CS99 Honors Thesis Proposal
Towards Ryser's Conjecture: An Algorithmic Approach to Min-Vertex Cover – Max-Matching Inequality in Hypergraphs

Anna Dodson
Advisor: Prof. Deeparnab Chakrabarty

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Background & Motivation.

A *maximum matching* of any graph $G = (V, E)$ is a selection of a subset of edges $\in E$ such that no two edges share an end vertex and the cardinality of the selection is maximized. A *minimum vertex cover* of any graph is a selection of vertices such that all edges in the graph include at least one vertex in the cover and the cardinality of the selection is minimized. The proposed work concerns the cardinalities of the maximum matching and the minimum vertex cover for partitioned graphs.

For general graphs, finding a minimum vertex cover is an NP-hard problem. However, in partitioned graphs, where all vertices are divided into groups such that none of the graph's edges contain multiple vertices from the same group, certain conclusions can be drawn about the minimum vertex cover. In particular, the cardinality of the maximum matching can be used to determine the cardinality of the minimum vertex cover by Ryser's Conjecture.

Let the maximum matching be denoted by M and the minimum vertex cover be denoted by C. Konig's famous theorem takes an algorithmic approach and concludes that for a bipartite graph, the maximum matching of a bipartite graph has the same cardinality as its minimum vertex cover; that is, $\|C\| = \|M\|$. Ryser's Conjecture follows from here.

Ryser's Conjecture. For any hypergraph of dimensionality n, the cardinality of the minimum vertex cover C is less than or equal to $(n-1)$ times the cardinality of its maximum matching M. I.e., in an $n-$partite hypergraph, —C—

$\leq (n-1)|M|$

In tripartite hypergraphs, $n = 3$. Thus we have the construction $\mid C \mid \leq 2 \mid M \mid$. While there are algorithms for finding minimum vertex covers from maximum matching in bipartite graphs, namely Konig's Theorem, and proofs of the tripartite case, from Ron Aharoni in 2001, there are no algorithmic proofs yet for the tripartite case. The goal of this work is to determine an algorithm for finding a maximum matching from a minimum vertex cover in hypergraphs, and use it to move towards a proof of Ryser's Conjecture.

Approach.

The proposed research concerns Ryser's Conjecture for a tripartite hypergraph. Ron Aharoni showed in 2001 that in any tripartite hypergraph, $\|C\| = 2\|M\|$. However, there is no known algorithm for finding the maximum matching or the minimum vertex cover for a given tripartite hypergraph. The project will aim to use the idea of local switching to determine an algorithm for finding a maximum matching in a tripartite hypergraph, and from there, the minimum vertex cover. A possible extension of the project is to consider the n-partite case and prove Ryser's Conjecture for higher dimensionality hypergraphs.

The current idea is to construct a working algorithm for the $M = 1$ tripartite graph case with the idea of local switching, prove its correctness, and then extend for $M = 2, 3 \ldots \infty$. The proof of correctness in the tripartite case may possibly be attainable by induction or by some insight derived from the results of the algorithm.

If the latter case, it is possible we will be able to extend the proof for higher-dimensionality hypergraphs, i.e., $n = 4, 5 \ldots \infty$. Further areas of research and topics of interest may arise over the course of the project, and related research topics will be explored as seen fit.