STATISTICS & DATA SCIENCE EDUCATOR

Probability explorations in a game context: The dice difference game

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Overview of lesson

Learning about probability poses difficulties for students at all levels. In this lesson students are asked to make predictions about the fairness of a dice difference game and then test them by gathering and examining data. Student predictions and conclusions are examined and re-examined in interactions among small group members and whole class or group and teacher. This lesson also addresses some common misconceptions relating to probability of simple and compound events.

Learning objectives

- Deriving and comparing experimental estimates with theoretical model probabilities for two-stage experiments (e.g., tossing two dice)
- Generating and comparing experimental probabilities from multiple samples
- Comparing experimental probability estimates with theoretical probabilities calculated from generating the sample space for the game and using theoretical probabilities to solve problems relating to "fair games"
- Conducting investigations using the PPDAC cycle from a probabilistic perspective

Suggested age range

With modifications, the game can be used with 11 to 14-year-olds.

Time required

Two 60-minute lessons may be required for the main part of the investigation.

Keywords

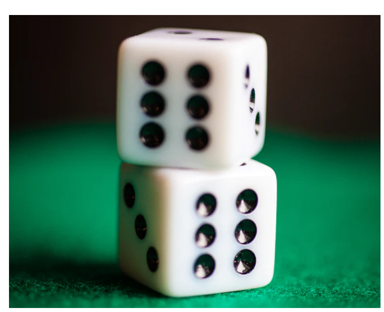
probability distributions, sample space, variation, law of large numbers, expected values

Introduction

My motivation for this lesson came from three sources. First of all, I teach mathematics education papers in both primary and secondary teacher education programmes. My student teachers are always keen to use games to teach mathematics. I decided to give them some more insight into using dice games to teach probability. I used the dice difference game which on the surface appears fair, but was unfair. Secondly, as part of one of the assignments, my students had to find and critique some activities. So to start this process, I modelled the dice difference game. At the end of the lesson, I asked my students to critique the game using the criteria for a rich mathematical task (see Breyfogle & Williams, 2008).

The students found that this activity fitted a range of criteria for a rich task. Thirdly, while there exists rich literature on students' misconceptions about probability, less attention has been paid to the development of students' probabilistic thinking in the classroom. Based on literature, I developed a teaching sequence for teaching probability (see Sharma, 2015). In particular, it demonstrates how a game context can be used to explore the bi-directional relationship between experimental and theoretical probabilities in a classroom setting. The approach integrates the content, processes and the language of probability and is grounded in socio-cultural theory. I decided to trial the lesson sequence with my junior secondary student teachers by adapting the sequence to suit their needs.

The game not only provides an interesting context for discussing probability concepts, such as distribution and variation but also requires students to think deeply about notions of sample space, representations of sample spaces, and experimental and theoretical probability. Teachers will be able to link to New Zealand Curriculum mathematical and statistical practices such as using tools to aid content exploration of experimental results converging towards theoretical results as the number of trials increases.



Lesson outline

1. Posing a problem

I engaged the students by posing the scenario below:

> Esha and Sarah decide to play a die rolling game. They take turns to roll two fair dice and calculate the difference (larger number minus smaller number) of the showing numbers. If the difference score is 0, 1, or 2, Esha wins, If the score is 3, 4 or 5, Sarah wins. Is this game fair? Explain your thinking.

I asked the students to read through the die rolling task and made sure they understood what was required. I had to clarify what “fair” and “difference” meant from probability and mathematical perspectives respectively. A **fair game** is a game in which there is an equal chance of winning or losing. **Difference** is larger number minus smaller number. We talked about how graphic organisers such as the one given below can help students compare and contrast the everyday meaning of “fair” with its probability meaning.

Probability meaning	Everyday meaning(s)
Example	Example

Next, I asked students to individually think about whether the game was fair and write down their prediction and explanation in their books. Students could use words and diagrams to explain their thinking. Making predictions is a good way to start the lesson as it can help find information about students' thinking, generate discussion and motivate students to want to explore the game. Next, in pairs, students discussed their ideas and tried to explain to each other their reasoning.

I circulated around observing how students made a start on the task, whether they were drawing diagrams, working with probabilities or simply writing a description. As they worked on the task, I listened to their reasoning carefully and noted misunderstandings that arose for later discussion with the whole class.

2. Playing the game and recording outcomes

The students played the dice difference game a few times in pairs. We discussed the rules of the game and the need to systematically record the data. In pairs, students discussed what data needed to be collected and designed recording sheets to keep track of results. Students played the game about 20 times with a partner, and tallied the results in a frequency table such as Table 1.

Table 1: Recording sheet of outcomes for dice difference game.

Outcome	Tally	Frequency
Player A wins		
Player B wins		

Writing down, explaining and evaluating their predictions and listening to others' predictions helped students to begin evaluating their own learning and constructing new meanings. One of my student teachers commented that this was a good strategy to use in other learning areas. Another student stated ”I think it is engaging, as the game element brings in the chance to see probability in real life, as well as providing a change from bookwork.”

The predictions element is also helped by the competition, as students want to get the right predictions when playing the game. In the probability context, “fair” means that each player has the same theoretical chance of winning a game. English as second language learners may face an even greater challenge when learning probability, for they must simultaneously learn and work with terms that have both ordinary and statistical meanings in which case the graphic organiser discussed above could be helpful.

Based on the data, students recorded their responses to the focus questions below and then discussed these with another group:

- *On the basis of your results, do you think the game is fair? Why, or why not?*
- *If you wanted to win this game, which player would you choose to be? Explain your answer.*
- *If you played the game 30 more times, would the results be the same as or different from the first game? If they would be different, how?*

You could introduce the idea of calculating the relative frequency at this point and compare relative frequencies of say Esha winning based on class results to get a feel for sample variation.

Playing the game helped my students get familiar with the rules of the game. Your students might think that the game is fair. After playing the game they may realise that it is not a fair game. Another concern is if your students do not believe that a six-sided die is fair, that is, has the same chance of coming up on each side. Hence discussion about the fairness of the dice difference game, which is at the next level of complexity, is likely to be challenging.

You may have to provide further activity such as using TinkerPlots™ to look at the variation in small and large trials of tossing a die. Students could record the variation and give prediction intervals, for example, (1-5) for the number of sixes say in 10 trials then for 100, 1000, 10000 trials. This will encourage students to examine their ideas about the likelihood of events occurring before embarking to the next phase of the lesson.

You may have to provide some phrases to help students write their responses, for example, from Table 1 it can be seen that .... because................. Students can also be encouraged to employ the "I expect, I notice, I wonder" strategy from their statistics learning.

3. Further data collection for dice differences

I posed the following questions and brainstormed responses:

- *Why does Esha win more often than Sarah?*
- *How can we determine if the game is fair by collecting more data?*
- *How can we record our results?*

The whole class shared and discussed the means by which they could collect more data to test their predictions. I got the students to suggest how they would record their group results. One possibility is given in Table 2.

Table 2: Group results recording sheet.

Outcome (Dice difference)	Frequency of occurrence (tally)	Relative frequency, or experimental probability
0		
1		
2		
3		
4		
5		

The students suggested collecting more data through either physical or computer simulations. We discussed the importance of both approaches. It is important that students first do a physical simulation. This is an appropriate moment to do that. I reminded the students of the need to record the number of times each score is rolled rather than just the number of times a player wins the game. My students also discussed how a die was rolled could affect the outcome (e.g., flipping die over, shaking in a container). We decided that it was important that the students roll the dice in the same manner in each trial.

4. Analysis of experimental data

In groups of three, data about the differences of two dice was collected and recorded. The data from a game played by three of my students in our class is shown in Table 3.

Table 3: Experimental Results for 30 trials.

Score	Frequency	Experimental probabilities
0	4	0.13
1	13	0.43
2	2	0.07
3	5	0.17
4	5	0.17
5	1	0.03

Next, group results were collated on the whiteboard. I put Table 4 on a chart for recording class data. Each group put their frequency data on the recording sheet and students analysed the pooled data for 90 trials.

Table 4: Class recording sheet.

Dice difference	Group 1	Group 2	Group 3	Total
0	10	4	2	16
1	6	13	10	29
2	4	2	8	14
3	9	5	4	18
4	1	5	4	10
5	0	1	2	3

Class results were compared with students' predictions and the pooled data led to the realization that Esha wins more often than Sarah. In groups, students answered the following questions:

- *What is the experimental probability of Esha winning?*

- *What is the experimental probability of Sarah winning?*
- *Is this game fair? Why?*
- *Draw a graph of the experimental probabilities for 90 trials. What patterns do you see in the graph?*
- *Why is the bar graph the best type of graph to use?*
- *Why is a histogram not appropriate here?*
- *How might the display look if we gathered more data?*
- *What do the experimental probabilities add up to?* Explain your answer.

I found recording the class results on a chart useful as I used the chart for the next session where we compared experimental and theoretical probabilities. Students could use grid paper or technology to produce graphs of data, which show the frequency of occurrences for each possible outcome. The graph in Figure 1 was created using Excel.

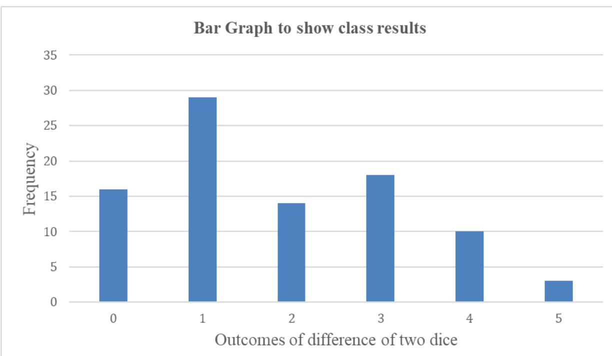


Figure 1: Graph display of the outcomes of the dice differences from a physical simulation of 90 trials.

5. Theoretical exploration

In groups, students analysed the game to determine theoretically why Esha wins more often than Sarah. They used their own methods for listing the possibilities. Students enumerated the sample space in systematic ways such as the grid method (see section), tree diagrams, listing possible outcomes, informal diagrams.

Once the students had completed their sample space diagrams, the diagrams were shared with the whole class and the responses to the focus question prompts were discussed:

- *Is the game fair? How do you know?*
- *Which player stands the best chance of winning?*
- *Can you explain what the theoretical probability is of players winning?*
- *Is there much difference between your experimental and theoretical results? Can you explain the difference?*
- *How do the experimental and theoretical probabilities compare with your original prediction?*
- *How could we make the game fair?*
- *What happens if we change the dice to ones with other faces, say 0-9 dice?*

My student teachers correctly listed 36 equally likely outcomes and noticed that the probabilities of obtaining the six scores (0–5) are not all the same. Having student teachers play the game first helped them see why this is the case. On reflection, I could have held a discussion comparing the advantages and limitations of these different ways of enumerating the sample space.

A tree diagram can be used to find the total possibilities although this could be a bit cumbersome at times. When listing outcomes, it is easy to miss outcomes without a diagram. A group of my students used probability trees to find all possible outcomes. I needed to point out to my students that this approach is used at level 7 of the curriculum and that it may be best to leave for later year levels or just use it with some students who need the challenge.

My students noticed discrepancies between the theoretical and the experimental probabilities, and attributed this to the smaller number of trials. This observation was a great opportunity for me to talk about the notion of law of large numbers according to which, for sufficiently larger sample sizes, the experimental probabilities will more closely reflect the theoretical estimates.

I also challenged the student teachers to design a fair game. This led to a lot of debate and explorations. A student commented: “I liked the way different people created their fair games.”

7. Discussion: What have we learned?

At the end of the class, I held a whole class discussion. This provided me with feedback on student learning. The following questions guided the discussion:

- *When you roll two dice and find their difference, what are the possible outcomes? Are these all equally likely? Explain your thinking.*
- *Would you use this game in your teaching?*

After playing and analysing the game, the students realised that the dice difference game was unfair. A student commented, “I will definitely use this game and other games for probability explorations because such rich activities engage and help maintain student interest and motivation and help teachers make connections between everyday life contexts and what happens in a mathematics classroom.” Another student said: “By changing the number of sides on the dice the task can be made more difficult or easy, to suit different year levels. Or by adding other conditions, such as player 1 only winning on a 1 or 2 as a difference.”

Adaptations

To increase the complexity of the lesson, students could be asked to explore a situation parallel to what they encountered in the lesson above. For example, how would they go about testing the fairness of “The Horse Race” game (see Sharma, 2015).

Another possibility is using the “Coin Toss” game. This is also a game for two players but has only four possible outcomes.

> Suppose you toss a coin twice. When you get heads, you score a 1 and when you get tails, you score 2. What are the possible outcomes for adding these results? Are these scores all equally likely? What are the possible results for finding the difference of two tosses? Are these differences all equally likely?

As an extension to the dice difference game, students could be asked to investigate dice games from diverse cultures. The probability lesson embedded in a cultural context can enable students to reflect on the connections between content (mathematics) and context (cultural) and as a result broaden their perceptions of mathematics.

To make the game easier, teachers could focus only on exploring experimental probability. Alternatively, when investigating theoretical probability students can be scaffolded, if necessary, with questions such as: *What numbers are on the first die? Can you write those possibilities across the top of the grid? What numbers are on the second die? Can you write those possibilities down the side of the grid? If you rolled a three with the first die and a five with the second, what is the difference in score? How many different outcomes are there? What is the probability of getting a score of zero etc.? How many of the outcomes belong to Sarah? What is the probability of Esha winning?*

Computer simulations can be used to demonstrate probability concepts, such as the shape of data, relative frequencies, model fit, and comparing simulated data with theoretical results. A simulation tool created by Anna Fergusson is freely available at learning.statistics-is-awesome.org/modelling-tool.
Figure 2 shows a screenshot of the tool and the outcome of one simulation for 100 values or trials or runs for the dice difference scenario.

Figure 2: Screenshot of dice difference simulation distribution for 100 values or trials.

Another computer simulation tool that can be used is TinkerPlotsTM. TinkerPlots was designed for middle school students and has the added advantage that students need to build the model in order to run simulations. Figure 3 shows a screenshot of the built model, which mimics and makes transparent the tossing of the two dice, the outcomes of the dice and the difference score, and the resultant model generated simulation distribution for 50 runs or trials.

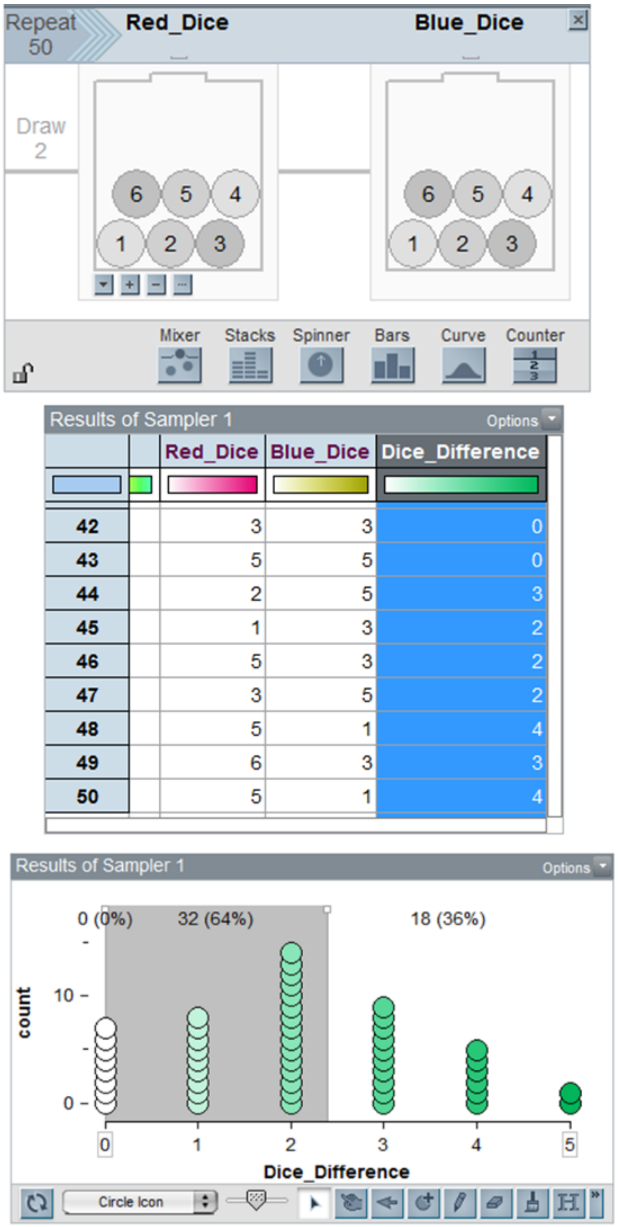


Figure 3: Screenshot of built model and dice difference simulation distribution for 50 trials in TinkerPlots. Dividers (shaded part in plot) show a difference of 0, 1 or 2 occurred for 32 of the 50 trials (64%).

Teacher notes

Multiple student experiences with chance and randomness at every level of the curriculum are critical to develop statistical thinking required for functional statistical literacy. This activity connects the subject areas of probability, mathematics and statistics.

Probabilities are dependent on the rules of the game. Combining simple events such as tossing two dice and writing the difference usually creates a much more complex sample space than the original event. A single fair die has equiprobable outcomes whereas for the difference of two fair dice the outcomes are not equally likely.

To address language challenges in multicultural settings, teachers may have to help students attend to the probability vocabulary in relation to this game: trial (one toss of the two dice), outcome (the way the dice landed), experimental data (scores of individual players), experimental probability (relative frequency of the collected data), relative frequency (the number of times an event occurred divided by the total number of trials).

The game is based on the difference in value of the two dice. Note that in this game six scores are possible (0–5). Each of the scores can be obtained in different ways. Therefore the theoretical probability of obtaining each of these scores will vary. For example, there are eight ways of getting a 2. Using the grid method, Table 5 shows all possible outcomes.

Table 5: Two-way table showing all possible differences.

		DICE 1					
		1	2	3	4	5	6
DICE 2	1	0	1	2	3	4	5
	2	1	0	1	2	3	4
	3	2	1	0	1	2	3
	4	3	2	1	0	1	2
	5	4	3	2	1	0	1
	6	5	4	3	2	1	0

The differences shown in Table 5 can be combined into a table to show the number of ways of obtaining each score. Table 6 shows that 24 of the 36 equally likely outcomes result in a win for Esha, and 12 result in a win for Sarah. If we play the game a large number of times, we can expect Esha to win twice as often as Sarah.

There are many different ways of devising a fair game. One is to say that Esha wins if the difference is 0, 2 or 4

Table 6: Number of ways of obtaining each scores.

Difference score	Number of ways of getting it	Who wins?
0	6	Esha
1	10	Esha
2	8	Esha
3	6	Sarah
4	4	Sarah
5	2	Sarah

and Sarah wins if it is 1, 3 or 5. Another is to say Esha wins if the score is 1 or 2, otherwise Sarah.

Computer simulations can be used to demonstrate what students are doing. The graphs below were created to show theoretical (Figure 4) and experimental probabilities of 1000 runs (Figure 5). Visual representations can easily show students that the difference game is not fair. Computer simulations can enable students to demonstrate both fair and unfair versions of games

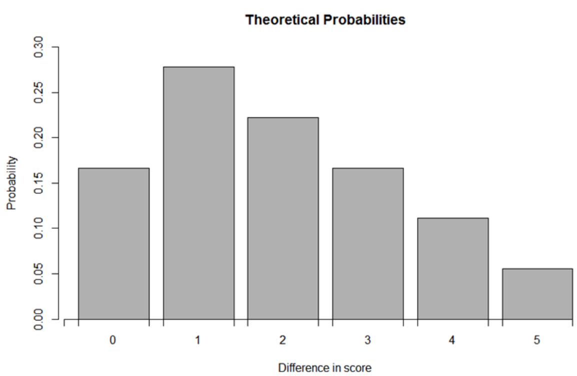


Figure 4: Theoretical or model probability distribution for dice differences.

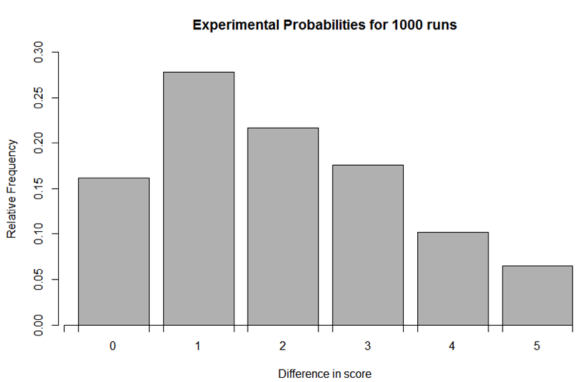


Figure 5: Model generated dice difference distribution for 1000 trials.

References

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Materials required

- Each pair of students needs two six-sided dice
- Recording sheet (see Table *)
- Whiteboard to record class results (see Table *)
- Data projector to show game instruction and plots
- Access to statistics software such as TinkerPlotsTM, CODAP or Excel for producing simulations and graphs.

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