Assignment: Wein's Displacement Law

Using the Planck function in terms of wavelength, derive Wein's displacement law.

Consider the Planck Function:

$$L(\lambda,T)=\frac{2hc^2}{\lambda^5}\cdot\frac{1}{e^{hc/\lambda kT}-1}$$

where h represents Planck's constant, c is the speed of light, and k is Boltzmann's constant.

Differentiating with respect to λ:

$$L(\lambda,T)=2hc^2\left(\frac{1}{\lambda^5}\cdot\frac{1}{e^{hc/\lambda kT}-1}\right)$$

factoring constants

$$\frac{\partial L}{\partial\lambda}=2hc^2\left(\frac{-5}{\lambda^6}\cdot\frac{1}{e^{hc/\lambda kT}-1}+\frac{1}{\lambda^5}\cdot\frac{(e^{hc/\lambda kT}-1)(0)-\left(\frac{-hce^{hc/\lambda kT}}{\lambda^2 kT}\right)(1)}{(e^{hc/\lambda kT}-1)^2}\right)$$

product rule
quotient rule
differentiation of exponentials

$$=2hc^2\left(\frac{-5}{\lambda^6(e^{hc/\lambda kT}-1)}+\frac{\frac{hce^{hc/\lambda kT}}{\lambda^2 kT}}{(e^{hc/\lambda kT}-1)^2}\right)$$

combining like terms

$$=\frac{2h^2c^3e^{\frac{hc}{k\lambda T}}}{\lambda^7 kT(e^{\frac{hc}{k\lambda T}}-1)^2}-\frac{10c^2h}{\lambda^6(e^{\frac{hc}{k\lambda T}}-1)}$$

distributing

Setting equal to 0:

$$\frac{2h^2c^3e^{\frac{hc}{k\lambda_{max}T}}}{\lambda_{max}^7 kT(e^{\frac{hc}{k\lambda_{max}T}}-1)^2}-\frac{10c^2h}{\lambda_{max}^6(e^{\frac{hc}{k\lambda_{max}T}}-1)}=0$$

$$\frac{hc}{\lambda_{max}kT}\left(\frac{2hc^2e^{hc/k\lambda_m T}}{\lambda_{max}^6(e^{hc/k\lambda_m T}-1)^2}\right)-\frac{10hc^2}{\lambda_{max}^6(e^{\frac{hc}{k\lambda_m T}}-1)}=0$$

factoring

Both terms share a common factor: $\dfrac{hc^2}{\lambda_{max}^6(e^{\frac{hc}{k\lambda_m T}}-1)}$

Factoring, we have:

$$\frac{hc}{\lambda_{max}kT}\cdot\frac{e^{hc/\lambda_m kT}}{e^{hc/\lambda_m kT}-1}-5=0$$

Using the approximation $x\equiv\dfrac{hc}{\lambda kT}\gg 1\longrightarrow\dfrac{e^x}{e^x-1}\sim 1$ the equation becomes:

$$\frac{xe^x}{e^x-1}-5=0$$

substituting $\dfrac{hc}{\lambda kT}$

$$x-5=0$$

approximating $\dfrac{e^x}{e^x-1}\sim 1$

$$x=5$$

Substituting back in:

$$\frac{hc}{\lambda_{max} kT} = 5$$

substitution

Rearranging, this gives:

$$\lambda_{max} T = \frac{hc}{5k}$$

Recall:

↳ h is Planck's Constant $= 6.626 \times 10^{-34}$ Js

→ c is the speed of light $= 3 \times 10^{8}$ m/s

→ k is Boltzmann's Constant $= 1.38 \times 10^{-23}$ J/K

$$\therefore \lambda_{max} T = \frac{(6.626E-34)(3E8)}{5(1.38E-23)} \approx 2.898 \times 10^{-3} \text{ mK}$$, as desired.

□