Is the wonky dice fair?

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Overview of lesson

In this activity, students explore the actual and expected outcomes of a six-sided "wonky" dice. Students use what they have learned about sampling variation to decide if there is any reason to think their wonky dice might not be fair.

Learning objectives

- Predicting with justification of whether the wonky dice is fair
- Testing prediction with data from their dice
- Using technology to visualise sampling variation from a probability situation
- Recognising small samples display lots of variability
- Experiencing sampling variability in distributions generated by probability models
- Comparing a distribution from real observed data with what is expected under a model taking sampling variation into account
- Drawing a conclusion comparing real observed data to model generated data

Suggested age range

This activity is appropriate for any high school age students (13 years to 17 years old) and has been used successfully with both junior students (Year 9 NZ, 13 year olds) and senior students (Year 12 NZ, 16 year olds).

Time required

Approximately one period (60 minutes)

Keywords

probability model, sampling variation, distribution, dice, fair, informal hypothesis testing

Introduction

The hook for the activity is a "wonky" dice that just seems a bit wrong. This gives the motivation for a different take on a classic probability activity. I first found the wonky dice when hunting online for weighted dice, with the idea of giving a class a set of dice with one that was weighted and asking them to decide who had the weighted dice. Instead, the wonky dice worked really well at capturing students' attention, as their whole experience with dice to date was with a regular cube. This activity has been developed to allow purposeful conversation around sampling variation generated from a random variable and distributions of outcomes. The activity leads to informal ideas of hypothesis testing – *If this dice is fair, what is likely or unlikely to happen with the distribution of outcomes just by chance alone?*

Lesson outline

1. Is there any reason to think my wonky dice is not fair?

Give pairs of students a wonky dice to examine. Ask students the following questions:

Is there any reason to think your wonky dice is not fair? Why? Why not?

> This question is worded in such a way to start students thinking about whether there is evidence to show that their wonky dice is not fair, rather than whether it is actually fair. We can never tell for sure if their wonky dice is actually fair, but, with observed data, we should be able to get a good idea if the dice is not fair.

What do we mean by fair with your wonky dice?

> Students' gut reaction is that their wonky dice is not fair, though they often struggle to come up with a

good reason to why they think this. Most reasons are related to the shape not being a regular cube. I have had one pair of very determined Year 9 students who carefully measured each face of their wonky dice and calculated their areas. Their conclusion was that their wonky dice was in fact fair as each face on it had the same area.

How could we check if your wonky dice is fair?

I want students to come up with the idea of rolling their wonky dice lots of times – occasionally I have had to lead students to this idea but usually someone in the class suggests it readily.

How many times do you think you need to roll the wonky dice to check that it is fair? Are 60 rolls sufficient to check that the wonky dice is fair?

Students often state a very low number (for example 6 rolls), and sometimes a ridiculously high number (for example 1000 rolls).

If your wonky dice is fair and you rolled your wonky dice 60 times, what do you think the distribution of outcomes will look like?

I made a deliberate choice to include formal language at this stage. I may need to unpack what I mean by "distribution of outcomes" with the students. If needed, I sketch a frequency graph on the board, add "frequency" on the vertical axis and ask students what the possible outcomes are when rolling their wonky dice. I remind them that this is the "sample space" and add the numbers 1, 2, 3, 4, 5, 6 along the horizontal axis of the graph.

Students can make a sketch in their book of what they think the distribution of outcomes will look like, or just use a hand (arm) gesture showing a uniform shape. This teaching choice is made based on the level of the class and their familiarity with probability distributions. Most students are comfortable at this stage with the frequency of all outcomes being the same (10), a discrete uniform distribution.

Rolling their wonky dice 60 times was chosen to make the maths easy and so students don't get bored just rolling a dice! It is enough rolls to start to see what might be going on with their wonky dice, but not enough rolls to be sure. Later on when we compare distributions from each wonky dice in the class there will be some quirky results (see Section refadaptations).

If your wonky dice is fair, how many times do you expect a 1 to come up? A 2? A 3? A 7? Why?

If sketching in their book I ask for a scale on the vertical axis showing the expected frequency is ten rolls for each outcome. By asking why they expect ten for each outcome we can discuss "expected value", and I can remind them that this is a theoretical concept under the assumption of "if your wonky dice is fair...". I often ask about the frequency of a 7 coming up on their wonky dice as a way of quickly assessing students' prior knowledge of probability.

2. Gathering data

In pairs, students roll their wonky dice 60 times and record the outcomes on the graph provided (see Figure 1).

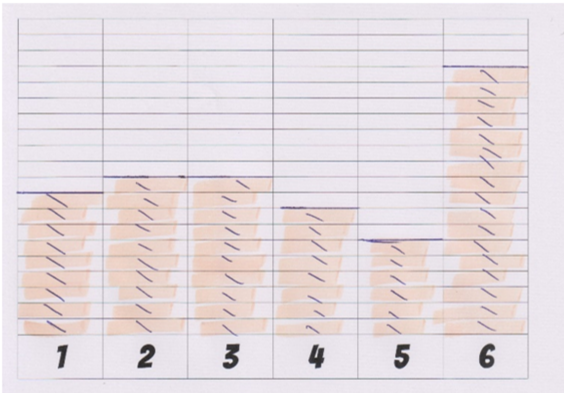
The graph grid is set up so one rectangle represents one outcome and students can just highlight it easily. I photocopy a few extra graphs as there will be students who make mistakes and need to re-do them. I always make a wall display with the graphs so I ask students to make sure that their graphs can be seen at the back of the class.

Students write one or two "I notice..." statements about their data for their wonky dice (see Figure 1). Remind students to be clear about what the variable is that they are discussing (the outcome of their wonky dice) and to include numbers (frequencies) to support their statement.

Ask students to compare their data from their wonky dice to what they predicted they would get.

Why do you think you are seeing similarities? Why do you think you are seeing differences? Based on your data, do you have any reason to think your wonky dice might not be fair? (see Figure 1)

I rove while students are answering these questions and ask why they are drawing that conclusion (probe more into their thinking). This gives me an idea of where students are at in their understanding of



I notice that six was the most common outcome on our wonky dice with 17 out of 60 rolls. Five was the least common outcome on our wonky dice with only 6 out of 60 rolls. Our wonky dice doesn't seem fair – we got a lot more sixes that we expected and a lot less fives.

Figure 1: Becky and Sophie's graph and comment on whether their wonky dice is fair.

sampling variation and ideas of random outcomes.

3. If your wonky dice is fair, what would you expect to see?

I utilise technology to see what the distribution of outcomes of a fair dice rolled 60 times looks like, and how different it can be each time we complete a set of 60 rolls. I use Anna Fergusson's probability modelling tool (learning.statistics-is-awesome.org/modelling-tool). The tool illustrates sampling variation in the frequency of outcomes in a distribution.

Start by asking students to sketch what they think the distribution of their wonky dice would look like if they rolled their dice 60 times again.

If needed I use prompts such as: *What features of the distribution are likely to stay the same? What features of the distribution are likely to be different? and why?* to start generating ideas and discussion around sampling variation and random outcomes.

I demonstrate the online modelling tool to the whole class using the data projector. I insert one pair of students' real observed data results on the left (Becky and Sophie's data in this case) and compare these to the known model (discrete uniform distribution) on the right (Figure 2) and generate simulated model data multiple times. Before selecting "start animation" I link back to the earlier discussion in class about what we would expect to see if the dice was fair, and explain that the online modelling tool allows us to view model generated data from a discrete uniform distribution; that is, if the dice is fair, each outcome is equally likely. This animation generates 60 random numbers from 1 to 6 where each outcome is equally likely – it essentially mimics what students' did when they rolled their dice 60 times (assuming their wonky dice is fair).

The modelling tool set up is shown in Figure 2. Note, if you scroll down below the graphs there is an option to change the dot size at the bottom, for this display I generally change the dots one size smaller than the default.

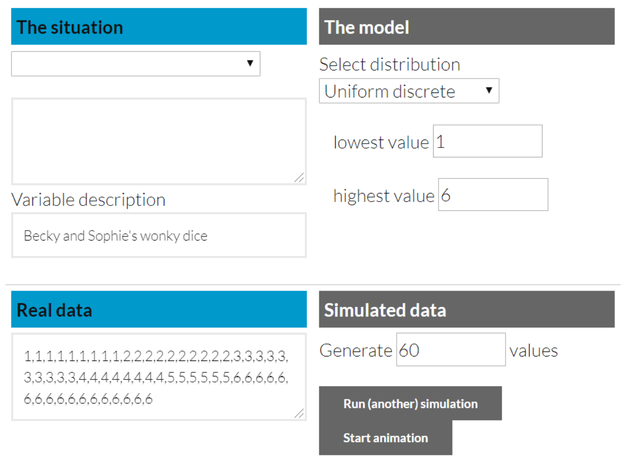


Figure 2: Modelling tool set up.

Ask students to observe how much the distribution changes by watching the tops of the bars on the right-hand side (see Figure 3).

Do any of these model generated distributions look like yours? Do all the stacks (outcomes of the dice) ever have a similar frequency? Very often? How much can they differ? Does that change your mind about whether you think Becky and Sophie's (or your) wonky dice might not be fair?

The modelling tool has the ability to capture the over-fitted shape of each simulation and transfer this onto the original real observed data (Figure 4). This helps students visualise the similarities and differences between each set of 60 dice rolls.

As shown in Figure 4 (top), a red line is drawn on the simulated model data on the right, highlighting the frequency of each outcome when "Track over-fitted shape for simulated data" is selected under the right-hand

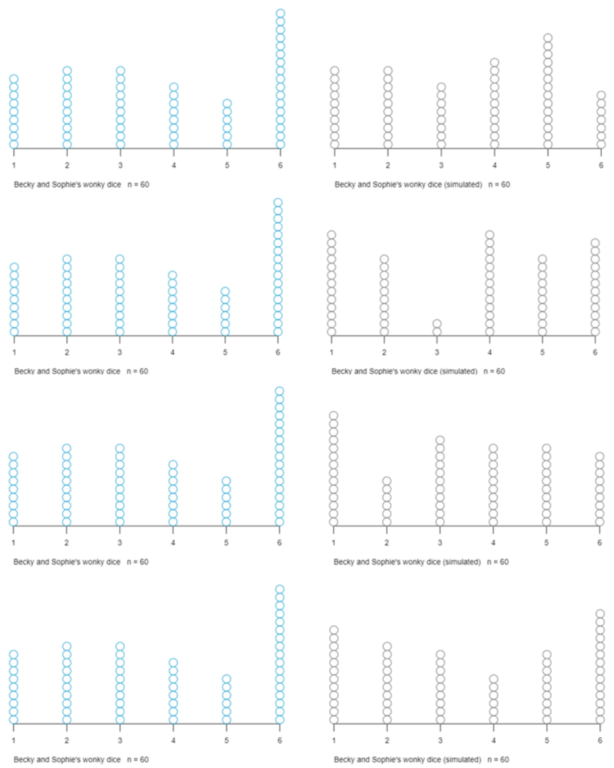


Figure 3: Screen shots of four simulated runs. The left hand side (in blue) shows Becky and Sophie's real observed data from their 60 rolls of their wonky dice. The right hand side (in grey) shows different sets of 60 values from a uniform distribution; that is model generated data to simulate 60 rolls of a fair dice.

Capture Features section. The red line for that set of model data then changes to grey and is kept. As more and more sets of model data are generated, more over-fitted shapes are sketched and the depth of grey lines highlights the frequency of those shapes occurring. Figure 4 (bottom) shows this after 87 sets of 60 simulations.

At this stage, after enough over-fitted shapes from the model data have been generated to build up a "fog of uncertainty", the "Transfer model data" option on the left is selected to transfer the captured over-fitted shapes onto the real observed data, Becky and Sophie's dice rolls (Figure 4 bottom). This allows an easy visual comparison between the real observed data, and the sampling variation from the model generated data. That is, we can visually see how much we would expect sets of 60 fair dice rolls to differ, and whether Becky and Sophie's wonky dice data looks unusual.

I sometimes use this part of the modelling tool package with my classes, depending on the level of the class and how much time I have left in the period.

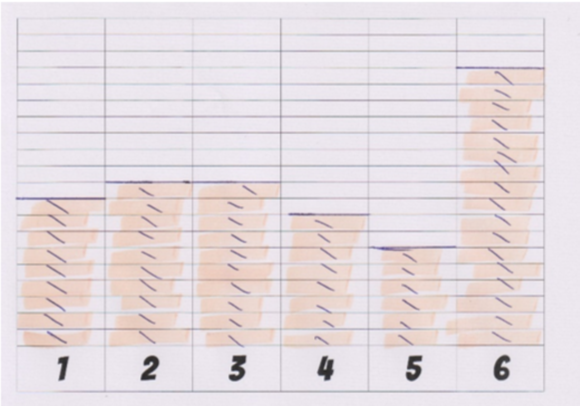
Figure 4: Screen shot of the tracked over-fitted shape for both the model generated data and original real observed data.

4. Final conclusion

Students should make a final conclusion about the fairness of Becky and Sophie's wonky dice based on both the girls' real observed data and the visualisations of repeatedly rolling a fair dice using the model generated data (see Figure 5). That is, *Do the real observed data from Becky and Sophie's dice look unusual now we have an idea of how different sets of 60 dice rolls can be?*

If the "Track over-fitted shape for simulated data" feature is used then it is very important that the projected online modelling tool is changed back to generating 60 values (see Figure 3) before students write their conclusion, and that students are reminded that the visualisations show how much sets of 60 dice rolls can differ each time. In other words, the online modelling tool is showing the sampling variation when generating multiple simulations of size 60 from a discrete uniform distribution, the known model.

> I am careful to try and avoid using "vary" (as in "… showing how much sets of 60 fair dice rolls can vary each time") as I have found it can confuse students when we are also talking about sampling variation, especially in the initial stages of developing these ideas.



We think our wonky dice is fair. We got more sixes that we expected initially and less fives, but after seeing how much sets of 60 dice rolls can change each time we don't think our wonky dice is unusual.

Figure 5: Becky and Sophie's graph and comments on the fairness of their wonky dice after seeing the simulation.

> Becky and Sophie's conclusion in Figure reffig:wonky-ref is reflecting on whether their dice is fair. Depending on the level of the class I am working with, I might discuss that actually, from this set of 60 rolls of their wonky dice, they have no evidence to say their dice is not fair; given what we have seen about how much different sets of 60 dice rolls from a fair dice differ, Becky and Sophie's dice does not look unusual – but it still could be.

Students should also make a final conclusion about their own wonky dice comparing the visualisations with what the real observed data from their wonky dice.

> Students could be asked to use the online modelling tool themselves at this stage to have a visual comparison of their own real observed data from their wonky dice they have rolled with the model generated data. I have not tried doing this.

5. Adaptations

This lesson is focused on the fairness of one wonky dice, with the real observed data from Becky and Sophie's dice used for comparison when visualising how much sets of 60 fair dice rolls differ. It is interesting to compare the real observed data of all the wonky dice in the class (see Figure 6).

Figure 6: A class set of wonky dice results.

There are two directions this can go. Firstly, IF we assume all the wonky dice are identical, then I ask questions such as:

Is there any evidence to say my (very expensive, especially imported just for you) wonky dice are not identical? Why? Why not? Is there any evidence to say my (very expensive, especially imported just for you, marketed as "fair") wonky dice might not all be fair? Why? Why not?

If we do not assume the wonky dice are all identical then the question I ask becomes subtlety different as we are comparing different wonky dice:

Is there anything to indicate that your wonky dice might not be fair based on what the rest of the class rolled? Can you tell from 60 rolls?

> I sometimes change this to a set up where I have been sold one dodgy wonky dice which isn't fair – *Which one might it be? Why?* When viewing the class set of results, I reinforce what we have seen through the online modelling tool – that the results of rolling their wonky dice 60 times can change quite a bit each time.

Some other ideas to extend this lesson plan are to:

- Make links to sampling variation from a sampling perspective. Students may have already seen situations where they take repeated samples from a population and looked at similarities and differences in the samples. The ideas covered here are the same except we are generating our samples from a probability perspective.
- Discuss with students how hard it is for us humans to have a good "gut" feeling for what random behaviour looks like, especially as we are designed

to be pattern-recognisers and our natural instinct is to hunt out patterns. There are many other small activities available that help students get a feel for the behaviour of "randomness" and what it looks like, for example Pure Randomness in Art (understandinguncertainty.org/node/1066) or Odd One Out (nrich.maths.org/5801). This can also lead into discussions around personal perceptions of randomness and coincidences and why it feels like some people are often lucky or unlucky playing games (see, for example, Metz (1997)).

- Link to other curriculum areas using simple activities such as calculating the surface area and volume of the wonky dice and creating nets, using more complex tasks such as creating another design for another fair wonky dice, or investigating the links between the geometry or physical properties of a dice and the random outcomes generated from rolling the dice.

Teacher notes

Pedagogy

The lesson plan is mainly a teacher-led discussion. My aim is to have all students engaged in the activity and hence I utilise techniques such as "think-pair-share" and "talk moves." I also remind myself to pose the question first, give students time to think about the answer, and then either select a student to respond or ask for volunteers.

"Think-pair-share" is a simple activity where a question is posed, students are given some time to think about what their answer is (30 seconds for example), then share their answer with their neighbour (pair) and finally sharing answers with the whole class. See theteachertoolkit.com/index.php/tool/think-pair-share, for example, for further information.

"Talk moves" are a variety of teacher strategies that can be implemented to encourage student engagement in classroom discussions. When using talk moves, the teacher will ask the class a question, give thinking time, then ask students to share their ideas either with their partner first (similar to "think-pair-share" above) or will move straight into selecting someone to share thoughts with the class. Talk moves are then used to continue the class discussion with the teacher usually selecting students to contribute. For example, "Mark, can you repeat for us what Michael just said in your own words please?" (re-voicing); "Mitchell, can you add on to what Matt just said for us please?" (adding on). Talk moves help keep students engaged in the dialogue as any student could be asked to contribute. It is therefore important to have a positive class culture and strong relationships, for students to know that if they are not sure of an answer it is okay to ask for help "umm, I'm not sure - Matt, can you please repeat what you just said?", and for the teacher to know if there are any students they should not force to speak in front of the class. See teach.conceptuamath.com/talk-moves, for example, for further information.

I also rove around the room whenever it is appropriate and listen in to conversations, capturing interesting comments, and asking questions such as "why do you think that....". I often feed back to the class some of these observations and discussions.

Framework informing the tool used in the lesson plan

The framework that underpins the modelling tool used and the modelling thinking encouraged in the task is shown in Figure 7.

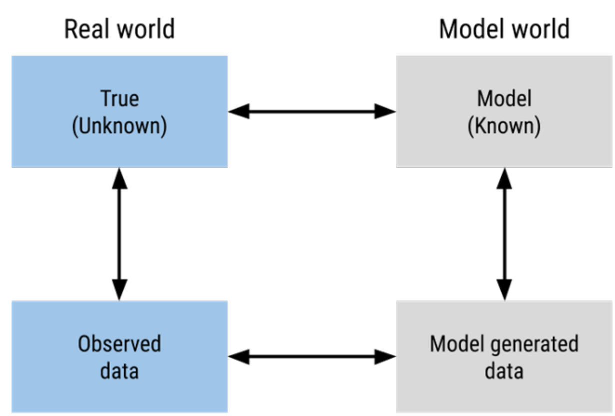


Figure 7: Statistical modelling framework developed by Anna Fergusson (Fergusson & Pfannkuch, 2019)

The framework consists of two parallel worlds: the real world and the model world. In both worlds, data are the output of a process, and the resulting data can be used to learn about this process. The framework allows for the separation but connection of data that are observed in the real world and data that are generated from a model, and the separation but connection of the true unknown process that is being modelled and the model itself. The purpose of the framework is to clearly show the different components of statistical modelling, to differentiate between the real world and the model world, and to theorise how one might move between the two worlds when engaging in modelling.

The framework consists of four components: The true (unknown), observed data, the model (known), and model generated data. Simulated data are a type of model generated data.

Note the tool used in this lesson plan separates the two worlds, with the real world on the left-hand side and the model world on the right-hand side as research has found that students can get confused between real observed data and model generated data. That is, they may think the model generated data is the real data (e.g., Gould, Davis, Patel & Esfandiari, 2010).

An important distinction for this framework compared to other modelling frameworks is the separation of data. Within probability learning contexts, it is common for both observed data and model generated data to be called *experimental* data, and calculations of probabilities using these data to be called *experimental probabilities*. However, viewing all data as *experimental* has the potential to limit development of a modelling perspective and can lead to students thinking a model is what is generating the real observed data. In this task, for example, it is important that students are aware of all four components of the modelling framework and how they relate to their use of the modelling tool as part of their investigation (see Figure 8).

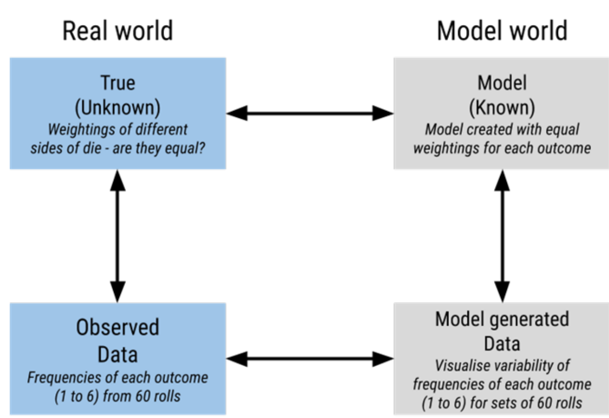


Figure 8: Modelling framework with details of task to illustrate four components.

References

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Materials required

The wonky dice are available through The Dice Lab (thedicelab.com) and are officially called "skew dice". You will need one wonky dice per pair of students.

A template for a graph to collect results is given on the next page, I print two to a page and use them as A5 sheets. The template gives consistency when comparing between student graphs and is used to make a wall display.

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