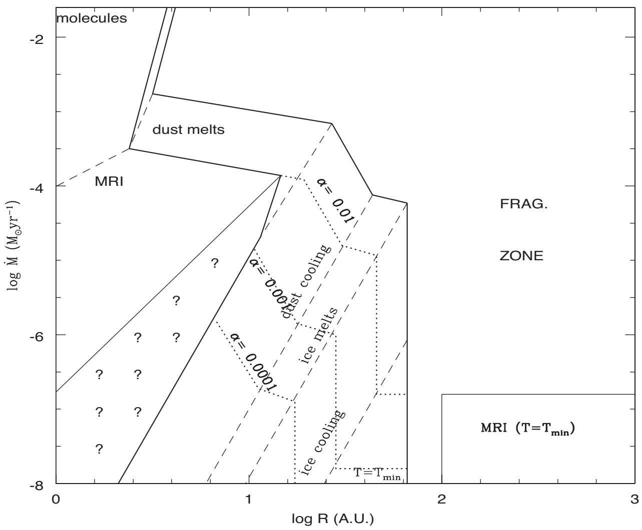

Figure 4. Contours of equal viscous α parameter.

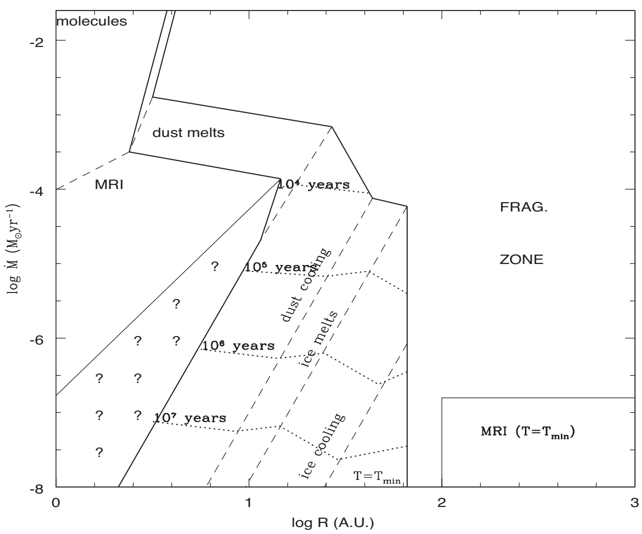

Figure 5. Contours of equal viscous time-scale in the case of a central object of mass 1 $M_{\odot}$.

is always optically thick in the regime $T > T_{\rm min}$, we calculate the (one-sided) local cooling rate per unit area as

$$Q^- = \frac{8acT^4}{3\tau}. \tag{3}$$

(Note that is effectively a one zone model vertically, inasmuch as it calculates the optical depth using opacity values characteristic of the mid-plane temperature; since the disc photosphere is cooler than the mid-plane, this will not be accurate in regions where the opacity is highly temperature dependent.)

In order to calculate the effective viscosity locally, we then apply the thermal equilibrium condition:

$$Q^+ = Q^-, \tag{4}$$

where Q^+ is the local rate of energy dissipation due to the effective viscosity, ν,

$$Q^+ = 9/8\nu\Sigma\Omega^2. \tag{5}$$

Hence, we have calculated the effective kinematic viscosity [$\nu(\Sigma, R)$] in the standard self-gravitating regime, and could use this to

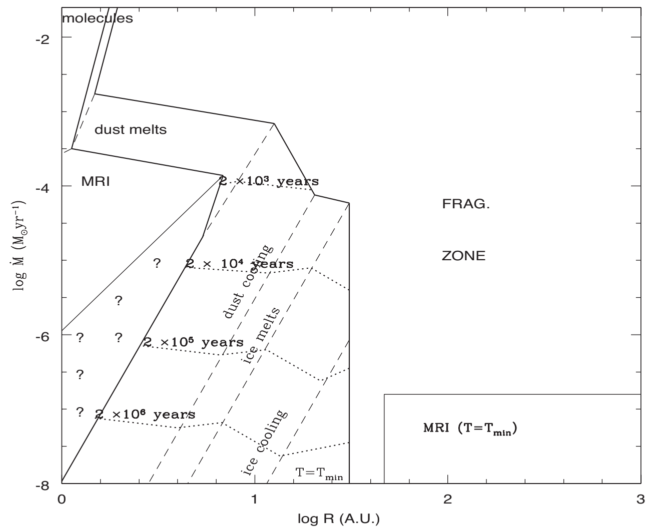

Figure 6. Contours of equal viscous time-scale in the case of a central object of mass 0.1 $M_{\odot}$.

evolve the disc by solving the time-dependent viscous diffusion equation. We can also determine the effective α value corresponding to this ν via the definition:

$$\alpha = \frac{3\nu\Omega}{2c_s^2}. \tag{6}$$

Comparison of this value with $\alpha_{\rm frag}$ and $\alpha_{\rm MRI}$ then establishes whether the solution is instead in the MRI or fragmenting regime. In the former case (i.e. where $\alpha < \alpha_{\rm MRI}$), then provided that the disc is potentially MRI active for this value of Σ and R (see Section 2.3), the condition $Q = 1$ is relaxed and is replaced by the restriction $\alpha = \alpha_{\rm MRI}$ (i.e. we compute standard fixed α-disc solutions, cf. Bell & Lin 1994). If, conversely, $\alpha > \alpha_{\rm frag}$ then the disc is deemed to be subject to fragmentation and is labelled as such in Figs 1–6.

2.5 Caveats

The association of fragmentation with a given value of α (or cooling time-scale) is mainly based on simulations that set up discs with simply parametrized cooling at a variety of rates. There are two possible objections to this approach. First, it is not immediately clear that the same fragmentation boundary would apply in the more realistic case where a state of rapid cooling is approached on a slow (secular) evolution time-scale. Clarke et al. 2007 however did not find that the location of the fragmentation boundary was unduly sensitive to the rate at which these conditions were approached, implying that it is a reasonable approximation (as assumed here) to associate fragmentation with a given, history independent, α value. More serious is the argument of Johnson & Gammie (2003) that a given α (or cooling time-scale) threshold is unreliable in the case that the cooling time-scale is a strong function of temperature: they showed in the regime where dust sublimes (and the opacity is a very steeply decreasing function of temperature) that the onset of fragmentation did not match their expectations based on the cooling time-scale implied by their initial conditions. The problem in this case hinges on the initial guess that one makes about the value of the Toomre Q parameter in the self-regulated state: if the cooling rate is highly temperature dependent, then a small adjustment in Q from that imposed initially can result in a large change in the cooling rate and thus a different fragmentation outcome from that expected. In practice, this means that there are large errorbars surrounding the location of the fragmentation boundary in the dust sublimation regime. We will however see (Fig. 1) that this corresponds to rather high accretion rates that are not normally encountered in low mass stars.

3 THERMAL EQUILIBRIUM SOLUTIONS

Through our calculation of $\nu(\Sigma, R)$, we can assign to each pair of parameters (Σ and R) the accretion rate that would correspond to this solution if the disc was in a steady state. In the case of a disc where the torque vanishes at the origin, we have

$$\dot{M}_{ss} = 3\pi\nu\Sigma. \tag{7}$$

We emphasize that our local solutions do not require the disc to be in a steady state and that we use $\dot{M}_{ss}$ only as a convenient way of parametrizing our solutions: in general, the actual accretion rate is related to $\dot{M}_{ss}$ via

$$\dot{M} = \dot{M}_{ss} + 2R\frac{\partial \dot{M}_{ss}}{\partial R}. \tag{8}$$

We can now (Fig. 1) classify all points in the parameter space of radius versus steady state accretion rate and draw regime boundaries in this space (see the Appendix for analytic expressions for solutions in the various regimes and for the location of regime boundaries).

The results of Fig. 1 can be summed up as follows. The regions between the bold lines represent the range of parameter space for which the disc is expected to be in the self-regulated, self-gravitating regime: in other words, angular momentum transport is dominated by gravitational torques, but the disc is not expected to fragment. For most of this region, the thermal input to the gas is also dominated by heating associated with gravitational modes. The exception to this is the lower right of this region (low accretion rates, large radius) where the gas is mainly heated by an assumed interstellar radiation field. Although the gas is assumed to be isothermal at 10 K in this region, it is deemed not to fragment because the cooling time-scale is long enough that perturbations behave quasi-adiabatically.

The topology of the boundaries of this region is set both by the fragmentation boundary (right-hand boundary) and the interface with regions where the MRI dominates the angular momentum transfer (left-hand boundary). A notable feature of the former is the vertical boundary at $\sim$70 au which coincides with regions of the disc where the opacity is dominated by ice grains. In this case $\kappa \propto T^2$, and therefore, for an optically thick disc in a state of marginal gravitational instability, the cooling time-scale is simply a function of radius (i.e. independent of accretion rate, provided one remains in the ice cooling regime). Consequently, the fragmentation boundary (which can be cast either in terms of a critical value of α or, equivalently, in terms of the ratio of cooling time-scale to dynamical time-scale) is encountered at a fixed radius, a feature first noted by Rafikov (2005) (see also Matzner & Levin 2005; Stamatellos et al. 2007). Outside this radius, therefore, a self-gravitating disc is *always* subject to fragmentation, regardless of the accretion rate.[4]

[4] This also applies, for self-gravitating discs at >70 au, in the case that the disc temperature is set by external irradiation, since the fact that the cooling time-scale is short compared implies that perturbations behave approximately isothermally. We note that the relationship between cooling time-scale and α (equation 2) no longer applies when the disc heating is not dominated by the gravitational modes, and that in this regime the disc will therefore fragment even at accretion rates corresponding to very low α values.

We also see that beyond the dead zone (i.e. at $r > R_{\rm dead}$ where we have assumed $R_{\rm dead} = 100$ au in Fig. 1) there is a region where the disc is non-self gravitating, with angular momentum transport effected by the MRI. We can readily compute the maximum value of the steady state accretion rate that is possible in this MRI region by noting that equations (1), (6) and (7) can be recast as

$$\dot{M}_{ss} = \frac{3\alpha c_{\rm s}^3}{GQ}. \tag{9}$$

Thus, for a disc that is isothermal at $T = T_{\rm min} = 10$ K and $\alpha = \alpha_{\rm MRI} = 0.01$, we see that the maximum value of $\dot{M}_{ss}$ for which the disc is self-consistently non-self gravitating is a few times $10^{-7}\,{\rm M}_\odot\,{\rm yr}^{-1}$. There is thus an island of parameter space in the lower right of the diagram where fragmentation is not expected.

Turning again to the self-regulated self-gravitating region of the disc, we see that at the lowest accretion rates the disc is isothermal with temperature $T = T_{\rm min} = 10$ K, but as the accretion rate is increased, the thermal equilibrium temperature rises above $T_{\rm min}$, with dissipative heating balancing optically thick radiative cooling, and opacity provided successively by ice and dust. Only at very high accretion rates ($>10^{-4}\,{\rm M}_\odot\,{\rm yr}^{-1}$) is the implied α value in excess of $\alpha_{\rm frag}$. We therefore confirm that fragmentation is not expected in the inner regions of discs around low mass stars unless they are subject to extremely high infall rates, a result that is consistent with radiation hydrodynamical simulations of self-gravitating discs (Boley et al. 2006; Stamatellos & Whitworth 2008). Inward of 33 au, the disc becomes hot enough for dust to start subliming before the point at which it fragments, and so there is a self-gravitating regime where the opacity is a steeply decreasing function of temperature and where the disc is consequently nearly isothermal. Inward of 11 au, there is a region where the mid-plane effective temperature is >1000 K and where the value of α delivered by a self-gravitating solution is less than $\alpha_{\rm MRI}$: consequently, we expect this region to be MRI dominated at intermediate accretion rates, but self-gravitating at higher or lower accretion rates. (Note that as discussed in Section 2.3, there is a region of parameter space where we cannot find self-consistent solutions, because the self-gravitating solution has $T > 1000$ K and the MRI solution has $T < 1000$ K.)

We stress that some of the above conclusions are dependent on our assumptions about the efficacy of the MRI (i.e. our assumed values of $R_{\rm dead}$ and $\alpha_{\rm MRI}$). For example, if we assumed $R_{\rm dead} <$ 70 au, then the isothermal, MRI dominated region in the lower right would join on to the self-regulated self-gravitating disc at smaller radius and it would then be possible for a disc fed at a low rate to avoid fragmentation altogether. Likewise, if $\alpha_{\rm MRI}$ was lower than we have assumed, then the thermally ionized MRI regime at a few au would become self-gravitating at lower accretion rate than shown in the figures.

Fig. 2 depicts contours of H/R in the $\dot{M}_{ss}, r$ plane for a solar mass central object and demonstrates that we are in the regime of lowishH/R over most of the self-regulated regime, thus justifying, a posteriori, the pseudo-viscous approach (see discussion in Section 1). We note that since all disc quantities (such as scaleheight H) relate to radius purely as a function of Ω, the Keplerian angular frequency, a given value of H (at fixed $\dot{M}_{ss}$) is associated with a value of R that scales with central object mass as $M^{1/3}$; therefore, the corresponding diagram for a lower mass star is shifted to smaller radii (by a factor of $M^{1/3}$) and each H/R contour is increased by a factor of $M^{-1/3}$. Fig. 3 demonstrates that, for given accretion rates, self-regulated discs around low mass stars are geometrically thicker and thus less well described by a pseudo-viscous (local description). Conversely for higher mass stars (and supermassive central objects) discs in the self-regulated regime (i.e. those that do not fragment) are geometrically thinner than in the solar mass case (Fig. 2a).

Fig. 4 depicts contours of α in the self-regulated regime. These contours are, by construction, parallel to the fragmentation boundary in all regimes where gravitational heating predominates. We draw attention to the fact that over a wide range of parameter space, the α values delivered by self-gravitating discs are very low; this presents a challenge to hydrodynamical modelling of disc formation since for α less than about 0.01, the angular momentum transport in current codes is dominated by numerical viscosity.

Figs 5 and 6 depict contours of constant viscous time-scale ($t_\nu = R^2/\nu$) for objects of central mass 1 $M_\odot$ and 0.1 $M_\odot$, respectively. This represents the characteristic time-scale for mass and angular momentum transport in the disc and will be used in our subsequent discussion of disc secular evolution. We draw attention to the fact that these time-scales are many orders of magnitude in excess of the dynamical time-scales at these radii and that it is therefore appropriate to discuss the secular evolution of such discs.

4 APPLICATIONS

4.1 Comparison with hydrodynamical infall models

Our a posteriori justification of the pseudo-viscous approach (see Fig. 2) means that we can use the solutions presented in the Appendix to compute the secular evolution of discs in the process of assembly from a collapsing core. This involves integration of the time-dependent viscous diffusion equation,

$$\frac{\partial \Sigma}{\partial t} = \frac{3}{R}\frac{\partial}{\partial R}\left[R^{1/2}\frac{\partial}{\partial R}(R^{1/2}\nu\Sigma)\right] + \dot{\Sigma}_{\rm inf}, \tag{10}$$

where $\dot{\Sigma}_{\rm inf}$ is the parametrized infall rate (mass per unit area per unit time). At given Σ and R, the value of $\nu\,\Sigma$ required for the solution of equation (10) can be simply derived from the $\Sigma(\dot{M}_{ss}, R)$ solutions given in the Appendix, noting the relationship between $\dot{M}_{ss}$ and $\nu\Sigma$ contained in equation (7). We have undertaken some trial experiments of this sort which confirm the statements we make below on analytic grounds. Nevertheless, the real utility of this approach will be its use in analysing the results of hydrodynamical simulations of collapsing cores, where one will obtain $\dot{\Sigma}_{\rm inf}$ numerically and can thus compare the secular evolution of the simulation with that predicted by a pseudo-viscous treatment. We emphasize the need for simulators to take note of Fig. 4, in order that they avoid regimes where the expected α delivered by gravitational instability is less than that associated with the numerical viscosity in their codes.

4.2 Fragmentation

In self-gravitating discs that are optically thick, with opacity provided by ice grains, the cooling time-scale is a function of radius only (Rafikov 2005) and is independent of temperature. This implies a particular radial location $r_{\rm frag}$ for disc fragmentation in this regime, independent of accretion rate. As has been noted by several previous authors (Matzner & Levin 2005; Rafikov 2005; Stamatellos et al. 2007) $r_{\rm frag}$ is around 70 au for solar mass stars; inward of $r_{\rm frag}$, fragmentation can only be expected for high accretion rates ($>10^{-4}\,M_\odot\,{\rm yr}^{-1}$).

Obviously, therefore, one expects disc fragmentation in the case of cores whose specific angular momentum is high enough for significant infall beyond ~70 au. In terms of the factor $\beta_{\rm J}$ (the ratio of the rotational kinetic energy of the core to its break-up value) this implies $\beta_{\rm J} > 0.01$ for solar mass cores collapsing from a Jeans scale at temperature ~10 K. However, our results furthermore imply that fragmentation at $r_{\rm frag}$ is inevitable for *any* plausible initial core rotation rate because of the possibility of outward transfer of angular momentum (and mass) even in the case where the maximum radius of infall ($r_{\rm inf}$) is restricted to radii $\ll r_{\rm frag}$. All that is required in this case is that there is enough angular momentum and mass in the infalling material for it to be self-gravitating at $>r_{\rm frag}$, following radial redistribution by gravitational torques. Since the specific angular momentum in a Keplerian disc scales as $R^{1/2}$, then we require that if a disc spreads so that mass $M_{\rm frag}$ ends up at radius larger than $r_{\rm frag}$, then we must have $M_{\rm disc}\, r_{\rm inf}^{1/2} > M_{\rm frag}\, r_{\rm frag}^{1/2}$. Given the typical parameters of self-gravitating discs at $r > r_{\rm frag}$, we require that $M_{\rm frag}$ is at least around 10 per cent of the central object mass. Since the entire central object mass is initially in the disc, we then require that the disc initial infall radius is at least around $0.01 r_{\rm frag}$. This radius is extremely small (less than 1 au) and would correspond to an implausibly low initial core rotation rate ($\beta_{\rm J} < 10^{-4}$). We thus conclude that, because of the efficient outward angular momentum transfer in self-gravitating discs, disc fragmentation at $>r_{\rm frag}$ is inevitable in just about any core with a realistic initial angular momentum content. This conclusion is not substantially changed if we relax the assumption that the MRI is effective only beyond 100 au. If instead we postulate that the disc is MRI active down to radius $r_{\rm frag}$ then we could in principle avoid fragmentation if the MRI took over as an angular momentum transfer mechanism once material reached $r_{\rm frag}$. We however see from Fig. 1 that this places an upper limit on the rate at which mass flows outwards at $r_{\rm frag}$. Our numerical experiments, involving the integration of equation (10) for a variety of disc infall histories, indicate that this condition is never met unless the rate of infall is very low ($<10^{-7}\,M_\odot\,{\rm yr}^{-1}$: see discussion following equation 9). Collapse models for low mass cores (e.g. Vorobyov & Basu 2005, and references therein) however suggest rates that are an order of magnitude higher than this and these higher values ($\sim10^{-6}\,M_\odot\,{\rm yr}^{-1}$) are corroborated by modelling of line profiles in such cores (e.g. Tafalla et al. 2000).

The two important elements to emerge from this discussion are (i) the thermodynamics of self gravitating discs only permits fragmentation at $r > r_{\rm frag} \sim 70\, M^{1/3}$ au and (ii) virtually any core with non-negligible angular momentum will produce a disc that will expand, due to the action of gravitational torques, to radii $> r_{\rm frag}$.[5] Both these points are relevant to binary star formation and it is tempting to make the association between $r_{\rm frag}$ and the median binary separation (which is observed to be similar to this e.g. Duquennoy & Mayor 1991; Fischer & Marcy 1992).

There are several aspects of this model which have not been captured by simulations of binary star formation to date. First of all, the model suggests that fragmentation should be the norm, since virtually all cores are likely to have enough angular momentum for their resulting discs to expand to $r_{\rm frag}$. In this scenario, therefore, the initial location of fragmentation does not depend, in the limit of low core angular momentum on the actual value of this angular momentum, since the characteristic fragmentation radius is instead set by the thermodynamics of ice dominated cooling. Secondly, where the disc expands out to $r_{\rm frag}$ before fragmentation occurs, the material involved in the fragmentation now has higher specific angular momentum than any other material in the core. This is

[5] This latter conclusion assumes that there is not substantial loss of angular momentum from the disc as a result of winds or outflows at this stage.

important for the mass ratio, q, of the resulting binary: not only is the companion initially of rather low mass (being composed of the high angular momentum tail of material that has attained radii $>r_{\rm frag}$) but it will not be driven to higher q by subsequent accretion of higher angular momentum material.

This therefore avoids a well-known problem of binary star formation, that is the tendency of binaries to end up with nearly equal mass ratios, in contrast to the observed situation in which low mass ratios are abundant (Duquennoy & Mayor 1991; Prato et al. 2002; Halbwachs et al. 2003). This tendency for simulations to produce nearly equal mass pairs is driven by the late accretion of high specific angular momentum material on to the proto-binary pair, which favours preferential accretion on to the secondary (Whitworth et al. 1995; Bate & Bonnell 1997; Bate 2000; Delgado et al. 2004). In most previous calculations, which employ an isothermal equation of state for all but the densest regions of the disc, fragmentation ensues as soon as the disc is formed and therefore the fragment cannot avoid interacting with high specific angular momentum material that continues to fall in from the outer regions of the core. The new element here is that fragmentation is delayed until the disc has re-expanded out to $r_{\rm frag}$. Fig. 4 shows that this occurs on a time-scale of $\sim10^5$– 10^6 yr, which is somewhat longer than the free fall time-scale of the collapsing core.[6] Although this scenario has to be substantiated by hydrodynamical calculations that incorporate the necessary physics, this *delayed fragmentation* provides a promising mechanism for the formation of low q binaries.

5 CONCLUSIONS

Our derivation of analytic expressions for the properties of self-gravitating self-regulated discs has allowed us to derive useful diagrams illustrating different physical regimes as a function of steady state accretion rate and radius (Figs 1–4). These delineate the fragmentation boundary and illustrate that for all but the lowest mass stars, one expects non-fragmenting discs to be well described by the local (pseudo-viscous) approach employed here. These plots also demonstrate, as noted by several previous authors, that fragmentation is only expected at radius $> r_{\rm frag} \sim 70\,(M/\,{\rm M}_{\odot})^{1/3}$ au provided that infall from the parent core is at less than $\sim10^{-4}\,{\rm M}_{\odot}\,{\rm yr}^{-1}$. Thus, it is only in high mass cores [$>10\,{\rm M}_{\odot}$, where such high infall rates are deduced (Cesaroni et al. 2007)] that one would expect fragmentation at smaller radius. On the other hand, fragmentation at $>r_{\rm frag}$ is pretty much inevitable for any plausible initial core rotation rates: the only scenario in which such fragmentation could be avoided is in the case both that the MRI extends in to $r_{\rm frag}$ and if the mass infall rate from the parent core is very low ($<10^{-7}\,{\rm M}_{\odot}\,{\rm yr}^{-1}$).

We point out that our analytic expressions provide a useful framework for comparison with future hydrodynamic collapse calculations that incorporate the necessary cooling physics. In particular, Fig. 4 raises a cautionary note about the very low viscous α values to be expected in some regions of parameter space for self-gravitating, self-regulated discs and shows that care must be taken that calculations are not instead dominated by the effect of numerical viscosity.

We also point out that our steady state expressions (listed in the Appendix) can be readily re-arranged so as to find the effective viscosity of self-gravitating discs as a function of surface density and radius and thus enable the secular evolution of the disc to be computed using the viscous diffusion equation. Such calculations may then be usefully compared with the results of hydrodynamical modelling.

We mainly apply our results to the issue of binary star formation, pointing out that, fortuitously or not, $r_{\rm frag}$ is close to the median binary separation. We show that the unlikelihood of fragmentation within $r_{\rm frag}$ offers the possibility, in low angular momentum cores, of *delayed* fragmentation, whereby the disc is assembled at small radii and fragmentation only then ensues later, one the disc has spread outwards, through the action of gravitational torques, to $> r_{\rm frag}$. This delayed fragmentation, that would not be seen in models that employ a mainly isothermal equation of state in the disc, has a distinct advantage when it comes to reproducing observed binary statistics. In this case, fragmentation is likely to occur *after* the bulk of the parent core has fallen in, and thus the fragment would avoid the accretion of infalling material which, in current models, drives binary mass ratios to high values. Since observations are unambiguous in requiring a large population of binaries (with separations of 10–100 s of au) that have low mass ratios ($q < 0.5$), this is an outstanding shortcoming of current models. We propose however that the different thermal regimes at $r <$ and $> r_{\rm frag}$ offer a way to solve the problem of the creation of low q binaries.

[6] It should be noted that the recent simulations of Attwood et al. (2009) also demonstrate delayed fragmentation: in this case, the delay is instead due to the time required to assemble a Toomre unstable disc, which is a shorter time-scale (1–3×10^4 yr) than the time required for angular momentum transport discussed here. This shorter time is now no longer much longer than the core infall time-scale, which probably explains why these simulations still show a tendency towards growth of q towards unity through accretion.

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APPENDIX A:

In order to derive 'standard self-regulated' solutions in thermal equilibrium (see Section 2), we solve equations (1)–(7), together with parametrizations for the opacity taken from Bell & Lin (1994). For Rosseland mean opacity parametrized in the form $\kappa = \kappa_0 \rho^a T^b$, the thermal equilibrium solution for optically thick cooling is given by

$$\nu \Sigma = C \Sigma^{7-2b} R^{15-3b+3a}, \tag{A1}$$

where

$$C = \frac{G^{3-b}\sigma}{M_*^{5-b+a}\kappa_0 \mathcal{R}^{4-b}}, \tag{A2}$$

where $\mathcal{R}$ is the gas constant, such that $c_s^2 = \mathcal{R}T$.

In the opacity regimes relevant to our calculations, we have $\kappa_0 = 2 \times 10^{-4}, a = 0, b = 2$ (for opacity provided by ice grains), $\kappa_0 = 2 \times 10^{16}, a = 0, b = -7$ (for the region where ice grains sublime) and $\kappa_0 = 0.1, a = 0, b = 0.5$ for the region where dust grains dominate the opacity.

These opacity regimes are therefore associated with the following optically thick, self-regulated solutions (for $M_* = 1\,\mathrm{M}_\odot$).

(a) *Ice grains*.

$$\Sigma = 14\dot{M}_{-6}^{1/3} R_{100}^{-3}\ \mathrm{g,\ cm^{-2}}, \tag{A3}$$

$$T = 3.2\dot{M}_{-6}^{2/3} R_{100}^{-3}\ \mathrm{K}, \tag{A4}$$

$$\alpha = 0.4 R_{100}^{9/2}, \tag{A5}$$

$$H/R = 0.055\dot{M}_{-6}^{1/3} R_{100}^{-1}, \tag{A6}$$

$$t_\nu = 1.7 \times 10^5 \dot{M}_{-6}^{-2/3} R_{100}^{-1}\ \mathrm{yr}, \tag{A7}$$

where R_{100} is radius normalized to 100 au and $\dot{M}_{-6}$ is the steady state accretion rate (equation 8) normalized to $10^{-6}\,\mathrm{M}_\odot\,\mathrm{yr}^{-1}$.

(b) *Ice grains sublime*.

$$\Sigma = 75\dot{M}_{-6}^{1/21} R_{100}^{-12/7}\ \mathrm{g\,cm^{-2}}, \tag{A8}$$

$$T = 95.\dot{M}_{-6}^{2/21} R_{100}^{-3/7}\ \mathrm{K}, \tag{A9}$$

$$\alpha = 2.5 \times 10^{-3} R_{100}^{9/14} \dot{M}_{-6}^{6/7}, \tag{A10}$$

$$H/R = 0.30\dot{M}_{-6}^{1/21} R_{100}^{2/7}, \tag{A11}$$

$$t_\nu = 9.4 \times 10^5 \dot{M}_{-6}^{-20/21} R_{100}^{2/7}\ \mathrm{yr}. \tag{A12}$$

(c) *Dust grains*.

$$\Sigma = 29\dot{M}_{-6}^{1/6} R_{100}^{-9/4}\ \mathrm{g\,cm^{-2}}, \tag{A13}$$

$$T = 14\dot{M}_{-6}^{1/3} R_{100}^{-1.5}\ \mathrm{K}, \tag{A14}$$

$$\alpha = 4.4 \times 10^{-2} R_{100}^{9/4} \dot{M}_{-6}^{1/2}, \tag{A15}$$

$$H/R = 0.11\dot{M}_{-6}^{1/6} R_{100}^{-1/4}, \tag{A16}$$

$$t_\nu = 3.4 \times 10^5 \dot{M}_{-6}^{-5/6} R_{100}^{-1/4}\ \mathrm{yr}. \tag{A17}$$

(d) *Dust grains sublime*.

$$\Sigma = 170\dot{M}_{-6}^{0.018} R_{100}^{-1.64}\ \mathrm{g\,cm^{-2}}, \tag{A18}$$

$$T = 500\dot{M}_{0.036}^{1/3} R_{100}^{-0.27}\ \mathrm{K}, \tag{A19}$$

$$\alpha = 2.1 \times 10^{-4} R_{100}^{0.41} \dot{M}_{-6}^{0.945}, \tag{A20}$$

$$H/R = 0.68\dot{M}_{-6}^{0.018} R_{100}^{0.36}, \tag{A21}$$

$$t_\nu = 2.1 \times 10^6 \dot{M}_{-6}^{-0.82} R_{100}^{0.36}\ \mathrm{yr}. \tag{A22}$$

These solutions are continuous at the zone boundaries, which are located at

$$R_{\mathrm{ice\,melt}} = 27\dot{M}_{-6}^{2/9}\ \mathrm{au} \tag{A23}$$

(where ice grains sublime)

$$R_{\mathrm{dust}} = 17\dot{M}_{-6}^{2/9}\ \mathrm{au} \tag{A24}$$

(where dust grains dominate the opacity) and

$$R_{\mathrm{dust\,melt}} = 5.5\dot{M}_{-6}^{0.24}\ \mathrm{au}, \tag{A25}$$

where dust grains sublime. In addition, the disc is isothermal (with $T = T_{\mathrm{min}} = 10\,\mathrm{K}$) for $R > R_{\mathrm{iso}}$ where

$$R_{\mathrm{iso}} = 70\dot{M}_{-6}^{2/9}\ \mathrm{au}, \tag{A26}$$

Throughout the isothermal regime, the disc properties are described by

$$\Sigma = 25 R_{100}^{-1.5}\ \mathrm{g\,cm^{-2}}, \tag{A27}$$

$$\alpha = 0.06\dot{M}_{-6}, \tag{A28}$$

$$H/R = 0.1 R_{100}^{0.5} \tag{A29}$$

and

$$t_\nu = 3 \times 10^5 R_{100}^{0.5} \dot{M}_{-6}^{-1}\ \mathrm{yr}. \tag{A30}$$

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